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Multipliers and Uniformly Continuous Functionals Over Fourier Algebras of Ultraspherical Hypergroups

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Abstract. Let H be an ultraspherical hypergroup associated to a locally compact group G and let $A(H)$ be the Fourier algebra of H. For a left Banach $A(H)$-submodule X of $VN(H)$, define Q_X to be the norm closure of the linear span of the set $\{uf : u \in A(H), f \in X\}$ in $B_{A(H)}(A(H), X^*)^*$. We will show that $B_{A(H)}(A(H), X^*)$ is a dual Banach space with predual Q_X. Applications obtained on the multiplier algebra $M(A(H))$ of the Fourier algebra $A(H)$. In particular, we prove that G is amenable if and only if $M(A(H)) = B_\lambda(H)$. We also study the uniformly continuous functionals associated with the Fourier algebra $A(H)$ and obtain some characterizations for H to be discrete. Finally, we establish a contractive and injective representation from $B_\lambda(H)$ into $B^\sigma_{A(H)}(B_\lambda(H))$. As an application of this result we show that the induced representation $\Phi : B_\lambda(H) \to B^\sigma_{A(H)}(B_\lambda(H))$ is surjective if and only if G is amenable.

1. Introduction

Let G be a locally compact group and let $A(G)$ and $B(G)$ be the Fourier and Fourier-Stieltjes algebras of G introduced by Eymard [4]. Let $M(A(G))$ denote the multiplier algebra of $A(G)$. Then we have the following inclusions

$$A(G) \subseteq B(G) \subseteq M(A(G))$$

and $\|v\|_{A(G)} = \|v\|_{B(G)} \geq \|v\|_M$ for all $v \in A(G)$. It is known that if G is amenable, then $B(G) = M(A(G))$ isometrically. Moreover, it is known from Losert [11] that G is amenable, or equivalently $A(G)$ has a bounded approximate identity, whenever the norms $\|\cdot\|_{B(G)}$ and $\|\cdot\|_M$ are equivalent on $A(G)$. As in the group case, the Fourier space $A(H)$ of a locally compact hypergroup H equiped with the left Haar measure, plays an important role in the harmonic analysis.

A class of hypergroups, called regular hypergroups, whose Fourier space forms a Banach algebra under pointwise multiplication appeared in [14]. Another class, called ultraspherical hypergroups, was studied by Muruganandam [15], which includes in particular all double coset hypergroups and hence all orbit hypergroups. In recent years several authors have looked at the Fourier algebra $A(H)$ of an ultraspherical hypergroup H. For example, Shravan Kumar showed in [18] that there is a unique topological invariant mean on $A(H)^*$ if and only if H is discrete. Degenfeld-Schonburg, Kaniuth and Lasser investigated spectral

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synthesis properties of $A(H)$ in [3]. In this work, we generalize some results on Fourier algebras of locally compact groups to the context of ultraspherical hupergroups.

Let $\mathcal{A}$ be a Banach algebra, and let X and Y be two right Banach $\mathcal{A}$-modules. Suppose that $B_{\mathcal{A}}(X,Y)$ is the Banach space of bounded right $\mathcal{A}$-module maps with the operator norm denoted by $\|\cdot\|_M$. In recent years, people have become interested in studying the properties of $B_{\mathcal{A}}(X,Y)$ for various classes of Banach algebras $\mathcal{A}$ and right Banach $\mathcal{A}$-modules X and Y; see for example [5–7, 13].

In this paper, for a left Banach $\mathcal{A}$-submodule X of $\mathcal{A}^*$ we study $B_{\mathcal{A}}(\mathcal{A},X^*)$ as a dual Banach space, paying special attention to the Fourier algebra $A(H)$ of an ultraspherical hypergroup H associated to a locally compact group G.

In Section 2, for a left Banach $\mathcal{A}$-submodule X of $\mathcal{A}^*$, we show that $B_{\mathcal{A}}(\mathcal{A},X^*)$ is a dual Banach space with predual Q_X, where Q_X denote the norm closure of the linear span of the set $\{af : a \in \mathcal{A}, f \in X\}$ in $B_{\mathcal{A}}(\mathcal{A},X^*)^*$. We will obtain a characterization of Q_X.

In Section 3, we apply these results to Fourier algebra $A(H)$ of an ultraspherical hypergroup H. For the case of $X = C^*_\lambda(H)$, we show that the predual $Q_{C^*_\lambda(H)}$ of multiplier algebra of $A(H)$, denoted $M(A(H))$, is equal to the closure of $L^1(H)$ in $M(A(H))$ under the multiplier norm. We also prove that G is amenable if and only if $M(A(H)) = B_\lambda(H)$, where $B_\lambda(H)$ is the reduced Fourier-Stieltjes algebra of H. In the case where $A(H)$ is weak*-dense in $M(A(H))$, we prove that G is amenable if and only if the norms $\|\cdot\|_{A(H)}$and $\|\cdot\|_M$ are equivalent on $A(H)$. For the case of $X = C_\delta(H)$, we study the predual of $B_{A(H)}(A(H), C_\delta(H)^*)$.

In Section 4, we shall define and study $UCB(\widehat{H})$, called uniformly continuous functionals on $A(H)$. We will focus in the relationship between $UCB(\widehat{H})$ and other subspaces of $VN(H)$. We extend various results of [9] to the context of ultraspherical hypergroups. For example, we prove that H is discrete if and only if $UCB(\widehat{H}) = C^*_\lambda(H)$. Finally, we establish a contractive and injective representation from $B_\lambda(H)$ into $B^\sigma_{A(H)}(B_\lambda(H))$. As an application of this result we show that the induced representation $\Phi : B_\lambda(H) \to B^\sigma_{A(H)}(B_\lambda(H))$ is surjective if and only if G is amenable.

2. The dual Banach space $B_{\mathcal{A}}(\mathcal{A}, X^*)$

Let $\mathcal{A}$ be a Banach algebra, and let X and Y be right and left Banach $\mathcal{A}$-modules, respectively. The *$\mathcal{A}$-module tensor product* of X and Y is the quotient space $X\widehat{\otimes}_{\mathcal{A}}Y = (X\widehat{\otimes}Y)/N$, where

$$N = \langle x \cdot a \otimes y - x \otimes a \cdot y : x \in X, y \in Y, a \in \mathcal{A}\rangle,$$

and $\langle\cdot\rangle$ denotes the closed linear span. It was shown in [16] that

$$B_{\mathcal{A}}(X,Y^*) \cong N^\perp \cong (X\widehat{\otimes}_{\mathcal{A}}Y)^*.$$

Let X be a left Banach $\mathcal{A}$-submodule of $\mathcal{A}^*$. In this section we show that $B_{\mathcal{A}}(\mathcal{A},X^*)$ is a dual Banach space and characterize its predual in terms of elements in $\mathcal{A}$ and X. For every $a \in \mathcal{A}$ and $f \in X$, define the bounded linear functional af on $B_{\mathcal{A}}(\mathcal{A},X^*)$ as follows:

$$\langle af, T\rangle = \langle f, T(a)\rangle \quad (T \in B_{\mathcal{A}}(\mathcal{A},X^*)).$$

Moreover, it is easy to see that $\|af\|_M \le \|a\|\|f\|_X$. Now, we denote the linear span of the set $\{af : a \in \mathcal{A}, f \in X\}$ by $\mathcal{A}X$ and define Q_X to be the norm closure of $\mathcal{A}X$ in $B_{\mathcal{A}}(\mathcal{A},X^*)^*$.

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra and let X be a left Banach $\mathcal{A}$-submodule of $\mathcal{A}^*$. Then $B_{\mathcal{A}}(\mathcal{A},X^*) = (Q_X)^*$.*

Proof. Let $J : \mathcal{A}\widehat{\otimes}X \to Q_X$ be defined by $J(\sum_{i=1}^{\infty} a_i \otimes f_i) = \sum_{i=1}^{\infty} a_i f_i$. Then it is clear that J is well defined and $\|J\| \le 1$. As $B(\mathcal{A},X^*) = (\mathcal{A}\widehat{\otimes}X)^*$, we have the adjoint operator $J^* : (Q_X)^* \to B(\mathcal{A},X^*)$ with $\|J^*\| \le 1$. Now, for each $T \in (Q_X)^*$, we show that $J^*(T) \in B_{\mathcal{A}}(\mathcal{A}, X^*)$. Let $a, b \in \mathcal{A}$ and $f \in X$. Then

$$\begin{aligned}\langle J^*(T)(ab), f\rangle &= \langle J^*(T), (ab) \otimes f\rangle = \langle T, (ab)f\rangle \\ &= \langle T, a(bf)\rangle = \langle T, J(a \otimes (bf))\rangle \\ &= \langle J^*(T), a \otimes (bf)\rangle = \langle J^*(T)(a), bf\rangle \\ &= \langle J^*(T)(a) \cdot b, f\rangle.\end{aligned}$$

Therefore, $J^*(T)(ab) = J^*(T)(a) \cdot b$ for all $a, b \in \mathcal{A}$. Thus, $J^*(T) \in B_{\mathcal{A}}(\mathcal{A}, X^*)$. Let $T \in B_{\mathcal{A}}(\mathcal{A}, X^*)$. Then the restriction of T to Q_X is in $(Q_X)^*$ and we have

$$\langle J^*(T), \sum_{i=1}^{\infty} a_i \otimes f_i\rangle = \langle T, \sum_{i=1}^{\infty} a_i f_i\rangle = \sum_{i=1}^{\infty}\langle T, a_i f_i\rangle = \sum_{i=1}^{\infty}\langle T(a_i), f_i\rangle = \langle T, \sum_{i=1}^{\infty} a_i \otimes f_i\rangle,$$

for all $\sum\limits_{i=1}^{\infty} a_i \otimes f_i \in \mathcal{A}\widehat{\otimes}X$. It follows that $J^*(T) = T$ and J^* is surjective. Since $J(\mathcal{A}\widehat{\otimes}X)$ is dense in Q_X, by [12, Theorem 3.1.17] J^* is injective. Therefore, J^* is a surjective isometry. □

Theorem 2.2. *Let $\mathcal{A}$ be a Banach algebra and let X be a left Banach $\mathcal{A}$-submodule of $\mathcal{A}^*$. Suppose that $f \in B_{\mathcal{A}}(\mathcal{A}, X^*)$. Then $f \in Q_X$ if and only if there are sequences $(a_i) \subseteq \mathcal{A}$ and $(f_i) \subseteq X$ with $\sum_{i=1}^{\infty} \|a_i\|\|f_i\| < \infty$ such that $f = \sum_{i=1}^{\infty} a_i f_i$ and*

$$\|f\|_M = \inf\left\{\sum_{i=1}^{\infty} \|a_i\|\|f_i\| : f = \sum_{i=1}^{\infty} a_i f_i,\ \sum_{i=1}^{\infty} \|a_i\|\|f_i\| < \infty\right\}.$$

Proof. By definition, each element of the form $\sum_{i=1}^{\infty} a_i f_i$, as in the proof of Theorem 2.1, lies in Q_X.

For the converse, let $I : \mathcal{A}\widehat{\otimes}_{\mathcal{A}}X \to Q_X$ be defined by

$$I(\sum_{i=1}^{\infty} a_i \otimes f_i + N) = \sum_{i=1}^{\infty} a_i f_i.$$

Then it is routine to check that I is well defined and $\|I\| \leq 1$. In fact, if $\sum_{i=1}^{\infty} a_i \otimes f_i \in N$, then for each $T \in B_{\mathcal{A}}(\mathcal{A}, X^*)$, we have

$$\langle\sum_{i=1}^{\infty} a_i f_i, T\rangle = \sum_{i=1}^{\infty}\langle T(a_i), f_i\rangle = \langle\sum_{i=1}^{\infty} a_i \otimes f_i, T\rangle = 0.$$

Hence, I is well defined by duality.

We know from Theorem 2.1 that $(\mathcal{A}\widehat{\otimes}_{\mathcal{A}}X)^* = B_{\mathcal{A}}(\mathcal{A}, X^*) = (Q_X)^*$. It follows that $I^* : (Q_X)^* \to (\mathcal{A}\widehat{\otimes}_{\mathcal{A}}X)^*$ is bijective. Hence, I is surjective by [12, Theorem 3.1.22]. This proves first part of the theorem.

For the second part, let $f \in Q_X$ and $\epsilon > 0$ be given. Then by first part of theorem, there are sequences $(a_i) \subseteq \mathcal{A}$ and $(f_i) \subseteq X$ such that $f = \sum_{i=1}^{\infty} a_i f_i$ with $\sum_{i=1}^{\infty} \|a_i\|\|f_i\| < \infty$. Let $\xi = \sum_{i=1}^{\infty} a_i \otimes f_i + N$. Then $\langle T, f\rangle = \langle T, \xi\rangle$ for all $T \in B_{\mathcal{A}}(\mathcal{A}, X^*)$, which implies that $\|f\|_M = \|\xi\|$. Now, as a consequence of the definition of quotient norm, there exist sequences $(b_i) \subseteq \mathcal{A}$ and $(h_i) \subseteq X$ such that $\sum_{i=1}^{\infty} \|b_i\|\|h_i\| < \|f\|_M + \epsilon$ and $\xi = \sum_{i=1}^{\infty} b_i \otimes h_i + N$. Hence, $f = \sum_{i=1}^{\infty} b_i h_i$ on $B_{\mathcal{A}}(\mathcal{A}, X^*)$, as required. This completes the proof. □

Suppose that X is a left Banach $\mathcal{A}$-module. Then X^* is a right Banach $\mathcal{A}$-module with the following module action

$$\langle m \cdot a, f\rangle = \langle m, a \cdot f\rangle \quad (m \in X^*, f \in X, a \in \mathcal{A}).$$

By the above notions it is not hard to see that, if X is a left Banach $\mathcal{A}$-submodule of $\mathcal{A}^*$, then the map

$$\iota : X^* \to B_{\mathcal{A}}(\mathcal{A}, X^*), \quad m \mapsto m_L$$

is a contractive linear map, where m_L is given by $m_L(a) = m \cdot a$ for all $a \in \mathcal{A}$ and $\|m_L\|_M \leq \|m\|_X$. Thus, we can assume that $X^* \subseteq B_{\mathcal{A}}(\mathcal{A}, X^*)$. Moreover, the adjoint map $\iota^* : B_{\mathcal{A}}(\mathcal{A}, X^*)^* \to X^{**}$ is simply the restriction map, say R and for every $a \in \mathcal{A}$, $f \in X$ and $m \in X^*$ we have

$$\langle R(af), m\rangle = \langle af, m_L\rangle = \langle f, m_L(a)\rangle = \langle f, m \cdot a\rangle = \langle a \cdot f, m\rangle,$$

which implies that $R(Q_X) \subseteq X$.

Proposition 2.3. *Let $\mathcal{A}$ be a Banach algebra and let X be a left Banach $\mathcal{A}$-submodule of $\mathcal{A}^*$. Then $R : Q_X \to X$ is surjective if and only if the norms $\|\cdot\|_X$ and $\|\cdot\|_M$ are equivalent on X^*.*

Proof. Let R be surjective. Then $R^* : X^* \to (Q_X)^*$ is injective and $R^*(X^*)$ is closed in $(Q_X)^*$ by [12, Theorem 3.1.22]. Since $\|\cdot\|_M \leq \|\cdot\|_X$ on X^*, the Open Mapping theorem shows that the norms $\|\cdot\|_M$ and $\|\cdot\|_X$ are equivalent on X^*.

Conversely, let the norms $\|\cdot\|_M$ and $\|\cdot\|_X$ are equivalent on X^*. Then R^* is injective and $R^*(X^*)$ is closed in $(Q_X)^*$. It follows from [12, Theorem 3.1.17] and [12, Theorem 3.1.21] that R is surjective. □

For every $a \in \mathcal{A}$ we can regard a as a functional on X. It follows that the map

$$\iota : \mathcal{A} \to B_{\mathcal{A}}(\mathcal{A}, X^*), \quad a \mapsto a_L$$

is a contractive linear map, where a_L is given by $a_L(b) = ab$ for all $b \in \mathcal{A}$ and $\|a_L\|_M \leq \|a\|_X \leq \|a\|_{\mathcal{A}}$. This implies that $\mathcal{A} \subseteq B_{\mathcal{A}}(\mathcal{A}, X^*)$.

Define $\widetilde{Q}_X$ to be the range of the linear map $\Gamma : \mathcal{A}\widehat{\otimes}X \to \mathcal{A}^*$ defined by $\Gamma(a \otimes f) = a \cdot f$. Then $\widetilde{Q}_X$ is a Banach space when equipped with the quotient norm from $\mathcal{A}\widehat{\otimes}X$. Moreover, $f \in \widetilde{Q}_X$ if and only if there are sequences $(a_i) \subseteq \mathcal{A}$ and $(f_i) \subseteq X$ with $\sum_{i=1}^{\infty} \|a_i\|\|f_i\| < \infty$ such that $f = \sum_{i=1}^{\infty} a_i \cdot f_i$.

Theorem 2.4. *Let $\mathcal{A}$ be a Banach algebra and let X be a left Banach $\mathcal{A}$-submodule of $\mathcal{A}^*$. Then $\mathcal{A}$ is weak*-dense in $B_{\mathcal{A}}(\mathcal{A}, X^*)$ if and only if $\widetilde{Q}_X$ is isometrically isomorphic to Q_X.*

Proof. Let $\mathcal{A}$ be weak*-dense in $B_{\mathcal{A}}(\mathcal{A}, X^*)$. Then it follows from [12, Proposition 2.6.6] that the annihilator $({}^{\perp}\mathcal{A})^{\perp}$ of ${}^{\perp}\mathcal{A}$ in $B_{\mathcal{A}}(\mathcal{A}, X^*)$ can be identified with $B_{\mathcal{A}}(\mathcal{A}, X^*) = (Q_X)^*$, where ${}^{\perp}\mathcal{A} = \{f \in Q_X : \langle a_L, f\rangle = 0, a \in \mathcal{A}\}$. Hence, $\mathcal{A}$ separates the points of Q_X. Now, define $\Lambda : Q_X \to \widetilde{Q}_X$ by

$$\Lambda(\sum_{i=1}^{\infty} a_i f_i) = \sum_{i=1}^{\infty} a_i \cdot f_i.$$

If $a \in \mathcal{A}$ is arbitrary, then for each sequences $(a_i) \subseteq \mathcal{A}$ and $(f_i) \subseteq X$ with $\sum_{i=1}^{\infty} \|a_i\|\|f_i\| < \infty$, we have

$$\langle a_L, \sum_{i=1}^{\infty} a_i f_i\rangle = \sum_{i=1}^{\infty}\langle aa_i, f_i\rangle = \sum_{i=1}^{\infty}\langle a, a_i \cdot f_i\rangle = \langle a, \sum_{i=1}^{\infty} a_i \cdot f_i\rangle.$$

From this and the fact that $\mathcal{A}$ separates the points of Q_X, we get that Λ is an isomorphism. Also, by Theorem 2.2 it is an isometry.

Conversely, let $\widetilde{Q}_X$ be isometrically isomorphic to Q_X. Then $\mathcal{A}$ separates the points of Q_X, which implies that $({}^{\perp}\mathcal{A})^{\perp} = B_{\mathcal{A}}(\mathcal{A}, X^*)$. Again by [12, Proposition 2.6.6], $\mathcal{A}$ is weak*-dense in $B_{\mathcal{A}}(\mathcal{A}, X^*)$. □

3. The multiplier algebra $M(A(H))$ and amenability

A bounded linear operator on commutative Banach algebra $\mathcal{A}$ is called a multiplier if it satisfies $aT(b) = T(ab)$ for all $a, b \in \mathcal{A}$. We denote by $\mathcal{M}(\mathcal{A})$ the space of all multipliers for $\mathcal{A}$. Clearly $\mathcal{M}(\mathcal{A})$ is a Banach algebra as a subalgebra of $B(\mathcal{A})$ and $\mathcal{M}(\mathcal{A}) = B_{\mathcal{A}}(\mathcal{A})$. For the general theory of multipliers we refer to Larsen [8]. It is known that for a semisimple commutative Banach algebra $\mathcal{A}$ every $T \in \mathcal{M}(\mathcal{A})$ can be identified uniquely with a bounded continuous function $\widehat{T}$ on $\Delta(\mathcal{A})$, the maximal ideal space of $\mathcal{A}$. Moreover, if we denote by $M(\mathcal{A})$ the normed algebra of all bounded continuous functions φ on $\Delta(\mathcal{A})$ such that $\varphi\widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}}$, then $M(\mathcal{A}) = \widehat{\mathcal{M}(\mathcal{A})}$; see [8, Corollary 1.2.1].

Let H be an ultraspherical hypergroup associated to a locally compact group G and a spherical projector $\pi : C_c(G) \to C_c(G)$ which was introduced and studied in [15]. Let $A(H)$ denote the Fourier algebra corresponding to the hypergroup H. A left Haar measure on H is given by $\int_H f(\dot{x})d\dot{x} = \int_G f(p(x))dx$, $f \in C_c(H)$, where $p : G \to H$ is the quotient map. The Fourier space $A(H)$ is an algebra and is isometrically isomorphic to the subalgebra $A_\pi(G) = \{u \in A(G) : \pi(u) = u\}$ of $A(G)$ [15, Theorem 3.10]. Recall that the character space $\Delta(A(H))$ of $A(H)$ can be canonically identified with H. The Fourier algebra $A(H)$ is semisimple, regular and Tauberian [15, Theorem 3.13]. As in the group case, let λ also denote the left regular representation of H on $L^2(H)$ given by

$$\lambda(\dot{x})(f)(\dot{y}) = f(\check{\dot{x}} * \dot{y}) \quad (\dot{x}, \dot{y} \in H, f \in L^2(H))$$

This can be extended to $L^1(H)$ by $\lambda(f)(g) = f * g$ for all $f \in L^1(H)$ and $g \in L^2(H)$. Let $C^*_\lambda(H)$ denote the completion of $\lambda(L^1(H))$ in $B(L^2(H))$ which is called the reduced C^*-algebra of H. The von Neumann algebra generated by $\{\lambda(\dot{x}) : \dot{x} \in H\}$ is called the von Neumann algebra of H, and is denoted by $VN(H)$. Note that $VN(H)$ is isometrically isomorphic to the dual of $A(H)$. Moreover, $A(H)$ can be considered as an ideal of $B_\lambda(H)$, where $B_\lambda(H)$ is the dual of $C^*_\lambda(H)$.

Remark 3.1. *As $A(H)$ is an ideal in $B_\lambda(H)$, there is a canonical $B_\lambda(H)$-bimodule structure on $VN(H)$. In particular, for $f \in L^1(H)$ and $\phi \in B_\lambda(H)$, we obtain*

$$\langle \phi \cdot \lambda(f), v\rangle = \langle \lambda(f), \phi v\rangle = \int f(\dot{x})\phi(\dot{x})v(\dot{x})d\dot{x} = \langle \lambda(\phi f), v\rangle$$

for all $v \in A(H)$. This shows that $\phi \cdot \lambda(f) = \lambda(\phi f) \in \lambda(L^1(H))$. Since $\lambda(L^1(H))$ is norm dense in $C^*_\lambda(H)$, we conclude that $C^*_\lambda(H)$ is a $B_\lambda(H)$-bimodule.

Theorem 3.2. *Let H ba an ultraspherical hypergroup. Then*

$$M(A(H)) = B_{A(H)}(A(H), C^*_\lambda(H)^*).$$

Proof. Since $A(H)$ is commutative and semisimple, it suffices to show that $\mathcal{M}(A(H)) = B_{A(H)}(A(H), B_\lambda(H))$. To prove this, first note that $\mathcal{M}(A(H)) \subseteq B_{A(H)}(A(H), B_\lambda(H))$. Conversely, assume that $u \in A(H)$ has compact support. By regularity of $A(H)$, there exists $v \in A(H)$ such that $v(x) = 1$ for all $x \in \text{supp}(u)$. Thus, for each $T \in B_{A(H)}(A(H), B_\lambda(H))$, we have

$$T(u) = T(vu) = vT(u).$$

Since $A(H)$ is an ideal in $B_\lambda(H)$, we conclude that $T(u) \in A(H)$. Moreover, since the set of compactly supported elements in $A(H)$ is dense in $A(H)$, a simple approximation argument shows that $T(u) \in A(H)$ for all $u \in A(H)$. Therefore, $T \in \mathcal{M}(A(H))$ as required. □

Let H ba an ultraspherical hypergroup and let $f \in L^1(H)$. Define a linear functional on $M(A(H))$ by

$$\langle f, \phi\rangle = \int f(\dot{x})\phi(\dot{x})d\dot{x} \quad (\phi \in M(A(H))).$$

Moreover, $|\langle f, \phi\rangle| \leq \|f\|_1\|\phi\|_\infty \leq \|f\|_1\|\phi\|_M$ for all $\phi \in M(A(H))$. Therefore, f is in $M(A(H))^*$ and $\|f\|_M = \sup\left\{\left|\langle f, \phi\rangle\right| : \phi \in M(A(H)), \|\phi\|_M \leq 1\right\} \leq \|f\|_1$. Put

$$Q(H) := \overline{L^1(H)}^{\|.\|_M} \subseteq M(A(H))^*.$$

Next we prove that $M(A(H))$ is a dual Banach space for any ultraspherical hypergroup H .

Theorem 3.3. *Let H ba an ultraspherical hypergroup. Then $Q_{C^*_\lambda(H)} = Q(H)$ and so $M(A(H)) = Q(H)^*$.*

Proof. Suppose that $f \in C_c(H)$. Using the regularity of $A(H)$, there exists $u \in A(H)$ such that $u|_{\mathrm{supp}(f)} \equiv 1$. Thus, $f = uf$ is in $Q_{C^*_\lambda(H)}$ and $\langle uf, \phi\rangle = \langle f, \phi\rangle = \int_H f(\dot{x})\phi(\dot{x})d\dot{x}$ for all $\phi \in M(A(H))$. Therefore, there is an isometry between the dense subspace of $Q_{C^*_\lambda(H)}$ and a dense subspace of $(L^1(H), \|\cdot\|_M)$. This shows that $Q_{C^*_\lambda(H)}$ is the completion of $L^1(H)$ with respect to the norm $\|\cdot\|_M$. □

For a locally compact group G, it is well-known that $M(A(G)) = B_\lambda(G)$ if and only if G is amenable. In what follows we prove a corresponding result for every ultraspherical hypergroup H.

Theorem 3.4. *Let H be an ultraspherical hypergroup on a locally compact group G. Then G is amenable if and only if $B_\lambda(H) = M(A(H))$.*

Proof. Suppose that G is amenable. Then $B_\lambda(H) = M(A(H))$ by [15, Theorem 4.2]. Conversely, assume that $B_\lambda(H) = M(A(H))$. Then the constant function 1 belongs to $B_\lambda(H)$. Since $A(H)$ is dense in $B_\lambda(H)$ with respect to the $\sigma(B_\lambda(H), C^*_\lambda(H))$-topology, there exists a net (u_α) in $A(H)$ such that $u_\alpha \to 1$ in the $\sigma(B_\lambda(H), C^*_\lambda(H))$-topology and $c = \sup_\alpha \|u_\alpha\|_{A(H)} < \infty$. Choose f in $C_c(H)$ with $f \geq 0$ and $\|f\|_1 = 1$. For each α, define $u'_\alpha = f * u_\alpha$. Notice first that $(u'_\alpha) \subseteq A(H)$ and

$$\|u'_\alpha\|_{A(H)} \leq \|f\|_1\|u_\alpha\|_{A(H)} \leq c$$

for all α. In fact, for each $g \in L^1(H)$ with $\|\lambda(g)\|_{C^*_\lambda(H)} \leq 1$, we have

$$\begin{aligned}
|\langle f * u_\alpha, \lambda(g)\rangle| &= |\int_H \int_H f(\dot{y})u_\alpha(\check{\dot{y}} * \dot{x})g(\dot{x})d\dot{y}d\dot{x}| \\
&= |\int_H f(\dot{y})\langle {}_{\dot{y}}u_\alpha, g\rangle d\dot{y}| \\
&\leq \int_H |f(\dot{y})|\|{}_{\dot{y}}u_\alpha\|_{A(H)}d\dot{y} \\
&\leq \|f\|_1\|u_\alpha\|_{A(H)} \leq c.
\end{aligned}$$

Let $K \subseteq H$ be compact. Then the set $\{\lambda({}_{\check{\dot{x}}}f) : \dot{x} \in K\}$ form a compact subset of $C^*_\lambda(H)$, where the function ${}_{\check{\dot{x}}}f$ on H is defined by ${}_{\check{\dot{x}}}f(\dot{y}) = f(\dot{x} * \dot{y})$ for all $\dot{y} \in H$. Since $u_\alpha \to 1$ in the $\sigma(B_\lambda(H), C^*_\lambda(H))$-topology and the net (u_α) is bounded in $B_\lambda(H)$, the convergence is uniform on compact subsets of $C^*_\lambda(H)$. Hence,

$$u'_\alpha(\dot{x}) = \langle \check{u_\alpha}, \lambda({}_{\check{\dot{x}}}f)\rangle \to \langle 1, \lambda({}_{\check{\dot{x}}}f)\rangle = \int_H {}_{\check{\dot{x}}}f(\dot{y})d\dot{y} = 1$$

uniformly on K, where $\check{u_\alpha}(\dot{x}) = u_\alpha(\check{\dot{x}})$ for all $\dot{x} \in H$, and noticing that $\check{u_\alpha} \in B_\lambda(H)$ by [14, Remark 2.9]. Again choose f in $C_c(H)$ with $f \geq 0$ and $\|f\|_1 = 1$ and put $w_\alpha = f * u'_\alpha$ for all α. Then $\|w_\alpha\|_{A(H)} \leq c$. Assume that $u \in A(H) \cap C_c(H)$. Next, we show that $\|w_\alpha u - u\|_{A(H)} \to 0$. In fact, if we put $K = \mathrm{supp}(f)\check{} * \mathrm{supp}(u)$, then for each $\dot{x} \in \mathrm{supp}(u)$ we have

$$w_\alpha(\dot{x}) = \int_H f(\dot{y})u'_\alpha(\check{\dot{y}} * \dot{x})d\dot{y} = \int_H f(\dot{y})(1_K u'_\alpha)(\check{\dot{y}} * \dot{x})d\dot{y} = (f * (1_K u'_\alpha))(\dot{x}).$$

Hence, $uw_\alpha = u(f * (1_K u'_\alpha))$, where 1_K denote the characteristic function of K. Similarly, $u = u(f * 1_K)$. Since $\|1_K u'_\alpha - 1_K\|_2 \to 0$, it follows that $\|uw_\alpha - u\|_{A(H)} \to 0$. Finally, since the net (w_α) is bounded and $A(H) \cap C_c(H)$ is dense in $A(H)$, a straightforward approximation argument shows that $\|uw_\alpha - u\|_{A(H)} \to 0$ for all u in $A(H)$. Thus, G is amenable by [1, Theorem 4.4]. □

Corollary 3.5. *Let H be an ultraspherical hypergroup on a locally compact group G. Then the following hold.*

(i) *Let* $f \in M(A(H))^*$. *Then* $f \in Q(H)$ *if and only if there exist sequences* $(u_i) \subseteq A(H)$ *and* $(f_i) \subseteq C^*_\lambda(H)$ *with* $\sum_{i=1}^{\infty} \|u_i\|_{A(H)} \|f_i\|_{C^*_\lambda(H)} < \infty$ *such that* $f = \sum_{i=1}^{\infty} u_i f_i$ *and*

$$\|f\|_M = \inf \left\{ \sum_{i=1}^{\infty} \|u_i\|_{A(H)} \|f_i\|_{C^*_\lambda(H)} : f = \sum_{i=1}^{\infty} u_i f_i, \ \sum_{i=1}^{\infty} \|u_i\|_{A(H)} \|f_i\|_{C^*_\lambda(H)} < \infty \right\}.$$

(ii) G *is amenable if and only if for any* $f \in C^*_\lambda(H)$ *and* $\epsilon > 0$ *there exist sequences* $(u_i) \subseteq A(H)$ *and* $(f_i) \subseteq C^*_\lambda(H)$ *such that* $f = \sum_{i=1}^{\infty} u_i f_i$ *on* $B_\lambda(H)$ *with*

$$\sum_{i=1}^{\infty} \|u_i\|_{A(H)} \|f_i\|_{C^*_\lambda(H)} < \|f\|_{C^*_\lambda(H)} + \epsilon.$$

Proof. (i). It is an immediate consequence of Theorem 2.2.

(ii). It follows from (i) that the condition of (ii) is equivalent to $C^*_\lambda(H) = Q(H)$ (equivalently, $B_\lambda(H) = M(A(H))$). However this is equivalent to G being amenable by Lemma 3.4. □

Proposition 3.6. *Let* H *be an ultraspherical hypergroup and let* X *be a Banach* $A(H)$*-submodule of* $VN(H)$ *with* $C^*_\lambda(H) \subseteq X$. *Then* $B_\lambda(H)$ *is a subalgebra of* $B_{A(H)}(A(H), X^*)$ *such that* $\|\phi\|_M \leq \|\phi\|_{B_\lambda(H)}$ *for all* $\phi \in B_\lambda(H)$.

Proof. Let $u \in A(H)$ and $\phi \in B_\lambda(H)$. Then $\phi u \in A(H) \subseteq VN(H)^*$. Thus $\phi u \in X^*$. From this and the fact that $C^*_\lambda(H) \subseteq X$, we get that

$$\|\phi u\|_{A(H)} = \|\phi u\|_{C^*_\lambda(H)} \leq \|\phi u\|_X \leq \|\phi\|_{C^*_\lambda(H)} \|u\|_{A(H)}.$$

Consequently, $\|\phi\|_M \leq \|\phi\|_{B_\lambda(H)}$. □

In general the restriction map $R : Q(H) \longrightarrow C^*_\lambda(H)$ is not necessarily injective. By [12, Theorem 3.1.17], it is easy to verified that R is injective if and only if $B_\lambda(H)$, or equivalently $A(H)$, is weak*-dense in $M(A(H))$.

Proposition 3.7. *Let* H *be an ultraspherical hypergroup on a locally compact group* G. *Then the following hold.*

(i) *The norms* $\|\cdot\|_{A(H)}$ *and* $\|\cdot\|_M$ *are equivalent on* $A(H)$ *if and only if the restriction map* $R : Q(H) \to C^*_\lambda(H)$ *is surjective.*

(ii) *If* $A(H)$ *is weak*-dense in* $M(A(H))$, *then* G *is amenable if and only if the norms* $\|\cdot\|_{A(H)}$ *and* $\|\cdot\|_M$ *are equivalent on* $A(H)$.

Proof. (i). Let $\|\cdot\|_{A(H)}$ and $\|\cdot\|_M$ be equivalent on $A(H)$. We first show that the norm on $B_\lambda(H)$ is equivalent to the multiplier norm. Let $i : A(H) \to M(A(H))$ be the inclusion map. Then i is bounded and has $\|\cdot\|_M$-closed range. It follows from [12, Theorem 3.1.21] that $i^*(M(A(H))^*)$ is weak*-closed in $A(H)^*$. Again, by [12, Theorem 3.1.21], $i^{**}(A(H)^{**})$ is norm-closed in $M(A(H))^{**}$. From this and the fact that $B_\lambda(H)$ is norm-closed in $A(H)^{**}$, we conclude that the $\|\cdot\|_{B_\lambda(H)}$-norm and the multiplier norm are equivalent on $B_\lambda(H)$. Therefore, R is surjective by Proposition 2.3. Conversely, suppose that R is surjective. Then by Proposition 2.3, the norms $\|\cdot\|_{B_\lambda(H)}$ and $\|\cdot\|_M$ are equivalent on $B_\lambda(H)$ and hence on $A(H)$.

(ii). Suppose first that G is amenable. Then $A(H)$ has a bounded approximate identity by [1, Theorem 4.4]. It follows easily that the norms $\|\cdot\|_{A(H)}$ and $\|\cdot\|_M$ are equivalent on $A(H)$. Conversely, assume that the norms $\|\cdot\|_{A(H)}$ and $\|\cdot\|_M$ are equivalent on $A(H)$. Suppose that $R(f) = 0$ for some $f \in Q(H)$. Then we have $\langle f, u\rangle = \langle R(f), u\rangle = 0$ for all $u \in A(H)$. Since $A(H)$ is weak*-dense in $M(A(H))$, we get that $\langle f, \phi\rangle = 0$ for all $\phi \in M(A(H))$. This shows that R is injective and so it is bijective by (i). Therefore, $Q(H)$ is isometrically isomorphic to $C^*_\lambda(H)$, which implies that $1 \in M(A(H)) = B_\lambda(H)$. Therefore, G is amenable by Theorem 3.4. □

Remark 3.8. *Identifying* $\ell^1(H)$ *with the subspace* $\lambda(\ell^1(H))$ *of* $VN(H)$, *we denote the norm closure of* $\ell^1(H)$ *in* $VN(H)$ *by* $C_\delta(H)$. *Let* $f = \sum \alpha_i \lambda(\dot{x}_i) \in \ell^1(H)$ *and* $u \in A(H)$. *Then*

$$u \cdot f = \sum \alpha_i u(\dot{x}_i) \lambda(\dot{x}_i) \in C_\delta(H),$$

and $\|u \cdot f\|_{C_\delta(H)} \leq \|u\|_\infty \|f\|_{C_\delta(H)} \leq \|u\|_{A(H)} \|f\|_{C_\delta(H)}$. *Hence,* $C_\delta(H)$ *is a Banach* $A(H)$*-submodule of* $VN(H)$. *Also, note that* $C_\delta(H)^* \subseteq \ell^\infty(H)$.

Proposition 3.9. *Let H be an ultraspherical hypergroup. Then the following hold.*

(i) $B_{A(H)}(A(H), C_\delta(H)^*)$ *consists of functions* $\phi \in \ell^\infty(H)$ *such that the pointwise multiplication map* $T_\phi : A(H) \to C_\delta(H)^*, u \mapsto \phi u$ *is a bounded operator.*

(ii) $Q_{C_\delta(H)}$ *is equal to the completion of* $\ell^1(H)$ *with respect to the norm*

$$\|f\|_M = \sup\left\{\left|\sum f(\dot{x})\phi(\dot{x})\right| : \phi \in B_{A(H)}(A(H), C_\delta(H)^*), \|\phi\| \le 1\right\}.$$

Furthermore, $M(A(H)) \subseteq B_{A(H)}(A(H), C_\delta(H)^*)$*, and the corresponding inclusion map is contractive.*

Proof. (i). Let $\phi \in \ell^\infty(H)$ be such that $T_\phi : A(H) \to C_\delta(H)^*$ is a bounded linear operator. Then since

$$T_\phi(uv) = \phi uv = uT_\phi(v) \quad (u, v \in A(H)),$$

it follows that $T_\phi \in B_{A(H)}(A(H), C_\delta(H)^*)$. For the reverse inclusion, let $\phi \in B_{A(H)}(A(H), C_\delta(H)^*)$. Define $\tilde{\phi} : H \to \mathbb{C}$ by $\tilde{\phi}(\dot{x}) = \langle \phi(u), \lambda(\dot{x})\rangle$, where u denotes a function in $A(H) \cap C_c(H)$ with $u(\dot{x}) = 1$. Then it is well defined. In fact, if v is another function in $A(H)\cap C_c(H)$ such that $v(\dot{x}) = 1$, then we put $K = \mathrm{supp}(u)\cup\mathrm{supp}(v)$ and choose $w \in A(H) \cap C_c(H)$ such that $w|_K \equiv 1$. Then

$$\begin{aligned}\langle \phi(u), \lambda(\dot{x})\rangle &= \langle \phi(uw), \lambda(\dot{x})\rangle = u(\dot{x})\langle \phi(w), \lambda(\dot{x})\rangle \\ &= v(\dot{x})\langle \phi(w), \lambda(\dot{x})\rangle = \langle \phi(vw), \lambda(\dot{x})\rangle \\ &= \langle \phi(v), \lambda(\dot{x})\rangle.\end{aligned}$$

Observe next that if $u \in A(H)$, $\dot{x} \in H$ and $v \in A(H) \cap C_c(H)$ with $v(\dot{x}) = 1$, then

$$\begin{aligned}\langle \phi(u), \lambda(\dot{x})\rangle &= v(\dot{x})\langle \phi(u), \lambda(\dot{x})\rangle = \langle \phi(uv), \lambda(\dot{x})\rangle \\ &= u(\dot{x})\langle \phi(v), \lambda(\dot{x})\rangle = u(\dot{x})\tilde{\phi}(\dot{x}).\end{aligned}$$

This shows that $\phi = T_{\tilde{\phi}}$.

(ii). Since $C_\delta(H)$ is a Banach $A(H)$-submodule of $VN(H)$, it follows from Theorem 2.1 that

$$B_{A(H)}(A(H), C_\delta(H)^*) = Q^*_{C_\delta(H)}.$$

Let $f \in \ell^1(H)$ be with finite support. Then $f = uf \in Q_{C_\delta(H)}$, where $u \in A(H)$ with $u|_{\mathrm{supp}(f)} \equiv 1$. Consequently,

$$\langle \phi, f\rangle = \langle \phi, uf\rangle = \langle \phi(u), f\rangle = \sum \phi(\dot{x})f(\dot{x}),$$

for all $\phi \in B_{A(H)}(A(H), C_\delta(H)^*)$. Hence, there is an isometry between the dense subspace of $\overline{\ell^1(H)}^{\|\cdot\|_M}$ and a dense subspace of $Q_{C_\delta(H)}$. Therefore, $Q_{C_\delta(H)} = \overline{\ell^1(H)}^{\|\cdot\|_M}$.

Since $A(H) \subseteq C_\delta(H)^*$ and $A(H)$ is an ideal in $M(A(H))$, it follows that $\phi u \in C_\delta(H)^*$ for all $\phi \in M(A(H))$ and $u \in A(H)$. This implies that $M(A(H)) \subseteq B_{A(H)}(A(H), C_\delta(H)^*)$. Furthermore, $\|\phi u\|_{C_\delta(H)} \le \|\phi u\|_{A(H)} \le \|\phi\|_M\|u\|_{A(H)}$. Hence, the inclusion map is contractive. □

4. Introverted subspaces of $VN(H)$ and discreteness

Let H be an ultraspherical hypergroup associated to a locally compact group G. The Arens product on $VN(H)^*$ is defined as following three steps. For u, v in $A(H)$, T in $VN(H)$ and $m, n \in VN(H)^*$, we define $u \cdot T$, $m \cdot T \in VN(H)$ and $m \odot n \in VN(H)^*$ as follows:

$$\langle u \cdot T, v\rangle = \langle T, uv\rangle, \quad \langle m \cdot T, u\rangle = \langle m, u \cdot T\rangle, \quad \langle m \odot n, T\rangle = \langle m, n \cdot T\rangle.$$

A linear subspace X of $VN(H)$ is called topologically invariant if $u \cdot X \subseteq X$ for all $u \in A(H)$. The topologically invariant subspace X of $VN(H)$ is called topologically introverted if $m \cdot T \in X$ for all $m \in X^*$ and $T \in X$. In this case, X^* is a Banach algebra with the multiplication induced by the Arens product $\odot$ inherited from

$VN(H)^*$. Let $W(\widehat{H})$ be the set of all T in $VN(H)$ such that the map $u \mapsto u \cdot T$ of $A(H)$ into $VN(H)$ is weakly compact. Let $UCB(\widehat{H})$ denote the closed linear span of

$$\{u \cdot T : u \in A(H), T \in VN(H)\}.$$

The elements in $UCB(\widehat{H})$ are called uniformly continuous functionals on $A(H)$. We also recall that, subspaces $W(\widehat{H})$ and $UCB(\widehat{H})$ of $VN(H)$ are both topologically introverted.

Proposition 4.1. *Let H be an ultraspherical hypergroup. Then $C^*_\lambda(H) \subseteq W(\widehat{H})$.*

Proof. It suffices to prove that if $f \in L^1(H)$, then $\lambda(f) \in W(\widehat{H})$. Let $f \in L^1(H)$ be fixed. Then by Remark 3.1, for each $\phi \in B_\lambda(H)$, we have $\phi \cdot \lambda(f) = \lambda(\phi f)$. Consider the map $\phi \mapsto \lambda(\phi f)$ from $B_\lambda(H)$ into $VN(H)$. This map is continuous when $B_\lambda(H)$ has the $\sigma(B_\lambda(H), C^*_\lambda(H))$-topology and $VN(H)$ has the weak topology. Indeed, let $\Psi \in VN(H)^*$ and $(\phi_\alpha) \subseteq B_\lambda(H)$ be a net such that $\langle \phi_\alpha, T\rangle \to \langle \phi, T\rangle$ for all $T \in C^*_\lambda(H)$. Then the restriction of Ψ to $C^*_\lambda(H)$ is in $C^*_\lambda(H)^* = B_\lambda(H)$. Thus, there exists $\psi \in B_\lambda(H)$ such that

$$\langle \Psi, \lambda(h)\rangle = \langle \psi, \lambda(h)\rangle = \int h(\dot{x})\psi(\dot{x})d\dot{x} \quad (h \in L^1(H)).$$

Hence,

$$\begin{aligned}\langle \Psi, \lambda(\phi_\alpha f)\rangle = \langle \psi, \lambda(\phi_\alpha f)\rangle &= \int \phi_\alpha(\dot{x}) f(\dot{x})\psi(\dot{x})d\dot{x} \\ &= \langle \phi_\alpha, \lambda(\psi f)\rangle \to \langle \phi, \lambda(\psi f)\rangle \\ &= \langle \psi, \lambda(\phi f)\rangle = \langle \Psi, \lambda(\phi f)\rangle.\end{aligned}$$

It follows that the set $\{\phi \cdot \lambda(f) : \phi \in B_\lambda(H), \|\phi\| \le 1\}$ is relatively compact in the weak topology of $VN(H)$. The rest of the proof follows from the fact that $A(H) \subseteq B_\lambda(H)$. □

Proposition 4.2. *Let H be an ultraspherical hypergroup. Then $C^*_\lambda(H) \subseteq UCB(\widehat{H})$.*

Proof. Let $f \in C_c(H)$. By regularity of $A(H)$, there exists $u \in A(H)$ such that $u|_{\mathrm{supp}(f)} \equiv 1$. Therefore,

$$\langle u \cdot \lambda(f), v\rangle = \langle \lambda(f), uv\rangle = \int f(\dot{x})u(\dot{x})v(\dot{x})d\dot{x} = \int f(\dot{x})v(\dot{x})dt = \langle \lambda(f), v\rangle$$

for all $v \in A(H)$. This implies that $u \cdot \lambda(f) = \lambda(f)$. Hence, $\lambda(f) \in UCB(\widehat{H})$. Consequently, $C^*_\lambda(H) \subseteq UCB(\widehat{H})$ by the density of $C_c(H)$ in $C^*_\lambda(H)$. □

Let X be a closed topologically invariant subspace of $VN(H)$ containing $\lambda(\dot{e})$. Then $m \in X^*$ is called a topologically invariant mean on X if:
(i) $\|m\| = \langle m, \lambda(\dot{e})\rangle = 1$;
(ii) $\langle m, u \cdot T\rangle = u(\dot{e})\langle m, T\rangle$ for all $T \in X$ and $u \in A(H)$.

We denote by $TIM(X)$ the set of all topologically invariant means on X. We also recall from Remark 3.1 that the space $C^*_\lambda(H)$ is an $A(H)$-submodule of $VN(H)$. The following proposition is a consequence of [2, Proposition 5.7] and [10, Proposition 6.3] and the fact that $A(H)$ is a commutative F-algebra.

Proposition 4.3. *Let H be an ultraspherical hypergroup. Then the following hold.*
(i) *The space $C^*_\lambda(H)$ is a topologically introverted subspace of $VN(H)$.*
(ii) *$W(\widehat{H})$ admits a unique topologically invariant mean.*

Corollary 4.4. *Let H be an ultraspherical hypergroup. Then H is discrete if and only if $\lambda(\dot{e}) \in C^*_\lambda(H)$.*

Proof. If H is discrete, then $\ell^1(H) = L^1(H)$. Therefore, $\lambda(\dot{e}) \in C^*_\lambda(H)$. Conversely, assume that $\lambda(\dot{e}) \in C^*_\lambda(H)$, and m denote the unique topologically invariant mean on $W(\widehat{H})$. Then $\langle m, \lambda(\dot{e})\rangle = 1$. It follows that H must be discrete by [17, Theorem 4.4(iv)]. □

Lemma 4.5. *Let H be an ultraspherical hypergroup and let $R : VN(H)^* \to UCB(\widehat{H})^*$ be the restriction map. Then $R : TIM(VN(H)) \to TIM(UCB(\widehat{H}))$ is a bijection.*

Proof. If $m_1, m_2 \in TIM(VN(H))$ with $m_1 \neq m_2$, then there exists $T \in VN(H)$ such that $\langle m_1, T\rangle \neq \langle m_2, T\rangle$. Given $u \in A(H)$ with $u(\dot{e}) = 1$, we have

$$\langle m_1, u \cdot T\rangle = \langle m_1, T\rangle \neq \langle m_2, T\rangle = \langle m_2, u \cdot T\rangle.$$

This implies that $R(m_1) \neq R(m_2)$, and hence R is injective.

Suppose that $\tilde{m} \in TIM(UCB(\widehat{H}))$. Choose $u \in A(H)$ with $\|u\|_{A(H)} = u(\dot{e}) = 1$; see [17, Proposition 3.4]. Define m on $VN(H)^*$ by

$$\langle m, T\rangle = \langle \tilde{m}, u \cdot T\rangle \quad (T \in VN(H)).$$

Since $\|u\|_{A(H)} = 1$, it follows that $\|m\| \leq 1$. Moreover,

$$\langle m, \lambda(\dot{e})\rangle = \langle \tilde{m}, u \cdot \lambda(\dot{e})\rangle = u(\dot{e})\langle \tilde{m}, \lambda(\dot{e})\rangle = \langle \tilde{m}, \lambda(\dot{e})\rangle = 1.$$

Therefore, $\|m\| = 1$. Furthermore, for each $v \in A(H)$ and $T \in VN(H)$, we have

$$\langle m, v \cdot T\rangle = \langle \tilde{m}, u \cdot (v \cdot T)\rangle = \langle \tilde{m}, v \cdot (u \cdot T)\rangle = v(\dot{e})\langle \tilde{m}, u \cdot T\rangle = v(\dot{e})\langle m, T\rangle.$$

Consequently, $m \in TIM(VN(H))$. Finally, if $T \in UCB(\widehat{H})$, then

$$\langle R(m), T\rangle = \langle m, T\rangle = \langle \tilde{m}, u \cdot T\rangle = \langle \tilde{m}, T\rangle.$$

Hence, R is surjective. □

Proposition 4.6. *Let H be an ultraspherical hypergroup. Then the following are equivalent.*
(i) *H is discrete.*
(ii) *$UCB(\widehat{H}) = C^*_\lambda(H)$.*
(iii) *There is a unique topologically invariant mean on $UCB(\widehat{H})$.*

Proof. (i)⇒(ii). Assume that H is discrete. Then for each $\dot{x} \in H$, the characteristic function $1_{\dot{x}}$ is in $A(H)$; see [14, Proposition 2.22]. Let $T \in VN(H)$ be fixed. Then for each $v \in A(H)$, we get

$$\langle 1_{\dot{x}} \cdot T, v\rangle = \langle T, v1_{\dot{x}}\rangle = \langle T, v(\dot{x})1_{\dot{x}}\rangle = v(\dot{x})\langle T, 1_{\dot{x}}\rangle.$$

Hence, $1_{\dot{x}} \cdot T = \langle T, 1_{\dot{x}}\rangle\lambda(\dot{x}) \in C^*_\lambda(H)$. Let $u \in A(H)$. Since $A(H) \cap C_c(H)$ is dense in $A(H)$, we can suppose that u has compact and hence finite support. Thus, u is a finite linear combination of characteristic functions on one point sets. Therefore, $u \cdot T \in C^*_\lambda(H)$. It follows from Proposition 4.2 that $UCB(\widehat{H}) = C^*_\lambda(H)$.

(ii)⇒(iii). If $UCB(\widehat{H}) = C^*_\lambda(H)$, then $UCB(\widehat{H}) \subseteq W(\widehat{H})$ by Proposition 4.1. Let m, n be topologically invariant means on $VN(H)$. Then $m = n$ when restricted to $W(\widehat{H})$ by Proposition 4.3(ii). Since $UCB(\widehat{H}) \subseteq W(\widehat{H})$, we conclude that $R(m) = R(n)$, and hence $m = n$ by Lemma 4.5. Again Lemma 4.5, implies that there is a unique topological invariant mean on $UCB(\widehat{H})$.

(iii)⇒(i). This follows from Lemma 4.5 and [18, Theorem 1.7]. □

It is shown in [15, Theorem 3.15] that $B_\lambda(H)$ is a Banach algebra under pointwise multiplication. As shown in Proposition 4.3, $C^*_\lambda(H)$ is topologically introverted. In particular, $C^*_\lambda(H)^* = B_\lambda(H)$ is a Banach algebra with the Arens Product. It is shown in [9, Proposition 5.3] that the Arens product on $B_\lambda(G)$ is precisely the pointwise product on it. Following we show that the same is also true for an ultraspherical hypergroup H.

Proposition 4.7. *Let H be an ultraspherical hypergroup. Then the Arens product and the pointwise multiplication on $B_\lambda(H)$ coincide.*

Proof. Let $\phi, \psi \in B_\lambda(H)$. Then for each $f \in L^1(H)$, we have

$$\langle \phi\psi, \lambda(f)\rangle = \langle \phi, \lambda(\psi f)\rangle = \langle \psi, \lambda(\phi f)\rangle.$$

This shows that the pointwise multiplication on $B_\lambda(H)$ is separately continuous in the weak*-topology. Furthermore, for each $\psi \in B_\lambda(H)$, the map $\phi \mapsto \phi \odot \psi$ from $B_\lambda(H)$ into $B_\lambda(H)$ is weak*-weak* continuous. Since $C^*_\lambda(H) \subseteq W(\widehat{H})$, it follows from [2, Proposition 3.11] that the map $\phi \mapsto \psi \odot \phi$ is continuous in the weak*-topology. Therefore, the Arens product also is separately continuous in the weak*-topology. Since the Arens product and the pointwise multiplication on $A(H)$ coincide and $A(H)$ is w^*-dense in $B_\lambda(H)$, we conclude that $\phi \odot \psi = \phi\psi$ for all $\phi, \psi \in B_\lambda(H)$. □

It is easily verified that the map $\Phi : B_\lambda(H) \longrightarrow B_{A(H)}(B_\lambda(H)), \phi \mapsto \phi_L$ induces a contractive, injective algebra homomorphism, where ϕ_L is given by $\phi_L(\psi) = \phi\psi$ for all $\psi \in B_\lambda(H)$. Finally, since the Arens product and the pointwise multiplication on $B_\lambda(H)$ coincide, we have $\Phi(B_\lambda(H)) \subseteq B^\sigma_{A(H)}(B_\lambda(H))$, where $B^\sigma_{A(H)}(B_\lambda(H))$ denote the subalgebra of $B_{A(H)}(B_\lambda(H))$ consisting of weak*-weak* continuous maps in $B_{A(H)}(B_\lambda(H))$.

Corollary 4.8. *Let H be an ultraspherical hypergroup on a locally compact group G. Then the map*

$$\Phi : B_\lambda(H) \longrightarrow B^\sigma_{A(H)}(B_\lambda(H)), \quad \phi \mapsto \phi_L$$

is surjective if and only if G is amenable.

Proof. Suppose that Φ is surjective. Since $id_{B_\lambda(H)} \in B^\sigma_{A(H)}(B_\lambda(H))$, there is a $u \in B_\lambda(H)$ such that $\Phi(u) = id_{B_\lambda(H)}$. A routine calculation shows that u is an identity for $B_\lambda(H)$ and hence the constant function 1 belongs to $B_\lambda(H)$. Therefore, G is amenable by Theorem 3.4.

For the converse, note that if G is amenable, then the constant function 1 belongs to $B_\lambda(H)$, by Theorem 3.4. Now let $\Lambda \in B^\sigma_{A(H)}(B_\lambda(H))$ and $\psi \in B_\lambda(H)$. Since $A(H)$ is weak*-dense in $B_\lambda(H)$, there is a net (u_α) in $A(H)$ such that $u_\alpha \xrightarrow{w^*} \psi$ and hence

$$\Lambda(u_\alpha) = \Lambda(u_\alpha 1) = \Lambda(1)u_\alpha \xrightarrow{w^*} \Lambda(1)\psi.$$

In particular, for each $T \in C^*_\lambda(H)$, by weak*-weak* continuity of Λ, we have

$$\langle \Phi(\Lambda(1))(\psi), T\rangle = \langle \Lambda(1)\psi, T\rangle = \lim_\alpha \langle \Lambda(u_\alpha), T\rangle = \langle \Lambda(\psi), T\rangle.$$

Therefore, $\Phi(\Lambda(1)) = \Lambda$ for all $\Lambda \in B^\sigma_{A(H)}(B_\lambda(H))$. □

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