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Substitution Vector-Valued Integral Type Operators on Orlicz Spaces

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Abstract. In this paper, we consider some fundamental properties of a substitution vector-valued integral operator T_φ^u from Orlicz space $L^\theta(\mu)$ to Hilbert space $\mathcal{H}$ by the language of conditional expectation operators. First, we present necessary and sufficient conditions for boundedness and compactness T_φ^u from $L^\theta(\mu)$ to $\mathcal{H}$. Next, we investigate the problem of conditions on the generating Young functions, the functions u, φ and $h = d(\mu \circ \varphi^{-1})/d\mu$, under which operator T_φ^u is of closed range or finite rank. Finally, we determine the lower and upper estimates for the essential norm of T_φ^u on Orlicz spaces under certain conditions.

1. Introduction and Preliminaries

Let $\theta : \mathbb{R} \to \mathbb{R}^+$ be a continuous convex function such that
(a) $\theta(x) = 0$ if and only if $x = 0$.
(b) $\lim_{x\to\infty} \theta(x) = \infty$.
(c) $\lim_{x\to\infty} \frac{\theta(x)}{x} = \infty$.
The convex function θ is called Young's function. With each Young's function θ, one can associate another convex function $\theta^* : \mathbb{R} \to \mathbb{R}^+$ having similar properties, which is defined by

$$\theta^*(y) = \sup\{x|y| - \theta(x) : x \geq 0\}.$$

The convex function θ^* is called complementary Young function to θ. Let $X = (X, \Sigma, \mu)$ be a σ-finite complete measure space. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. If θ is a Young function, then the set of Σ-measurable functions

$$L^\theta(\mu) = \left\{ f : X \to \mathbb{C} : \exists \alpha > 0, \int_X \theta(\alpha|f|)d\mu < \infty \right\}$$

is a Banach space, with respect to the Luxemburg norm defined by

$$\|f\|_\theta = \inf\left\{ \delta > 0 : \int_X \theta(\frac{|f|}{\delta})d\mu \leq 1 \right\}.$$

$(L^\theta(\mu), \|\cdot\|_\theta)$ is called Orlicz space. A Young function θ is said to satisfy the Δ_2-condition(globally) if $\theta(2x) \leq k\theta(x), x > x_0 (x_0 = 0)$, for some constant $k > 0$. constant

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Proposition 1.1. *[14](Holder's inequality). For all $f \in L^{\theta}(\mu)$ and $g \in L^{\theta^*}(\mu)$,*

$$\int_X |fg| d\mu \leq 2\|f\|_\theta \|g\|_{\theta^*}.$$

Let θ_1, θ_2 be two Young functions, then θ_1 is called stronger than θ_2, which $\theta_1 \succ \theta_2$[or $\theta_2 \prec \theta_1$], if

$$\theta_2(x) \leq \theta_1(ax), \quad x \geq x_0 \geq 0,$$

for some $a \geq 0$ and x_0, if $x_0 = 0$ then this condition is said to hold globally. Throughout this note, we assume that θ satisfies Δ_2-condition.

The Orlicz norm on $L^{\theta}(\mu)$ is given by

$$\|f\|_\theta := \Big\{ \int_X fg d\mu; g \in L^{\theta^*}(\mu), \|g\|_{\theta^*} \leq 1 \Big\}.$$

Theorem 1.2 ([1], Theorem 8.14). *if θ is a Young function. Then Luxemburg and Orlicz norms are equivalent:*

$$\|f\|_\theta \leq |f|_\theta \leq 2\|f\|_\theta.$$

If the Young function satisfies Δ_2-condition, then the dual space of the Orlicz space equipped with the Luxemburg norm is isometrically isomorphic th the Orlics space generated by the complementary function and equipped with the Orlicz norm, see [1]. Let θ and θ^* be a pair of complementary Orlicz functions. Then each $g \in L^{\theta^*}(\mu)$ defines a bounded linear functional F_g on $L^{\theta}(\mu)$ by $F_g(f) = \int fg d\mu$, $f \in L^{\theta}(\mu)$.

Simple functions are not necessarily dense in $L^{\theta}(\mu)$. But, if θ satisfies Δ_2-condition, then simple functions are dense in $L^{\theta}(\mu)$. It is well-known that if $A \in \Sigma$ and $0 < \mu(A) < \infty$ then $\|\chi_A\|_\theta = 1/\theta^{-1}(1/\mu(A))$ where $\theta^{-1}(t) = \inf\{\delta > 0, \theta(s) > t\}$ is the right continuous inverse of θ. The usual convergence in the Orlicz space $L^{\theta}(\mu)$ can be introduced in term of the Orlicz norm $\|\cdot\|_\theta$ as $x_n \to x$ in $L^{\theta}(\mu)$ means $\|x_n - x\|_\theta \to 0$. Also, a sequence $\{x_n\}_{n=1}^{\infty}$ in $L^{\theta}(\mu)$ is said to converges in θ-mean to $x \in L^{\theta}(\mu)$, if

$$\lim_{n\to\infty} I_\theta(x_n - x) = \lim_{n\to\infty} \int_X \theta(|x_n - x|) d\mu = 0.$$

For further information on Orlicz spaces, see [12–14].

Let $\varphi : X \to X$ be a non-singular measurable transformation; i.e. $\mu \circ \varphi^{-1} \ll \mu$. It is assumed that the Radon-Nikodym derivative $h = d\mu \circ \varphi^{-1}/d\mu$ is almost everywhere finite-valued, or equivalently $\varphi^{-1}(\Sigma) \subseteq \Sigma$ is a sub-σ-finite algebra [16]. We have the following change of variable formula:

$$\int_{\varphi^{-1}(A)} f \circ \varphi d\mu = \int_A hf d\mu \qquad A \in \Sigma, f \in L^0(\Sigma).$$

Any nonsingular measurable transformation φ induces a linear operator (composition operator) C_φ from $L^0(\mu)$ into itself defined by

$$C_\varphi(f)(x) = f(\varphi(x)); \; x \in X, \; f \in L^0(\mu),$$

where $L^0(\Sigma)$ denotes the linear space of all equivalence classes of Σ-measurable functions on X. Here non-singularity of φ guarantees that the operator C_φ is well defined as a mapping from $L^0(\Sigma)$ into itself. If C_φ maps on Orlicz space $L^{\theta}(\mu)$ into itself , then C_φ is called composition operator on $L^{\theta}(\mu)$. Note that, in this case C_φ is bounded. The support of a measurable function f is defined by $\sigma(f) = \{x \in X; f(x) \neq 0\}$. For a given complex Hilbert space $\mathcal{H}$, let $u : X \to \mathcal{H}$ be a mapping. We say that u is weakly measurable if for each $g \in \mathcal{H}$ the mapping $x \mapsto \langle u(x), g\rangle$ of X to $\mathbb{C}$ is measurable. We will denote this map by $\langle u, g\rangle$.

Definition 1.3. *Let $\varphi : X \to X$ be a non-singular measurable transformation and C_φ be a composition operator on $L^\theta(X)$. Also let $u : X \to \mathcal{H}$ be a weakly measurable function. Then the pair (u,φ) induces a substitution vector-valued integral operator $T_u^\varphi : L^\theta(\mu) \to \mathcal{H}$ defined by*

$$\langle T_u^\varphi f, g\rangle = \int_X \langle u, g\rangle f \circ \varphi d\mu, \quad f \in L^\theta(\mu).$$

It is easy to see that T_u^φ is well defined and linear.

From [4], we have that if $u : X \to \mathcal{H}$ be a weakly measurable function. We say that $(u,\varphi,\mathcal{H})$ has absolute property, if for each $f \in L^\theta(X)$, there exists $g_f \in \mathcal{D}$ such that $\sup_{g\in\mathcal{D}} \int_X |\langle u, g\rangle||C_\varphi f|d\mu = \int_X |\langle u, g_f\rangle||C_\varphi f|d\mu$, and $\langle u, g_f\rangle = e^{i(-\arg C_\varphi f+\beta_f)}|\langle u, g_f\rangle|$, for a constant β_f.

Proposition 1.4 ([4]). *Assume that $(u,\varphi,\mathcal{H})$ has the absolute property. Then*

$$\sup_{g\in\mathcal{D}} |\int_X \langle u, g\rangle C_\varphi f d\mu| = \sup_{g\in\mathcal{D}} \int_X |\langle u, g\rangle||C_\varphi f|d\mu.$$

Throughout of this paper, we assume that $(u,\varphi,\mathcal{H})$ has the absolute property.

For a sub-σ-finite algebra $\mathcal{A} \subseteq \Sigma$, the conditional expectation operator associated with $\mathcal{A}$ is the mapping $f \to E^{\mathcal{A}} f$, defined for all non-negative f as well as for all $f \in L^p(\Sigma), 1 \le p \le \infty$, where $E^{\mathcal{A}} f$, by Radon-Nikodym Theorem, is the unique $\mathcal{A}$-measurable function satisfying

$$\int_A f d\mu = \int_A E^{\mathcal{A}} f d\mu, \quad \forall A \in \mathcal{A}.$$

For more details on the properties of $E^{\mathcal{A}}$ see [11]. Throughout this paper, we assume that $\mathcal{A} = \varphi^{-1}(\Sigma)$ and $E^{\varphi^{-1}(\Sigma)} = E$.

Recall that an atom of the measure μ is an element $A \in \Sigma$ with $\mu(A) > 0$, such that for each $B \in \Sigma$, if $B \subset A$ then either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. A measure with no atoms is called non-atomic. We can easily check the following well-known facts (see[19]):

(a) Every σ-finite measure space (X,Σ,μ) can be partitioned uniquely as

$$X = (\cup_{n\in\mathbb{N}} A_n) \cup B, \tag{1}$$

where $\{A_n\}_{n\in\mathbb{N}} \subseteq \Sigma$ is a countable collection of pairwise disjoint atoms and B, being disjoint from each A_n, is non-atomic. Since (X,Σ,μ) is σ-finite, it follows that $\mu(A_n) < \infty$ for every $n \in \mathbb{N}$.

(b) Let E be a non-atomic set with $\mu(E) > 0$. Then there exists a sequence of positive disjoint Σ-measurable subsets of E, $\{E_n\}_{n\in\mathbb{N}}$ such that $\mu(E_n) > 0$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mu(E_n) = 0$.

The basic properties of composition and weighted composition operators on measurable function spaces are studied by more mathematicians. For more details on these operators see [2, 5, 16, 17]. The multiplication and weighted composition operators are studied on Orlicz spaces in [3, 6–9]. Also, the fundamental properties of substitution vector-valued integral operator are studied by the author et al in [4] and the essential norm these operators investigated by the author in [10]. In this paper, we are going to present some assertions about boundedness, compactness and essential norm of substitution vector-valued integral operator on Orlicz spaces. In section 2, we present some necessary and sufficient conditions for boundedness and compactness of the substitution vector-valued integral operator on Orlicz spaces. Then in section 3, we characterize substitution vector-valued integral operators on Orlicz spaces that have closed ranges. Also in this section by using the compactness assertions , that is proved in section 2, we estimate the essential norm of substitution vector-valued integral operators. Then we present two examples to illustrate our results.

2. Boundeness and Compactness of Substitution vector-valued integral operators

In this section, we state various necessary and sufficient conditions under which the substitution vector-valued integral operators T_u^{φ} on Orlicz spaces is bounded and also, we present necessary and sufficient conditions for compactness these type operators.

Definition 2.1. *Let $u : X \to \mathcal{H}$ be a weakly measurable function. We say that u is a semi-weakly bounded function if for some $M > 0$,*

$$\|\langle u, \lambda\rangle\|_{\theta^*} \leq M\|\lambda\|; \quad \textit{for each } \lambda \in \mathcal{H}.$$

Theorem 2.2. *Let $u : X \to \mathcal{H}$ be a weakly measurable function. If u is a semi-weakly bounded function and $h \in L^{\infty}(\Sigma)$, then $T_u^{\varphi} : L^{\theta}(\mu) \to \mathcal{H}$ is bounded.*

proof Let $f \in L^{\theta}(\mu)$. By Holder inequality, we have

$$\begin{aligned}
\|T_u^{\varphi} f\| &= \sup_{\lambda\in\mathcal{H}_1} \Big| \int \langle u, \lambda\rangle f \circ \varphi d\mu \Big| \\
&\leq \sup_{\lambda\in\mathcal{H}_1} \int |\langle u, \lambda\rangle||f \circ \varphi| d\mu \\
&\leq \sup_{\lambda\in\mathcal{H}_1} 2\|\langle u, \lambda\rangle\|_{\theta^*}\|f \circ \varphi\|_{\theta} \\
&\leq \sup_{\lambda\in\mathcal{H}_1} 2M\|\lambda\|\|f \circ \varphi\|_{\theta} \\
&= 2M \inf\Big\{\delta > 0, \int_X \theta\Big(\frac{|f \circ \varphi|}{\delta}\Big) d\mu \leq 1\Big\} \\
&= 2M \inf\Big\{\delta > 0, \int_X h\theta\Big(\frac{|f|}{\delta}\Big) d\mu \leq 1\Big\} \\
&\leq 2M\|h\|_{\infty} \inf\Big\{\delta > 0, \int_X \theta\Big(\frac{|f|}{\delta}\Big) d\mu \leq 1\Big\} \\
&= 2M\|h\|_{\infty}\|f\|_{\theta}.
\end{aligned}$$

This shows that T_u^{φ} is bounded.

Proposition 2.3. *Let $u : X \to \mathcal{H}$ be a weakly measurable function. Then*

(i) If for each $\lambda \in \mathcal{H}_1$, the functions $\langle u, \lambda\rangle$ are conditionable and T_u^{φ} is bounded. Hence for all $\lambda \in \mathcal{H}_1$, $hE(\langle u, \lambda\rangle) \circ \varphi^{-1} \in L^{\theta^*}(\mu)$.

(ii) T_u^{φ} is bounded if and only if $\sup_{\lambda\in\mathcal{H}_1} \|hE(\langle u, \lambda\rangle) \circ \varphi^{-1}\|_{\theta^*} < \infty$.

proof (i) Since T_u^{φ} is bounded, so there exists $K > 0$ such that for each $f \in L^{\theta}(\mu)$, $\|T_u^{\varphi} f\| \leq K\|f\|_{\theta}$. For an arbitrary and fixed $\lambda \in \mathcal{H}_1$, we define a linear functional Ω_λ on $L^{\theta}(\mu)$ by $\Omega_\lambda(f) = \int_X hE(\langle u, \lambda\rangle) \circ \varphi^{-1} f d\mu$. We have

$$|\Omega_\lambda(f)| = \sup_{\lambda\in\mathcal{H}_1} \Big| \int_X hE(\langle u, \lambda\rangle) \circ \varphi^{-1} f d\mu \Big| = \|T_u^{\varphi} f\| \leq K\|f\|_{\theta},$$

Therefore, Ω_λ is a bounded linear functional on $L^{\theta}(\mu)$. By Theorem 1 in [13] there exists a unique function $g \in L^{\theta^*}(\mu)$ such that for every $f \in L^{\theta}(\mu)$, ${}_{\lambda}(f) = \int_X f g d\mu$. This implies that $g = hE(\langle u, \lambda\rangle) \circ \varphi^{-1} \in L^{\theta^*}(\mu)$.

(ii) Let $K := \sup_{\lambda\in\mathcal{H}_1} \|hE(\langle u,\lambda\rangle)\circ\varphi^{-1}\|_{\theta^*} < \infty$ and $f \in L^\theta(\mu)$. Hence, by Holder inequality and change of variable formula we have

$$\begin{aligned}\|T_u^\varphi f\| &= \sup_{\lambda\in\mathcal{H}_1}\int_X hE(\langle u,\lambda\rangle)\circ\varphi^{-1}|f|d\mu\\ &\le 2\sup_{\lambda\in\mathcal{H}_1}\|hE(\langle u,\lambda\rangle)\circ\varphi^{-1}\|_{\theta^*}\|f\|_\theta\end{aligned}$$

and this implies that T_u^φ is bounded.

Conversely, we let that T_u^φ be bounded. Define $\Omega_\lambda(f) = \int_X hE(\langle u,\lambda\rangle)\circ\varphi^{-1}fd\mu$ for an arbitrary and fixed $\lambda\in\mathcal{H}_1$ and $f\in L^\theta(\mu)$. Then by a similar argument as in proof of (i), it is easy to see that $hE(\langle u,\lambda\rangle)\circ\varphi^{-1}\in L^{\theta^*}(\mu)$ for each $\lambda\in\mathcal{H}_1$.

Theorem 2.4. *The substitution vector-valued integral operator T_u^φ is a compact operator on $L^\theta(\mu)$ if and only if for any $\varepsilon > 0$ the set*

$$N_\varepsilon := \Big\{x\in X : \sup_{\lambda\in\mathcal{H}_1} hE|\langle u,\lambda\rangle|\circ\varphi^{-1}\ge\varepsilon\Big\}$$

consists of finitely many atoms.

proof Assume the contrary. Then N_ε either contains a non-atomic subset or has infinitely many atoms. Hence there exist $\delta > 0$ and $\lambda_1\in\mathcal{H}_1$ such that the set $C := \{x\in N_\epsilon : hE|\langle u,\lambda_1\rangle|\circ\varphi^{-1}\ge\delta\}$ has positive measure. In both cases, we can find a sequence of pairwise disjoint measurable subsets $\{E_n\}$ with $0 < \mu(E_n) < \infty$ for each $n\in\mathbb{N}$. Put $f_n = \frac{\chi_{E_n}}{\|\chi_{E_n}\|_\theta}$. Note that $I_\theta(f_n) = \int_X \theta(|f_n|)d\mu = 1$, whence $f_n\in L^\theta(\mu)$ and $\|f_n\|_\theta = 1$. For each $n, m\in\mathbb{N}$, we have

$$\begin{aligned}\|T_u^\varphi f_m - T_u^\varphi f_n\| &= \sup_{\lambda\in\mathcal{H}_1}\int_X |\langle u,\lambda\rangle||f_m - f_n|\circ\varphi d\mu\\ &\ge \int_X |\langle u,\lambda_1\rangle||f_m - f_n|\circ\varphi d\mu\\ &= \int_X hE(|\langle u,\lambda_1\rangle|)\circ\varphi^{-1})||f_m - f_n|d\mu\\ &\ge \int_{E_m\cup E_n} hE(|\langle u,\lambda_1\rangle|)\circ\varphi^{-1}||f_m - f_n|d\mu\\ &= \int_{E_m}\frac{hE(|\langle u,\lambda_1\rangle|)\circ\varphi^{-1}}{\|\chi_{E_m}\|_\theta}d\mu\\ &+ \int_{E_n}\frac{hE(|\langle u,\lambda_1\rangle|)\circ\varphi^{-1}}{\|\chi_{E_n}\|_\theta}d\mu\\ &\ge 2\delta\Big(\mu\Big(E_m\theta^{-1}\Big(\frac{1}{\mu(E_m)}\Big)\Big) + \mu\Big(E_n\theta^{-1}\Big(\frac{1}{\mu(E_n)}\Big)\Big)\Big) > 2\delta M\end{aligned}$$

where $M = \mu(E_m\theta^{-1}(\frac{1}{\mu(E_m)}) + \mu(E_n\theta^{-1}(\frac{1}{\mu(E_n)}) > 0$. This implies that the sequence $\{T_u^\varphi f_n\}_n$ does not contain a convergent subsequence and this follows that $T_u^\varphi f_m$ is not compact.
Conversely, by Theorem 2.10 in [4], it is easy to see that if for each $\varepsilon > 0$ the set $N_\varepsilon = \{x\in X : \sup_{\lambda\in\mathcal{H}_1} hE(|\langle u,\lambda\rangle|)\circ\varphi^{-1}\ge\varepsilon\}$ consists of finitely many atoms. Consequently T_u^φ is a compact operator on $L^\theta(\mu)$.

3. Substitution vector-valued integral operators with closed-range and their essential norm

In this section, first we are going to investigate closed-range substitution vector-valued integral operators on Orlicz spaces. Next, we determine the essential norm these type operators.

Let $(B_1, \|\cdot\|_{B_1})$ and $(B_2, \|\cdot\|_{B_2})$ be two Banach spaces and T be a bounded linear operator from B_1 into B_2. There exists a constant $c > 0$ such that $\|Tx\|_{B_2} \geq c\|x\|_{B_1}$ for all $x \in B_1$ if and only if $\ker T = \{0\}$ and $T(B_1)$ is closed in B_2.

Now, we characterize the closedness of range of a substitution vector-valued integral operator from $L^\theta(\mu)$ to $\mathcal{H}$.
We start by the following lemma. Put $J := \cup_{\lambda\in\mathcal{H}_1}\sigma(hE(|\langle u,\lambda\rangle| \circ \varphi^{-1})$.

Lemma 3.1. *Let T_u^φ be a bounded substitution vector-valued integral operator from $L^\theta(\mu)$ to $\mathcal{H}$ and there is a constant $c > 0$ such that $\sup_{\lambda\in\mathcal{H}_1} hE(|\langle u,\lambda\rangle| \circ \varphi^{-1} \geq c$ on J, then $T_u^\varphi|_J$ is injective.*

proof Let f be a non-zero element in $\ker T_u^\varphi|_J$. Then, we have

$$
\begin{aligned}
0 = \|T_u^\varphi\| &= \sup_{\lambda\in\mathcal{H}_1}\int_X hE(|\langle u,\lambda\rangle| \circ \varphi^{-1})|f|d\mu \\
&\geq \sup_{\lambda\in\mathcal{H}_1}\int_J hE(|\langle u,\lambda\rangle| \circ \varphi^{-1})|f|d\mu \\
&\geq c\int_J |f|d\mu.
\end{aligned}
$$

This means that $f = 0$ on J and this completes the proof.

Theorem 3.2. *Let T_u^φ be a bounded substitution vector-valued integral operator from $L^\theta(\mu)$ to $\mathcal{H}$. Then, the following statements are hold.*

(i) Suppose T_u^φ from $L^\theta(\mu)$ to $\mathcal{H}$ has closed range and $x \prec \theta$ then there is a constant $c > 0$ such that $\sup_{\lambda\in\mathcal{H}_1} hE(|\langle u,\lambda\rangle| \circ \varphi^{-1} \geq c$ on J.

(ii) If there is a constant $c > 0$ such that $\sup_{\lambda\in\mathcal{H}_1} hE(|\langle u,\lambda\rangle| \circ \varphi^{-1} \geq c$ on J and $\theta \prec x$, then T_u^φ from $L^\theta(\mu)$ to $\mathcal{H}$ has closed range

proof (i) Suppose that T_u^φ has closed range. Since $x \prec \theta$, so for some $a > 0$, $x \leq \theta(ax)$. Then there exists a constant $\delta > 0$ such that $\|T_u^\varphi f\| \geq \delta\|f\|_\theta$, for any $f \in L^\theta(\mu)$. Take $c = \frac{\delta}{a}$. Let $E = \{x \in J : \sup_{\lambda\in\mathcal{H}_1} hE(|\langle u,\lambda\rangle|) \circ \varphi^{-1} < c\}$. If $\mu(E) > 0$, we can find a measurable set $F \subseteq E$ such that $\chi_F \in L^\theta|_J(\mu)$. It is easy to see that $x < \frac{a}{\theta^{-1}(\frac{1}{x})}$ It is known that $\|\chi_F\|_\theta = \frac{1}{\theta^{-1}(\frac{1}{\mu(F)})}$. Hence, we get that

$$
\begin{aligned}
\|T_u^\varphi \chi_F\| &= \sup_{\lambda\in\mathcal{H}_1}\int_J hE(|\langle u,\lambda\rangle|) \circ \varphi^{-1}\chi_F d\mu \\
&\leq c\mu(F) < c\frac{a}{\theta^{-1}(\frac{1}{\mu(F)})} = \delta\|\chi_F\|_\theta
\end{aligned}
$$

and this is a contradiction. Hence $\mu(E) = 0$ and this completes the proof.
(ii) Assume that there is a constant $c > 0$ such that

$$\sup_{\lambda\in\mathcal{H}_1} hE(|\langle u,\lambda\rangle| \circ \varphi^{-1} \geq c$$

on J. Since $\theta \prec x$ so for some $a > 0$ we have $\theta(x) \leq ax$. Hence

$$\int_X \theta\Big(\frac{|f|}{a\int_X |f|d\mu}\Big)d\mu \leq \int_X \Big(\frac{a|f|}{a\int_X |f|d\mu}\Big)d\mu = 1.$$

This implies that $\|f\|_\theta \leq a \int_X |f| d\mu$. Therefore, for every $f \in L^\theta|_J(\mu)$, we have

$$\begin{aligned}\|T_u^\varphi\| &= \sup_{\lambda\in\mathcal{H}_1}\int_X hE(|\langle u,\lambda\rangle|)\circ\varphi^{-1}d\mu\\ &= \sup_{\gamma\in\mathcal{H}_1}\int_J hE(|\langle u,\lambda\rangle|)\circ\varphi^{-1}d\mu + \sup_{\lambda\in\mathcal{H}_1}\int_{X\setminus J} hE(|\langle u,\gamma\rangle|)\circ\varphi^{-1}d\mu\\ &\geq c\int_J |f|d\mu \geq \frac{c}{a}\|f\|_\theta\end{aligned}$$

Now, consider $T_u^\varphi|_J$. By using Lemma 3.1, $T_u^\varphi|_J$ is injective. In view of the above calculation , we conclude that $T_u^\varphi|_J(\mu)$ is closed in $\mathcal{H}$. Since $\ker T_u^\varphi = L^\theta|_{X\setminus J}(\mu)$, $T_u^\varphi(L^\theta(\mu))$ must be closed in $\mathcal{H}$.

In following, we give equivalent conditions with conditions of Theorem 3.2.

Lemma 3.3. *Let $\mathcal{B}$ be the collection of all Σ-measurable sets E such that*

(i) $\mu(E) < \infty$ and

(ii) whenever $F \in \Sigma$ satisfies $F \subseteq E$ and $\sup_{\lambda\in\mathcal{H}_1}\int_{\varphi^{-1}(F)}|\langle u,\lambda\rangle|d\mu = 0$, then $\mu(F) = 0$.
Suppose that $E \in \Sigma$ and $\mu(E) < \infty$. Hence, $E \in \mathcal{B}$ if and only if $E \subseteq J$.

proof ($\Rightarrow$) Obviously, $F := E \setminus \cup_{\lambda\in\mathcal{H}_1}\sigma(hE(|\langle u,\lambda\rangle|\circ\varphi^{-1})$ is a Σ-measurable subset of E. Also,

$$\sup_{\lambda\in\mathcal{H}_1}\int_{\varphi^{-1}(F)}|\langle u,\lambda\rangle| = \sup_{\lambda\in\mathcal{H}_1}\int_F hE(|\langle u,\lambda\rangle|)\circ\varphi^{-1} = 0.$$

Since $E \in \mathcal{B}$, property (ii) implies that $\mu(F) = 0$.
($\Leftarrow$) It suffices to show that (ii) holds. Assume that $F \in \Sigma$ with $F \subseteq E$ and $\sup_{\lambda\in\mathcal{H}_1}\int_{\varphi^{-1}(F)} hE(|\langle u,\lambda\rangle|\circ\varphi^{-1} = 0$. We claim that $\mu(F) = 0$. Suppose not, then it follows from $F \subseteq E \subseteq \cup_{\lambda\in\mathcal{H}_1}\sigma(hE(|\langle u,\lambda\rangle|)\circ\varphi^{-1})$ that for some $\delta > 0$,

$$\mu(\{x \in F : \sup_{\lambda\in\mathcal{H}_1} hE(|\langle u,\lambda\rangle|)\circ\varphi^{-1} \geq \delta\}) > 0.$$

Put $M_{\lambda_1} := \{x \in F : hE(|\langle u,\lambda_1\rangle|)\circ\varphi^{-1} \geq \delta\}$ and also $\mu(M_{\lambda_1}) > 0$. Consequently, we get that

$$\begin{aligned}\sup_{\lambda\in\mathcal{H}_1}\int_{\varphi^{-1}(F)}|\langle u,\lambda\rangle|d\mu &= \sup_{\lambda\in\mathcal{H}_1}\int_F hE(|\langle u,\lambda\rangle|)\circ\varphi^{-1}d\mu\\ &\geq \int_{M_\lambda} hE(|\langle u,\lambda_1\rangle|)\circ\varphi^{-1}d\mu\\ &\geq \delta\mu(M_{\lambda_1}) > 0.\end{aligned}$$

and this contradicts our assumption on F. Therefore $\mu(F) = 0$.

Proposition 3.4. *The following statements are equivalent.*

(i) There is a constant $c > 0$ such that $\sup_{\lambda\in\mathcal{H}_1} hE(|\langle u,\lambda\rangle|)\circ\varphi^{-1} \geq c$ on J.

(ii) There is a constant $\alpha > 0$ such that $\sup_{\lambda\in\mathcal{H}_1}\int_{\varphi^{-1}(E)}|\langle u,\lambda\rangle|d\mu \geq \alpha\mu(E)$ for all $E \in \mathcal{B}$

proof (i) $\Rightarrow$ (ii) Assume that there is some $c > 0$ such that $\sup_{\lambda\in\mathcal{H}_1} hE(|\langle u,\lambda\rangle| \circ \varphi^{-1} \geq c$ on J. Take $E \in \mathcal{B}$, we have $E \subseteq J$ by Lemma 3.1. It implies that

$$\sup_{\lambda\in\mathcal{H}_1} \int_{\varphi^{-1}(E)} |\langle u,\lambda\rangle| d\mu = \sup_{\lambda\in\mathcal{H}_1} \int_E hE(|\langle u,\lambda\rangle|) \circ \varphi^{-1} d\mu \geq c\mu(E).$$

This proves (ii).
(ii) $\Rightarrow$ (i) Suppose (ii) holds. We claim that the set $E := \{x \in J, \sup_{\lambda\in\mathcal{H}_1} hE(|\langle u,\lambda\rangle|) \circ \varphi^{-1} < \frac{\alpha}{2}\}$ has zero μ-measure. If not, then $\mu(E) > 0$. Since (X, Σ, μ) is σ-finite space, we may assume that $\mu(E) < \infty$. Now, by Lemma 3.1, we have $E \in \mathcal{B}$. However,

$$\sup_{\lambda\in\mathcal{H}_1} \int_{\varphi^{-1}(E)} |\langle u,\lambda\rangle| d\mu = \sup_{\lambda\in\mathcal{H}_1} \int_E hE(|\langle u,\lambda\rangle|) \circ \varphi^{-1} d\mu \leq \frac{\alpha}{2}\mu(E) < \alpha\mu(E),$$

which contradicts our original assumption. This shows that $\sup_{\lambda\in\mathcal{H}_1} hE(|\langle u,\lambda\rangle|) \circ \varphi^{-1} \geq \frac{\alpha}{2}$, *a.e.* on J, and (i) is proved.

Let $\mathcal{B}$ be a Banach space and $\mathcal{K}$ be the set of all compact operators on $\mathcal{B}$, the essential norm of T means the distance from T to $\mathcal{K}$ in the operator norm, namely

$$\|T\|_e = \inf\{\|T - S\| : S \in \mathcal{K}\}.$$

Clearly, T is compact if and only if $\|T\|_e = 0$. As is seen [15], the essential norm plays an interesting role in the compact problem of concrete operators. Many researchers have computed the essential norm of various concrete operators see [6, 10, 15, 17].

Theorem 3.5. *Let T_u^φ be a bounded operator on $L^\theta(\mu)$. Also let $\alpha = \inf\{r > 0, N_r$ consists of finitely many atoms$\}$ and $N_r = \{x \in X := \sup_{\lambda\in\mathcal{H}_1} \|hE(|\langle u,\lambda\rangle|) \circ \varphi^{-1} \geq r\}$. Then we obtain that*
(i) $\|T_u^\varphi\|_e = 0$ if and only if $\alpha = 0$.
(ii) $\|T_u^\varphi\|_e \geq \frac{1}{a}\alpha$ where $\theta \prec x$(i.e. for some $a > 0$ we have $\theta(x) \leq ax$. In particular if $a \leq 1$ we have $\|T_u^\varphi\|_e \geq \alpha$
(iii) $\|T_u^\varphi\|_e \leq a\alpha$ where $x \prec \theta$(i.e. for some $a > 0$ we have $\theta(x) \leq ax$.In particular if $a \leq 1$ we have $\|T_u^\varphi\|_e \leq \alpha$

proof Theorem 2.4 implies that T_u^φ is compact if and only if $\alpha = 0$. So (i) is a direct consequence of Theorem 2.4.

(ii) Take $\varepsilon > 0$ arbitrary. The definition of α implies that $G = N_{\alpha-\frac{\varepsilon}{2}}$ either contains a non-atomic subset or has infinitely many atoms. If G contains a non-atomic subset, then there are measurable sets $G_n, n \in \mathbb{N}$, such that $G_{n+1} \subseteq G_n \subseteq G, \mu(G_{n+1}) = \frac{1}{2^n}$. For $n \in \mathbb{N}$ define $f_n = \frac{\chi_{G_n}}{\|\chi_{G_n}\|_\theta}$. Then $\|f_n\|_\theta = 1$ for all $n \in \mathbb{N}$. We claim that $f_n \to 0$ weakly. For this we show that $\int_X f_n g \to 0$ for all $g \in L^{\theta^*}(\mu)$, where θ^* is the complementary function to θ. Let $E \subseteq G$ with $0 < \mu(E) < \infty$ and $g = \chi_E$. Then we have

$$\Big|\int_X f_n \chi_E d\mu\Big| = \theta^{-1}(\frac{1}{\mu(G_n})(\mu(G_n) \cap E) \leq \frac{1}{2^n}\theta^{-1}(\frac{1}{\mu(G_n)}) \longrightarrow 0,$$

as $n \to \infty$. Since simple functions are dense in $L^{\theta^*}(\mu)$, thus f_n is proved to converge to 0 weakly. Now assume that G consists of infinitely many atoms. Let $\{G_n\}_n$ be disjoints in G. Again put f_n as above. If $\mu(G_n) \to 0$, then by using the similar argument we had $\int_X f_n \chi_E d\mu \to 0$. Otherwise, $\mu(G) \geq \mu(\cup G_n) = \sum \mu(G_n) = +\infty$ and it implies that for each $E \subseteq G$ with $0 < \mu(E) < \infty$ we have $\mu(E \cap G_n) \to 0$, as $n \to \infty$. Consequently in both cases $\int_X f_n g d\mu \to 0$. Now, take a compact operator T on $L^\theta(\mu)$ such that $\|T_u^\varphi - T\| < \|T_u^\varphi\|_e + \frac{\varepsilon}{2}$. The definition of $G = N_{\alpha-\frac{\varepsilon}{2}}$ implies that there exists $\lambda \in \mathcal{H}_1$ such as λ_1 such that $hE|\langle u,\lambda_1\rangle| \circ \varphi^{-1} > \alpha - \varepsilon$ on G. Since $\theta \prec x$

so for some $a > 0$ we have $\theta(x) \leq ax$. Thus $x \leq a\theta^{-1}(x)$. Therefore for each $n \in \mathbb{N}$ and $\lambda_1 \in \mathcal{H}_1$, we get that

$$\begin{aligned}
\|T_u^{\varphi}\|_e &> \|T_u^{\varphi} - T\| - \varepsilon \geq \|T_u^{\varphi} f_n - Tf_n\| - \varepsilon \\
&\geq \|T_u^{\varphi} f_n\| - \|Tf_n\| - \varepsilon \\
&= \sup_{\lambda \in \mathcal{H}_1} \int_X hE(|\langle u, \lambda\rangle|) \circ \varphi^{-1} |f_n| d\mu - \|Tf_n\| - \varepsilon \\
&\geq \int_X hE(|\langle u, \lambda_1\rangle|) \circ \varphi^{-1} \frac{\chi_{G_n}}{\|\chi_{G_n}\|_\theta} d\mu - \|Tf_n\| - \varepsilon \\
&\geq (\alpha - \varepsilon) \int_{G_n} \theta^{-1}\Big(\frac{1}{\mu(G_n)}\Big) d\mu - \|Tf_n\| - \varepsilon \\
&> (\alpha - \varepsilon)\theta^{-1}\Big(\frac{1}{\mu(G_n)}\Big)\mu(G_n)\Big) - \|Tf_n\| - \varepsilon \\
&> \frac{(\alpha - \varepsilon)}{a} - \|Tf_n\| - \varepsilon.
\end{aligned}$$

Since a compact operator maps weakly convergent sequences into norm convergent ones, it follows $\|Tf_n\| \to 0$. Therefore $\|T_u^{\varphi}\|_e \geq \frac{1}{a}(\alpha - \varepsilon) - \varepsilon$. Since ε was arbitrary, we get that $\|T_u^{\varphi}\|_e \geq \frac{1}{a}\alpha$.

(iii)Take ε arbitrary. Put $A = N_{\alpha+\varepsilon}$ and $v = \chi_A u$. The definition of α implies that A consists of finitely many atoms. So we can write $A = \{A_1, A_2, ..., A_m\}$ are distinct. By a similar argument as in proof of [[10],Theorem 2.4], it is easy to see that T_v^{φ} is a finite rank operator on $L^{\theta}(\mu)$. Notice that T_v^{φ} is a compact operator. Thus, we have

$$\begin{aligned}
\|T_u^{\varphi} - T_v^{\varphi}\| = \|T_{u-v}^{\varphi}\| &= \sup_{\|f\|_\theta \leq 1} \|T_{u-v}^{\varphi} f\| \\
&= \sup_{\|f\|_\theta \leq 1} \sup_{\lambda \in \mathcal{H}_1} \int_X h_E|\langle (1 - \chi_A)u, \lambda\rangle| \circ \varphi^{-1} |f| d\mu \\
&= \sup_{\|f\|_\theta \leq 1} \sup_{\lambda \in \mathcal{H}_1} \int_{X-A} hE|\langle u, \lambda\rangle| \circ \varphi^{-1} |f| d\mu \\
&\leq (\alpha + \varepsilon) \sup_{\|f\|_\theta \leq 1} \int_{X\setminus A} |f| d\mu \\
&\leq a(\alpha + \varepsilon) \sup_{\|f\|_\theta \leq 1} \int_{X\setminus A} |\theta(f)| d\mu \\
&\leq a(\alpha + \varepsilon)
\end{aligned}$$

Since ε was arbitrary, so $\|T_u^{\varphi}\|_e \leq a\alpha$.

Example 3.6. *Let $X = [1, 100]$ and μ be the Lebesque measure on X. Define $\varphi : X \to X$ as $\varphi(x) = \frac{1}{2}x$. If we set $\theta(x) = (1 + x)\log_{0.1}(1 + x) - x$ for each $x \in [1, 100]$. It is easy to verify that $\theta \prec x$ for $a = 1$. Also, put $\mathcal{H} = \mathbb{R}$ and Let $u : X \to \mathbb{R}$ be defined by $u(x) = 1$. Then, for any $\lambda \in \mathbb{R}$, we have*

$$\|\langle u, \lambda\rangle\|_{\theta^*} = \|\lambda\|_{\theta^*} = \inf\{\delta > 0, \int_X \theta^*(\frac{|\lambda|}{\delta}) d\mu \leq 1\} \leq M|\lambda|;$$

where $M = \mu(X)\theta^*(1)$. Hence, by Theorem 2.2, we deduce that $T_u^{\varphi} : L^{\theta}(\mu) \to \mathbb{R}$ is a bounded operator. Direct computation shows that $h = 2$. Moreover, $\sup_{\lambda \in \mathbb{R}_1} hE|\langle u, \lambda\rangle| \circ \varphi^{-1} = h$, where $\mathbb{R}_1$ is the closed unit ball of $\mathbb{R}$. Now, by using Theorem 3.2(ii), we conclude that T_u^{φ} has closed range and also by using Theorem 3.5 (ii), $\|T_u^{\varphi}\|_e \geq 2$.

Example 3.7. *Let* $X = (-1, 0] \cup \{1, 2, 3, 4, 5, 6\}$ *and* μ *be the Lebesque measure on* $(-1, 0]$ *and* $\mu(\{n\}) = \frac{1}{2^n}$*, if* $n = 1, 2, 3, 4, 5, 6$*. Define* $\varphi : X \to X$ *as*

$$\varphi = 3\chi_{\{1\}} + 4\chi_{\{2,3\}} + 5\chi_{\{4,5\}} + 6\chi_{\{6,7\}} + \frac{1}{3}x\chi_{(-1,0]}.$$

If we set $\theta(x) = e^x - x - 1$ *for each* $x \in X$*. It is easy to verify that* $x \prec \theta$ *for* $a = 1$*. Also, put* $\mathcal{H} = \mathbb{R}^2$ *and Let* $u : X \to \mathbb{R}^2$ *be defined by* $u(x) = 1$*. Then, since* $\theta^*(1) \leq 1$ *so for any* $\lambda \in \mathbb{R}^2$*, we have*

$$\|\langle u, \lambda\rangle\|_{\theta^*} = \|\lambda\|_{\theta^*} = \inf\{\delta > 0, \int_X \theta^*(\frac{|\lambda|}{\delta})d\mu \leq 1\} \leq M|\lambda|;$$

where $M = \mu(X) > 0$*. Hence, by Theorem 2.2, we deduce that* $T_u^\varphi : L^\theta(\mu) \to \mathbb{R}$ *is a bounded operator. Direct computation shows that*

$$h = 4\chi_{\{3\}} + 6\chi_{\{4\}} + 3\chi_{\{5\}} + \frac{5}{2}\chi_{\{6\}} + 3\chi_{(-1,0]}.$$

Moreover, $\sup_{\lambda \in \mathbb{R}_1^2} hE|\langle u, \lambda\rangle| \circ \varphi^{-1} = h$*, where* $\mathbb{R}_1^2$ *is the closed unit ball of* $\mathbb{R}^2$*. Now by using Theorem 3.5(iii), we obtain that* $\|T_u^\varphi\|_e \leq \frac{5}{2}$*.*

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