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Inequalities Involving Operator Superquadratic Functions

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Abstract. In this paper, related to the well-known operator convex functions, we study a class of operator functions, the operator superquadratic functions. We present some Jensen-type operator inequalities for these functions. In particular, we show that $f : [0,\infty) \to \mathbb{R}$ is an operator midpoint superquadratic function if and only if

$$f(C^*AC) \leq C^*f(A)C - f\left(\sqrt{C^*A^2C - (C^*AC)^2}\right)$$

holds for every positive operator $A \in \mathcal{B}(\mathcal{H})^+$ and every contraction C. As applications, some inequalities for quasi-arithmetic operator means are given.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be C^*-algebra of all bounded linear operators defined on a complex Hilbert space $\mathcal{H}$ with the identity operator $1_{\mathcal{H}}$. Let $\mathcal{B}(\mathcal{H})^+$ be the set of bounded positive operators on $\mathcal{H}$, and $\mathcal{B}(\mathcal{H})^{++}$ be the set of invertible $A \in \mathcal{B}(\mathcal{H})^+$. We also write $A \geq 0$ when $A \in \mathcal{B}(\mathcal{H})^+$, and $A > 0$ when $A \in \mathcal{B}(\mathcal{H})^{++}$. Let $f : I \to \mathbb{R}$ be a continuous function. We denote by $\sigma(A)$ the spectrum of an operator A. For $\sigma(A) \subseteq I$, we mean by $f(A)$, the continuous functional calculus of f at A.

A continuous function $f : J \subseteq \mathbb{R} \to \mathbb{R}$ is said to be operator convex if $f((1-\lambda)A+\lambda B) \leqslant (1-\lambda)f(A)+\lambda f(B)$ holds for all operators A, B with $\sigma(A), \sigma(B) \subseteq J$ and every $\lambda \in [0,1]$. Every operator convex function is a real convex function, while the converse is not true in general. Typical examples of operator convex functions are the power functions $f(t) = t^p$, where $p \in [-1,0] \cup [1,2]$. If $-f$ is operator convex, then f is called operator concave. For $p \in [0,1]$, then $f(t) = t^p$ is operator concave. The Jensen type operator inequality $f(\Phi(A)) \leq \Phi(f(A))$, known as Choi–Davis operator inequality, holds true for every unital positive linear map Φ and every self-adjoint operator A with spectrum in J if and only if $f : J \subseteq \mathbb{R} \to \mathbb{R}$ is operator convex.

Superquadratic functions have been introduced as a modification of convex functions in [2]. A function $f : [0,\infty) \to \mathbb{R}$ is said to be *superquadratic* whenever for every $s \geq 0$ there exists a constant $C_s \in \mathbb{R}$ such that

$$f(t) \geq f(s) + C_s(t-s) + f(|t-s|), \qquad \forall t \geq 0. \tag{1}$$

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We say that f is *subquadratic* if $-f$ is a superquadratic function.

Now we quote some basic properties of superquadratic functions established in [1] and [2]. Let f be a superquadratic function with C_s as in (1). Then:

(i) $f(0) \leq 0$.
(ii) If f is differentiable and $f(0) = f'(0) = 0$ then $C_s = f'(s)$ for all $s \geq 0$.
(iii) If $f \geq 0$, then f is convex and $f(0) = f'(0) = 0$.

The converse of (iii) is not true: if $1 < p \leq 2$, then $f(x) = x^p$ is convex and subquadratic.

Banić and Varošanec in [3, Theorem 9] gave an important result with characterizations of the superquadratic functions, which are analogous to the well known characterizations of the convex functions: For the function $f : [0, \infty) \to \mathbb{R}$ the following conditions are equivalent:

1. f is a superquadratic function, i.e., there exists a constant C_x such that

$$f(y) \geq f(x) + C_x(y - x) + f(|y - x|), \qquad \forall x, y \geq 0.$$

2. For any two non-negative n-tuples $(x_1, \ldots, x_n)$ and $(p_1, \ldots, p_n)$ such that $P_n = \sum_{i=1}^n p_i > 0$ the inequality

$$f(\bar{x}) \leq \frac{1}{P_n} \sum_{i=1}^n p_i f(x_i) - \frac{1}{P_n} \sum_{i=1}^n p_i f\left(|x_i - \bar{x}|\right) \tag{2}$$

holds, where $\bar{x} = \frac{1}{P_n} \sum_{i=1}^n p_i x_i$.

3. The inequality

$$\begin{aligned} f\left(\lambda y_1 + (1 - \lambda) y_2\right) &\leqslant \lambda f(y_1) + (1 - \lambda) f(y_2) \\ &- \lambda\, f((1 - \lambda)\, |y_1 - y_2|) - (1 - \lambda)\, f(\lambda\, |y_1 - y_2|) \end{aligned} \tag{3}$$

holds for all $y_1, y_2 \geq 0$ and $\lambda \in [0, 1]$.

4. For all $y_1, y_2 \geq 0$, such that $y_1 < x < y_2$ we have

$$\frac{f(y_1) - f(x) - f(x - y_1)}{y_1 - x} \leq \frac{f(y_2) - f(x) - f(x - y_2)}{y_2 - x}.$$

By applying (1) Kian in [6] (see also [7, Corollary 2.6]) gives the Jensen type operator inequality for superquadratic functions:

Corollary A [6, Corollary 2.3]. *Let $A_1, \ldots, A_n$ be positive operators in $\mathcal{B}(\mathcal{H})^+$ and let $x_1, \ldots, x_n \in \mathcal{H}$ be such that $\sum_{i=1}^n \|x_i\| = 1$. If $f : [0, \infty) \to \mathbb{R}$ is a superquadratic function, then*

$$f\Big(\sum_{i=1}^n \langle A_i x_i, x_i \rangle\Big) \leq \sum_{i=1}^n \langle f(A_i) x_i, x_i \rangle - \sum_{i=1}^n \Big\langle f\Big(\Big|A_i - \sum_{k=1}^n \langle A_k x_k, x_k \rangle\Big|\Big) x_i, x_i \Big\rangle.$$

In this paper, we are interested in the class of operator superquadratic functions to obtain operator inequalities for not operator convex functions and to obtain refinement of operator inequalities for some operator functions. We show some Jensen type inequalities for operator superquadratic functions. As applications, some inequalities for quasi-arithmetic operator means are given.

2. Operator superquadratic functions

In what follows, assume that J is an interval of the form $J = [0, M]$ $(M > 0)$ or $J = [0, \infty)$.

As an operator version of superquadratic functions defined by (1), we give the following definition based on (3) and the definition of operator convex function.

Definition 2.1. *A continuous function $f : J \to \mathbb{R}$ is said to be operator superquadratic if*

$$\begin{aligned} f(\lambda A + (1-\lambda) B) \quad &\le \lambda f(A) + (1-\lambda) f(B) \\ &\quad - \lambda f((1-\lambda)|B-A|) - (1-\lambda) f(\lambda |B-A|) \end{aligned} \tag{4}$$

for every $\lambda \in [0,1]$ and all $A, B \in \mathcal{B}(\mathcal{H})^+$ with $\sigma(A), \sigma(B) \subseteq J$.

We say that f is operator subquadratic if $-f$ is operator superquadratic.

We remark that this definition is correct. If $J = [0, M]$ or $J = [0, \infty)$ and $\sigma(A), \sigma(B) \subseteq J$ then $\sigma((1-\lambda)|B - A|), \sigma(\lambda |B - A|) \subseteq J$ for every $\lambda \in [0,1]$.

Remark 2.2.

(i) If f is operator superquadratic on J, then $f(0) \le 0$. Really, setting $A = B = 0$ in the definition (4) *we get that*

$$f(0) = f(\lambda\, 0 + (1-\lambda)\, 0) \le \lambda f(0) + (1-\lambda) f(0) - \lambda f((1-\lambda)\, 0) - (1-\lambda) f(\lambda\, 0) = 0.$$

By using the definition of operator convexity and the definition (4) *it is easy to prove that the following statements are true: If f is operator superquadratic and $f \ge 0$, then f is operator convex and $f(0) = 0$. If $f : J \to (-\infty, 0]$ is an operator convex function, then f is operator superquadratic.*

(ii) It is obvious that if f is operator superquadratic, then it is an operator midpoint-superquadratic function, i.e., the inequality

$$f\left(\frac{A+B}{2}\right) \le \frac{f(A)+f(B)}{2} - f\left(\frac{|A-B|}{2}\right) \tag{5}$$

holds for all $A, B \in \mathcal{B}(\mathcal{H})^+$ with $\sigma(A), \sigma(B) \subseteq J$.

Now, we give some examples.

Example 2.3.

1. *Since a function $f(t) = at + b$ is non-positive on $[0, \infty)$ for $a, b \le 0$ and operator convex, then f is operator superquadratic. Also, f is operator subquadratic for $a, b \ge 0$.*
2. *The power functions $f(t) = t^p$ is real subquadratic on $[0, \infty)$ for $p \in [0, 2]$ and real superquadratic for $p \in [2, \infty)$. What about operator superquadratic and operator subquadratic functions?*
   It is well-known that $f(t) = t^p$ is operator convex if and only if $p \in [-1, 0] \cup [1, 2]$ and it is operator concave if and only if $p \in [0, 1]$. In the case of $t \ge 0$, we have:
   - *(2.1) If $p < 2$, f is not a real superquadratic, so it will not be an operator superquadratic.*
   - *(2.2) If $p > 2$, f is not operator convex, so f is not operator superquadratic, since every non-negative operator superquadratic function must be operator convex.*
   - *(2.3) If $p = 2$, f is operator superquadratic and operator subquadratic, since*
     $\lambda A^2 + (1-\lambda)B^2 - (\lambda A + (1-\lambda)B)^2 - \lambda(1-\lambda)^2|A-B|^2 - (1-\lambda)\lambda^2|A-B|^2 = 0.$
   - *(2.4) If $0 \le p \le 1$, f is operator subquadratic, since every non-negative operator concave function is operator subquadratic.*
3. *It is known that every non-negative operator superquadratic function is operator convex. The converse is not true, since $f(t) = t^p$, $p \in (1, 2)$, are operator convex and non-negative, but these are not operator superquadratic nor operator subquadratic, see (2.1)–(2.4) above.*
4. *Now, we will give an example of a non-negative operator superquadratic function. It is known that the function $f(t) = t \log t$ for $t > 0$ and $f(0) = 0$ is operator convex on $[0, \infty)$. So, $f(t) = t \log t$ is non-positive operator superquadratic on $[0, 1]$. Let $g_\varepsilon(t) = \frac{t^2}{\varepsilon} + t \log t$ for $t \in [\varepsilon \cdot W(1/\varepsilon), 1]$ and $g_\varepsilon(t) = 0$ for $t \in [0, \varepsilon \cdot W(1/\varepsilon)]$, where $0 < \varepsilon \ll 1$ and $W(t)$ is the Lambert W function. We remark that $\varepsilon \cdot W(1/\varepsilon)$ is a null point of a function g_ε and $\lim_{\varepsilon \to 0} \varepsilon \cdot W(1/\varepsilon) = 0$. Also, the function g_ε is operator convex on $[0, \infty)$ and non-negative operator superquadratic on $[0, 1]$.*

5. *For every $p \in [0,1]$, the function $t \mapsto -t^p$ is a non-positive operator convex function and so it is operator superquadratic on $[0,\infty)$. In addition, the function $t \mapsto t^2$ is operator superquadratic on $[0,\infty)$. Consequently, the function $f(t) = t^2 - t^p$ is operator convex and operator superquadratic on $[0,\infty)$.*
6. *It follows from (5) that $f(t) = t^p - t^2$, $p \in [0,1]$, is operator subquadratic on $[0,\infty)$.*
7. *One might think that operator superquadraticity is a stronger criteria than operator convexity. But, if our function takes negative values, then it may be considerably weaker. E.g. any function f satisfying $-2 \leqslant f(t) \leqslant -1$ for all $t \in J$ is operator superquadratic. Really, for all $A, B \in \mathcal{B}(\mathcal{H})^+$ with $\sigma(A), \sigma(B) \subseteq J$ we have $-2\,1_{\mathcal{H}} \leq f(A), f(B) \leq -1_{\mathcal{H}}$ and $1_{\mathcal{H}} \leq -f((1-\lambda)\,|B-A|), -f(\lambda\,|B-A|) \leq 2\,1_{\mathcal{H}}$. Then*

$$-\lambda 1_{\mathcal{H}} \leq \lambda\Big[f(A) - f((1-\lambda)\,|B-A|)\Big] \leq \lambda 1_{\mathcal{H}}$$

$$-(1-\lambda)1_{\mathcal{H}} \leq (1-\lambda)\Big[f(B) - f(\lambda\,|B-A|)\Big] \leq (1-\lambda)1_{\mathcal{H}}$$

for every $\lambda \in [0,1]$. Summing the above two inequalities, we obtain

$$-1_{\mathcal{H}} \leq \lambda f(A) + (1-\lambda) f(B) - \lambda f((1-\lambda)\,|B-A|) - (1-\lambda) f(\lambda\,|B-A|) \leq 1_{\mathcal{H}}.$$

Using that $-2\,1_{\mathcal{H}} \leq f(\lambda\,A + (1-\lambda)\,B) \leq -1_{\mathcal{H}}$ and the above inequality, we get

$$\begin{aligned} &f(\lambda\,A + (1-\lambda)\,B) \\ &\quad \leq \lambda f(A) + (1-\lambda) f(B) - \lambda f((1-\lambda)\,|B-A|) - (1-\lambda) f(\lambda\,|B-A|), \end{aligned}$$

which by definition means that f is operator superquadratic.
According to the above discussion, for example, the functions $f(t) = \frac{1}{2}(\sin(t) - 3)$ and $g(t) = -2\frac{1+t^2}{1+2t^2}$ are operator superquadratic on $[0,\infty)$, while they are not operator convex nor operator concave.

3. Jensen-type operator inequalities

In this section we observe Jensen operator inequality for operator superquadratic functions. Let $\mathcal{B}_h^J(\mathcal{H})$ denote the convex set of self-adjoint operators in $\mathcal{B}(\mathcal{H})$, whose spectra are contained in J. It is well-known that (see [5]) if $f : J \to \mathbb{R}$ is an operator convex function, then the Hansen-Pedersen-Jensen inequality $f(C^*AC) \leq C^* f(A) C$ holds for every isometry $C \in \mathcal{B}(\mathcal{H})$ and every $A \in \mathcal{B}_h^J(\mathcal{H})$. If in addition $0 \in J$ and $f(0) \leq 0$, then the inequality remains valid for every contraction C, in particular, for every projection P. In their ingenious proof of this inequality, Hansen and Pedersen utilized unitary dilations. We want to present a Jensen operator inequality for operator superquadratic functions. To this end, we have to interpret the two operators

$$\lambda f((1-\lambda)|A-B|) \quad \text{and} \quad (1-\lambda) f(\lambda|A-B|) \tag{6}$$

in the definition 2.1. Assume that A, B are positive operators in $\mathcal{B}(\mathcal{H})^+$. For every $\lambda \in [0,1]$, the operator matrix $U_\lambda = \begin{bmatrix} \sqrt{\lambda}1_{\mathcal{H}} & -\sqrt{1-\lambda}1_{\mathcal{H}} \\ \sqrt{1-\lambda}1_{\mathcal{H}} & \sqrt{\lambda}1_{\mathcal{H}} \end{bmatrix}$ is a unitary operator in $\mathbb{M}_2(\mathcal{B}(\mathcal{H}))$. Let $X = A \oplus B$ and $P = 1_{\mathcal{H}} \oplus 0$ so that P is a projection in $\mathbb{M}_2(\mathcal{B}(\mathcal{H}))$. Then

$$\Delta = PU_\lambda^* X U_\lambda P + (I-P)U_{1-\lambda}^* X U_{1-\lambda}(I-P) = (\lambda A + (1-\lambda)B) \oplus (\lambda A + (1-\lambda)B)$$

where $I = 1_{\mathcal{H}} \oplus 1_{\mathcal{H}}$ and $(I-P)$ is the orthogonal projection to P. Our motivation for interpreting the two operators in (6) comes from

$$|X - \Delta| = \left|X - PU_\lambda^* X U_\lambda P + (I-P)U_{1-\lambda}^* X U_{1-\lambda}(I-P)\right| = \begin{bmatrix} (1-\lambda)|A-B| & 0 \\ 0 & \lambda|A-B| \end{bmatrix}$$

and so

$$\begin{aligned} &PU_\lambda^* f(|X-\Delta|) U_\lambda P + (I-P)U_{1-\lambda}^* f(|X-\Delta|) U_{1-\lambda}(I-P) \\ &\quad = \begin{bmatrix} \lambda f((1-\lambda)|A-B|) + (1-\lambda) f(\lambda|A-B|) & 0 \\ 0 & \lambda f((1-\lambda)|A-B|) + (1-\lambda) f(\lambda|A-B|) \end{bmatrix}. \end{aligned}$$

Therefore, we can rewrite inequality (4) as

$$\begin{aligned} & f(PU_\lambda^* X U_\lambda P + (I-P)U_{1-\lambda}^* X U_{1-\lambda}(I-P)) \\ & \quad \le PU_\lambda^* f(X) U_\lambda P + (I-P)U_{1-\lambda}^* f(X) U_{1-\lambda}(I-P) \\ & \qquad - PU_\lambda^* f(|X-\Delta|) U_\lambda P - (I-P)U_{1-\lambda}^* f(|X-\Delta|) U_{1-\lambda}(I-P). \end{aligned}$$

Hence, it is natural to expect that the inequality

$$f\left(\sum_{j=1}^k C_j^* A_j C_j\right) \le \sum_{j=1}^k C_j^* f(A_j) C_j - \sum_{j=1}^k C_j^* f\left(\left|A_j - \sum_{\ell=1}^k C_\ell^* A_\ell C_\ell\right|\right) C_j \tag{7}$$

holds, where $\sum_{j=1}^k C_j^* C_j = 1_{\mathcal{H}}$.

However, when f is operator midpoint superquadratic, we have the following Jensen operator inequality.

Theorem 3.1. *Assume that $f : [0,\infty) \to \mathbb{R}$ is a continuous function. The function f is operator midpoint superquadratic if and only if*

$$f(C^*AC) \le C^* f(A) C - f\left(\sqrt{C^*A^2C - (C^*AC)^2}\right) \tag{8}$$

holds for every positive operator $A \in \mathcal{B}(\mathcal{H})^+$ and every contraction C.

Proof. First note that the operator $C^*A^2C - (C^*AC)^2$ is positive due to the Jensen operator inequality for the operator convex function $t \mapsto t^2$. Secondly, f is defined on $[0,\infty)$ and so the operator $f\left(\sqrt{C^*A^2C - (C^*AC)^2}\right)$ is well-defined. Assume that f is an operator midpoint superquadratic function. We apply an argument similar to that of [5, Theorem 1.9]. Assume that A, B are positive operators. First assume that $C \in \mathcal{B}(\mathcal{H})$ is an isometry, say $C^*C = I$. The block matrices $U = \begin{bmatrix} C & D \\ 0 & -C^* \end{bmatrix}$ and $V = \begin{bmatrix} C & -D \\ 0 & C^* \end{bmatrix}$ are unitary operator matrices in $\mathbb{M}_2(\mathcal{B}(\mathcal{H}))$, provided that $D = (I - CC^*)^{1/2}$. With $\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ we compute

$$\frac{U^*\tilde{A}U + V^*\tilde{A}V}{2} = (C^*AC) \oplus (DAD + CBC^*) \tag{9}$$

and

$$\left|\frac{U^*\tilde{A}U - V^*\tilde{A}V}{2}\right| = |DAC| \oplus |C^*AD|. \tag{10}$$

Now

$$\begin{aligned} & f(C^*AC) \oplus f(DAD + CBC^*) = f(C^*AC \oplus (DAD + CBC^*)) \\ & = f\left(\frac{U^*\tilde{A}U + V^*\tilde{A}V}{2}\right) \qquad \text{(by (9))} \\ & \le \frac{f\left(U^*\tilde{A}U\right) + f\left(V^*\tilde{A}V\right)}{2} - f\left(\left|\frac{U^*\tilde{A}U - V^*\tilde{A}V}{2}\right|\right) \qquad \text{(by (5))} \\ & = \frac{U^*f(\tilde{A})U + V^*f(\tilde{A})V}{2} - f\left(\left|\frac{U^*\tilde{A}U - V^*\tilde{A}V}{2}\right|\right) \\ & = [(C^*f(A)C) \oplus (Df(A)D + Cf(B)C^*)] - [f(|DAC|) \oplus f(|C^*AD|)], \end{aligned}$$

where the last equality follows from (9) and (10). Comparing the $(1,1)$-blocks of both sides we learn that $f(C^*AC) \le C^*f(A)C - f(|DAC|)$. However,

$$|DAC|^2 = (C^*A(I - CC^*)AC)^{1/2} = \left(C^*A^2C - (C^*AC)^2\right)^{1/2}$$

and so (8) holds for every isometry C.

To see that (8) is valid even if C is a contraction, note that if (8) is valid, then a multivariate version holds as well. If $A_1, \ldots, A_k$ are positive operators and $C_1, \ldots, C_k \in \mathcal{B}(\mathcal{H})$ with $\sum_{j=1}^k C_j^*C_j = I$, then

$$f\left(\sum_{j=1}^k C_j^*A_jC_j\right) \le \sum_{j=1}^k C_j^*f(A_j)C_j - f\left(\sqrt{\sum_{j=1}^k C_j^*A_j^2C_j - \left(\sum_{j=1}^k C_j^*A_jC_j\right)^2}\right). \tag{11}$$

Now note that if C is a contraction, i.e., $C^*C \le I$, then there exists an operator D such that $C^*C + D^*D = I$. Use (11) with $C_1 := C$, $C_2 := D$, $A_1 := A$ and $A_2 := 0$ to get

$$f(C^*AC) = f(C^*AC + D^*0D) \le C^*f(A)C + D^*f(0)D - f\left(\sqrt{C^*A^2C - (C^*AC)^2}\right). \tag{12}$$

Since $f(0) \le 0$ for every operator midpoint superquadratic function f, we find out that (8) holds, when C is a contraction.

Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators and $\lambda \in [0,1]$. Put $C_1 = I \oplus 0$ and $C_2 = 0 \oplus I$ so that $C_1^*C_1 + C_2^*C_2 = I$. With $X = A \oplus B \in \mathbb{M}_2(\mathcal{B}(\mathcal{H}))$ and the unitary $U = 1/2\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ we have

$$\begin{aligned}
&f\left(\frac{A+B}{2}\right) \oplus f\left(\frac{A+B}{2}\right) \\
&\quad = f\left(\frac{A+B}{2} \oplus \frac{A+B}{2}\right) \\
&\quad = f(C_1^*U^*XUC_1 + C_2^*U^*XUC_2) \\
&\quad \le C_1^*f(U^*XU)C_1 + C_2^*f(U^*XU)C_2 \\
&\qquad - f\left(\sqrt{C_1^*(U^*XU)^2C_1 + C_2^*(U^*XU)^2C_2 - \left(C_1^*U^*XUC_1 + C_2^*U^*XUC_2\right)^2}\right).
\end{aligned} \tag{13}$$

By calculating we have

$$\sqrt{C_1^*(U^*XU)^2C_1 + C_2^*(U^*XU)^2C_2 - \left(C_1^*U^*XUC_1 + C_2^*U^*XUC_2\right)^2} = \frac{|A-B|}{2} \oplus \frac{|A-B|}{2},$$

whence we conclude from (13) that

$$\begin{aligned}
f\left(\frac{A+B}{2}\right) \oplus f\left(\frac{A+B}{2}\right) &\le C_1^*U^*f(X)UC_1 + C_2^*U^*f(X)UC_2 - f\left(\frac{|A-B|}{2}\right) \oplus f\left(\frac{|A-B|}{2}\right) \\
&= \frac{f(A)+f(B)}{2} \oplus \frac{f(A)+f(B)}{2} - f\left(\frac{|A-B|}{2}\right) \oplus f\left(\frac{|A-B|}{2}\right).
\end{aligned}$$

Hence we obtain (4) for $\lambda = 1/2$ so that f is operator midpoint superquadratic. □

It is remarkable that the equality holds in (8) if and only if $f(t) = t^2$.

Remark 3.2. *As a special case of Theorem 3.1 we obtained that*

$$f\left(\sum_{j=1}^k C_j^*A_jC_j\right) \le \sum_{j=1}^k C_j^*f(A_j)C_j - f\left(\sqrt{\sum_{j=1}^k C_j^*A_j^2C_j - \left(\sum_{j=1}^k C_j^*A_jC_j\right)^2}\right). \tag{14}$$

holds for every operator superquadratic function f, where $A_1, \ldots, A_k$ are positive operators and $C_1, \ldots, C_k \in \mathcal{B}(\mathcal{H})$ with $\sum_{j=1}^k C_j^*C_j = I$ (see (11)).

Corollary 3.3. *Let $f : [0,\infty) \to \mathbb{R}$ be operator superquadratic and A, B be invertible positive operators. If $A \leq B$, then*

$$A^{-1}f(A) \leq B^{-1}f(B) - A^{-1}\sigma_{f(\sqrt{t})}(B-A),$$

where $\sigma_{f(\sqrt{t})}$ is defined similar to the operator mean corresponding to the function $t \mapsto g(t) \equiv f(\sqrt{t})$:

$$X\sigma_g Y := X^{\frac{1}{2}} g\left(X^{-\frac{1}{2}} Y X^{-\frac{1}{2}}\right) X^{\frac{1}{2}} \quad \text{for } X, Y \in \mathcal{B}(\mathcal{H})^+ \text{ and } X \text{ is invertible.} \tag{15}$$

Proof. Assume that $A \leq B$ so that the operator $C = B^{-1/2}A^{1/2}$ is a contraction. It follows from (8) that

$$\begin{aligned} f(A) = f(C^*BC) &\leq C^*f(B)C - f\left(\sqrt{C^*B^2C - (C^*BC)^2}\right) \\ &= A^{1/2}B^{-1/2}f(B)B^{-1/2}A^{1/2} - f\left(\sqrt{A^{1/2}BA^{1/2} - A^2}\right) \\ &= A^{1/2}\left(B^{-1}f(B) - A^{-1/2}f\left(\sqrt{A^{1/2}(B-A)A^{1/2}}\right)A^{-1/2}\right)A^{1/2}. \end{aligned}$$

Therefore

$$A^{-1}f(A) \leq B^{-1}f(B) - A^{-1}\sigma_{f(\sqrt{t})}(B-A)$$

□

Remark 3.4.

(i) If $g : [0,\infty) \to [0,\infty)$ is operator monotone, then σ_g defined by (15) *is a connection. This connection is an operator mean if and only if g is normalized in the sense $g(1) = 1$, see the theory of operator means established by Kubo and Ando [8].*

(ii) Consider the function $f(t) = t^2 - t^p$, $p \in (0,1]$, which is operator superquadratic on $[0,\infty)$. Corollary 3.3 implies that

$$A^{-1}(A^2 - A^p) \leq B^{-1}(B^2 - B^p) - A^{-1}\sigma_{t-t^{p/2}}(B-A).$$

Noting that $A^{-1}\sigma_{t-t^{p/2}}(B-A) = (B-A) - A^{-1}\sharp_{p/2}(B-A)$, so we have

$$B^{p-1} - A^{p-1} \leq A^{-1}\sharp_{p/2}(B-A) \leq B - A \qquad \text{for every } p \in (0,1].$$

Next, we define Jensen's operator, deduced from Jensen's functional, see e.g. [10, Definition 2]. Let $\mathcal{F}_o(J)$ denote the set of all continuous real-valued functions on an interval J. Let $\mathcal{B}_h^J(\mathcal{H})$ denote the convex set of self-adjoint operators in $\mathcal{B}(\mathcal{H})$, whose spectra are contained in J.

We define Jensen's operator $\mathcal{J}_n : \mathcal{F}_o(J) \times [\mathcal{B}_h^J(\mathcal{H})]^n \times (0,\infty)^n \to \mathcal{B}(\mathcal{H})$ by

$$\mathcal{J}_n(f, \mathbf{X}, \mathbf{p}) = \frac{1}{P_n}\sum_{i=1}^n p_i f(X_i) - f\left(\frac{1}{P_n}\sum_{i=1}^n p_i X_i\right), \tag{16}$$

where $\mathbf{X} = (X_1, X_2, \dots, X_n)$, $\mathbf{p} = (p_1, p_2, \dots, p_n)$ and $P_n = \sum_{i=1}^n p_i$.

Note that the operator $\mathcal{J}_n$ is well-defined. If f is an operator convex function, then Jensen's operator inequality implies that $\mathcal{J}_n(f, \mathbf{X}, \mathbf{p})$ is a positive operator $\mathcal{J}_n(f, \mathbf{X}, \mathbf{p}) \geq 0$.

Putting $C_j = \sqrt{\frac{p_j}{P_n}}1_{\mathcal{H}}$, $j = 1, 2, \dots, n \equiv k$, in Remark 3.2 we obtain the following corollary.

Corollary 3.5. *Suppose $\mathcal{J}_n$ is an operator defined by* (16) *and $J = [0, M]$ or $J = [0, \infty)$. If f is an operator superquadratic function on J, then*

$$\mathcal{J}_n(f, \mathbf{X}, \mathbf{p}) \geq f\left(\sqrt{\frac{1}{P_n}\sum_{i=1}^{n} p_i X_i^2 - \bar{X}_n^2}\right) = f\left(\sqrt{\mathcal{J}_n(t^2, \mathbf{X}, \mathbf{p})}\right) \tag{17}$$

holds, where $\bar{X}_n = \frac{1}{P_n}\sum_{i=1}^{n} p_i X_i$.

If f is operator subquadratic, then the reverse inequality is valid in (17).

By using mathematical induction we get the following lower bound of Jensen's operator.

Proposition 3.6. *Suppose $\mathcal{J}_n$ is an operator defined by* (16) *and $J = [0, M]$ or $J = [0, \infty)$. If f is an operator superquadratic function on J, then*

$$\mathcal{J}_n(f, \mathbf{X}, \mathbf{p}) \geq \frac{p_1}{P_n} f(|X_1 - \bar{X}_2|) + \frac{1}{P_n}\sum_{i=2}^{n} p_i\, f(|X_i - \bar{X}_i|) + \frac{1}{P_n}\sum_{i=3}^{n} P_{i-1}\, f\Big(\frac{p_i}{P_{i-1}}\, |X_i - \bar{X}_i|\Big), \tag{18}$$

where $\bar{X}_i = \frac{1}{P_i}\sum_{k=1}^{i} p_k\, X_k$ and $P_i = \sum_{k=1}^{i} p_k$, $i = 2, \dots, n$.

If f is operator subquadratic, then the reverse inequality is valid in (18).

Proof. We prove (18) using mathematical induction. Since f is operator superquadratic, then setting $\lambda = \frac{p_1}{P_2}$, $1 - \lambda = \frac{p_2}{P_2}$ in (4) gives (18) for $n = 2$, where we note that $(1 - \frac{p_i}{P_2})\, |X_1 - X_2| = |X_i - \bar{X}_2|$, $i = 1, 2$. Now assume that (18) is valid for a natural number $n \geq 2$. Then for a $(n + 1)$-tuples $(X_1, \dots, X_{n+1})$ and $(p_1, \dots, p_{n+1})$ we have

$$\begin{aligned} f\left(\sum_{i=1}^{n+1} \frac{p_i}{P_{n+1}}\, X_i\right) &= f\left(\frac{P_n}{P_{n+1}}\left(\sum_{i=1}^{n} \frac{p_i}{P_n}\, X_i\right) + \frac{p_{n+1}}{P_{n+1}}\, X_{n+1}\right) \\ &\leq \frac{P_n}{P_{n+1}}\, f\left(\sum_{i=1}^{n} \frac{p_i}{P_n}\, X_i\right) + \frac{p_{n+1}}{P_{n+1}}\, f(X_{n+1}) \\ &\quad - \frac{P_n}{P_{n+1}}\, f\left(\left|\sum_{i=1}^{n} \frac{p_i}{P_n}\, X_i - \bar{X}_{n+1}\right|\right) - \frac{p_{n+1}}{P_{n+1}}\, f\left(|X_{n+1} - \bar{X}_{n+1}|\right), \end{aligned} \tag{19}$$

where we use the (18) for $n = 2$. Moreover, by the hypothesis of induction in step n, we can write

$$\begin{aligned} f\left(\sum_{i=1}^{n} \frac{p_i}{P_n}\, X_i\right) &\leq \sum_{i=1}^{n} \frac{p_i}{P_n}\, f(X_i) - \frac{p_1}{P_n}\, f(|X_1 - \bar{X}_2|) \\ &\quad - \sum_{i=2}^{n} \frac{p_i}{P_n}\, f(|X_i - \bar{X}_i|) - \sum_{i=3}^{n} \frac{P_{i-1}}{P_n}\, f\left(\frac{p_i}{P_{i-1}}\, |X_i - \bar{X}_i|\right). \end{aligned} \tag{20}$$

It follows from (19) and (20) that

$$\begin{aligned} f\left(\sum_{i=1}^{n+1} \frac{p_i}{P_{n+1}}\, X_i\right) &\leq \sum_{i=1}^{n+1} \frac{p_i}{P_{n+1}}\, f(X_i) - \frac{p_1}{P_{n+1}}\, f(|X_1 - \bar{X}_2|) - \sum_{i=2}^{n+1} \frac{p_i}{P_{n+1}}\, f(|X_i - \bar{X}_i|) \\ &\quad - \sum_{i=3}^{n} \frac{P_{i-1}}{P_{n+1}}\, f\left(\frac{p_i}{P_{i-1}}\, |X_i - \bar{X}_i|\right) - \frac{P_n}{P_{n+1}}\, f\left(\left|\bar{X}_n - \bar{X}_{n+1}\right|\right). \end{aligned} \tag{21}$$

Noting that $\left|\bar{X}_n - \bar{X}_{n+1}\right| = \frac{p_{n+1}}{P_n}\left|X_{n+1} - \bar{X}_{n+1}\right|$, we have

$$-\sum_{i=3}^{n} \frac{P_{i-1}}{P_{n+1}}\, f\left(\frac{p_i}{P_{i-1}}\, |X_i - \bar{X}_i|\right) - \frac{P_n}{P_{n+1}}\, f\left(\left|\bar{X}_n - \bar{X}_{n+1}\right|\right) = -\sum_{i=3}^{n+1} \frac{P_{i-1}}{P_{n+1}}\, f\left(\frac{p_i}{P_{i-1}}\, |X_i - \bar{X}_i|\right)$$

and so (21) concludes the desired inequality (18) for $n + 1$. □

Next, we obtain another lower bound for Jensen's operator if f is an operator midpoint-superquadratic function. We need the following lemma, which is interesting in its own.

Lemma 3.7. *Let $\mathbf{X} = (X_1, X_2)$, where $X_1, X_2 \in \mathcal{B}(\mathcal{H})^+$ be positive operators on $\mathcal{H}$ with spectra contained in J, and $\mathbf{q} = (\frac{1}{2^m}, \frac{2^m-1}{2^m})$ for $m \in \mathbb{N}$. If f is an operator midpoint-superquadratic function on $J = [0, M]$ or on $J = [0, \infty)$, then*

$$\mathcal{J}_2(f, \mathbf{X}, \mathbf{q}) \geq \sum_{i=1}^{m} 2^{i-m} f\left(\frac{|X_1 - X_2|}{2^i}\right). \tag{22}$$

If f is an operator midpoint-subquadratic, then the reverse inequality is valid in (22).

Proof. The proof is based on mathematical induction. For $m = 1$ (22) is just (5). Fix $m \in \mathbb{N}$, $m > 1$ and suppose that (22) is true. Then

$$\begin{aligned}
& f\left(\tfrac{1}{2^{m+1}} X_1 + \tfrac{2^{m+1}-1}{2^{m+1}} X_2\right) = f\left(\tfrac{1}{2}\left(\tfrac{X_1+(2^m-1)X_2}{2^m} + X_2\right)\right) \\
\leq\ & \tfrac{1}{2} f\left(\tfrac{X_1+(2^m-1)X_2}{2^m}\right) + \tfrac{1}{2} f(X_2) - f\left(\tfrac{1}{2}\left|\tfrac{X_1+(2^m-1)X_2}{2^m} - X_2\right|\right) \\
\leq\ & \tfrac{1}{2}\left[\tfrac{1}{2^m} f(X_1) + \tfrac{2^m-1}{2^m} f(X_2) - \sum_{i=1}^{m} \tfrac{1}{2^m} 2^i f\left(\tfrac{|X_1-X_2|}{2^i}\right)\right] + \tfrac{1}{2} f(X_2) - f\left(\tfrac{|X_2-X_1|}{2^{m+1}}\right) \\
=\ & \tfrac{1}{2^{m+1}} f(X_1) + \tfrac{2^{m+1}-1}{2^{m+1}} f(X_2) - \sum_{i=1}^{m+1} \tfrac{1}{2^{m+1-i}} f\left(\tfrac{|X_1-X_2|}{2^i}\right).
\end{aligned}$$

In other words

$$\mathcal{J}_2(f, \mathbf{X}, \mathbf{q}) \geq \sum_{i=1}^{m+1} 2^{i-m+1} f\left(\frac{|X_1 - X_2|}{2^i}\right)$$

holds for $\mathbf{q} = (\frac{1}{2^{m+1}}, \frac{2^{m+1}-1}{2^{m+1}})$, i.e. (22) is true for $m + 1$. □

Corollary 3.8. *Let $\mathcal{J}_n$ be an operator defined by* (16), *$J = [0, M]$ or $J = [0, \infty)$, and $m \in \mathbb{N}$. If f is an operator midpoint-superquadratic function on J, then*

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i \mathcal{J}_2\left(f, (X_i, \bar{X}_n), \left(\frac{1}{2^m}, \frac{2^m-1}{2^m}\right)\right) \geq \frac{1}{P_n} \sum_{i=1}^{n} p_i \sum_{j=1}^{m} 2^{j-m} f\left(\frac{|X_i - \bar{X}_n|}{2^j}\right). \tag{23}$$

If f is non-negative operator superquadratic, then

$$\mathcal{J}_n(f, \mathbf{X}, \mathbf{p}) \geq \frac{1}{P_n} \sum_{i=1}^{n} p_i \sum_{j=1}^{m} 2^j f\left(\frac{|X_i - \bar{X}_n|}{2^j}\right) \geq 0. \tag{24}$$

If f is operator midpoint-subquadratic, then the reverse inequality is valid in (23). *If f is non-positive operator subquadratic, then the reverse inequality is valid in* (24).

Proof. By replacing X_1 with X_i and X_2 with $\bar{X}_n := \frac{1}{P_n} \sum_{i=1}^{n} p_i X_i$ in (22), and then multiplying by $\frac{p_i}{P_n}$ and finally summing over i we obtain (23).

If $f \geq 0$ operator superquadratic, then f is operator convex and Jensen's operator inequality gives

$$f(\bar{X}_n) = f\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i \left(\frac{1}{2^m} X_i + \frac{2^m-1}{2^m} \bar{X}_n\right)\right) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i f\left(\frac{1}{2^m} X_i + \frac{2^m-1}{2^m} \bar{X}_n\right).$$

By using the reverse of the above inequality we have

$$\begin{aligned}\frac{1}{2^m}\mathcal{J}_n(f,\mathbf{X},\mathbf{p}) &= \frac{1}{2^m}\frac{1}{P_n}\sum_{i=1}^{n} p_i f(X_i) - \frac{1}{2^m} f(\bar{X}_n) = \frac{1}{2^m}\frac{1}{P_n}\sum_{i=1}^{n} p_i f(X_i) + \frac{2^m-1}{2^m} f(\bar{X}_n) - f(\bar{X}_n) \\ &\geq \frac{1}{2^m}\frac{1}{P_n}\sum_{i=1}^{n} p_i f(X_i) + \frac{2^m-1}{2^m} f(\bar{X}_n) - \frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(\frac{1}{2^m}X_i + \frac{2^m-1}{2^m}\bar{X}_n\right) \\ &= \frac{1}{P_n}\sum_{i=1}^{n} p_i \mathcal{J}_2\left(f,(X_i,\bar{X}_n),\left(\frac{1}{2^m},\frac{2^m-1}{2^m}\right)\right).\end{aligned}$$

Finally, by using (23) and the above inequality we obtain

$$\frac{1}{2^m}\mathcal{J}_n(f,\mathbf{X},\mathbf{p}) \geq \frac{1}{P_n}\sum_{i=1}^{n} p_i \sum_{j=1}^{m} 2^{j-m} f\left(\frac{|X_i-\bar{X}_n|}{2^j}\right),$$

which gives (24). □

Remark 3.9. *Let f be non-negative operator superquadratic. As a special case of* (24) *we obtain the following improvement of the lower bound of Jensen's operator:*

a) $$\mathcal{J}_n(f,\mathbf{X},\mathbf{p}) \geq \frac{2}{P_n}\sum_{i=1}^{n} p_i f\left(\frac{|X_i-\bar{X}_n|}{2}\right) \geq 0,$$

b) $$\mathcal{J}_n(f,\mathbf{X},\mathbf{1/n}) \geq \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m} 2^j f\left(\frac{|X_i-\bar{X}_n|}{2^j}\right) \geq 0,$$

c) $$\mathcal{J}_n(f,\mathbf{X},\mathbf{1/n}) \geq \frac{2}{n}\sum_{i=1}^{n} f\left(\frac{|X_i-\bar{X}_n|}{2}\right) \geq 0,$$

where $\mathbf{1/n} = (\frac{1}{n},\ldots,\frac{1}{n})$.

4. Some inequalities for quasi-arithmetic operator means

By applying results obtained in the previous section, we give some inequalities for quasi-arithmetic operator means.

We recall the definition of weighted quasi-arithmetic operator means:

$$\mathcal{M}_\varphi(\mathbf{X},\mathbf{p},n) := \varphi^{-1}\left(\sum_{j=1}^{n}\frac{p_j}{P_n}\varphi(X_j)\right), \tag{25}$$

where $\mathbf{X} = (X_1,\ldots,X_n)$ is an n-tuple of self-adjoint operators in $\mathcal{B}_h(\mathcal{H})$ with spectra in an interval J, $\mathbf{p} = \frac{1}{P_n}(p_1,\ldots,p_n)$ is a weight vector, i.e. $p_1,\ldots,p_n \geq 0$ with $\sum_{j=1}^{n} p_j = P_n > 0$, and $\varphi : J \to \mathbb{R}$ is a strictly monotone function.

The power operator mean is a special case of the weighted quasi-arithmetic mean

$$\mathcal{M}_q(\mathbf{X},\mathbf{p},n) := \left(\sum_{j=1}^{n}\frac{p_j}{P_n}X_j^q\right)^{1/q}, \quad q \in \mathbb{R}\backslash\{0\}, \quad \text{for positive operators } X_1,\ldots,X_n.$$

By $C(J)$ we mean the space of continuous real-valued functions on the interval J. It is known that quasi-arithmetic means enjoy a monotonicity property as follows.

Theorem B [9, Theorem 2.1]. *Let $\mathbf{X} = (X_1,\ldots,X_n)$, $\mathbf{p} = (\frac{p_1}{P_n},\ldots,\frac{p_n}{P_n})$ be as in the definition of the quasi-arithmetic mean* (25) *and let $\psi,\varphi \in C(J)$ be strictly monotone functions.*
If one of the following conditions is satisfied:

(i) $\psi \circ \varphi^{-1}$ *is operator convex and* ψ^{-1} *is operator monotone,*

(i') $\psi \circ \varphi^{-1}$ *is operator concave and* $-\psi^{-1}$ *is operator monotone,*

then

$$M_\varphi(\mathbf{X}, \mathbf{p}, n) \le M_\psi(\mathbf{X}, \mathbf{p}, n). \tag{26}$$

If one of the following conditions is satisfied:

(ii) $\psi \circ \varphi^{-1}$ *is operator concave and* ψ^{-1} *is operator monotone,*

(ii') $\psi \circ \varphi^{-1}$ *is operator convex and* $-\psi^{-1}$ *is operator monotone,*

then the reverse inequality is valid in (26).

First, we give some general results for superquadratic functions.

Theorem 4.1. *Let* $\mathbf{X} = (X_1, \ldots, X_n)$, $\mathbf{p} = (\frac{p_1}{P_n}, \ldots, \frac{p_n}{P_n})$ *be as in the definition of quasi-arithmetic mean* (25) *and let* $\varphi, \psi \in C(J)$ *be strictly monotone functions and* φ *be non-negative.*

If one of the following conditions is satisfied:

(i) $\psi \circ \varphi^{-1}$ *is operator superquadratic and* ψ^{-1} *is operator monotone,*

(i') $\psi \circ \varphi^{-1}$ *is operator subquadratic and* $-\psi^{-1}$ *is operator monotone,*

then

$$\begin{aligned} & \mathcal{M}_\varphi(\mathbf{X}, \mathbf{p}, n) \\ \le \;& \psi^{-1}\Big(\psi\Big(\mathcal{M}_\psi(\mathbf{X}, \mathbf{p}, n)\Big) - \psi \circ \varphi^{-1}\Big(\sqrt{\varphi^2 \mathcal{M}_{\varphi^2}(\mathbf{X}, \mathbf{p}, n) - (\varphi \mathcal{M}_\varphi(\mathbf{X}, \mathbf{p}, n))^2}\Big)\Big). \end{aligned} \tag{27}$$

But, if one of the following conditions is satisfied:

(ii) $\psi \circ \varphi^{-1}$ *is operator subquadratic and* ψ^{-1} *is operator monotone,*

(ii') $\psi \circ \varphi^{-1}$ *is operator superquadratic and* $-\psi^{-1}$ *is operator monotone,*

then the reverse inequality is valid in (27).

Proof. We will only prove case (*i*). If we put $f = \psi \circ \varphi^{-1}$ in Corollary 3.5 and replace X_j with $\varphi(X_j)$, we obtain

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i \psi(X_j) - \psi \circ \varphi^{-1}\Big(\frac{1}{P_n} \sum_{i=1}^{n} p_i \varphi(X_j)\Big) \ge \mathcal{D}_0, \tag{28}$$

where

$$\begin{aligned} \mathcal{D}_0 &= \psi \circ \varphi^{-1}\left(\sqrt{\tfrac{1}{P_n}\textstyle\sum_{i=1}^n p_i \varphi(X_i)^2 - \left(\tfrac{1}{P_n}\sum\limits_{i=1}^n p_i\, \varphi(X_i)\right)^2}\right). \\ &= \psi \circ \varphi^{-1}\left(\sqrt{\varphi^2\left(\mathcal{M}_{\varphi^2}(\mathbf{X}, \mathbf{p}, n)\right) - (\varphi\left(\mathcal{M}_\varphi\left(\mathbf{X}, \mathbf{p}, n\right)\right))^2}\right). \end{aligned}$$

We can concisely write (28) as

$$\begin{aligned} & \psi\left(\mathcal{M}_\varphi(\mathbf{X}, \mathbf{p}, n)\right) \\ \le \;& \psi\left(\mathcal{M}_\psi(\mathbf{X}, \mathbf{p}, n)\right) - \psi \circ \varphi^{-1}\left(\sqrt{\varphi^2\left(\mathcal{M}_{\varphi^2}(\mathbf{X}, \mathbf{p}, n)\right) - (\varphi\left(\mathcal{M}_\varphi\left(\mathbf{X}, \mathbf{p}, n\right)\right))^2}\right). \end{aligned} \tag{29}$$

Applying operator monotonicity of ψ^{-1} we obtain (27). □

It is known that [4, Corollary 3.2] if A, B are positive operators on a finite dimensional Hilbert space and $f : [0,\infty) \to [0,\infty)$ is a monotone convex function with $f(0) \le 0$, then there exist unitaries U and V such that

$$f(A+B) \ge U^* f(A) U + V^* f(B) V. \tag{30}$$

Parallel to this, if f is monotone increasing, then $A \le B$ implies that $f(A) \le U^* f(B) U$ for some unitary U, see [4]. Now assume that $\mathbf{X}, \mathbf{p}$ are as in Theorem 4.1. If $\psi \circ \varphi^{-1}$ is operator superquadratic, then we have from (40) that

$$\begin{aligned} &\psi\left(\mathcal{M}_\varphi(\mathbf{X},\mathbf{p},n)\right) + \psi\circ\varphi^{-1}\left(\sqrt{\varphi^2\left(\mathcal{M}_{\varphi^2}(\mathbf{X},\mathbf{p},n)\right) - \left(\varphi\left(\mathcal{M}_\varphi(\mathbf{X},\mathbf{p},n)\right)\right)^2}\right) \\ \le\ &\psi\left(\mathcal{M}_\psi(\mathbf{X},\mathbf{p},n)\right). \end{aligned} \tag{31}$$

Then suppose that ψ^{-1} is monotone increasing and convex. Applying (30) we get

$$\begin{aligned} &U^*\mathcal{M}_\varphi(\mathbf{X},\mathbf{p},n)U + V^*\varphi^{-1}\left(\sqrt{\varphi^2\left(\mathcal{M}_{\varphi^2}(\mathbf{X},\mathbf{p},n)\right) - \left(\varphi\left(\mathcal{M}_\varphi(\mathbf{X},\mathbf{p},n)\right)\right)^2}\right)V \\ \le\ &\psi^{-1}\left\{\psi\left(\mathcal{M}_\varphi(\mathbf{X},\mathbf{p},n)\right) + \psi\circ\varphi^{-1}\left(\sqrt{\varphi^2\left(\mathcal{M}_{\varphi^2}(\mathbf{X},\mathbf{p},n)\right) - \left(\varphi\left(\mathcal{M}_\varphi(\mathbf{X},\mathbf{p},n)\right)\right)^2}\right)\right\} \end{aligned}$$

for some unitaries U and V. On the other hand, from the monotonicity of ψ^{-1} and (31) we conclude that

$$\begin{aligned} &\psi^{-1}\left\{\psi\left(\mathcal{M}_\varphi(\mathbf{X},\mathbf{p},n)\right) + \psi\circ\varphi^{-1}\left(\sqrt{\varphi^2\left(\mathcal{M}_{\varphi^2}(\mathbf{X},\mathbf{p},n)\right) - \left(\varphi\left(\mathcal{M}_\varphi(\mathbf{X},\mathbf{p},n)\right)\right)^2}\right)\right\} \\ \le\ &W^*\mathcal{M}_\psi(\mathbf{X},\mathbf{p},n)W \end{aligned}$$

for some unitary W. Accordingly, two unitaries U and V can be found in such a way that

$$\begin{aligned} &\mathcal{M}_\varphi(\mathbf{X},\mathbf{p},n) + V^*\varphi^{-1}\left(\sqrt{\varphi^2\left(\mathcal{M}_{\varphi^2}(\mathbf{X},\mathbf{p},n)\right) - \left(\varphi\left(\mathcal{M}_\varphi(\mathbf{X},\mathbf{p},n)\right)\right)^2}\right)V \\ \le\ &U^*\mathcal{M}_\psi(\mathbf{X},\mathbf{p},n)U. \end{aligned} \tag{32}$$

The above discussion brings us to the next corollary.

Corollary 4.2. *Let $\mathcal{H}$ be a finite dimensional Hilbert space, $\mathbf{X}, \mathbf{p}$ be as in Theorem 4.1 and let $\varphi, \psi \in C([0,\infty))$ be positive strictly monotone functions. If $\psi\circ\varphi^{-1}$ is operator superquadratic and ψ^{-1} is monotone increasing and convex, then* (32) *holds for some unitaries U and V. If $\psi\circ\varphi^{-1}$ is operator subquadratic and ψ^{-1} is monotone increasing and concave, then the reverse inequality holds in* (32)*.*

We give an example to clarify (32).

Example 4.3. *Let $\mathcal{H}$ be a finite dimensional Hilbert space. Assume that $q \ge 1$ so that the function $\psi(t) = t^q$ is monotone increasing and convex on $[0,\infty)$. If we consider $\varphi(t) = t^{q/2}$, then $\psi\circ\varphi^{-1}(t) = t^2$ is operator superquadratic and* (32) *gives*

$$\mathcal{M}_{q/2}(\mathbf{X},\mathbf{p},n) + V^*\left[\frac{1}{P_n}\sum_{i=1}^n p_i X_i^q - \left(\frac{1}{P_n}\sum_{i=1}^n p_i X_i^{q/2}\right)^2\right]^{1/q} V \le U^*\mathcal{M}_q(\mathbf{X},\mathbf{p},n)U$$

for some unitaries U and V. Moreover, $\psi\circ\varphi^{-1}(t) = t^2$ is operator convex and Theorem B gives $\mathcal{M}_{q/2}(\mathbf{X},\mathbf{p},n) \le \mathcal{M}_q(\mathbf{X},\mathbf{p},n)$.

The next theorem follows from Proposition 3.6. The proof is similar to that of Theorem 4.1.

Theorem 4.4. *Let the assumptions of Theorem 4.1 hold. If one of the conditions (i) or (i') is satisfied then*

$$\mathcal{M}_\varphi(\mathbf{X},\mathbf{p},n) \le \psi^{-1}\Big(\psi\Big(\mathcal{M}_\psi(\mathbf{X},\mathbf{p},n)\Big) - \mathcal{D}\Big) \tag{33}$$

holds, where

$$\begin{aligned} \mathcal{D} \;\equiv\; & \mathcal{D}(\varphi,\psi,\mathbf{X},\mathbf{p},n) = \tfrac{p_1}{P_n}\psi\circ\varphi^{-1}\Big(\Big|\varphi(X_1) - \varphi\Big(\mathcal{M}_\varphi(\mathbf{X},\mathbf{p},2)\Big)\Big|\Big) \\ + & \sum_{j=2}^{n}\frac{p_j}{P_n}\psi\circ\varphi^{-1}\Big(\Big|\varphi(X_j) - \varphi\Big(\mathcal{M}_\varphi(\mathbf{X},\mathbf{p},j)\Big)\Big|\Big) \\ + & \sum_{j=3}^{n}\frac{p_{j-1}}{P_n}\psi\circ\varphi^{-1}\Big(\frac{p_j}{P_{j-1}}\Big|\varphi(X_j) - \varphi\Big(\mathcal{M}_\varphi(\mathbf{X},\mathbf{p},j)\Big)\Big|\Big). \end{aligned} \tag{34}$$

But, if one of the conditions (ii) or (ii') is satisfied, then the reverse inequality is valid in (33).

Let $f \in C(J)$ be a strictly monotone function and let f^{-1} be operator superquadratic. By applying Theorem 4.4 with $\varphi \equiv f$ and $\psi \equiv \mathcal{I}$ (the identity function) we conclude from (33) that $\mathcal{M}_f(\mathbf{X},\mathbf{p},n) \le \mathcal{M}_1(\mathbf{X},\mathbf{p},n) - \mathcal{D}_1$ holds, i.e.

$$\mathcal{M}_f(\mathbf{X},\mathbf{p},n) + \mathcal{D}_1 \le \mathcal{M}_1(\mathbf{X},\mathbf{p},n), \tag{35}$$

where $\mathcal{M}_1(\mathbf{X},\mathbf{p},n)$ is the weighted arithmetic mean, $\mathcal{D}_1 = \mathcal{D}(f,\mathcal{I},\mathbf{X},\mathbf{p},n)$ and $\mathcal{D}$ is defined by (34).

Similarly, if $g \in C(J)$ is a strictly monotone function and g^{-1} is operator subquadratic, then the reverse inequality in (33) gives

$$\mathcal{M}_g(\mathbf{X},\mathbf{p},n) + \mathcal{D}_2 \ge \mathcal{M}_1(\mathbf{X},\mathbf{p},n) \tag{36}$$

where $\mathcal{D}_2 = \mathcal{D}(g,\mathcal{I},\mathbf{X},\mathbf{p},n)$.

By combining (35) and (36) we get $\mathcal{M}_f(\mathbf{X},\mathbf{p},n) + \mathcal{D}_1 \le \mathcal{M}_g(\mathbf{X},\mathbf{p},n) + \mathcal{D}_2$.

So we reach the following result.

Corollary 4.5. *Let* $\mathbf{X} = (X_1,\dots,X_n)$, $\mathbf{p} = (\frac{p_1}{P_n},\dots,\frac{p_n}{P_n})$ *be as in the definition of the quasi-arithmetic mean,* $f, g \in C(J)$ *be be strictly monotone non-negative functions. If* f^{-1} *is operator superquadratic and* g^{-1} *is operator subquadratic, then*

$$\mathcal{M}_f(\mathbf{X},\mathbf{p},n) + \mathcal{D}_1 \le \mathcal{M}_g(\mathbf{X},\mathbf{p},n) + \mathcal{D}_2, \tag{37}$$

where

$$\begin{aligned} \mathcal{D}_1 \;=\; & \tfrac{p_1}{P_n}f^{-1}\Big(\Big|f(X_1) - f\Big(\mathcal{M}_f(\mathbf{X},\mathbf{p},2)\Big)\Big|\Big) + \sum_{j=2}^{n}\tfrac{p_j}{P_n}f^{-1}\Big(\Big|f(X_j) - f\Big(\mathcal{M}_f(\mathbf{X},\mathbf{p},j)\Big)\Big|\Big) \\ + & \sum_{j=3}^{n}\tfrac{p_{j-1}}{P_n}f^{-1}\Big(\tfrac{p_j}{P_{j-1}}\Big|f(X_j) - f\Big(\mathcal{M}_f(\mathbf{X},\mathbf{p},j)\Big)\Big|\Big) \end{aligned} \tag{38}$$

and $\mathcal{D}_2$ *is obtained from* (38) *when we replace* f *by* g.

Remark 4.6. *Let the assumptions of Corollary 4.5 hold. Set* $n = 2$, $p_1 = p_2 = \frac{1}{2}$. *Then* (37) *gives*

$$\begin{aligned} & \mathcal{M}_f(A,B) + \tfrac{1}{2}f^{-1}\Big(\Big|f(A) - f\Big(\mathcal{M}_f(A,B)\Big)\Big|\Big) + \tfrac{1}{2}f^{-1}\Big(\Big|f(B) - f\Big(\mathcal{M}_f(A,B)\Big)\Big|\Big) \\ \le\; & \mathcal{M}_g(A,B) + \tfrac{1}{2}g^{-1}\Big(\Big|g(A) - g\Big(\mathcal{M}_g(A,B)\Big)\Big|\Big) + \tfrac{1}{2}g^{-1}\Big(\Big|g(B) - g\Big(\mathcal{M}_g(A,B)\Big)\Big|\Big). \end{aligned} \tag{39}$$

Since $f\Big(\mathcal{M}_f(A,B)\Big) = \frac{f(A)+f(B)}{2}$, *then*

$$\Big|f(A) - f\Big(\mathcal{M}_f(A,B)\Big)\Big| = \Big|f(B) - f\Big(\mathcal{M}_f(A,B)\Big)\Big| = \Big|\frac{f(A) - f(B)}{2}\Big|,$$

so (39) *becomes*

$$\mathcal{M}_f(A,B) + f^{-1}\Big(\Big|\frac{f(A)-f(B)}{2}\Big|\Big) \le \mathcal{M}_g(A,B) + g^{-1}\Big(\Big|\frac{g(A)-g(B)}{2}\Big|\Big). \tag{40}$$

We now give an example of some functions for which (40) holds, but (26) in Theorem B does not hold.

Example 4.7. *Assume that* $f^{-1}(t) = t^2 - t$ *and* $g^{-1}(t) = 1 + \frac{1}{1+2t^2}$. *Then* $f^{-1} : [0,\infty) \to [-1/4,\infty)$ *is operator superquadratic (see (2.3) and (1) in Example 2.3) and* $g^{-1} : [0,\infty) \to (1,2]$ *is operator subquadratic (see (7) in Example 2.3), but* g^{-1} *is not operator convex nor operator concave. Let* $J = (1,2]$. *The function* $f(t) = \frac{1+\sqrt{1+4t}}{2}$ *is monotone increasing and positive on* J *and* $g(t) = \frac{\sqrt{1-t/2}}{\sqrt{t-1}}$ *is monotone decreasing and positive on* J.

Let $1_{\mathcal{H}} < A, B \le 2\cdot 1_{\mathcal{H}}$. *Applying* (40), *we obtain*

$$\begin{aligned}&\mathcal{M}_f(A,B) + \tfrac{1}{16}\left[(1_{\mathcal{H}}+4A)^{1/2} - (1_{\mathcal{H}}+4B)^{1/2}\right]^2 - \tfrac{1}{4}\left|(1_{\mathcal{H}}+4A)^{1/2} - (1_{\mathcal{H}}+4B)^{1/2}\right|\\ &\leqslant \mathcal{M}_g(A,B) + 1_{\mathcal{H}}\\ &+\left\{1_{\mathcal{H}} + \tfrac{1}{2}\left[(1_{\mathcal{H}}-A/2)^{1/2}(A-1_{\mathcal{H}})^{-1/2} - (1_{\mathcal{H}}-B/2)^{1/2}(B-1_{\mathcal{H}})^{-1/2}\right]^2\right\}^{-1},\end{aligned}$$

where

$$\begin{aligned}&\mathcal{M}_f(A,B)\\ =\ &\tfrac{1}{16}\left[2\cdot 1_{\mathcal{H}} + (1_{\mathcal{H}}+4A)^{1/2} + (1_{\mathcal{H}}+4B)^{1/2}\right]^2 - \tfrac{1}{4}\left[2\cdot 1_{\mathcal{H}} + (1_{\mathcal{H}}+4A)^{1/2} + (1_{\mathcal{H}}+4B)^{1/2}\right]\end{aligned}$$

and

$$\begin{aligned}&\mathcal{M}_g(A,B)\\ =\ &1_{\mathcal{H}} + \left\{1_{\mathcal{H}} + \tfrac{1}{2}\left[(1_{\mathcal{H}}-A/2)^{1/2}(A-1_{\mathcal{H}})^{-1/2} + (1_{\mathcal{H}}-B/2)^{1/2}(B-1_{\mathcal{H}})^{-1/2}\right]^2\right\}^{-1}.\end{aligned}$$

But, $\mathcal{M}_f(A,B) \nleq \mathcal{M}_g(A,B)$ *in general.*
E.g. let $A = \begin{pmatrix} 1.2 & -0.1 \\ -0.1 & 1.8 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1.4 & -0.1 \\ -0.1 & 1.9 \end{pmatrix}$.
Then (rounded to 3 decimal places)

$$\mathcal{M}_f(A,B) = \begin{pmatrix} 1.298 & -0.1 \\ -0.1 & 1.85 \end{pmatrix} \nleq\ngeq \begin{pmatrix} 1.277 & -0.103 \\ -0.103 & 1.853 \end{pmatrix} = \mathcal{M}_g(A,B),$$

$$\begin{aligned}\mathcal{D}_1 &= f^{-1}\Big(\Big|\frac{f(A)-f(B)}{2}\Big|\Big) = \begin{pmatrix} -0.037 & -0.001 \\ -0.001 & -0.017 \end{pmatrix},\\ \mathcal{D}_2 &= g^{-1}\Big(\Big|\frac{g(A)-g(B)}{2}\Big|\Big) = \begin{pmatrix} 1.595 & -0.044 \\ -0.044 & 1.965 \end{pmatrix}\end{aligned}$$

and

$$\mathcal{M}_f(A,B) + \mathcal{D}_1 = \begin{pmatrix} 1.26 & -0.101 \\ -0.101 & 1.833 \end{pmatrix} < \begin{pmatrix} 2.872 & -0.147 \\ -0.147 & 3.817 \end{pmatrix} = \mathcal{M}_g(A,B) + \mathcal{D}_2.$$

By using Corollary 3.8 we obtain another variant of (27).

Theorem 4.8. *Let the assumptions of Theorem 4.1 hold. If* $\psi\circ\varphi^{-1} \ge 0$ *is operator superquadratic and* ψ^{-1} *is operator monotone then*

$$\mathcal{M}_\varphi(\mathbf{X},\mathbf{p},n) \le \psi^{-1}\Big(\psi\Big(\mathcal{M}_\psi(\mathbf{X},\mathbf{p},n)\Big) - \mathcal{D}_3\Big) \le \mathcal{M}_\psi(\mathbf{X},\mathbf{p},n), \tag{41}$$

where

$$\mathcal{D}_3 \equiv \mathcal{D}_1(\varphi,\psi,\mathbf{X},\mathbf{p},n,m) = \frac{1}{P_n}\sum_{i=1}^{n} p_i \sum_{j=1}^{m} 2^j\, \psi\circ\varphi^{-1}\Bigg(\frac{\left|\varphi(X_i) - \varphi\left(\mathcal{M}_\varphi(\mathbf{X},\mathbf{p},n)\right)\right|}{2^j}\Bigg) \tag{42}$$

for any $m \in \mathbb{N}$.

But, if $\psi \circ \varphi^{-1} \leq 0$ *is operator subquadratic and* ψ^{-1} *is operator monotone then the reverse inequality is valid in* (41).

Using Theorem 4.4 (resp. Theorem 4.1) and Theorem 4.8 we can obtain similar results as in Corollary 4.2 (resp. Corollary 4.5). We leave that to the interested reader.

Open questions

- Is there a non trivial example of positive operator convex function on $[0, \infty)$ which is also operator superquadratic?

- Whether or not Jensen's operator inequality (7) is valid as a generalisation of the classical Jensen's inequality for operator superquadratic functions?

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