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Viscosity Approximation Methods for Quasi-Nonexpansive Mappings in Banach Spaces

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Abstract. In this article, we present viscosity approximation methods for finding a common point of the set of solutions of a variational inequality problem and the set of fixed points of a multi-valued quasi-nonexpansive mapping in a Banach space. We also discuss some examples to illustrate facts and study the convergence behaviour of the iterative schemes presented herein, numerically.

1. Introduction

The study of existence and convergence of fixed points of nonexpansive type mappings is an interesting and important subject in nonlinear analysis, particularly in fixed point theory. In general, the sequence of iterates of a nonexpansive mapping need not converge to a fixed point of the mapping in a Banach space. Also, some of the well-known classical iteration processes give only weak convergence. In order to obtain strong convergence for a sequence of iterates of nonexpansive mapping, Moudafi [9] proposed new methods known as viscosity approximation methods. These methods are beneficial and effective tools to solve variational inequality, split and common split feasibility, convex optimization, and many other problems arising in nonlinear analysis [3, 8, 16].

Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space and $\mathcal{K}$ a nonempty convex closed subset of $\mathcal{X}$. A mapping $T : \mathcal{K} \to \mathcal{K}$ is said to be contraction if there exists $r \in [0, 1)$ such that

$$\|T(u) - T(v)\| \leq r\|u - v\|, \quad \text{for all } u, v \in \mathcal{K}.$$

If $r = 1$, then the mapping T is called nonexpansive. We denote the set of all fixed points of T by $\mathcal{F}(T)$. The mapping T is said to be quasi-nonexpansive if for all $u \in \mathcal{K}$ and $u^\dagger \in \mathcal{F}(T)$,

$$\|T(u) - u^\dagger\| \leq \|u - u^\dagger\|.$$

The variational inequality problem is to find a point $u \in \mathcal{K}$ such that

$$\langle T(u), v - u\rangle \geq 0, \quad \text{for all } v \in \mathcal{K}.$$

The problem of finding common elements of the set of fixed points for mappings and the set of solutions for variational inequalities is a closely related subject of current interest [14, 18]. Viscosity approximation

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methods are interesting because under appropriate conditions, the iteration converges strongly to the unique solution of the variational inequality problem in the set of fixed points. Because of this fact, we can apply these methods in linear programming, convex optimization, monotone inclusions, and many other problems.

In a Hilbert space H, Moudafi [9] defined the sequence of iterates of a nonexpansive mapping T as follows:

$$u_{n+1} = \frac{\zeta_n}{1+\zeta_n}\mathcal{G}(u_n) + \frac{1}{1+\zeta_n}T(u_n), \text{ for all } n \geq 0, \tag{1}$$

where $\mathcal{G}$ is a contraction, $u_0 \in H$ is an initial guess, $r \in (0,1)$ and $\{\zeta_n\}$ is a sequence in $(0,1)$ with some conditions. Under some conditions the sequence $\{u_n\}$ defined by (1) converges strongly to unique solution $u^\dagger \in \mathcal{F}(T)$ of the following variational inequality:

$$\langle (I-\mathcal{G})u^\dagger, u-u^\dagger\rangle, \ \forall\, u \in \mathcal{F}(T).$$

In 2003, Xu [16] defined the sequence $\{u_n\}$ for all $n \geq 0$ as follows:

$$u_{n+1} = \zeta_n b + (I-\zeta_n A)T(u_n), \tag{2}$$

and proved that, the sequence $\{u_n\}$ defined above is converging strongly to the unique solution of a minimization problem. Marino and Xu [8] defined a new sequence by combining (1) and (2) as follows:

$$u_{n+1} = \zeta_n\gamma\mathcal{G}(u_n) + (I-\zeta_n A)T(u_n), \text{ for all } n \geq 0. \tag{3}$$

They proved that the sequence $\{u_n\}$ generated by (3) with certain conditions on sequence $\{\zeta_n\}$ converges strongly to the unique solution of the variational inequality:

$$\langle (\gamma\mathcal{G}-A)u^\dagger, u-u^\dagger\rangle \leq 0, \text{ for all } u \in \mathcal{K}.$$

Xu [17] generalized the Moudafi's results from Hilbert spaces to Banach spaces and presented the following result:

Theorem 1.1. [17]. *Suppose $\mathcal{X}$ be a uniformly smooth Banach space and $\mathcal{K}$ a nonempty closed and convex subset of $\mathcal{X}$. Let $\mathcal{G} : \mathcal{K} \to \mathcal{K}$ be a contraction and $T : \mathcal{K} \to \mathcal{K}$ a nonexpansive mapping with $\mathcal{F}(T) \neq \emptyset$. Let the sequence $\{u_n\}$ defined as $u_{n+1} = \zeta_n\mathcal{G}(u_n) + (1-\zeta_n)T(u_n)$, where $\zeta_n \in (0,1)$ satisfies*

(a_1) $\zeta_n \to 0$,

(a_2) $\sum_{n=0}^{\infty} \zeta_n = \infty$,

(a_3) *either* $\sum_{n=0}^{\infty} |\zeta_{n+1} - \zeta_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\zeta_{n+1}}{\zeta_n} = 1$.

Then, u_n converges to $u^\dagger \in \mathcal{F}(T)$ which is the solution of the following variational inequality

$$\langle (I-\mathcal{G})u^\dagger, j(u^\dagger - u)\rangle \leq 0, \text{ for all } u \in \mathcal{F}(T).$$

Mainge [7] considered the viscosity approximation method (1) and proved the strong convergence results for quasi-nonexpansive mappings in a Hilbert space. Motivated by the results of Moudafi [9], Xu [17], Mainge [7] and others, we study some strong convergence results for multi-valued quasi-nonexpansive mappings in Banach spaces. The main purpose of this paper is to present some strong convergence theorems to find a common element of the set of fixed points of a multi-valued quasi-nonexpansive mapping and the set of solutions of a variational inequality problem in Banach spaces. We illustrate our results by presenting some useful examples. We also present numerical convergence behaviour for different choices of coefficients and initial guesses.

2. Preliminaries

Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space, $CB(\mathcal{X})$ the collection of all nonempty bounded and closed subsets of $\mathcal{X}$, and $C(\mathcal{X})$ the collection of all nonempty compact subsets of $\mathcal{X}$. Suppose d is the metric induced by the norm $\|\cdot\|$, i.e., $d(u,v) = \|u - v\|$ for all $u, v \in \mathcal{K}$. The Hausdorff metric on $CB(\mathcal{X})$ is defined as,

$$\mathcal{H}(\mathcal{A},\mathcal{B}) = \max\left\{\sup_{u\in\mathcal{A}} d(u,\mathcal{B}), \sup_{v\in\mathcal{B}} d(v,\mathcal{A})\right\}$$

for all $\mathcal{A},\mathcal{B} \in CB(\mathcal{X})$, where $d(u,\mathcal{B}) = \inf_{v\in\mathcal{B}} \|u - v\|$. A point $u^\dagger \in \mathcal{X}$ is said to be a fixed point of a multi-valued mapping $T : \mathcal{X} \to CB(\mathcal{X})$ if $u^\dagger \in T(u^\dagger)$. We also denote the set of fixed points of multi-valued mapping T by $\mathcal{F}(T)$. A multi-valued mapping $T : \mathcal{X} \to CB(\mathcal{X})$ is said to be quasi-nonexpansive if for all $u \in \mathcal{X}$ and $u^\dagger \in \mathcal{F}(T)$, we have

$$\mathcal{H}(T(u), T(u^\dagger)) \le \|u - u^\dagger\|.$$

Example 2.1. *[12]. Let $\mathcal{X} = [0,\infty)$ with the usual metric and $T : \mathcal{X} \to CB(\mathcal{X})$ be defined as*

$$T(u) = \begin{cases} \{0\}, \ u \le 1, \\ \left[u - \frac{3}{4}, u - \frac{1}{3}\right], u > 1. \end{cases}$$

Then T is a quasi-nonexpansive mapping. However, T is not a nonexpansive mapping.

Definition 2.2. *[6]. The Banach space $\mathcal{X}$ is said to be uniformly convex if for each ε, $0 < \varepsilon \le 2$, there exists a $\delta(\varepsilon) > 0$ such that the conditions*

$$\|u\| = \|v\| = 1, \quad \|u - v\| \ge \varepsilon, \text{ imply } \left\|\frac{u+v}{2}\right\| \le 1 - \delta(\varepsilon) \text{ for all } u, v \in \mathcal{X}.$$

The Banach space $\mathcal{X}$ is said to be strictly convex if

$$\left\|\frac{u+v}{2}\right\| < 1, \text{ whenever } u, v \in \mathcal{X} \text{ with } \|u\| = \|v\| = 1, u \ne v.$$

Definition 2.3. *[1]. Suppose $\mathcal{X}$ be a Banach space and $\mathcal{X}'$ its dual. Then the normalized duality mapping is a multi-valued mapping $J : \mathcal{X} \to 2^{\mathcal{X}'}$ defined as*

$$J(u) := \{\mathcal{G} \in \mathcal{X}' : \langle u, \mathcal{G}\rangle = \|u\|^2 = \|\mathcal{G}\|^2\},$$

and we denote the single-valued duality mapping by j.

Definition 2.4. *[5]. The modulus of smoothness of the Banach space $\mathcal{X}$ is given by*

$$\rho_{\mathcal{X}(t)} = \sup\left\{\frac{\|u + tv\| + \|u - tv\|}{2} - 1 : \|u\| = \|v\| = 1\right\}, t \ge 0.$$

The Banach space $\mathcal{X}$ is called uniformly smooth if $\lim_{t\to 0} \frac{\rho_{\mathcal{X}(t)}}{t} = 0$. The mapping J is single-valued and uniformly continuous on each bounded subset of $\mathcal{X}$ if and only if the Banach space $\mathcal{X}$ is uniformly smooth.

Definition 2.5. *[11]. The Banach space $\mathcal{X}$ satisfies Opial property if, for each weakly convergent sequence $\{u_n\}$ with the weak limit $u \in \mathcal{X}$ holds:*

$$\liminf_{n\to\infty} \|u_n - u\| < \liminf_{n\to\infty} \|u_n - v\|$$

for all $v \in \mathcal{X}$ with $u \ne v$.

All Hilbert spaces, all finite dimensional Banach spaces, and ℓ^p $(1 \leq p < \infty)$ satisfy the Opial property. A Banach space with a weakly sequentially continuous duality mapping also satisfies the Opial property. But L_p $(1 < p < \infty,\ p \neq 2)$ do not have the Opial property. We say that the Banach space $\mathcal{X}$ has a weakly sequentially continuous duality mapping $J : \mathcal{X} \to \mathcal{X}'$ if J is single-valued and is weak-to-weak* sequentially continuous, that is, if $\{u_n\} \subset \mathcal{X}$, $u_n \xrightarrow{w} u$, then $J(u_n) \xrightarrow{w^*} J(u)$. A Banach space $\mathcal{X}$ that has a weakly continuous duality mapping satisfies the Opial property [2, 4].

Lemma 2.6. [15]. *Let J be a normalized duality mapping of a Banach space $\mathcal{X}$. Then, for each $u, v \in \mathcal{X}$, the following inequality holds*

$$\|u + v\|^2 \leq \|u\|^2 + 2\langle v, J(u + v)\rangle.$$

Lemma 2.7. [15]. *Assume $\{\tau_n\}$ be a sequence of non-negative real numbers satisfying*

$$\tau_{n+1} \leq (1 - \sigma_n)\tau_n + \xi_n + \eta_n$$

for all $n \geq 0$, where $\{\sigma_n\}$ is a subsequence in $(0, 1)$, $\{\xi_n\}$ and $\{\eta_n\}$ are real sequences. Suppose that:

(1) $\sum_{n=1}^{\infty} \sigma_n = \infty$,

(2) $\sum_{n=1}^{\infty} |\xi_n| < \infty$ or $\limsup_{n\to\infty} \frac{\xi_n}{\sigma_n} \leq 0$,

(3) $\sum_{n=1}^{\infty} \eta_n < \infty$.

Then, $\lim_{n\to\infty} \tau_n = 0$.

3. Variational Inequality Problem

Theorem 3.1. *Let $\mathcal{X}$ be a Banach space and $\mathcal{K}$ a nonempty closed and convex subset of $\mathcal{X}$. Let $T : \mathcal{K} \to C(\mathcal{K})$ be a quasi-nonexpansive mapping and $\mathcal{G} : \mathcal{K} \to \mathcal{K}$ a contraction. Let there exists $u_t \in \mathcal{K}$ for any $t \in (0, 1)$ such that*

$$u_t \in t\mathcal{G}(u_t) + (1 - t)T(u_t).$$

If $\mathcal{F}(T) \neq \emptyset$ then we get the following results:

(1) *$\{u_t\}$ is bounded and $\lim_{t\to 0} d(u_t, T(u_t)) = 0$,*

(2) *for $u^\dagger \in \mathcal{F}(T)$, $\|u_t - u^\dagger\|^2 \leq \langle \mathcal{G}(u_t) - u^\dagger, j(u_t - u^\dagger)\rangle$,*

(3) *for $u^\dagger \in \mathcal{F}(T)$, $\langle u_t - \mathcal{G}(u_t), j(u_t - u^\dagger)\rangle \leq 0$.*

Proof. By the assumptions for all $t \in (0, 1)$, we have $u_t \in \mathcal{K}$ such that $u_t \in t\mathcal{G}(u_t) + (1 - t)T(u_t)$, thus there exists $v_t \in T(u_t)$ for each $u_t \in \mathcal{K}$ such that

$$u_t = t\mathcal{G}(u_t) + (1 - t)v_t.$$

For any $u^\dagger \in \mathcal{F}(T)$, we have,

$$\begin{aligned}
\|u_t - u^\dagger\| &= \|t\mathcal{G}(u_t) + (1 - t)v_t - u^\dagger\| \\
&\leq t\|\mathcal{G}(u_t) - u^\dagger\| + (1 - t)\|v_t - u^\dagger\| \\
&\leq t\|\mathcal{G}(u_t) - u^\dagger\| + (1 - t)\mathcal{H}(T(u_t), T(u^\dagger)) \\
&\leq t\|\mathcal{G}(u_t) - u^\dagger\| + (1 - t)\|u_t - u^\dagger\| \\
&= \|\mathcal{G}(u_t) - u^\dagger\| \\
&\leq \|\mathcal{G}(u_t) - \mathcal{G}(u^\dagger)\| + \|\mathcal{G}(u^\dagger) - u^\dagger\| \\
&\leq r\|u_t - u^\dagger\| + \|\mathcal{G}(u^\dagger) - u^\dagger\|.
\end{aligned}$$

This implies that

$$\|u_t - u^\dagger\| \leq \frac{1}{(1-r)}\|\mathcal{G}(u^\dagger) - u^\dagger\|.$$

Hence $\{u_t\}$ is a bounded sequence. So that $\{\mathcal{G}(u_t)\}$ and $\{v_t\}$ are bounded. Since $v_t \in T(u_t)$, $d(u_t, T(u_t)) \leq \|u_t - v_t\|$, so we have

$$\lim_{t\to 0} d(u_t, T(u_t)) \leq \lim_{t\to 0}\|u_t - v_t\| = \lim_{t\to 0} t\|\mathcal{G}(u_t) - v_t\| = 0.$$

This proves (1). Again, we have

$$\begin{aligned}\|u_t - u^\dagger\|^2 &= \langle u_t - u^\dagger, j(u_t - u^\dagger)\rangle \\ &= t\langle \mathcal{G}(u_t) - u^\dagger, j(u_t - u^\dagger)\rangle + (1-t)\langle v_t - u^\dagger, j(u_t - u^\dagger)\rangle \\ &\leq t\langle \mathcal{G}(u_t) - u^\dagger, j(u_t - u^\dagger)\rangle + (1-t)\|v_t - u^\dagger\|\|j(u_t - u^\dagger)\| \\ &\leq t\langle \mathcal{G}(u_t) - u^\dagger, j(u_t - u^\dagger)\rangle + (1-t)d(T(u_t), u^\dagger)\|u_t - u^\dagger\| \\ &\leq t\langle \mathcal{G}(u_t) - u^\dagger, j(u_t - u^\dagger)\rangle + (1-t)\|u_t - u^\dagger\|^2.\end{aligned}$$

So, we get

$$\|u_t - u^\dagger\|^2 \leq \langle \mathcal{G}(u_t) - u^\dagger, j(u_t - u^\dagger)\rangle. \tag{4}$$

Finally,

$$\begin{aligned}\langle u_t - \mathcal{G}(u_t), j(u_t - u^\dagger)\rangle &= \langle u_t - u^\dagger, j(u_t - u^\dagger)\rangle + \langle u^\dagger - \mathcal{G}(u_t), j(u_t - u^\dagger)\rangle \\ &= \|u_t - u^\dagger\|^2 - \langle \mathcal{G}(u_t) - u^\dagger, j(u_t - u^\dagger)\rangle.\end{aligned}$$

Using (4) we get,

$$\langle u_t - \mathcal{G}(u_t), j(u_t - u^\dagger)\rangle \leq 0.$$

Hence the proof is now completed. □

Theorem 3.2. *Let $\mathcal{X}$ be a reflexive Banach space and $\mathcal{K}$ a nonempty closed and convex subset of $\mathcal{X}$. Let j be a weakly continuous normalized duality mapping on $\mathcal{X}$, $T : \mathcal{K} \to C(\mathcal{K})$ a quasi-nonexpansive mapping, $\mathcal{G} : \mathcal{K} \to \mathcal{K}$ a contraction and $\mathcal{F}(T) \neq \emptyset$. Then the net $u_t = t\mathcal{G}(u_t) + (1-t)v_t$ converges strongly to a point in $\mathcal{F}(T)$ as $t \to 0$. Also the mapping $G : \psi \to \mathcal{F}(T)$ defined as*

$$G(\mathcal{G}) = \lim_{t\to 0} u_t,$$

satisfies the following variational inequality

$$\langle (I - \mathcal{G})G(\mathcal{G}), j(G(\mathcal{G}) - u^\dagger)\rangle, \ \forall\, u^\dagger \in \mathcal{F}(T).$$

Proof. Since the net $\{u_t\}$ is bounded, by Theorem 3.1 and reflexivity of $\mathcal{X}$, a weakly convergent subsequence $\{u_{t_n}\}$ of $\{u_t\}$ exists, where $\{t_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} t_n = 0$. Let $\{u_{t_n}\}$ is converging weakly to a point $u^\dagger \in \mathcal{K}$. Since $T(u^\dagger)$ is compact so we can find a subsequence $\{v_n\}$ in $T(u^\dagger)$ converging to a point $v \in T(u^\dagger)$ such that

$$\|u_{t_n} - v_n\| = d(u_{t_n}, T(u^\dagger)). \tag{5}$$

Let $v \neq u^{\dagger}$. Using triangle inequality and (5), we get

$$\begin{aligned}\limsup_{n\to\infty} \|u_{t_n} - v\| &\leq \limsup_{n\to\infty} (\|u_{t_n} - v_n\| + \|v_n - v\|) \\ &= \limsup_{n\to\infty} d(u_{t_n}, T(u^{\dagger})) \\ &\leq \limsup_{n\to\infty} d(u_{t_n}, T(u_{t_n})) + \mathcal{H}(T(u_{t_n}), T(u^{\dagger})).\end{aligned}$$

Since $u^{\dagger} \in \mathcal{F}(T)$ and T is a quasi-nonexpansive mapping,

$$\mathcal{H}(T(u_{t_n}), T(u^{\dagger})) \leq \|u_{t_n} - u^{\dagger}\|.$$

By Theorem 3.1, $\limsup_{n\to\infty} d(u_{t_n}, T(u_{t_n})) = 0$. Using Opial property, we get

$$\limsup_{n\to\infty} \|u_{t_n} - v\| \leq \limsup_{n\to\infty} \|u_{t_n} - u^{\dagger}\| < \limsup_{n\to\infty} \|u_{t_n} - v\|,$$

a contradiction unless $v = u^{\dagger}$. Now using Theorem 3.1 (2), we get

$$\|u_{t_n} - u^{\dagger}\|^2 \leq \langle \mathcal{G}(u_{t_n}) - u^{\dagger}, j(u_{t_n} - u^{\dagger})\rangle.$$

Since, j is weakly sequentially continuous, the subsequence $\{u_{t_n}\}$ converges strongly to $u^{\dagger}$. Now, to prove the whole net $\{u_t\}$ converges strongly to $u^{\dagger}$, we will assume another subsequence $\{u_{s_k}\}$ of $\{u_t\}$ such that $\lim_{s_k\to 0} u_{s_k} \to u^*$. Let $u^{\dagger} \neq u^*$ and $w_k = u_{s_k}$. Now for $w \in \mathcal{F}(T)$, we get

$$\begin{aligned}&|\langle w_k - \mathcal{G}(w_k), j(w_k - w)\rangle - \langle u^* - \mathcal{G}(w_k), j(u^* - w)\rangle| \\ &\quad = |\langle w_k - u^*, j(w_k - w)\rangle + \langle u^* - \mathcal{G}(w_k), j(w_k - w)\rangle - \langle u^* - \mathcal{G}(w_k), j(u^* - w)\rangle| \\ &\quad \leq \|w_k - u^*\|\|j(w_k - w)\| + |\langle u^* - \mathcal{G}(w_k), j(w_k - w) - j(u^* - w)\rangle|.\end{aligned}$$

Since $\{u_t\}$ is bounded and j is single-valued normalized duality mapping as well as weakly sequentially continuous, applying $k \to \infty$ and $w_k \to u^*$, we get

$$\|w_k - u^*\|\|j(w_k - w)\| + |\langle u^* - \mathcal{G}(w_k), j(w_k - w) - j(u^* - w)\rangle| = 0.$$

Now, for each $w \in \mathcal{F}(T)$, using Theorem 3.1 (3), we get

$$\langle u^* - \mathcal{G}(u^*), j(u^* - w)\rangle = \limsup_{n\to\infty} \langle w_k - \mathcal{G}(w_k), j(w_k - w)\rangle \leq 0. \tag{6}$$

Similarly,

$$\langle u^{\dagger} - \mathcal{G}(u^{\dagger}), j(u^{\dagger} - w)\rangle \leq 0. \tag{7}$$

Now replacing w by $u^{\dagger}$ in (6) and u^* in (7), we get

$$\begin{aligned}&\langle u^* - \mathcal{G}(u^*), j(u^* - u^{\dagger})\rangle \leq 0, \\ &\langle u^{\dagger} - \mathcal{G}(u^{\dagger}), j(u^{\dagger} - u^*)\rangle \leq 0.\end{aligned}$$

Now adding both the above inequalities, we get

$$\langle u^{\dagger} - u^* - (\mathcal{G}(u^{\dagger}) - \mathcal{G}(u^*)), j(u^{\dagger} - u^*)\rangle \leq 0.$$

It implies that

$$\begin{aligned}&\|(u^{\dagger} - u^*) - (\mathcal{G}(u^{\dagger}) - \mathcal{G}(u^*))\|\|u^{\dagger} - u^*\| \leq 0 \\ &\|u^{\dagger} - u^*\|^2 - \|u^{\dagger} - u^*\|\|\mathcal{G}(u^{\dagger}) - \mathcal{G}(u^*)\| \leq 0 \\ &\|u^{\dagger} - u^*\|^2 - r\|u^{\dagger} - u^*\|\|u^{\dagger} - u^*\| \leq 0 \\ &\|u^{\dagger} - u^*\|^2 \leq r\|u^{\dagger} - u^*\|^2,\end{aligned}$$

which is a contradiction to our hypothesis that $u^\dagger \neq u^*$. Thus the sequence $\{u_t\}$ converges strongly to $u^\dagger$, and $u^\dagger$ is the unique solution of the variational inequality

$$\langle u^\dagger - \mathcal{G}(u^\dagger), j(u^\dagger - w)\rangle \leq 0, \text{ for all } w \in \mathcal{F}(T).$$

Now $\forall\, \mathcal{G} \in \psi$ define

$$G(\mathcal{G}) = \lim_{t\to 0} u_t.$$

Thus,

$$\langle (I - \mathcal{G})G(\mathcal{G}), j(G(\mathcal{G}) - w)\rangle \leq 0, \text{ for all } w \in \mathcal{F}(T).$$

□

Lemma 3.3. [10]. *Let $\mathcal{X}$ be a metric space and $\mathcal{A}, \mathcal{B}$ are nonempty and compact subsets of metric space $\mathcal{X}$. Then for each $a \in \mathcal{A}$, $\exists\ b \in \mathcal{B}$ such that*

$$d(a, b) \leq \mathcal{H}(\mathcal{A}, \mathcal{B}).$$

Theorem 3.4. *Let $\mathcal{X}$ be a reflexive Banach space and $\mathcal{K}$ a nonempty, closed and convex subset of $\mathcal{X}$. Let j be a weakly continuous normalized duality mapping on $\mathcal{X}$, $\mathcal{G} : \mathcal{K} \to \mathcal{K}$ a contraction and $T : \mathcal{K} \to C(\mathcal{K})$ a quasi-nonexpansive mapping with $\mathcal{F}(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in $\mathcal{K}$ defined as*

$$u_{n+1} = \zeta_n \mathcal{G}(u_n) + (1 - \zeta_n) v_n, \tag{8}$$

where $v_n \in T(u_n)$ for all $n \in \mathbb{N} \cup \{0\}$ and satisfies conditions:

- (a_1) $\zeta_n \to 0$ *as* $n \to \infty$,
- (a_2) $\sum_{n=0}^{\infty} \zeta_n = \infty$,
- (a_3) *either* $\sum_{n=0}^{\infty} |\zeta_{n+1} - \zeta_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\zeta_{n+1}}{\zeta_n} = 1$,
- (a_4) $\lim_{n\to\infty} d(u_n, T(u_n)) = 0$.

Then the sequence $\{u_n\}$ converges strongly to $G(\mathcal{G})$, where $G : \psi \to \mathcal{F}(T)$ is defined as $G(\mathcal{G}) = \lim_{t\to 0} u_t$.

Proof. Let $u^\dagger \in \mathcal{F}(T)$. Using triangle inequality and definition of T, we have

$$\begin{aligned}
\|u_{n+1} - u^\dagger\| &= \|\zeta_n \mathcal{G}(u_n) + (1 - \zeta_n)v_n - u^\dagger\| \\
&\leq \zeta_n \|\mathcal{G}(u_n) - u^\dagger\| + (1 - \zeta_n)\|v_n - u^\dagger\| \\
&\leq (1 - \zeta_n)\mathcal{H}(T(u_n), T(u^\dagger)) + \zeta_n\|\mathcal{G}(u_n) - \mathcal{G}(u^\dagger)\| + \zeta_n\|\mathcal{G}(u^\dagger) - u^\dagger\| \\
&\leq (1 - \zeta_n)\|u_n - u^\dagger\| + r\zeta_n\|u_n - u^\dagger\| + \zeta_n\|\mathcal{G}(u^\dagger) - u^\dagger\| \\
&= (1 - \zeta_n + r\zeta_n)\|u_n - u^\dagger\| + \zeta_n\|\mathcal{G}(u^\dagger) - u^\dagger\| \\
&= (1 - \zeta_n + r\zeta_n)\|u_n - u^\dagger\| + \frac{(\zeta_n - r\zeta_n)}{(1 - r)}\|\mathcal{G}(u^\dagger) - u^\dagger\| \\
&\leq \max\left\{\|u_n - u^\dagger\|, \frac{\|\mathcal{G}(u^\dagger) - u^\dagger\|}{(1 - r)}\right\}.
\end{aligned}$$

Hence $\{u_n\}$ is a bounded sequence so that $\{v_n\}$, $\{\mathcal{G}(u_n)\}$ are bounded.
Now, we prove that

$$\limsup_{n\to\infty} \langle (I - \mathcal{G})G(\mathcal{G}), j(G(\mathcal{G}) - u_n)\rangle \leq 0.$$

Since $\{u_n\}$ is a bounded sequence and space $\mathcal{X}$ is reflexive, a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ can be found, such that it converges weakly to some $u^* \in X$. Let $\{u_{n_k}\}$ be such that

$$\limsup_{n\to\infty}\langle (I-\mathcal{G})G(\mathcal{G}), j(G(\mathcal{G})-u_n)\rangle = \lim_{k\to\infty}\langle (I-\mathcal{G})G(\mathcal{G}), j(G(\mathcal{G})-u_{n_k})\rangle.$$

Now from (a_4), we get

$$\lim_{n\to\infty} d(u_{n_k}, T(u_{n_k})) = 0,$$

and $u^* \in \mathcal{F}(T)$. Since normalized duality mapping $j : \mathcal{X} \to \mathcal{X}'$ is weakly sequentially continuous, using Theorem 3.2

$$\lim_{n\to\infty}\langle (I-\mathcal{G})G(\mathcal{G}), j(G(\mathcal{G})-u_{n_k})\rangle = \langle (I-\mathcal{G})G(\mathcal{G}), j(G(\mathcal{G})-u^*)\rangle \leq 0,$$

and

$$\limsup_{n\to\infty}\langle (I-\mathcal{G})G(\mathcal{G}), j(G(\mathcal{G})-u_n)\rangle \leq 0. \tag{9}$$

Next, we prove the sequence $\{u_n\}$ converges strongly to $u^\dagger = G(\mathcal{G}) \in \mathcal{F}(T)$. For all $n \in \mathbb{N}$, define

$$\Gamma_n = \max\{\langle u^\dagger - \mathcal{G}(u^\dagger), j(u^\dagger - u_n)\rangle, 0\} \geq 0.$$

By (9), $\lim_{n\to\infty}\Gamma_n = 0$. Now from Lemma 2.6, we have

$$\begin{aligned}
\|u_{n+1}-u^\dagger\|^2 &= \|\zeta_n(\mathcal{G}(u_n)-u^\dagger) + (1-\zeta_n)(v_n-u^\dagger)\|^2 \\
&\leq (1-\zeta_n)^2\|v_n-u^\dagger\|^2 + 2\zeta_n\langle \mathcal{G}(u_n)-u^\dagger, j(u_{n+1}-u^\dagger)\rangle \\
&\leq (1-\zeta_n)^2\|T(u_n)-u^\dagger\|^2 + 2\zeta_n\langle \mathcal{G}(u_n)-\mathcal{G}(u^\dagger), j(u_{n+1}-u^\dagger)\rangle \\
&\quad + 2\zeta_n\langle \mathcal{G}(u^\dagger)-u^\dagger, j(u_{n+1}-u^\dagger)\rangle \\
&\leq (1-\zeta_n)^2\|u_n-u^\dagger\|^2 + 2r\zeta_n\|u_n-u^\dagger\|\|u_{n+1}-u^\dagger\| \\
&\quad + 2\zeta_n\langle \mathcal{G}(u^\dagger)-u^\dagger, j(u_{n+1}-u^\dagger)\rangle \\
&\leq (1-\zeta_n)^2\|u_n-u^\dagger\|^2 + r\zeta_n(\|u_n-u^\dagger\|^2 + \|u_{n+1}-u^\dagger\|^2) \\
&\quad + 2\zeta_n\langle \mathcal{G}(u^\dagger)-u^\dagger, j(u_{n+1}-u^\dagger)\rangle
\end{aligned}$$

$$\|u_{n+1}-u^\dagger\|^2 \leq \frac{1-(2-r)\zeta_n+\zeta_n^2}{1-r\zeta_n}\|u_n-u^\dagger\|^2 + \frac{2\zeta_n}{1-r\zeta_n}\Gamma_{n+1}.$$

For a fixed constant $C > 0$, we have

$$\begin{aligned}
\|u_{n+1}-u^\dagger\|^2 &\leq \frac{1-(2-r)\zeta_n}{1-r\zeta_n}\|u_n-u^\dagger\|^2 + \frac{2\zeta_n}{1-r\zeta_n}\Gamma_{n+1} + C\zeta_n^2 \\
&\leq \left(1-\frac{2\zeta_n(1-r)}{1-r\zeta_n}\right)\|u_n-u^\dagger\|^2 + \zeta_n\left(\frac{2}{1-r\zeta_n}\Gamma_{n+1} + C\zeta_n\right).
\end{aligned}$$

Taking $\tau_n = \|u_n-u^\dagger\|^2$, $\sigma_n = \frac{2\zeta_n(1-r)}{1-r\zeta_n}$, $\xi_n = \zeta_n\left(\frac{2}{1-r\zeta_n}\Gamma_{n+1} + C\zeta_n\right)$ and $\eta_n = 0$.
So,

$$\sum_{n=0}^{\infty}\sigma_n = \sum_{n=0}^{\infty}\frac{2\zeta_n(1-r)}{1-r\zeta_n} > \sum_{n=0}^{\infty}2\zeta_n(1-r) = \infty,$$

and

$$\limsup_{n\to\infty} \frac{\xi_n}{\sigma_n} = \limsup_{n\to\infty} \frac{1-r\zeta_n}{2(1-r)} \left(\frac{2}{1-r\zeta_n} \Gamma_{n+1} + C\zeta_n \right)$$
$$\leq \limsup_{n\to\infty} \left(\frac{1}{(1-r)} \Gamma_{n+1} + \frac{C\zeta_n}{2(1-r)} \right) = 0.$$

Applying Lemma 2.7, we get $\lim_{n\to\infty} \|u_n - u^\dagger\| = 0$. Hence the proof is now completed. □

Theorem 3.5. *Let $\mathcal{X}$ be a reflexive Banach space and $\mathcal{K}$ a nonempty closed and convex subset of $\mathcal{X}$. Let j be a weakly continuous normalized duality mapping on $\mathcal{X}$, $\mathcal{G} : \mathcal{K} \to \mathcal{K}$ a contraction and $T : \mathcal{K} \to C(\mathcal{K})$ a quasi-nonexpansive mapping with $\mathcal{F}(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in $\mathcal{K}$ defined as*

$$u_{n+1} = \mu_n \mathcal{G}(u_n) + \zeta_n u_n + (1 - \zeta_n - \mu_n) v_n,$$

where $v_n \in T(u_n)$ for all $n \in \mathbb{N} \cup \{0\}$ and satisfies conditions:

(1) $\mu_n \to 0$ *as* $n \to \infty$,
(2) $\sum_{n=1}^{\infty} \mu_n = \infty$,
(3) $0 < \liminf_{n\to\infty} \zeta_n \leq \limsup_{n\to\infty} \zeta_n < 1$,
(4) $\lim_{n\to\infty} d(u_n, T(u_n)) = 0$.

Then the sequence $\{u_n\}$ converges strongly to a fixed point of T.

Proof. It can be completed following the same line of proof of Theorem 3.4. □

4. Numerical examples

In this section, we present some examples for mappings which are quasi-nonexpansive but not nonexpansive. Further, we illustrate our results by showing convergence behaviour for different choices of initial guesses and coefficients.

Example 4.1. *Suppose $\mathcal{X} = \mathbb{R}$ and $T : \mathcal{X} \to \mathcal{X}$ be a mapping defined as*

$$T(u) = \begin{cases} \frac{u}{2} \cos \frac{1}{u}, u \neq 0, \\ 0, u = 0. \end{cases}$$

Clearly, only 0 is the fixed point of above mapping T. So we have,

$$\|T(u) - u^\dagger\| = \|T(u) - 0\| = \left|\frac{u}{2}\right| \left|\cos \frac{1}{u}\right| \leq \frac{|u|}{2} < |u| = \|u - u^\dagger\|,$$

and hence T is a quasi-nonexpansive mapping. For $u = \frac{2}{3\pi}$, $v = \frac{1}{\pi}$,

$$\|T(u) - T(v)\| = \frac{1}{2\pi} > \frac{1}{3\pi} = \|u - v\|.$$

Hence T is not a nonexpansive mapping.

Example 4.2. *Suppose $\mathcal{X} = [-\pi, \pi]$ and define the mapping $T : X \to X$ as $T(u) = u \cos u$.*
Clearly, only 0 is the fixed point of mapping T. So we have,

$$\|T(u) - u^\dagger\| = \|T(u) - 0\| = |u| |\cos u| \leq |u| = \|u - u^\dagger\|.$$

Therefore T is a quasi-nonexpansive mapping. For $u = \frac{\pi}{2}$, $v = \pi$,

$$\|T(u) - T(v)\| = \pi > \frac{\pi}{2} = \|u - v\|.$$

Hence T is not a nonexpansive mapping.

Example 4.3. *Suppose $\mathcal{X} = \left(\mathbb{R}^2, \|\cdot\|_{\frac{3}{2}}\right)$ and $\mathcal{K} = [-1,1]\times[-1,1]$. Define the mapping $T : \mathcal{K} \to \mathcal{K}$ as*

$$T(u_1,u_2) = \begin{cases} \left(\frac{|u_1|}{2}, u_2\right), |u_1| < 1, \\ \left(\frac{-1}{2}, u_2\right), |u_1| = 1. \end{cases}$$

Here, only $(0,0)$ is the fixed point of above mapping T. For all $u = (u_1,u_2)$, we have

$$\|T(u) - u^\dagger\|_{\frac{3}{2}} = \left\|\left(\frac{|u_1|}{2}, u_2\right) - (0,0)\right\|_{\frac{3}{2}} \leq \|u - u^\dagger\|_{\frac{3}{2}},$$

and T is a quasi-nonexpansive mapping. For $u = \left(\frac{2}{3}, u_2\right)$, $v = (1,u_2)$

$$\|T(u) - T(v)\|_{\frac{3}{2}} = \left\|\left(\frac{1}{3}, u_2\right) - \left(\frac{-1}{2}, u_2\right)\right\|_{\frac{3}{2}} = \frac{5}{6} > \frac{1}{3} = \left\|\left(\frac{2}{3}, u_2\right) - (1,u_2)\right\|_{\frac{3}{2}}.$$

Thus T is not a nonexpansive mapping.

Example 4.4. *[13] Suppose $\mathcal{X} = \mathbb{R}^2$ and $\mathcal{K} = \{u = (u_1,u_2) \in [0,1]\times[0,1]\}$ be the subset of $\mathcal{X}$ with the norm $\|u\| = \|(u_1,u_2)\| = (|u_1|^2 + |u_2|^2)^{1/2}$. Let $T : \mathcal{K} \to \mathcal{K}$ is defined by*

$$T(u_1,u_2) = \begin{cases} (1-u_1, 1-u_2),\ (u_1,u_2) \in \left[0, \frac{1}{2}\right] \times [0,1], \\ \frac{1}{3}(1+u_1, 1+u_2),\ (u_1,u_2) \in \left(\frac{1}{2}, 1\right] \times [0,1]. \end{cases}$$

Clearly, only $\left(\frac{1}{2}, \frac{1}{2}\right)$ is the fixed point of mapping T. So we have,

$$\|T(u) - u^\dagger\| = \left\|\frac{1}{2} - u_1, \frac{1}{2} - u_2\right\| \leq \|u - u^\dagger\|,$$

and T is a quasi-nonexpansive mapping. On the other hand, for $u = (0,0)$, $v = \left(\frac{51}{100}, \frac{25}{100}\right)$, we have

$$\|T(u) - T(v)\| = 0.766 > 0.567 = \|u - v\|.$$

Therefore T is not a nonexpansive mapping.

Now we present the convergence behaviour of sequence (8) for different choices of ζ_n and initial guesses for the mapping T considered in Example 4.4, our stopping criterion is $\|u_n - u^\dagger\| \leq 10^{-2}$.

Table 1: Influence of coefficient ζ_n.

Number of iterations	For $\zeta_n = \frac{9n}{(n+1)^2}$, $(u_1,u_2) = (0.7, 0.3)$, $u_n = (u_1^n, u_2^n)$
1	(0.700000000000000, 0.300000000000000)
2	(−0.183333333333333, −0.316666666666667)
3	(−1.30555555555556, −1.52777777777778)
4	(−2.31944444444444, −2.59722222222222)
5	(−2.57388888888889, −2.82944444444444)
···	··· ··· ··· ··· ··· ··· ··· ··· ··· ··· ···
101	(0.484452968812232, 0.484452968812232)
102	(0.484612486608801, 0.484612486608801)
103	(0.484768763613574, 0.484768763613574)
104	(0.484921897612977, 0.484921897612977)
105	(0.485071982497771, 0.485071982497771)

...	... ...
246	$(0.00000000000000, 0.00000000000000)$
247	$(0.00000000000000, 0.00000000000000)$
248	$(0.00000000000000, 0.00000000000000)$
249	$(0.00000000000000, 0.00000000000000)$
250	$(0.00000000000000, 0.00000000000000)$
Number of iterations	For $\zeta_n = \frac{9n}{(n+1)^2}$, $(u_1, u_2) = (0.5, 0.2)$, $u_n = (u_1^n, u_2^n)$
1	$(0.500000000000000, 0.200000000000000)$
2	$(-0.250000000000000, -0.850000000000000)$
3	$(-1.41666666666667, -2.41666666666667)$
4	$(-2.45833333333333, -3.70833333333333)$
5	$(-2.70166666666667, -3.85166666666667)$
...	... ...
101	$(0.484452968812232, 0.484452968812232)$
102	$(0.484612486608801, 0.484612486608801)$
103	$(0.484768763613574, 0.484768763613574)$
104	$(0.484921897612977, 0.484921897612977)$
105	$(0.485071982497771, 0.485071982497771)$
...	... ...
246	$(0.00000000000000, 0.00000000000000)$
247	$(0.00000000000000, 0.00000000000000)$
248	$(0.00000000000000, 0.00000000000000)$
249	$(0.00000000000000, 0.00000000000000)$
250	$(0.00000000000000, 0.00000000000000)$
Number of iterations	For $\zeta_n = \frac{17n}{(2n+1)^2}$, $(u_1, u_2) = (0.7, 0.3)$, $u_n = (u_1^n, u_2^n)$
1	$(0.700000000000000, 0.300000000000000)$
2	$(-0.0629629629629629, -0.196296296296296)$
3	$(-0.411209876543210, -0.519654320987655)$
4	$(-0.200265054169816, -0.242314940791131)$
5	$(0.136593882424178, 0.131575583197766)$
...	... ...
101	$(0.492820266400416, 0.492820266399056)$
102	$(0.492892317336455, 0.492892317337738)$
103	$(0.492962936385898, 0.492962936384685)$
104	$(0.000000000000, 0.00000000000)$
105	$(0.000000000000, 0.00000000000)$
...	... ...
496	$(0.000000000000, 0.00000000000)$
497	$(0.000000000000, 0.00000000000)$
498	$(0.000000000000, 0.00000000000)$
499	$(0.000000000000, 0.00000000000)$
500	$(0.000000000000, 0.00000000000)$
Number of iterations	For $\zeta_n = \frac{17n}{(2n+1)^2}$, $(u_1, u_2) = (0.5, 0.2)$, $u_n = (u_1^n, u_2^n)$
1	$(0.500000000000000, 0.200000000000000)$
2	$(-0.129629629629630, -0.585185185185185)$
3	$(-0.465432098765432, -0.835950617283951)$
4	$(-0.221289997480474, -0.364960443436634)$
5	$(0.134084732810972, 0.116938877120731)$

···	··· ··· ··· ··· ··· ··· ··· ··· ··· ···
101	(0.492820266399736, 0.492820266395092)
102	(0.492892317337096, 0.492892317341482)
103	(0.492962936385292, 0.492962936381147)
104	(0.000000000000, 0.00000000000)
105	(0.000000000000, 0.00000000000)
···	··· ··· ··· ··· ··· ··· ··· ··· ··· ···
496	(0.000000000000, 0.00000000000)
497	(0.000000000000, 0.00000000000)
498	(0.000000000000, 0.00000000000)
499	(0.000000000000, 0.00000000000)
500	(0.000000000000, 0.00000000000)

Table 2: Influence of coefficient ζ_n with initial guess $(u_1, u_2) = (0.5, 0.2)$.

Number of iterations	For $\zeta_n = \frac{n}{(n+1)^2}$, $\|u_n - u^\dagger\|$
1	0.150923085635624
2	0.069843162641588
3	0.0875379596281369
4	0.0434111327212228
5	0.0594583952217913
6	0.0309815458857161
7	0.0444149687052897
8	0.0239364759473162
9	0.035194777331846
10	0.019447752373618
···	··· ··· ··· ···
15	0.021321543813537
16	0.012388548656388
17	0.018778124499166
18	0.011045421088126
19	0.016756447380614
Number of iterations	$\zeta_n = \frac{17n}{(2n+1)^2}$, $\|u_n - u^\dagger\|$
1	0.895408191778464
2	1.367478838380575
3	1.020491360773425
4	0.517494499668932
5	0.298363883969987
6	0.226255955414445
7	0.182617007336525
8	0.153803634748247
9	0.132902274358429
10	0.117127848229374
···	··· ··· ··· ···
97	0.010472147912589
98	0.010363793824320
99	0.010257659215580
100	0.010153676631719

101	0.010051781218922

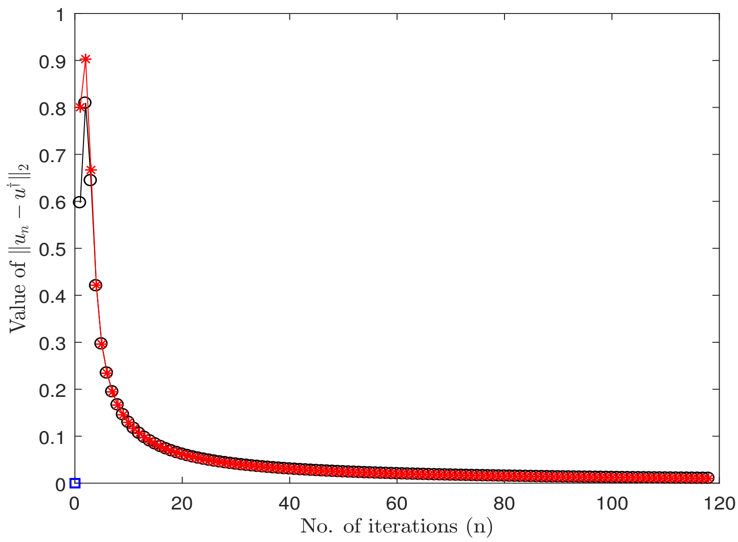

Figure 1: Convergence behaviour for fixed $\zeta_n = \frac{5n}{(n+1)^2}$ and different initial guesses $(u_1, u_2) = (0.7, 0.3), (u_1, u_2) = (0.5, 1)$.

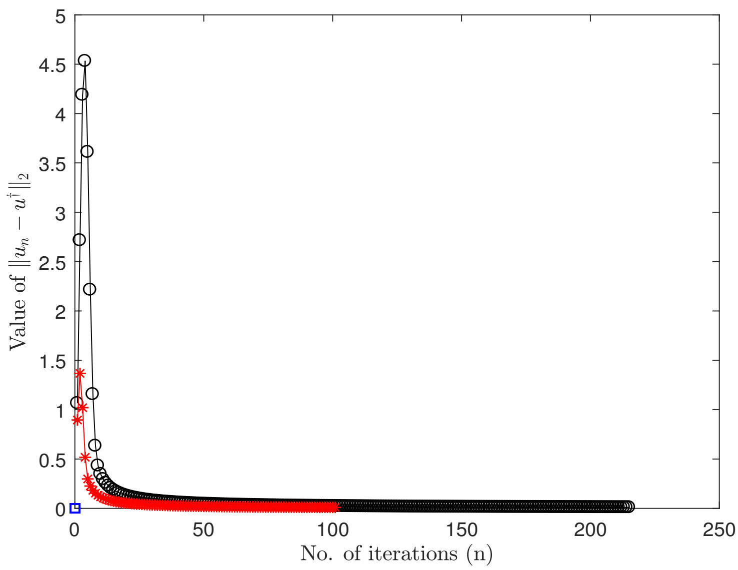

Figure 2: Convergence behaviour for different choices of $\zeta_n = \frac{9n}{(n+1)^2}, \zeta_n = \frac{17n}{(2n+1)^2}$ and fixed initial guess $(u_1, u_2) = (0.7, 0.3)$.

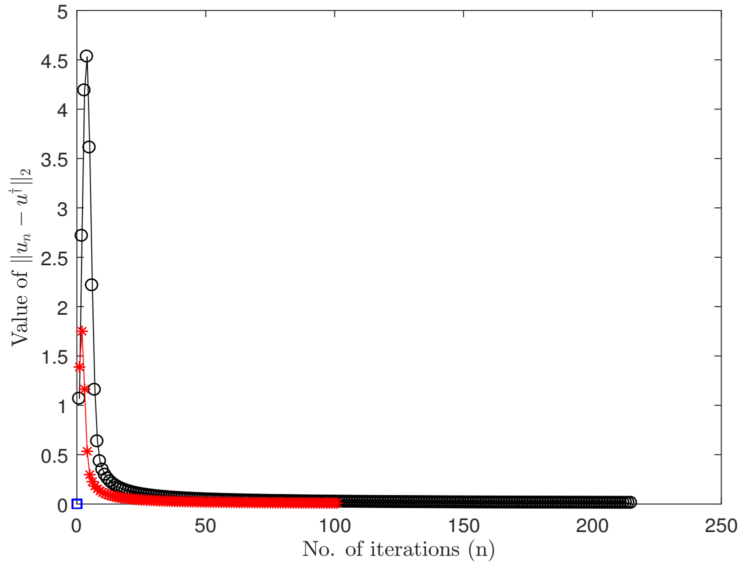

Figure 3: Convergence behaviour for different choices of $\zeta_n = \frac{9n}{(n+1)^2}, \zeta_n = \frac{17n}{(2n+1)^2}$ and different initial guesses $(u_1, u_2) = (0.7, 0.3)$, $(u_1, u_2) = (0.5, 1)$.

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