



All Projection-Based Commuting Solutions of the Yang-Baxter-like Matrix Equation

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Abstract. We determine all the commuting solutions of the quadratic matrix equation $AXA = XAX$, which can be expressed in terms of projection matrices. All such solutions have special structures and their building blocks satisfy a system of matrix equations, and our general result extends the previous ones in *J. Math. Anal. Appl.* 402 (2013), 567-573, *Applied Math. Lett.* 35 (2014), 86-89, *Applied Math. Lett.* 64 (2017), 231-234, and *Applied Math. Lett.* 79 (2018), 155-161.

1. Introduction

Let A be an $n \times n$ complex matrix. We consider the following problem: Find all solutions of the form $X = AP$ to the Yang-Baxter-like matrix equation

$$AXA = XAX, \tag{1}$$

where P is a projection matrix such that $AP = PA$. These solutions are called *projection-based commuting solutions* or simply *projection-based solutions*.

The Yang-Baxter-like matrix equation (1) is a matrix equation analog of the classic Yang-Baxter equation [1, 15] in format, and it was first studied in [3] via Brouwer's fixed point theorem. Since then the nonlinear matrix equation has been investigated for various classes of matrix A in the literature (see, e.g., [4, 5, 7, 13] and the references therein). The Yang-Baxter equation has been a hot research topic in the past decades because of its relation and applications to knot theory, quantum groups, and statistical mechanics [10, 16], and the solutions of (1) have also found applications in those areas.

The projection-based solutions were first studied by [4], in which a so-called *spectral solution* of (1) was constructed for each eigenvalue of A , based on the spectral projection [11] that maps all C^n onto the generalized eigenspace associated with the eigenvalue along a special complementary subspace of the generalized eigenspace. The matrix representation of the above spectral projection associated with eigenvalue λ of A is $P_\lambda = U_\lambda V_\lambda^H$, where $V_\lambda^H U_\lambda = I$ with the columns of U_λ and V_λ a basis of $N((A - \lambda I)^{\nu(\lambda)})$ and $N((A^H - \bar{\lambda}I)^{\nu(\lambda)})$, respectively. Here $\nu(\lambda)$ is the index of λ [11].

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The basic idea behind finding the spectral solutions in [4] is a simple fact that if P is a projection matrix satisfying the commutativity equality $AP = PA$, then the matrix AP is a commuting solution of (1). The technique of [4] was generalized in [6] to find more projection-based solutions when the eigenvalue λ of A is semi-simple, that is, the corresponding generalized eigenspace is the same as the eigenspace, which means that $v(\lambda) = 1$. More specifically, it was proved that if the dimension of the eigenspace is m , then for any $m \times m$ projection matrix M , the matrix $AU_\lambda M V_\lambda^H$ is a projection-based commuting solution of (1). The trick of using an arbitrary projection matrix of suitable size to construct more projection-based solutions from [6] was followed by [7], extending the result of [6] to the case when the Jordan form of A has several 1×1 Jordan blocks associated with eigenvalue λ . Recently, the authors of [18] used the same approach of [6] combined with the Jordan form decomposition of A , resulting in a unified expression of such special projections that contain the previous ones under various sufficient conditions.

However, most projection-based commuting solutions still cannot be obtained via the results of [4, 6, 7, 18]. As a simple example, let A be a 2×2 diagonal matrix with its diagonal entries 1 and 2, so its Jordan form is itself. Then the previous methods will only give two trivial projection-based solutions, that is, the zero matrix and matrix A itself. On the other hand, a simple computation finds all the nontrivial projection-based solutions

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

A better example, which is 3×3 in size, will be demonstrated in the next section and show how limited the previous methods are.

It is worthwhile to point out that the problem of finding all commuting solutions (1) has been explored in [9, 12]. The important paper [12] gave an extensive investigation and a detailed analysis of solving the system $AXA = XAX$ and $AX = XA$, and proposed a numerical scheme for solving all the solutions of the above system for a general matrix A . An equivalent problem was obtained and the structure of the commuting solutions was found in terms of some special Toeplitz matrices. The key point of this method is to construct various intermediate matrices related to the commuting solutions, which transform the complicated problem into simpler matrix equations. All commuting solutions can be obtained step by step by recursively solving such matrix equations. On the other hand, following the idea of [8] for all commuting solutions of (1) when A is a nilpotent matrix, the paper [9] gave an independent approach for finding all the commuting solutions of (1) and obtained another equivalent system of matrix equations. But in the above papers no attempt was given to find those commuting solutions that are projection-based ones. It seems difficult to find all the projection-based commuting solutions from their methods for all general commuting solutions. Therefore, a further exploration is in need to find such desired solution through a careful structural analysis.

So far to our knowledge, not all projection-based solutions of the Yang-Baxter-like matrix equation have been found in the general case. The purpose of this paper is to determine all projection-based commuting solutions of (1) for an arbitrary given matrix A .

In the next section we introduce some concepts and present an inspiring and motivating simple example of size 3×3 . The general construction of all required solutions and the main theorem will be given in Section 3. We illustrate our main result in Section 4 by analyzing a particular case, and we conclude in Section 5.

2. Preliminaries

To construct all projection-based commuting solutions of (1), as pointed out below, we need to study the following closely related problem: Determine all projection matrices that commute with a given matrix with only one eigenvalue.

Projection matrices are complex matrices whose squares equal themselves. An $n \times n$ projection matrix P projects every n -dimensional complex vector to the range of P along the null space of P . If a projection matrix commutes with A , that is, $AP = PA$, then

$$A(AP)A = A^3P = (AP)A(AP),$$

so the matrix AP solves (1). This is the basic idea that gave different sufficient conditions in [4, 6, 7, 18] for a projection matrix to commute with A , resulting in the construction of various projection-based commuting solutions in those papers.

Let $\lambda_1, \dots, \lambda_d$ be all the distinct eigenvalues of A , and let

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_d \end{bmatrix}$$

be the Jordan canonical form of A such that for $i = 1, \dots, d$, the i th diagonal block J_i of J is itself a block diagonal matrix consisting of all the Jordan blocks associated with eigenvalue λ_i . The $j \times j$ Jordan block $J_j(\lambda)$ associated with eigenvalue λ is

$$J_j(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & \lambda & 1 \\ 0 & 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}.$$

Suppose that U is the nonsingular matrix such that $A = UJU^{-1}$. Let two matrices X and Y be related by $X = UYU^{-1}$. Then solving (1) is equivalent to solving the “simplified” Yang-Baxter-like matrix equation

$$JYJ = YJY, \tag{2}$$

in the sense that X is a solution of (1) if and only if $Y = U^{-1}XU$ is a solution of (2). In addition, X is a commuting solution of (1) if and only if Y is a commuting solution of (2). Furthermore, X is a projection-based commuting solution of (1) if and only if Y is a projection-based commuting solution of (2). The last assertion can be seen as follows: Suppose that $X = AP = PA$ solves (1) for a projection matrix P . Then $Y \equiv U^{-1}XU = U^{-1}APU = JU^{-1}PU$. Denote $\hat{P} = U^{-1}PU$. Then $\hat{P}^2 = \hat{P}$ and $J\hat{P} = JU^{-1}PU = U^{-1}APU = U^{-1}PAU = U^{-1}PUJ = \hat{P}J$. The truth of the converse can also be proved in the same way. Therefore we can focus on solving (2) for its projection-based commuting solutions Y and then get projection-based commuting solutions $X = UYU^{-1}$ of the original matrix equation (1).

Since the eigenvalue of each block J_i of J is different from that of another block J_j , from a well known result of matrix theory [2, 17], any matrix that commutes with J must be block diagonal with the same size of the diagonal blocks (see also Lemma 2.3 of [5]). In other words, if Y is a commuting solution of (2), then $Y = \text{diag}(Y_1, \dots, Y_d)$ in which Y_i has the same size as J_i for all i . Thus, finding all projection-based commuting solutions of (2) is equivalent to finding all projection-based commuting solutions of a system of d mutually independent Yang-Baxter-like matrix equations:

$$J_i Y_i J_i = Y_i J_i Y_i, \quad i = 1, \dots, d.$$

Therefore, we just need to solve (2) with J being an $n \times n$ Jordan canonical form associated with only one eigenvalue, which will be denoted as λ .

If P is a projection matrix such that $JP = PJ$, then $Y = JP$ is a projection-based commuting solution of (2). We want to find all projection matrices that commute with J . Before we solve this challenging problem in the next section, we give a simple example to illustrate the difficulty for dealing with the general case.

Let

$$J = \text{diag}(1, J_2(1)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = I_3 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

where I_3 is the 3×3 identity matrix. Denote by N the last matrix in the above, which is a nilpotent matrix of order 2. Let $P = [p_{ij}]$ be a 3×3 projection matrix. Then $JP = PJ$ if and only if $NP = PN$. Now

$$NP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ p_{31} & p_{32} & p_{33} \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$PN = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & p_{12} \\ 0 & 0 & p_{22} \\ 0 & 0 & p_{32} \end{bmatrix}.$$

So $NP = PN$ if and only if $p_{12} = 0, p_{31} = 0, p_{32} = 0$, and $p_{22} = p_{33}$. Thus the matrices P that commute with J are

$$P = \begin{bmatrix} p_{11} & 0 & p_{13} \\ p_{21} & p_{22} & p_{23} \\ 0 & 0 & p_{22} \end{bmatrix}.$$

Since P is a projection matrix, the additional condition $P^2 = P$ is equivalent to the system

$$\begin{cases} p_{11}^2 = p_{11}, \\ p_{22}^2 = p_{22}, \\ p_{21}(p_{11} + p_{22}) = p_{21}, \\ p_{13}(p_{11} + p_{22}) = p_{13}, \\ p_{21}p_{13} + 2p_{22}p_{23} = p_{23}. \end{cases} \tag{3}$$

The first two equations of (3) give four solutions for p_{11} and p_{22} : (i) $p_{11} = 0$ and $p_{22} = 0$; (ii) $p_{11} = 0$ and $p_{22} = 1$; (iii) $p_{11} = 1$ and $p_{22} = 0$; (iv) $p_{11} = 1$ and $p_{22} = 1$. Case (i) implies that $P = 0$ and case (iv) gives that $P = I$, which provides the two trivial solutions $Y = 0$ and $Y = J$ of (2). The system (3) for case (ii) is reduced to one equation $p_{21}p_{13} + p_{23} = 0$, so all projection matrices P with $p_{11} = 0$ and $p_{22} = 1$ such that $PJ = JP$ are

$$P = \begin{bmatrix} 0 & 0 & p_{13} \\ p_{21} & 1 & -p_{21}p_{13} \\ 0 & 0 & 1 \end{bmatrix}, \quad \forall p_{21}, p_{13}.$$

Similarly, In case (iii), (3) becomes $p_{21}p_{13} - p_{23} = 0$, so all projection matrices P with $p_{11} = 1$ and $p_{22} = 0$ such that $PJ = JP$ are

$$P = \begin{bmatrix} 1 & 0 & p_{13} \\ p_{21} & 0 & p_{21}p_{13} \\ 0 & 0 & 0 \end{bmatrix}, \quad \forall p_{21}, p_{13}.$$

Hence, we have found all the projection-based commuting solutions for this particular example, namely ($a = p_{13}$ and $b = p_{21}$)

$$\begin{bmatrix} 0 & 0 & a \\ b & 1 & 1-ab \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & a \\ b & 0 & ab \\ 0 & 0 & 0 \end{bmatrix}, \quad \forall a, b.$$

which form a two dimensional nonlinear manifold. Note that the past works on projection-based commuting solutions could only find the two trivial solutions.

However, when J has a general structure, the above direct method will lead to complicated systems of scalar equations, so we shall develop a unified approach below.

The following result will be used in the next section, whose proof is referred to the book [2] (Theorem 5.15).

Lemma 2.1. *Let t and s be two positive integers. Then all the solutions of the equation $J_t(0)Z = ZJ_s(0)$ are*

$$Z = [0 \ T] \text{ if } t \leq s \text{ or } Z = \begin{bmatrix} T \\ 0 \end{bmatrix} \text{ if } t > s,$$

where T is an upper triangular Toeplitz matrix of the form

$$T = \begin{bmatrix} t_1 & t_2 & \cdots & \cdots & t_{u-1} & t_u \\ 0 & t_1 & t_2 & \cdots & \cdots & t_{u-1} \\ 0 & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & t_2 \\ 0 & 0 & \cdots & \cdots & 0 & t_1 \end{bmatrix}, \quad u = \min\{t, s\}$$

in which t_1, \dots, t_u are arbitrary complex numbers.

3. Projection-Based Commuting Solutions

Now we solve (2) to obtain all projection-based commuting solutions. As pointed out in the previous section, we can assume that the matrix A is a Jordan canonical form J corresponding to only one eigenvalue λ , and without loss of generality we can assume that its Jordan blocks are arranged in the order of increasing size, so

$$J = \begin{bmatrix} \lambda I_m & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix}, \quad J_j = \begin{bmatrix} J_j(\lambda) & & & \\ & J_j(\lambda) & & \\ & & \ddots & \\ & & & J_j(\lambda) \end{bmatrix}, \quad j = 2, \dots, k,$$

in which the $j \times j$ Jordan block $J_j(\lambda)$ appears r_j times in J_j . We note that the largest size k of the Jordan blocks is exactly the index $\nu(\lambda)$ of eigenvalue λ , which is the smallest positive integer p such that $N((A - \lambda I)^p) = N((A - \lambda I)^{p+1})$ [11].

Before we deal with the general situation, we first give an answer to the simplest case.

Proposition 3.1. *For each $j = 2, \dots, \nu(\lambda)$, the only projection-based commuting solutions of the equation*

$$J_j(\lambda)Y_jJ_j(\lambda) = Y_jJ_j(\lambda)Y_j$$

are $Y_j = 0$ and $Y_j = J_j(\lambda)$.

Proof Let P be a projection matrix that commutes with $J_j(\lambda)$. Since $J_j(\lambda)$ is an upper triangular Toeplitz matrix, by a simple fact in linear algebra [2, 14], P is also an upper triangular Toeplitz matrix, so its common diagonal element is the only eigenvalue of P . But P has only one eigenvalue if and only if it is either the zero matrix or the identity matrix. \square

As mentioned in the previous section, all the projection-based commuting solutions of (2) are JP in which the projection matrices P commute with J . Since $J = \lambda I + N$, where the nilpotent matrix N of order k is

$$N = \begin{bmatrix} 0_m & & & \\ & N_2 & & \\ & & \ddots & \\ & & & N_k \end{bmatrix}, \quad N_j = \begin{bmatrix} J_j(0) & & & \\ & J_j(0) & & \\ & & \ddots & \\ & & & J_j(0) \end{bmatrix}, \quad j = 2, \dots, k,$$

we see that $JP = PJ$ if and only if $NP = PN$.

Our first task is to find all matrices P that commute with N . By partitioning P as

$$P = \begin{bmatrix} P_{11} & \cdots & P_{1k} \\ \vdots & \vdots & \vdots \\ P_{k1} & \cdots & P_{kk} \end{bmatrix} \tag{4}$$

that matches the block matrix structure of N , we see that

$$NP = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ N_2P_{21} & N_2P_{22} & \cdots & N_2P_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ N_kP_{k1} & N_kP_{k2} & \cdots & N_kP_{kk} \end{bmatrix}, \quad PN = \begin{bmatrix} 0 & P_{12}N_2 & \cdots & P_{1k}N_k \\ 0 & P_{22}N_2 & \cdots & P_{2k}N_k \\ \vdots & \vdots & \vdots & \vdots \\ 0 & P_{k2}N_2 & \cdots & P_{kk}N_k \end{bmatrix}.$$

So $NP = PN$ is equivalent to the system

$$\begin{cases} N_2P_{21} = 0, & \dots, & N_kP_{k1} = 0, \\ P_{12}N_2 = 0, & \dots, & P_{1k}N_k = 0, \\ N_2P_{22} = P_{22}N_2, & \dots, & N_2P_{2k} = P_{2k}N_k, \\ \vdots & \vdots & \vdots \\ N_kP_{k2} = P_{k2}N_2, & \dots, & N_kP_{kk} = P_{kk}N_k. \end{cases} \tag{5}$$

Note that P_{11} does not appear in the above system, so it can be any $m \times m$ matrix.

To solve the first line equations of (5), $N_tP_{t1} = 0$ for $t = 2, \dots, k$, we partition P_{t1} into an $r_t \times 1$ block matrix with blocks $P_{t1}^{(1)}, \dots, P_{t1}^{(r_t)}$ of the same size $t \times m$, so the equations are

$$N_tP_{t1} = \begin{bmatrix} J_t(0) & & \\ & \ddots & \\ & & J_t(0) \end{bmatrix} \begin{bmatrix} P_{t1}^{(1)} \\ \vdots \\ P_{t1}^{(r_t)} \end{bmatrix} = \begin{bmatrix} J_t(0)P_{t1}^{(1)} \\ \vdots \\ J_t(0)P_{t1}^{(r_t)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Denote $P_{t1}^{(s)} = [p_{ij}^{(t,s)}]$ for $s = 1, \dots, r_t$. Then the s th equation in the above becomes

$$\begin{aligned} J_t(0)P_{t1}^{(s)} &= \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \begin{bmatrix} p_{11}^{(t,s)} & \cdots & \cdots & p_{1m}^{(t,s)} \\ p_{21}^{(t,s)} & \cdots & \cdots & p_{2m}^{(t,s)} \\ \vdots & \cdots & \cdots & \vdots \\ p_{t1}^{(t,s)} & \cdots & \cdots & p_{tm}^{(t,s)} \end{bmatrix} \\ &= \begin{bmatrix} p_{21}^{(t,s)} & \cdots & \cdots & p_{2m}^{(t,s)} \\ \vdots & \cdots & \cdots & \vdots \\ p_{t1}^{(t,s)} & \cdots & \cdots & p_{tm}^{(t,s)} \\ 0 & \cdots & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}, \end{aligned}$$

so its solution is

$$P_{t1}^{(s)} = \begin{bmatrix} p_{11}^{(t,s)} & \cdots & \cdots & p_{1m}^{(t,s)} \\ 0 & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad \forall p_{11}^{(t,s)}, \dots, p_{1m}^{(t,s)}; \quad s = 1, \dots, r_t. \tag{6}$$

Similarly, the solutions of the second line equations of (5), $P_{1t}N_t = 0$ for $t = 2, \dots, k$ are $P_{1t} = [P_{1t}^{(1)} \cdots P_{1t}^{(r_t)}]$, where the $m \times t$ blocks

$$P_{1t}^{(s)} = \begin{bmatrix} 0 & \cdots & 0 & q_{1t}^{(t,s)} \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & q_{mt}^{(t,s)} \end{bmatrix}, \quad \forall q_{1t}^{(t,s)}, \dots, q_{mt}^{(t,s)}; \quad s = 1, \dots, r_t. \tag{7}$$

Now we solve the remaining equations (5), $N_iP_{ij} = P_{ij}N_j$ for $i, j = 2, \dots, k$. For this purpose, we partition P_{ij} into an $r_i \times r_j$ block matrix

$$P_{ij} = \begin{bmatrix} P_{11}^{(i,j)} & \cdots & P_{1r_j}^{(i,j)} \\ \vdots & \cdots & \vdots \\ P_{r_i1}^{(i,j)} & \cdots & P_{r_i r_j}^{(i,j)} \end{bmatrix}$$

that matches the block structure of N_i and N_j . Each block $P_{uv}^{(i,j)}$ is $i \times j$ for $u = 1, \dots, r_i$ and $v = 1, \dots, r_j$. Then $N_iP_{ij} = P_{ij}N_j$ if and only if

$$J_i(0)P_{uv}^{(i,j)} = P_{uv}^{(i,j)}J_j(0), \quad \forall u = 1, \dots, r_i, v = 1, \dots, r_j.$$

By Lemma 2.1, the solutions of the above equations are

$$P_{uv}^{(i,j)} = \begin{bmatrix} p_{uv1}^{(i,j)} & p_{uv2}^{(i,j)} & \cdots & p_{uvj}^{(i,j)} \\ 0 & p_{uv1}^{(i,j)} & \ddots & \vdots \\ \vdots & 0 & \ddots & p_{uv2}^{(i,j)} \\ 0 & 0 & \cdots & p_{uv1}^{(i,j)} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \forall p_{uv1}^{(i,j)}, \dots, p_{uvj}^{(i,j)} \tag{8}$$

if $i > j$ or

$$P_{uv}^{(i,j)} = \begin{bmatrix} 0 & \cdots & 0 & p_{uv1}^{(i,j)} & p_{uv2}^{(i,j)} & \cdots & p_{uvi}^{(i,j)} \\ \vdots & & \vdots & 0 & p_{uv1}^{(i,j)} & \ddots & \vdots \\ \vdots & & \vdots & 0 & 0 & \ddots & p_{uv2}^{(i,j)} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & p_{uv1}^{(i,j)} \end{bmatrix}, \quad \forall p_{uv1}^{(i,j)}, \dots, p_{uvi}^{(i,j)} \tag{9}$$

if $i \leq j$. This completes the task of finding all the solutions of $JP = PJ$.

Under the commuting condition satisfied by P so that its structure is given by (6), (7), (8), and (9), we further require that P be a projection matrix. This means that we need to solve the additional equation $P^2 = P$, which can be written as

$$(P_{ii} - I_{n_i})P_{ij} + \sum_{s=1, s \neq i}^k P_{is}P_{sj} = 0, \quad \forall i, j = 1, \dots, k \tag{10}$$

from the partition (4) of P . Here $n_1 = m$ and $n_i = ir_i$ for $i = 2, \dots, k$.

The remaining thing is to solve the above system of $k \times k$ equations with the blocks of P given by (6), (7), (8), and (9). We need several lemmas to simplify (10).

Lemma 3.2. For $s = 2, \dots, k$, we have $P_{1s}P_{s1} = 0$.

Proof Suppose $2 \leq s \leq k$. By (6) and (7),

$$P_{1s}P_{s1} = \sum_{u=1}^{r_s} P_{1s}^{(u)}P_{s1}^{(u)} = \sum_{u=1}^{r_s} \begin{bmatrix} 0 & \cdots & 0 & q_{1s}^{(s,u)} \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & q_{ms}^{(s,u)} \end{bmatrix} \begin{bmatrix} p_{11}^{(s,u)} & \cdots & \cdots & p_{1m}^{(s,u)} \\ 0 & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} = 0. \quad \square$$

Lemma 3.2 reduces the first equation of (10) to $(P_{11} - I_m)P_{11} = 0$, so P_{11} is any $m \times m$ projection matrix.

Lemma 3.3. *If $s > j \geq 2$, then $P_{1s}P_{sj} = 0$.*

Proof By (7) and (8), the v th block of the $1 \times r_j$ block matrix of $P_{1s}P_{sj}$ is

$$\sum_{u=1}^{r_s} P_{1s}^{(u)}P_{uv}^{(s,j)} = \sum_{u=1}^{r_s} \begin{bmatrix} 0 & \cdots & 0 & q_{1s}^{(s,u)} \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & q_{ms}^{(s,u)} \end{bmatrix} \begin{bmatrix} p_{uv1}^{(s,j)} & p_{uv2}^{(s,j)} & \cdots & p_{uvj}^{(s,j)} \\ 0 & p_{uv1}^{(s,j)} & \ddots & \vdots \\ \vdots & 0 & \ddots & p_{uv2}^{(s,j)} \\ 0 & 0 & \cdots & p_{uvj}^{(s,j)} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0$$

for $v = 1, \dots, r_j$, so $P_{1s}P_{sj} = 0$. \square

Lemma 3.3 simplifies the equations of (10) with $i = 1$ and $j = 2, \dots, k$ to

$$(P_{11} - I_m)P_{1j} + \sum_{s=2}^j P_{1s}P_{sj} = 0.$$

Lemma 3.4. *If $s > i \geq 2$, then $P_{is}P_{s1} = 0$.*

Proof From (6) and (9), the v th block of the $r_i \times 1$ block matrix of $P_{is}P_{s1}$ is

$$\sum_{u=1}^{r_s} P_{vu}^{(i,s)}P_{s1}^{(u)} = \sum_{u=1}^{r_s} \begin{bmatrix} 0 & \cdots & 0 & p_{vu1}^{(i,s)} & p_{vu2}^{(i,s)} & \cdots & p_{vui}^{(i,s)} \\ \vdots & & \vdots & 0 & p_{vu1}^{(i,s)} & \ddots & \vdots \\ \vdots & & \vdots & 0 & 0 & \ddots & p_{vu2}^{(i,s)} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & p_{vu1}^{(i,s)} \end{bmatrix} \begin{bmatrix} p_{11}^{(s,u)} & \cdots & \cdots & p_{1m}^{(s,u)} \\ 0 & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} = 0$$

for $v = 1, \dots, r_i$, so $P_{is}P_{s1} = 0$. \square

Lemma 3.4 reduces the equations of (10) with $j = 1$ and $i = 2, \dots, k$ to

$$(P_{ii} - I_{n_i})P_{i1} + \sum_{s=1}^{i-1} P_{is}P_{s1} = 0.$$

Lemma 3.5. *If $s \geq i + j$ with $i, j = 2, \dots, k$, then $P_{is}P_{sj} = 0$.*

Proof From (8) and (9), the (u, v) entry of the $r_i \times r_j$ block matrix of $P_{is}P_{sj}$ is

$$\sum_{w=1}^{r_s} P_{uw}^{(i,s)} P_{wv}^{(s,j)} = \sum_{w=1}^{r_s} \begin{bmatrix} 0 & \cdots & 0 & p_{uw1}^{(i,s)} & p_{uw2}^{(i,s)} & \cdots & p_{uwi}^{(i,s)} \\ \vdots & & \vdots & 0 & p_{uw1}^{(i,s)} & \ddots & \vdots \\ \vdots & & \vdots & 0 & 0 & \ddots & p_{uw2}^{(i,s)} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & p_{uw1}^{(i,s)} \end{bmatrix} \begin{bmatrix} p_{wv1}^{(s,j)} & p_{wv2}^{(s,j)} & \cdots & p_{wvj}^{(s,j)} \\ 0 & p_{wv1}^{(s,j)} & \ddots & \vdots \\ \vdots & 0 & \ddots & p_{wv2}^{(s,j)} \\ 0 & 0 & \cdots & p_{wv1}^{(s,j)} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0$$

for $u = 1, \dots, r_i$ and $v = 1, \dots, r_j$, so $P_{is}P_{sj} = 0$. \square

By Lemma 3.5, the equations of (10) with $i, j = 2, \dots, k$ are

$$(P_{ii} - I_{n_i})P_{ij} + \sum_{s=1, s \neq i}^{\min\{i+j-1, k\}} P_{is}P_{sj} = 0.$$

In summary, we have the following theorem.

Theorem 3.6. Let J be a Jordan matrix given by (4) with eigenvalue λ . Then all the projection-based commuting solutions of the Yang-Baxter-like matrix equation $JYJ = YJY$ are $Y = JP$, where P is partitioned as (4) the blocks of which have the structure given by (6), (7), (8), (9) and solve the following equations

$$\begin{cases} P_{11}^2 = P_{11}, \\ (P_{11} - I_m)P_{1j} + \sum_{s=2}^j P_{1s}P_{sj} = 0, \quad j = 2, \dots, k, \\ (P_{ii} - I_{n_i})P_{i1} + \sum_{s=1}^{i-1} P_{is}P_{s1} = 0, \quad i = 2, \dots, k, \\ (P_{ii} - I_{n_i})P_{ij} + \sum_{s=1, s \neq i}^{\min\{i+j-1, k\}} P_{is}P_{sj} = 0, \quad i, j = 2, \dots, k. \end{cases} \tag{11}$$

4. The Case of $k = 2$

When $k = 2$, the system (11) becomes

$$\begin{cases} P_{11}^2 = P_{11}, \\ (P_{11} - I_m)P_{12} + P_{12}P_{22} = 0, \\ (P_{22} - I_{n_2})P_{21} + P_{21}P_{11} = 0, \\ (P_{22} - I_{n_2})P_{22} + P_{21}P_{12} = 0. \end{cases} \tag{12}$$

The first equation indicates that P_{11} is a projection matrix. So, after choosing it, we solve for P_{12}, P_{21} , and P_{22} from the remaining equations as follows. For simplicity we let $r = r_2$,

$$P_{21} = \begin{bmatrix} P_1 \\ \vdots \\ P_r \end{bmatrix}, P_s = \begin{bmatrix} p_1^{(s)} & \cdots & p_m^{(s)} \\ 0 & \cdots & 0 \end{bmatrix}, P_{12} = [Q_1 \cdots Q_r], Q_s = \begin{bmatrix} 0 & q_1^{(s)} \\ \vdots & \vdots \\ 0 & q_m^{(s)} \end{bmatrix}, s = 1, \dots, r,$$

$$P_{22} = \begin{bmatrix} R_{11} & \cdots & R_{1r} \\ \vdots & \vdots & \vdots \\ R_{r1} & \cdots & R_{rr} \end{bmatrix}, R_{ij} = \begin{bmatrix} a_{ij} & b_{ij} \\ 0 & a_{ij} \end{bmatrix}, i, j = 1, \dots, r.$$

Then

$$P_{12}P_{22} = \left[\sum_{s=1}^r Q_s R_{s1} \quad \sum_{s=1}^r Q_s R_{s2} \quad \cdots \quad \sum_{s=1}^r Q_s R_{sr} \right],$$

in which

$$\sum_{s=1}^r Q_s R_{sj} = \begin{bmatrix} 0 & \sum_{s=1}^r q_1^{(s)} a_{sj} \\ \vdots & \vdots \\ 0 & \sum_{s=1}^r q_m^{(s)} a_{sj} \end{bmatrix}, \quad j = 1, \dots, r.$$

Similarly,

$$P_{22}P_{21} = \begin{bmatrix} \sum_{s=1}^r R_{1s} P_s \\ \vdots \\ \sum_{s=1}^r R_{rs} P_s \end{bmatrix}, \quad \sum_{s=1}^r R_{is} P_s = \begin{bmatrix} \sum_{s=1}^r a_{is} p_1^{(s)} & \cdots & \sum_{s=1}^r a_{is} p_m^{(s)} \\ 0 & \cdots & 0 \end{bmatrix}, \quad i = 1, \dots, r,$$

$$P_{21}P_{12} = \begin{bmatrix} P_1 Q_1 & \cdots & P_1 Q_r \\ \vdots & \vdots & \vdots \\ P_r Q_1 & \cdots & P_r Q_r \end{bmatrix}, \quad P_i Q_j = \begin{bmatrix} 0 & \sum_{s=1}^m p_s^{(i)} q_s^{(j)} \\ 0 & 0 \end{bmatrix}, \quad i, j = 1, \dots, r,$$

and

$$P_{22}^2 = \begin{bmatrix} \sum_{s=1}^r R_{1s} R_{s1} & \cdots & \sum_{s=1}^r R_{1s} R_{sr} \\ \vdots & \vdots & \vdots \\ \sum_{s=1}^r R_{rs} R_{s1} & \cdots & \sum_{s=1}^r R_{rs} R_{sr} \end{bmatrix},$$

where

$$\sum_{s=1}^r R_{is} R_{sj} = \begin{bmatrix} \sum_{s=1}^r a_{is} a_{sj} & \sum_{s=1}^r (a_{is} b_{sj} + b_{is} a_{sj}) \\ 0 & \sum_{s=1}^r a_{is} a_{sj} \end{bmatrix}, \quad i, j = 1, \dots, r.$$

Let the $m \times m$ matrix projection matrix $P_{11} = [p_{ij}]$. Denote the $m \times r$ matrix $Q = [q_i^{(j)}]$, the $r \times r$ matrices $T = [a_{ij}]$ and $S = [b_{ij}]$, the $r \times m$ matrix $W = [p_j^{(i)}]$, and let $Z = [z_{ij}] = QT$, $C = [c_{ij}] = TW$, $D = [d_{ij}] = WQ$, $E = [e_{ij}] = T^2$, $F = [f_{ij}] = TS + ST$, $G = WP_{11}$, and $H = P_{11}Q$. Then $P_{21}P_{12} = [d_{ij}]_2(0)$, $P_{22}^2 = [e_{ij}]_2 + [f_{ij}]_2(0)$, and

$$P_{12}P_{22} = \begin{bmatrix} 0 & z_{11} & \cdots & 0 & z_{1r} \\ 0 & z_{21} & \cdots & 0 & z_{2r} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & z_{m1} & \cdots & 0 & z_{mr} \end{bmatrix}, \quad P_{22}P_{21} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rm} \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

It follows that $(P_{22} - I_{n_2})P_{22} = [(e_{ij} - a_{ij})I_2 + (f_{ij} - b_{ij})]_2(0)$,

$$(P_{22} - I_{n_2})P_{21} = \begin{bmatrix} c_{11} - p_1^{(1)} & c_{12} - p_2^{(1)} & \cdots & c_{1m} - p_m^{(1)} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ c_{r1} - p_1^{(r)} & c_{r2} - p_2^{(r)} & \cdots & c_{rm} - p_m^{(r)} \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$P_{11}P_{12} = \begin{bmatrix} 0 & h_{11} & \cdots & 0 & h_{1r} \\ 0 & h_{21} & \cdots & 0 & h_{2r} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & h_{m1} & \cdots & 0 & h_{mr} \end{bmatrix}, \quad P_{21}P_{11} = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1m} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ g_{r1} & g_{r2} & \cdots & g_{rm} \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

So the last three equations of (12) are respectively

$$(P_{11} - I_m)P_{12} + P_{12}P_{22} = \begin{bmatrix} 0 & z_{11} + h_{11} - q_1^{(1)} & \cdots & 0 & z_{1r} + h_{1r} - q_1^{(r)} \\ 0 & z_{21} + h_{21} - q_2^{(1)} & \cdots & 0 & z_{2r} + h_{2r} - q_2^{(r)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & z_{m1} + h_{m1} - q_m^{(1)} & \cdots & 0 & z_{mr} + h_{mr} - q_m^{(r)} \end{bmatrix} = 0,$$

$$(P_{22} - I_{n_2})P_{21} + P_{21}P_{11} = \begin{bmatrix} c_{11} - p_1^{(1)} + g_{11} & c_{12} - p_2^{(1)} + g_{12} & \cdots & c_{1m} - p_m^{(1)} + g_{1m} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ c_{r1} - p_1^{(r)} + g_{r1} & c_{r2} - p_2^{(r)} + g_{r2} & \cdots & c_{rm} - p_m^{(r)} + g_{rm} \\ 0 & 0 & \cdots & 0 \end{bmatrix} = 0,$$

and $(P_{22} - I_{n_2})P_{22} + P_{21}P_{12} = [(e_{ij} - a_{ij})I_2 + (d_{ij} + f_{ij} - b_{ij})J_2(0)] = 0$.

The above equations can be written as

$$\begin{cases} z_{ij} + h_{ij} = q_i^{(j)}, & i = 1, \dots, m, j = 1, \dots, r, \\ c_{ij} + g_{ij} = p_i^{(j)}, & i = 1, \dots, r, j = 1, \dots, m, \\ a_{ij} = e_{ij}, & i = 1, \dots, r, j = 1, \dots, r, \\ d_{ij} + f_{ij} = b_{ij}, & i = 1, \dots, r, j = 1, \dots, r. \end{cases},$$

which is equivalent to the system of four matrix equations

$$\begin{cases} QT = (I - P_{11})Q, \\ TW = W(I - P_{11}), \\ T = T^2, \\ WQ = S(I - T) - TS. \end{cases} \tag{13}$$

We give a simple example to illustrate the above system. Let A be a 3×3 matrix that is exactly the same as its Jordan form $J = \text{diag}(0, J_2(0))$. Then $\lambda = 0, m = 1, k = 2$, and $r = 1$. The 1×1 projection sub-matrix P_{11} is either $[1]$ or $[0]$, and all the other matrices in (13) are also 1×1 . Denote

$$Q = [p], \quad T = [a], \quad S = [b], \quad W = [q].$$

Then in the case that $P_{11} = [1]$, the system (13) becomes

$$\begin{cases} pa = 0, \\ qa = 0, \\ a = a^2, \\ qp = b(1 - a) - ab. \end{cases}$$

Hence $a = 0$ or $a = 1$. If $a = 0$, then

$$P = \begin{bmatrix} 1 & 0 & q \\ p & 0 & pq \\ 0 & 0 & 0 \end{bmatrix}, \quad \forall p, q,$$

so $Y = JP = 0$. When $a = 1$, we have $P = I$ and thus $Y = JP = J$.

For the other case that $P_{11} = 0$, the corresponding system is

$$\begin{cases} pa = p, \\ aq = q, \\ a = a^2, \\ qp = b(1 - a) - ab. \end{cases}$$

If $a = 0$, then $p = q = b = 0$, so $P = 0$ and thus $Y = 0$. If $a = 1$, then

$$P = \begin{bmatrix} 0 & 0 & q \\ p & 1 & -pq \\ 0 & 0 & 1 \end{bmatrix}, \quad \forall p, q,$$

and so $Y = JP = J$. In summary, the only projection-based solutions of (2) are the trivial ones $Y = 0$ and $Y = J$, which can be verified directly from the example in Section 2.

Now we consider $A = J = \text{diag}(J_2(0), J_2(0))$. For this example, $m = 0, k = 2$ and $r = 2$. Also there are no P_{11}, P_{12} , and P_{21} , so $P = P_{22}$. The system (12) is then reduced to $P_{22} = P_{22}^2$, where

$$P_{22} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}, \quad R_{ij} = \begin{bmatrix} a_{ij} & b_{ij} \\ 0 & a_{ij} \end{bmatrix}, \quad i, j = 1, 2.$$

Since P_{11}, Q , and W disappear, the corresponding system (13) is nothing but

$$T = T^2 \quad \text{and} \quad TS - S(I - T) = 0, \quad T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad S = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

The first equation above means that T is a projection matrix, so given any such matrix, we can solve the second linear equation for S , which is of Sylvester’s type. From the theory of Sylvester’s equation (see, e.g., [2, 11]), the homogeneous equation $TS - S(I - T) = 0$ has a solution $S \neq 0$ if and only if T and $I - T$ have a common eigenvalue. So when $T = 0$ or $T = I$, the only solution is $S = 0$, resulting in $P = 0$ or $P = I$ that give rise to the trivial solutions $Y = 0$ or $Y = J$. On the other hand, if T is any other projection matrix so that 0 and 1 are both the eigenvalues of T , then there are nonzero solutions S , since 0 and 1 are also the eigenvalues of the projection matrix $I - T$. Now we solve them out.

For this purpose, denote $a_{11} = a, a_{12} = b, a_{21} = c, a_{22} = d, b_{11} = x, b_{12} = y, b_{21} = z, b_{22} = w$. Then $a + d = 0 + 1 = 1$ and $ad - bc = 0 \cdot 1 = 0$, from which we have the general expression of all 2×2 projection matrices $T \neq 0$ or I :

$$T = \begin{bmatrix} \frac{1 \pm \sqrt{1-4bc}}{2} & b \\ c & \frac{1 \mp \sqrt{1-4bc}}{2} \end{bmatrix}, \quad \forall b, c.$$

For a given T as above, the equation $TS - S(I - T) = 0$ can be rewritten as

$$\begin{bmatrix} 2a - 1 & c & b & 0 \\ b & 0 & 0 & b \\ c & 0 & 0 & c \\ 0 & c & b & 2d - 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{14}$$

We solve (14) for different cases. First assume that $b \neq 0$ and $c \neq 0$. Then applying the Gaussian elimination to (14) gives

$$\begin{bmatrix} b & 0 & 0 & b \\ 0 & c & b & 2d - 1 \\ c & 0 & 0 & c \\ 0 & c & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

whose solutions are $x = -w, y = [(1 - 2d)w - bz]/c$ with z and w arbitrary numbers, so under the condition $b \neq 0$ and $c \neq 0$, all the solutions of $TS - S(I - T) = 0$, where T is a projection matrix and $T \neq 0$ or I , are

$$S = \begin{bmatrix} -w & \frac{(1-2d)w-bz}{c} \\ z & w \end{bmatrix} = \begin{bmatrix} -w & \frac{\pm\sqrt{1-4bc}\cdot w-bz}{c} \\ z & w \end{bmatrix}, \quad \forall z, w.$$

Now suppose $b \neq 0$ and $c = 0$. Then

$$\begin{bmatrix} b & 0 & 0 & b \\ 0 & 0 & b & 1-2a \\ c & 0 & 0 & c \\ 0 & c & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

the solutions of which are $x = -w$ and $z = (2a - 1)w/b$ with arbitrary numbers y and w , so under the condition $b \neq 0$ and $c = 0$, all the solutions of $TS - S(I - T) = 0$, where T is a projection matrix and $T \neq 0$ or I , are

$$S = \begin{bmatrix} -w & y \\ \frac{(2a-1)w}{b} & w \end{bmatrix} = \begin{bmatrix} -w & y \\ \pm\frac{w}{b} & w \end{bmatrix}, \quad \forall y, w.$$

if $b = 0$ and $c \neq 0$, then

$$\begin{bmatrix} c & 0 & 0 & c \\ 0 & c & 0 & 1-2a \\ c & 0 & 0 & c \\ 0 & c & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus $x = -w$ and $y = (2a - 1)w/c$, and the resulting solutions are

$$S = \begin{bmatrix} -w & \frac{(2a-1)w}{c} \\ z & w \end{bmatrix} = \begin{bmatrix} -w & \pm\frac{w}{c} \\ z & w \end{bmatrix}, \quad \forall z, w.$$

Finally, when $b = c = 0$, then

$$\begin{bmatrix} 2a-1 & 0 & 0 & 0 \\ 0 & c & 0 & 2a-1 \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

so $x = 0$ and $w = 0$ with y and z arbitrary. Hence,

$$S = \begin{bmatrix} 0 & y \\ z & 0 \end{bmatrix}, \quad \forall y, z.$$

In summary, we have the following conclusions:

(i) $b \neq 0$ and $c \neq 0$. Then

$$P = \begin{bmatrix} \frac{1 \pm \sqrt{1-4bc}}{2} & -w & b & \frac{\pm\sqrt{1-4bc}\cdot w-bz}{c} \\ 0 & \frac{1 \pm \sqrt{1-4bc}}{2} & 0 & b \\ c & z & \frac{1 \mp \sqrt{1-4bc}}{2} & w \\ 0 & c & 0 & \frac{1 \mp \sqrt{1-4bc}}{2} \end{bmatrix}, \quad \forall z, w.$$

(ii) $b \neq 0$ and $c = 0$. Then

$$P = \begin{bmatrix} \frac{1 \pm 1}{2} & -w & b & y \\ 0 & \frac{1 \pm 1}{2} & 0 & b \\ 0 & \pm\frac{w}{b} & \frac{1 \mp 1}{2} & w \\ 0 & 0 & 0 & \frac{1 \mp 1}{2} \end{bmatrix}, \quad \forall y, w.$$

(iii) $b = 0$ and $c \neq 0$. Then

$$P = \begin{bmatrix} \frac{1 \pm 1}{2} & -w & 0 & \pm \frac{w}{c} \\ 0 & \frac{1 \pm 1}{2} & 0 & 0 \\ c & z & \frac{1 \mp 1}{2} & w \\ 0 & c & 0 & \frac{1 \mp 1}{2} \end{bmatrix}, \quad \forall z, w.$$

(iv) $b = 0$ and $c = 0$. Then

$$P = \begin{bmatrix} \frac{1 \pm 1}{2} & 0 & 0 & y \\ 0 & \frac{1 \pm 1}{2} & 0 & 0 \\ 0 & z & \frac{1 \mp 1}{2} & 0 \\ 0 & 0 & 0 & \frac{1 \mp 1}{2} \end{bmatrix}, \quad \forall y, z.$$

Proposition 4.1. Let $J = \text{diag}(J_2(0), J_2(0))$. Then all the projection-based solutions of (2) are $Y = 0, J$, and

$$Y = \begin{bmatrix} 0 & \frac{1 \pm \sqrt{1-4bc}}{2} & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & c & 0 & \frac{1 \mp \sqrt{1-4bc}}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \forall b, c. \tag{15}$$

Proof. All projection-based solutions of (2) are $Y = JP$ with $P = 0, I$, and the those projection matrices from (i)-(iv) above for the different cases of b and c . After a direct computation, all the obtained solutions Y have the single expression (15). \square

Remark: For the above example, the methods of [4, 18] can only find the two trivial projection-based solutions, 0 and $\text{diag}(J_2(0), J_2(0))$, while the results of [6, 7] are not applicable here. But our method finds all projection-based commuting solutions, which constitute a two dimensional manifold.

5. Conclusions

Among all commuting solutions of the Yang-Baxter-like matrix equation (1), we have characterized those that can be expressed as the product of the given matrix A and a projection matrix P that commutes with A . Our general result gives all the projection-based commuting solutions and extends previous ones that only provide special projection-based commuting solutions for various classes of matrices, such as the spectral solutions for general matrices and projection-based solutions for matrices with special Jordan canonical forms.

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