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Measure of Noncompactness and JS-Prešić Fixed Point Theorems and Its Applications to a System of Integral Equations

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Abstract. In this paper, we gain a new extension of well-known Darbo's fixed point theorem in a Banach space. Our results provide several expressions which all are generalizations of Darbo's fixed point theorem. As applications, we obtain some Prešić type extensions of Darbo fixed point theorem which help us in the studying of the existence of solution for a system of functional integral equations in $BC(\mathbb{R}^+)$. Finally, we provided an example to demonstrate the efficacy of our results.

1. Introduction and preliminaries

The theory of integral equations is a crucial component of mathematical analysis, with numerous applications in real-world problems. Noncompactness measures, on the other hand, are a powerful tool in the field of functional analysis. Functional equations, ODEs, PDEs, fractional partial differential equations, integral and integro-differential equations, and optimal control theory are all studied using them.

In reality, Kuratowski [14] proposed the intriguing concept of the measure of non compactness (MNC) in 1930. For more information, see [3], [2], [5], [6], [7] and [22].

Measures of Noncompactness, their properties, and certain applications are discussed in Chapter 7 of [15]. The authors offered various results from fixed point theory and compact operators, as well as an axiomatic introduction to measures of noncompactness and their most well-known properties, such as monotonicity and the generalized Cantor intersection property.

Malkowsky and V. Rakočević reviewed some of the conclusions in [16], including an axiomatic introduction, a discussion of the Kuratowski, Hausdorff, inner Hausdorff, and separation measures of noncompactness, as well as their roles in fixed-point theory and operator theory. The Hausdorff measure of noncompactness of matrix operators and bounded linear operators was also determined in [16].

We propose various notations and definitions that will be used throughout this work. Let $(Q, \|\cdot\|)$ be a real Banach space. As well as, $\overline{B}(\sigma, r)$ marks the closed ball centered at σ with radius r. The symbol $\overline{B}_r$ subsists for the ball $\overline{B}(0, r)$. For $\Delta \subseteq Q$, let $\overline{\Delta}$ and $Conv\Delta$ be the closure and the closed convex hull of Δ, respectively. As well as, let us denote by $\mathfrak{M}_Q$ the family of nonempty bounded subsets of Q and by $\mathfrak{N}_Q$ its subfamily subtending of all relatively compact subsets of Q.

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Definition 1.1. *[8] Let $\tau : \mathfrak{M}_Q \longrightarrow \mathbb{R}_+$ be a mapping. It is declared to be a measure of noncompactness in Q if:*

1° *The subset $ker\tau = \{\Delta \in \mathfrak{M}_Q : \tau(\Delta) = 0\}$ of $\mathfrak{M}_Q$ is nonempty and $ker\tau \subset \mathfrak{N}_Q$;*

2° $\Delta \subset \Lambda \Longrightarrow \tau(\Delta) \leq \tau(\Lambda)$;

3° $\tau(\overline{\Delta}) = \tau(\Delta) = \tau(Conv\Delta)$;

4° $\tau(\eta\Delta + (1-\eta)\Lambda) \leq \eta\tau(\Delta) + (1-\eta)\tau(\Lambda)$ *for all* $\eta \in [0,1]$;

5° *If $\{\Delta_n\} \subseteq \mathfrak{M}_Q$ is a sequence of closed sets such that $\Delta_{n+1} \subset \Delta_n$ for all $n = 1, 2, \cdots$, and $\lim_{n\to\infty} \tau(\Delta_n) = 0$, then $\Delta_\infty = \cap_{n=1}^{\infty}\Delta_n \neq \emptyset$.*

Now, we'll look at two important theorems in fixed point theory.

Theorem 1.2. *(Schauder fixed point theorem) ([1]) Having Ω as a nonempty, bounded, closed and convex subset of a Banach space Q, a mapping $\Upsilon : \Omega \to \Omega$ admits at least one fixed point in the set Ω provided that Υ be continuous and compact.*

The Darbo fixed point theorem, which is described below, generalizes the aforementioned result.

Theorem 1.3. *(Darbo[12]). Having $\mathfrak{R}$ as a nonempty, bounded, closed and convex subset of a Banach space Q and $T : \Omega \to \Omega$ as a continuous mapping, Υ possesses at least a fixed point in Ω provided that there exists a constant $K \in [0,1)$ such that $\tau(T\Delta) \leq K\tau(\Delta)$ for any nonempty subset Δ of Ω, where τ is an MNC defined in Q.*

Theorem 1.4. *[13, Corollary 2.1] Let $(\mathcal{W}, d)$ be a complete metric space and let $\Gamma : \mathcal{W} \to \mathcal{W}$ be a given map and*

$$\theta(d(\Gamma x, \Gamma y)) \leq \theta(d(x,y))^k,$$

for all $x, y \in \mathcal{W}$ so that $d(\Gamma x, \Gamma y) \neq 0$ where $\theta : (0,\infty) \to (1,\infty)$ is increasing, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$, for each sequence $\{t_n\} \subseteq (0,\infty)$, and there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = \ell$ and $k \in (0,1)$. Then there is a unique fixed point for Γ.

We use Jleli-Samet type contractions to achieve certain generalizations of Darbo's fixed point theorem in this study. The results of fixed points of the Prešić type are presented. As a result, we provide a program for solving a system of functional integral equations. These findings generalize a number of similar ones seen in the literature.

2. Main Results

Denote by Ξ the set of all functions $\Gamma : (0,\infty) \to (1,\infty)$ so that:

(Υ_1) Γ is continuous and increasing;

(Υ_2) $\lim_{n\to\infty} t_n = 0$ iff $\lim_{n\to\infty} \Gamma(t_n) = 1$ for all $\{t_n\} \subseteq (0,\infty)$,

The following are some examples of elements in Ξ:
(i) $\Gamma_1(t) = 1 + \ln(1+t)$,
(ii) $\Gamma_2(t) = 1 + \ln(t + 1 - \frac{1}{t+1} + 1)$,
(iii) $\Gamma_3(t) = -\frac{1}{\sqrt{t+1}} + 2$,
(iv) $\Gamma_4(t) = -\frac{1}{\sqrt{t+1}} + t + 2$,
(v) $\Gamma_5(t) = -\frac{1}{t+1} + t + 2$.
(vi) $\Gamma_6(t) = -\frac{1}{t+1} + 2$.
Denote by Ψ the set of all functions $\psi : (1,\infty) \to (1,\infty)$ so that:

(ψ_1) ψ is continuous and increasing;

(ψ_2) $\lim_{n\to\infty} \psi^n(t) = 1$ for all $t \in (1, \infty)$,

Here are some examples of elements in Ψ:
(*i*) $\psi_1(t) = 1 + \ln(t)$,
(*ii*) $\psi_2(t) = 1 + \ln(t - \frac{1}{t} + 1)$,
(*iii*) $\psi_3(t) = -\frac{1}{\sqrt{t}} + 2$,
(*iv*) $\psi_4(t) = -\frac{1}{\sqrt{t}} + t + 1$,
(*v*) $\psi_5(t) = -\frac{1}{t} + t + 1$.
(*vi*) $\psi_6(t) = -\frac{1}{t+1} + \frac{1}{2} + 1$.
(*vi*) $\psi_7(t) = t^k$ where $k \in (0, 1)$.
Now we'll present the study's primary result, which expands and generalizes Darbo's well-known fixed point theorem.

Theorem 2.1. *Having $\Re$ as a nonempty, bounded, closed and convex subset of a Banach space Q and $\Upsilon : \Re \to \Re$ as a continuous operator such that*

$$\Gamma(\tau(\Upsilon\Delta)) \leq \psi(\Gamma(\tau(\Delta))) \tag{1}$$

for all $\Delta \subseteq \Re$, where $\Gamma \in \Xi$, $\psi \in \Psi$ and τ is an arbitrary MNC, then Υ possesses at least one fixed point in $\Re$.

Proof. Let $\{\Re_n\}$ be such that $\Re_0 = \Re$ and $\Re_{n+1} = \overline{Conv}(\Upsilon(\Re_n))$ for all $n \in \mathbb{N}$.

If for an integer $N \in \mathbb{N}$ one has $\tau(\Re_N) = 0$, then $\Re_N$ is relatively compact and so Schauder Theorem guarantees the existence of a fixed point for Υ. So, let $\tau(\Re_n) > 0$ for all $n \in \mathbb{N} \cup \{0\}$.

Evidently, $\{\Re_n\}_{n\in\mathbb{N}}$ is a sequence of nonempty, bounded, closed and convex subsets such that

$$\Re_0 \supseteq \Re_1 \supseteq \cdots \supseteq \Re_n \supseteq \Re_{n+1}.$$

On the other hand

$$\Gamma(\tau(\Re_{n+1})) = \Gamma(\tau(\Upsilon\Re_n)) \leq \psi(\Gamma(\tau(\Re_n))) \leq \cdots \leq \psi^{n+1}(\Gamma(\tau(\Re_0))). \tag{2}$$

Thus, $\{\tau(\Re_n)\}_{n\in\mathbb{N}}$ is a convergent sequence. Suppose that $\lim_{n\to\infty} \tau(\Re_n)) = r$.
Now, we show that $r = 0$.
Passing to the limit through (2),

$$\Gamma(\tau(\Re_{n+1})) \to 1.$$

Therefore, we have $\lim_{n\to\infty} \tau(\Re_{n+1}) = 0$.

Axiom (5°) of Definition 1.1 yields that the set $\Re_\infty = \bigcap_{n=1}^{\infty} \Re_n$ is a nonempty, closed and convex set and it is stable under the operator Υ and is an element of $Ker\tau$. Then according to the Schauder theorem Υ possesses a fixed point. □

Taking $\psi(t) = t^{\varphi(t)}$ where $\varphi : [0, \infty) \to [0, 1)$ so that $\lim_{n\to\infty} (\varphi(t))^n = 0$ for all $t \in [0, \infty)$, in Theorem 2.1, we have,

Corollary 2.2. *Having $\Re \subseteq Q$ as a nonempty, bounded, closed and convex subset and $\Upsilon : \Re \to \Re$ as a continuous operator such that*

$$\Gamma(\tau(\Upsilon\Delta)) \leq (\Gamma(\tau(\Delta)))^{\varphi(\tau(\Delta))} \tag{3}$$

for all $\Delta \subseteq \Re$, and τ is an arbitrary MNC, then Υ admits at least one fixed point in $\Re$.

Taking $\psi(t) = t^k$, $k \in (0,1)$ in Theorem 2.1, we have,

Corollary 2.3. *Having $\Re \subseteq Q$ as a nonempty, bounded, closed and convex subset and $\Upsilon : \Re \to \Re$ as a continuous operator such that*

$$\Gamma(\tau(\Upsilon\Delta)) \leq (\Gamma(\tau(\Delta)))^k \tag{4}$$

for all $\Delta \subseteq \Re$, and τ is an arbitrary MNC, then Υ admits at least one fixed point in $\Re$.

Taking $\Gamma(t) = 1 + \ln(1+t)$ and $\psi(t) = 1 + \ln(t)$ in Theorem 2.1, we have,

Corollary 2.4. *Let $\Re \subseteq Q$ be a nonempty, bounded, closed and convex and let $\Upsilon : \Re \to \Re$ be continuous such that*

$$1 + \ln(1 + \tau(\Upsilon(\Delta)) \leq 1 + \ln(1 + \ln(1 + \tau(\Delta))) \tag{5}$$

for all $\Delta \subseteq \Re$, $\mathcal{D} \in \mathfrak{D}$ and τ is an arbitrary MNC. Then Υ has at least one fixed point in $\Re$.

Remark 2.5. *Obviously, the Darbo's fixed point theorem is a special case of Corollary 2.3 if we take $\Gamma(t) = e^t$.*

3. Presić type fixed point results

One of the most powerful results in nonlinear analysis is the Banach contraction principle (BCP) [4]. In the background of ODE and PDE, it has numerous applications.

Theorem 3.1. *[4] Let (Δ, d) be a complete metric space and let $\Upsilon : \Delta \to \Delta$ so that*

$$d(\Upsilon\iota, \Upsilon\kappa) \leq \gamma d(\iota, \kappa) \text{ for all } \iota, \kappa \in \Delta,$$

where $\gamma \in [0,1)$. Then, there is a unique σ in Δ such that $\sigma = \Upsilon\sigma$. Also, for each $\zeta_0 \in \Delta$, the sequence $\zeta_{n+1} = \Upsilon\zeta_n$ converges to σ.

The BCP has been expanded and generalized in a variety of ways (see, for example, [18], [17] and [21]). Prešić [20] came up with the following outcome.

Theorem 3.2. *[20] Let (Δ, d) be a complete metric space and let $\Upsilon : \Delta^k \to \Delta$ (k is a positive integer). Suppose that*

$$d(\Upsilon(\zeta_1, ..., \zeta_k), \Upsilon(\zeta_2, ..., \zeta_{k+1})) \leq \sum_{i=1}^{k} \lambda_i d(\zeta_i, \zeta_{i+1}) \tag{6}$$

for all $\zeta_1, ..., \zeta_{k+1}$ in Δ, where $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i \in [0,1)$. Then Υ has a unique fixed point ζ^* (that is, $\Upsilon(\zeta^*, ..., \zeta^*) = \zeta^*$). Moreover, for all arbitrary points $\zeta_1, ..., \zeta_{k+1}$ in Δ, the sequence $\{\zeta_n\}$ defined by $\zeta_{n+k} = \Upsilon(\zeta_n, \zeta_{n+1}, ..., \zeta_{n+k-1})$, converges to ζ^*.

It is obvious that for $k = 1$, Theorem 3.2 coincides with the BCP.
The above theorem generalized by Ćirić and Presić [11] as follows.

Theorem 3.3. *[11] Let (Δ, d) be a complete metric space and $\Upsilon : \Delta^k \to \Delta$ (k is a positive integer). Suppose that*

$$d(\Upsilon(\zeta_1, ..., \zeta_k), \Upsilon(\zeta_2, ..., \zeta_{k+1})) \leq \lambda \max\{d(\zeta_i, \zeta_{i+1}) : 1 \leq i \leq k\}, \tag{7}$$

for all $\zeta_1, ..., \zeta_{k+1}$ in Δ, where $\lambda \in [0,1)$. Then Υ has a fixed point $\zeta^* \in \Delta$. Also, for all points $\zeta_1, ..., \zeta_{k+1} \in \Delta$, the sequence $\{\zeta_n\}$ defined by $\zeta_{n+k} = \Upsilon(\zeta_n, \zeta_{n+1}, ..., \zeta_{n+k-1})$, converges to ζ^*. The fixed point of Υ is unique if

$$d(\Upsilon(\rho, ..., \rho), \Upsilon(\varrho, ..., \varrho)) < d(\rho, \varrho),$$

for all $\rho, \varrho \in \Delta$ with $\rho \neq \varrho$.

For more details on Presić type contractions, we refer the reader to [10, 18, 20].

Theorem 3.4. *[7] Suppose that $\tau_1, \tau_2, \ldots, \tau_n$ be some MNC in Banach spaces $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n$, respectively, and let the function $\upsilon : [0,\infty)^n \longrightarrow [0,\infty)$ be convex and $\upsilon(\sigma_1,\ldots,\sigma_n) = 0$ if and only if $\sigma_i = 0$ for $i = 1,2,\ldots,n$. Then*

$$\tilde{\tau}(\Delta) = \upsilon(\tau_1(\Delta_1), \tau_2(\Delta_2), \ldots, \tau_n(\Delta_n)),$$

is a measure of noncompactness in $\mathcal{Q}_1 \times \mathcal{Q}_2 \times \ldots \times \mathcal{Q}_n$, where Δ_i denotes the natural projection of Δ into $\mathcal{Q}_i$, for $i = 1,2,\ldots,n$.

From now on, let Γ be a subadditive mapping.

Theorem 3.5. *Having $\Re \subseteq \mathcal{Q}$ as a nonempty, bounded, closed and convex subset and $\Upsilon : \Re^n \to \Re$ as a continuous function such that*

$$\Gamma\Big(\tau\Big(\Upsilon(\Delta_1 \times \ldots \times \Delta_n)\Big)\Big) \leq \frac{1}{n}\psi\Big(\Big[\Gamma\Big(\tau(\Delta_1) + \ldots + \tau(\Delta_n)\Big)\Big]\Big) \tag{8}$$

for all $\Delta_1, \ldots, \Delta_n \subseteq \Re$, where $\Gamma \in \Xi$ is a subadditiv mapping, $\psi \in \Psi$ and τ is an arbitrary MNC, then Υ has at least a Presić type fixed point.

Proof. We define the mapping $\widetilde{\Upsilon} : \Re^n \to \Re^n$ by

$$\widetilde{\Upsilon}(\sigma_1, \ldots, \sigma_n) = (\Upsilon(\sigma_1, \ldots, \sigma_n), \ldots, \Upsilon(\sigma_1, \ldots, \sigma_n)).$$

Clearly, $\widetilde{\Upsilon}$ is continuous. We demonstrate that $\widetilde{\Upsilon}$ satisfies all the conditions of Theorem 2.1. Let $\Delta \subset \Re^n$ be a nonempty subset. We know that $\widetilde{\tau}(\Delta) = \tau(\Delta_1) + \ldots + \tau(\Delta_n)$ is a (MNC) [8], where $\Delta_1, \ldots, \Delta_n$ denote the natural projections of Δ into $\mathcal{Q}$. From (8) we have

$$\begin{aligned}
\Gamma\Big(\widetilde{\tau}(\widetilde{\Upsilon}(\Delta))\Big) &= \Gamma\Big(\widetilde{\tau}(\Upsilon(\Delta_1 \times \ldots \times \Delta_n) \times \ldots \times \Upsilon(\Delta_1 \times \ldots \times \Delta_n))\Big) \\
&= \Gamma\Big(n\tau(\Upsilon(\Delta_1 \times \ldots \times \Delta_n))\Big) \\
&\leq n\Gamma\Big(\tau(\Upsilon(\Delta_1 \times \ldots \times \Delta_n))\Big) \\
&\leq \psi\Big(\Gamma\Big(\tau(\Delta_1) + \ldots + \tau(\Delta_n)\Big)\Big) \\
&= \psi\Big(\Gamma\Big(\widetilde{\tau}(\Delta)\Big)\Big).
\end{aligned}$$

Now, according to Theorem 2.1 we deduce that $\widetilde{\Upsilon}$ admits at least a fixed point which implies that there exists $\sigma_1, \ldots, \sigma_n$ such that $\Upsilon(\sigma_1, \ldots, \sigma_n) = \sigma_1 = \ldots = \sigma_n$, that is, Υ possesses at least a Presić type fixed point. □

Theorem 3.6. *Having $\Re \subseteq \mathcal{Q}$ as a nonempty, bounded, closed and convex subset and $\Upsilon : \Re^n \to \Re$ as a continuous function so that*

$$\Gamma\Big(\tau(\Upsilon(\Delta_1 \times \ldots \times \Delta_n))\Big) \leq \psi\Big(\Gamma(\max\{\tau(\Delta_1), \ldots, \tau(\Delta_n)\})\Big) \tag{9}$$

for all subsets $\Delta_1, \ldots, \Delta_n \subseteq \Re$, where $\Gamma \in \Xi$, $\psi \in \Psi$ and τ is an arbitrary MNC, then Υ has at least a Presić type fixed point.

Proof. Let $\widetilde{\Upsilon} : \Re^n \to \Re^n$ be defined by

$$\widetilde{\Upsilon}(\sigma_1, \ldots, \sigma_n) = (\Upsilon(\sigma_1, \ldots, \sigma_n), \ldots, \Upsilon(\sigma_1, \ldots, \sigma_n)).$$

Evidently, $\widetilde{\Upsilon}$ is continuous. It will be demonstrated that $\widetilde{\Upsilon}$ satisfies all the conditions of Theorem 2.1. Clearly, $\widetilde{\tau}(\Delta) = \max\{\tau(\Delta_1), ..., \tau(\Delta_n)\}$ is a (MNC)[8], where Δ_1, ... and Δ_n denote the natural projections of Δ into Q. Let $\Delta \subset \Re^n$ be a nonempty subset. According to (12) we have

$$\begin{aligned}
\Gamma\Big(\widetilde{\tau}(\widetilde{\Upsilon}(\Delta))\Big) &= \Gamma\Big(\widetilde{\tau}(\Upsilon(\Delta_1 \times ... \times \Delta_n) \times ... \times \Upsilon(\Delta_1 \times ... \times \Delta_n))\Big) \\
&= \Gamma\Big(\max\{\tau(\Upsilon(\Delta_1 \times ... \times \Delta_n)), ..., \tau(\Upsilon(\Delta_1 \times ... \times \Delta_n))\}\Big) \\
&= \Gamma\Big(\tau(\Upsilon(\Delta_1 \times ... \times \Delta_n))\Big) \\
&\leq \psi\Big(\Gamma\Big(\max\{\tau(\Delta_1), ..., \tau(\Delta_n)\}\Big)\Big) \\
&= \psi\Big(\Gamma\Big(\widetilde{\tau}(\Delta)\Big)\Big).
\end{aligned}$$

So, from Theorem 2.1 we obtain that $\widetilde{\Upsilon}$ has at least a fixed point which implies that Υ has at least a Presić type fixed point. □

Corollary 3.7. *Having $\Re \subseteq Q$ as a nonempty, bounded, closed and convex subset and $\Upsilon : \Re^n \to \Re$ as a continuous function so that*

$$\Gamma\Big(\tau(\Upsilon(\Delta_1 \times ... \times \Delta_n))\Big) \leq \psi(\Gamma\Big(\sum_{i=1}^{n} \lambda_i \tau(\Delta_i)))\tag{10}$$

for all subsets $\Delta_1, ..., \Delta_n \subseteq \Re$, where $\Gamma \in \Xi$, $\psi \in \Psi$, τ is an arbitrary MNC, $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i \in [0, 1)$ then Υ has at least a Presić type fixed point.

Corollary 3.8. *Having $\Re \subseteq Q$ as a nonempty, bounded, closed and convex subset and $\Upsilon : \Re^n \to \Re$ as a continuous function so that*

$$\Gamma\Big(\tau(\Upsilon(\Delta_1 \times ... \times \Delta_n))\Big) \leq (\Gamma\Big(\sum_{i=1}^{n} \lambda_i \tau(\Delta_i)))^{\varphi(\sum_{i=1}^{n} \lambda_i \tau(\Delta_i))}\tag{11}$$

for all subsets $\Delta_1, ..., \Delta_n \subseteq \Re$, where $\Gamma \in \Xi$, $\varphi : [0, \infty) \to [0, 1)$ so that $\lim_{n\to\infty}(\varphi(t))^n = 0$ for all $t \in [0, \infty)$, τ is an arbitrary MNC, $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i \in [0, 1)$ then Υ has at least a Presić type fixed point.

Corollary 3.9. *Having $\Re \subseteq Q$ as a nonempty, bounded, closed and convex subset and $\Upsilon : \Re^n \to \Re$ as a continuous function so that*

$$\Gamma\Big(\tau(\Upsilon(\Delta_1 \times ... \times \Delta_n))\Big) \leq (\Gamma\Big(\sum_{i=1}^{n} \lambda_i \tau(\Delta_i)))^{k}\tag{12}$$

for all subsets $\Delta_1, ..., \Delta_n \subseteq \Re$, where $\Gamma \in \Xi$, $k \in [0, 1)$, τ is an arbitrary MNC, $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i \in [0, 1)$ then Υ has at least a Presić type fixed point.

4. Application

Studying the existence of solutions for the following system is the aim of this section.

$$\begin{cases} \mathcal{G}_1(t) = \varpi\Big(t, h(t, \mathcal{G}_1(\alpha(t)), ..., \mathcal{G}_n(\alpha(t)), \int_0^{\zeta(t)} \pi\Big(t, s, \mathcal{G}_1(\gamma(s)), ..., \mathcal{G}_n(\gamma(s))\Big)ds\Big) \\ \mathcal{G}_2(t) = \varpi\Big(t, h(t, \mathcal{G}_1(\alpha(t)), ..., \mathcal{G}_n(\alpha(t)), \int_0^{\zeta(t)} \pi\Big(t, s, \mathcal{G}_1(\gamma(s)), ..., \mathcal{G}_n(\gamma(s))\Big)ds\Big) \\ \qquad\qquad\qquad\qquad ... \\ \mathcal{G}_n(t) = \varpi\Big(t, h(t, \mathcal{G}_1(\alpha(t)), ..., \mathcal{G}_n(\alpha(t)), \int_0^{\zeta(t)} \pi\Big(t, s, \mathcal{G}_1(\gamma(s)), ..., \mathcal{G}_n(\gamma(s))\Big)ds\Big) \end{cases} \tag{13}$$

in the space $BC(\mathbb{R}_+)$ consisting of all bounded and continuous real functions on $\mathbb{R}_+$. Let $\varpi : \mathbb{R}^3 \to \mathbb{R}$ be such that

$$\varpi\Big(t_1, t_2, t_3\Big) \geq 0,$$

for all $t_1, t_2, t_3 \in \mathbb{R}_+$.

We will use a measure of noncompactness in the space $BC(R_+)$ which was constructed in the paper [9].

Let $BC(\mathbb{R}_+)$ be endowed with the norm

$$\|\mathcal{G}\| = \sup\Big\{|\mathcal{G}(t)| : t \geq 0\Big\}.$$

The modulus of continuity of a function $\mathcal{G} \in BC(R_+)$ is defined by

$$\omega^T(\mathcal{G}, \epsilon) = \sup\Big\{|\mathcal{G}(t) - \mathcal{G}(s)| : t, s \in [0, T], |t - s| \leq \epsilon\Big\}.$$

Let

$$\omega^T(\Delta, \epsilon) = \sup\Big\{\omega^T(\mathcal{G}, \epsilon) : \mathcal{G} \in \Delta\Big\},$$

$$\omega_0^T(\Delta) = \lim_{\varepsilon \to 0} \omega^T(\Delta, \epsilon),$$

and

$$\omega_0(\Delta) = \lim_{T \to \infty} \omega_0^T(\Delta).$$

Let

$$\Delta(t) = \Big\{\mathcal{G}(t) : \mathcal{G} \in \Delta\Big\},$$

and

$$diam\Delta(t) = \sup\Big\{|\mathcal{G}_1(t) - \mathcal{G}_2(t)| : \mathcal{G}_1, \mathcal{G}_2 \in \Delta\Big\}$$

for a fixed number $t \in R_+$.

Define

$$m(\Delta) = \omega_0(\Delta) + \limsup_{t \to \infty} diam\Delta(t).$$

Banas [9] proved that the above function is a measure of noncompactness in the space $BC(R_+)$.

Theorem 4.1. *Assume that*

(i) $\alpha, \zeta, \gamma : R_+ \longrightarrow R_+$ *are continuous functions and* $\alpha(t) \to \infty$ *as* $t \to \infty$.

(ii) *The function* $\varpi : \mathbb{R}_+ \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is continuous and*

$$\Big|\varpi(t_1,\sigma_1,\sigma_2) - \varpi(t_2,\varsigma_1,\varsigma_2)\Big| \le \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\Big|t_1-t_2\Big| + \max\Big\{\Big|\sigma_1-\varsigma_1\Big|, \Big|\sigma_2-\varsigma_2\Big|\Big\}\Big)\Big)\Big),$$

for all $t_1, t_2 \in \mathbb{R}_+$ *and* $\sigma_1,\sigma_2,\varsigma_1,\varsigma_2 \in C(R)$.

(ii') *The function* $h : R_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *is continuous so that*

$$\Big|h(t_1,\sigma_1,...,\sigma_n) - h(t_2,\varsigma_1,...,\varsigma_n)\Big| \le \Big|t_1-t_2\Big| + \max\Big\{\Big|\sigma_1-\varsigma_1\Big|,...,\Big|\sigma_n-\varsigma_n\Big|\Big\},$$

for all $t_1, t_2 \in \mathbb{R}_+$ *and* $\sigma_1,...,\sigma_n,\varsigma_1,...,\varsigma_n \in BC(R_+)$.

(iii) $M := \sup\Big\{\Big|\varpi\Big(t,0,0\Big)\Big| : t \in R_+\Big\}$ *and* $N := \sup\Big\{\Big|h\Big(t,0,...,0\Big)\Big| : t \in \mathbb{R}_+\Big\}$.

(iv) $\pi : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *is continuous. Moreover, there exist continuous functions* $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$|\pi(t,s,\mathcal{G}_1,...,\mathcal{G}_n)| \le a(t)b(s)$$

and $\lim_{t\to\infty}\int_0^{\zeta(t)} a(t)b(s)ds = 0$ *for all* $\mathcal{G}_1,...,\mathcal{G}_n \in BC(\mathbb{R}_+)$ *and* $t, s \in \mathbb{R}_+$.

(v) *There exists a positive solution* r_0 *to the inequality*

$$\Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\max\Big\{r+N, Q\Big\}\Big)\Big)\Big) + M \le r,$$

where $Q = \sup_{t\ge 0}\Big\{\Big|\int_0^{\zeta(t)} a(t)b(s)ds\Big|\Big\}$.

Then the system of integral equations (13) *has at least one solution in the space* $(BC(\mathbb{R}_+))^n$.

Proof. Let $\Upsilon : BC(\mathbb{R}_+) \times ... \times BC(\mathbb{R}_+) \longrightarrow BC(\mathbb{R}_+)$ be defined by

$$\Upsilon(\mathcal{G}_1,...,\mathcal{G}_n)(t) = \varpi\Big(t, h(t,\mathcal{G}_1(\alpha(t)),...,\mathcal{G}_n(\alpha(t)), \int_0^{\zeta(t)} \pi\Big(t,s,\mathcal{G}_1(\gamma(s)),...,\mathcal{G}_n(\gamma(s))\Big)ds\Big). \tag{14}$$

Let $t > 0$ be fixed and $\{t_n\}$ be a sequence such that $t_n \to t$ as $n \to \infty$. Without loss of generality, we can choose $t_n \ge t$. Then

$$\begin{aligned}
&\Big|\Upsilon(\mathcal{G}_1,...,\mathcal{G}_n)(t_n) - \Upsilon(\mathcal{G}_1,...,\mathcal{G}_n)(t)\Big| \\
&\le \Big|\varpi\Big(t_n, h(t_n,\mathcal{G}_1(\alpha(t_n)),...,\mathcal{G}_n(\alpha(t_n)), \int_0^{\zeta(t_n)} \pi(t_n,s,\mathcal{G}_1(\gamma(s)),...,\mathcal{G}_n(\gamma(s))ds\Big) \\
&- \varpi\Big(t, h(t,\mathcal{G}_1(\alpha(t)),...,\mathcal{G}_n(\alpha(t)), \int_0^{\zeta(t)} \pi(t,s,\mathcal{G}_1(\gamma(s)),...,\mathcal{G}_n(\gamma(s))ds\Big)\Big| \\
&\le \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\Big|t_n-t\Big| + \max\Big\{|h(t_n,\mathcal{G}_1(\alpha(t_n)),...,\mathcal{G}_n(\alpha(t_n)) - h(t,\mathcal{G}_1(\alpha(t)),\mathcal{G}_n(\alpha(t))|, \\
&|\int_0^{\zeta(t_n)} \pi\Big(t_n,s,\mathcal{G}_1(\gamma(s)),...,\mathcal{G}_n(\gamma(s))\Big)ds - \int_0^{\zeta(t)} \pi\Big(t,s,\mathcal{G}_1(\gamma(s)),...,\mathcal{G}_n(\gamma(s))\Big)ds|\Big\}\Big)\Big)\Big) \\
&\le \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\Big|t_n-t\Big| + \max\{|t_n-t| + \max\{|\mathcal{G}_1(\alpha(t_n)) - \mathcal{G}_1(\alpha(t))|,...,|\mathcal{G}_n(\alpha(t_n)) - \mathcal{G}_n(\alpha(t))|, \\
&|\int_0^{\zeta(t_n)} \pi\Big(t_n,s,\mathcal{G}_1(\gamma(s)),...,\mathcal{G}_n(\gamma(s))\Big)ds - \int_0^{\zeta(t)} \pi\Big(t,s,\mathcal{G}_1(\gamma(s)),...,\mathcal{G}_n(\gamma(s))\Big)ds|\Big\}\Big)\Big)\Big) \\
&\le \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\Big|t_n-t\Big| + \max\{|t_n-t| + \max\{|\mathcal{G}_1(\alpha(t_n)) - \mathcal{G}_1(\alpha(t))|,...,|\mathcal{G}_n(\alpha(t_n)) - \mathcal{G}_n(\alpha(t))|, \\
&|\int_{\zeta(t)}^{\zeta(t_n)} \pi\Big(t_n,s,\mathcal{G}_1(\gamma(s)),...,\mathcal{G}_n(\gamma(s))\Big)ds| \\
&+ |\int_0^{\zeta(t)} \Big(\pi\Big(t_n,s,\mathcal{G}_1(\gamma(s)),...,\mathcal{G}_n(\gamma(s))\Big)ds - \pi\Big(t,s,\mathcal{G}_1(\gamma(s)),...,\mathcal{G}_n(\gamma(s))\Big)\Big)ds|\Big\}\Big)\Big)\Big).
\end{aligned}$$

Since $\mathcal{G}_i$ $(1 \leq i \leq n)$, α, ζ and π are continuous functions, therefore it is observed that the above inequality tends to 0, as $n \to \infty$.

We observe that the function Υ is bounded. Applying the assumptions (i) – (iv) we have

$$\begin{aligned}
&\Big|\Upsilon(\mathcal{G}_1, ..., \mathcal{G}_n)(t)\Big| \\
&\leq \Big|\varpi\Big(t, h(t, \mathcal{G}_1(\alpha(t)), ..., \mathcal{G}_n(\alpha(t)), \int_0^{\zeta(t)} \pi(t, s, \mathcal{G}_1(\gamma(s)), ..., \mathcal{G}_n(\gamma(s))ds\Big) - \varpi\Big(t, 0, 0\Big)\Big| + \Big|\varpi\Big(t, 0, 0\Big)\Big| \\
&\leq \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\max\Big\{\Big|h(t, \mathcal{G}_1(\alpha(t)), ..., \mathcal{G}_n(\alpha(t))\Big|, \Big|\int_0^{\zeta(t)} \pi(t, s, \mathcal{G}_1(\gamma(s)), ..., \mathcal{G}_n(\gamma(s))ds\Big|\Big\}\Big)\Big)\Big) + \Big|\varpi\Big(t, 0, 0\Big)\Big| \\
&\leq \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\max\Big\{\max\Big\{|\mathcal{G}_1(\alpha(t))|, ..., |\mathcal{G}_n(\alpha(t))|\Big\} + |h(t, 0, ..., 0)|, Q\Big\}\Big)\Big)\Big) + \Big|\varpi\Big(t, 0, 0\Big)\Big| \\
&\leq \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\max\Big\{\max\Big\{\|\mathcal{G}_1\|, ..., \|\mathcal{G}_n\|\Big\} + N, Q\Big\}\Big)\Big)\Big) + M.
\end{aligned}$$

Therefore,

$$\|\Upsilon(\mathcal{G}_1, ..., \mathcal{G}_n)\| \leq \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\max\Big\{\max\Big\{\|\mathcal{G}_1\|, ..., \|\mathcal{G}_n\|\Big\} + N, Q\Big\}\Big)\Big)\Big) + M. \tag{15}$$

From (15) and by (v), it can be shown that Υ brings $(\bar{B}_{r_0})^n$ into $(\bar{B}_{r_0})$.

Now we prove that the operator Υ is a continuous operator on $(\bar{B}_{r_0})^n$. Let us fix arbitrarily $\varepsilon > 0$. Take $(u_1, ..., u_n), (v_1, ..., v_n) \in (\bar{B}_{r_0})^n$ such that $\|u_1 - v_1\| + \|u_n - v_n\| < \varepsilon$. Then for all $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
&\Big|\Upsilon(u_1, ..., u_n)(t) - \Upsilon(v_1, v_n)(t)\Big| \\
&= \Big|\varpi\Big(t, h(t, u_1(\alpha(t)), ..., u_n(\alpha(t)), \textstyle\int_0^{\zeta(t)} \pi(t, s, u_1(\gamma(s)), ..., u_n(\gamma(s))ds\Big) \\
&-\varpi\Big(t, h(t, v_1(\alpha(t)), ..., v_n(\alpha(t)), \textstyle\int_0^{\zeta(t)} \pi(t, s, v_1(\gamma(s)), ..., v_n(\gamma(s))ds\Big)\Big| \\
&\leq \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\max\Big\{|h(t, u_1(\alpha(t)), ..., u_n(\alpha(t)) - h(t, v_1(\alpha(t)), ..., v_n(\alpha(t))|, \\
&\textstyle\int_0^{\zeta(t)} \pi(t, s, u_1(\gamma(s)), ..., u_n(\gamma(s))ds - \int_0^{\zeta(t)} \pi(t, s, v_1(\gamma(s)), ..., v_n(\gamma(s))ds|\Big\}\Big)\Big)\Big) \\
&\leq \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\max\Big\{\max\{|u_1(\alpha(t)) - v_1(\alpha(t))|, ..., |u_n(\alpha(t)) - v_n(\alpha(t))|\}, \\
&|\textstyle\int_0^{\zeta(t)} \pi(t, s, u_1(\gamma(s)), ..., u_n(\gamma(s))ds - \int_0^{\zeta(t)} \pi(t, s, v_1(\gamma(s)), ..., v_n(\gamma(s))ds|\Big\}\Big)\Big)\Big).
\end{aligned}$$

Let

$$\begin{aligned}
&\omega_{r_0}(\pi, \varepsilon) \\
&= \sup\Big\{|\pi(t, s, u_1, ..., u_n) - \pi(t, s, v_1, ..., v_n)| : t \in [0, T], s \in [0, \zeta_T], u_1, ..., u_n, v_1, ..., v_n \in [-r_0, r_0], \\
&\quad \|u_1 - v_1\| + ... + \|u_n - v_n\| < \varepsilon\Big\}
\end{aligned}$$

where

$$\zeta_T = \sup\Big\{\zeta(t) : t \in [0, T]\Big\}.$$

Therefore, for an arbitrary fixed $t \in [0, T]$ we have

$$\Big|\Upsilon(u_1, ..., u_n)(t) - \Upsilon(v_1, ..., v_n)(t)\Big| \leq \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\max\Big\{\max\Big\{\|u_1 - v_1\|, ..., \|u_n - v_n\|\Big\}, \zeta_T \omega_{r_0}(\pi, \varepsilon)\Big\}\Big)\Big)\Big).$$

Applying the continuity of π on $[0, T] \times [0, \zeta_T] \times [-r_0, r_0]^n$, we have $\omega_{r_0}(\pi, \varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, Υ is a continuous function on $(\bar{B}_{r_0})^n$.

Now, we prove that Υ satisfies all conditions of Theorem 3.6. So, let $\Delta_1, ..., \Delta_n$ are nonempty and bounded subsets of $\bar{B}_{r_0}$, and assume that $T > 0$ and $\varepsilon > 0$ are arbitrary constants. Let $t_1, t_2 \in [0, T]$, with $|t_1 - t_2| \leq \varepsilon$, $\zeta(t_1) \leq \zeta(t_2)$ and $(\mathcal{G}_1, ..., \mathcal{G}_n) \in \Delta_1 \times ... \times \Delta_n$. Then we have

$$\begin{aligned}
&\Big|\Upsilon(\mathcal{G}_1, ..., \mathcal{G}_n)(t_2) - \Upsilon(\mathcal{G}_1, ..., \mathcal{G}_n)(t_1)\Big| \\
&\leq \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\Big|\varpi\Big(t_2, h(t_2, \mathcal{G}_1(\alpha(t_2)), ..., \mathcal{G}_n(\alpha(t_2)), \int_0^{\zeta(t_2)} \pi(t_2, s, \mathcal{G}_1(\alpha(s)), ..., \mathcal{G}_n(\alpha(s))ds) \\
&-\varpi\Big(t_1, h(t_2, \mathcal{G}_1(\alpha(t_2)), ..., \mathcal{G}_n(\alpha(t_2)), \int_0^{\zeta(t_2)} \pi(t_2, s, \mathcal{G}_1(\alpha(s)), ..., \mathcal{G}_n(\alpha(s))ds)\Big| \\
&+\Big|\varpi\Big(t_1, h(t_2, \mathcal{G}_1(\alpha(t_2)), ..., \mathcal{G}_n(\alpha(t_2)), \int_0^{\zeta(t_2)} \pi(t_2, s, \mathcal{G}_1(\alpha(s)), ..., \mathcal{G}_n(\alpha(s))ds) \\
&-\varpi\Big(t_1, h(t_1, \mathcal{G}_1(\alpha(t_1)), ..., \mathcal{G}_n(\alpha(t_1)), \int_0^{\zeta(t_2)} \pi(t_2, s, \mathcal{G}_1(\alpha(s)), ..., \mathcal{G}_n(\alpha(s))ds)\Big| \\
&+\Big|\varpi\Big(t_1, h(t_1, \mathcal{G}_1(\alpha(t_1)), ..., \mathcal{G}_n(\alpha(t_1)), \int_0^{\zeta(t_2)} \pi(t_2, s, \mathcal{G}_1(\alpha(s)), ..., \mathcal{G}_n(\alpha(s))ds) \\
&-\varpi\Big(t_1, h(t_1, \mathcal{G}_1(\alpha(t_1)), ..., \mathcal{G}_n(\alpha(t_1)), \int_0^{\zeta(t_1)} \pi(t_2, s, \mathcal{G}_1(\alpha(s)), ..., \mathcal{G}_n(\alpha(s))ds)\Big| \\
&+\Big|\varpi\Big(t_1, h(t_1, \mathcal{G}_1(\alpha(t_1)), ..., \mathcal{G}_n(\alpha(t_1)), \int_0^{\zeta(t_1)} \pi(t_2, s, \mathcal{G}_1(\alpha(s)), ..., \mathcal{G}_n(\alpha(s))ds) \\
&-\varpi\Big(t_1, h(t_1, \mathcal{G}_1(\alpha(t_1)), ..., \mathcal{G}_n(\alpha(t_1)), \int_0^{\zeta(t_1)} \pi(t_1, s, \mathcal{G}_1(\alpha(s)), ..., \mathcal{G}_n(\alpha(s))ds)\Big|\Big)\Big)\Big) \\
&\leq \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(|t_2 - t_1| + \Big|h(t_2, \mathcal{G}_1(\alpha(t_2)), ..., \mathcal{G}_n(\alpha(t_2))) - h(t_1, \mathcal{G}_1(\alpha(t_1)), ..., \mathcal{G}_n(\alpha(t_1)))\Big| \\
&+\Big|\int_{\zeta(t_1)}^{\zeta(t_2)} \pi(t_2, s, \mathcal{G}_1(\alpha(s)), ..., \mathcal{G}_n(\alpha(s))ds\Big| + \zeta(T)\omega^T_{r_0,G}(\pi, \varepsilon)\Big)\Big)\Big) \\
&\leq \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(|t_2 - t_1| + |t_2 - t_1| + \max\Big\{|\mathcal{G}_1(\alpha(t_2)) - \mathcal{G}_1(\alpha(t_1))|, ..., |\mathcal{G}_n(\alpha(t_2)) - \mathcal{G}_n(\alpha(t_1))|\Big\} \\
&+(\zeta(t_2) - \zeta(t_1))U_{r_0}(\pi, \varepsilon) + \zeta(T)\omega^T_{r_0}(\pi, \varepsilon)\Big)\Big)\Big)
\end{aligned}$$

where

$$\omega^T(\rho, \varepsilon) = \sup\Big\{|\rho(t_2) - \rho(t_1)| : t_1, t_2 \in [0, T], |t_2 - t_1| \leq \varepsilon\Big\},$$

$$\omega^T(\varrho, \omega^T(\rho, \varepsilon)) = \sup\Big\{|\varrho(t_2) - \varrho(t_1)| : t_1, t_2 \in [0, T], |t_2 - t_1| \leq \omega^T(\rho, \varepsilon)\Big\},$$

$$\zeta(T) = \sup\Big\{\zeta(t) : t \in [0, T]\Big\},$$

$$U_{r_0} = \sup\Big\{|\pi(t, \varsigma, \mathcal{G}_1, ..., \mathcal{G}_n)| : t \in [0, T], \varsigma \in [0, \zeta(T)], \mathcal{G}_1, ..., \mathcal{G}_n \in [-r_0, r_0]\Big\},$$

and

$$\begin{aligned}
\omega^T_{r_0}(\pi, \varepsilon) = \sup\Big\{&|\pi(t_2, \varsigma, \mathcal{G}_1, ..., \mathcal{G}_n) - \pi(t_1, \varsigma, \mathcal{G}_1, ..., \mathcal{G}_n)| : t_1, t_2 \in [0, T], \\
&|t_2 - t_1| \leq \varepsilon, \mathcal{G}_1, ..., \mathcal{G}_n) \in [-r_0, r_0], \varsigma \in [0, \zeta(T)]\Big\},
\end{aligned}$$

Since $(\mathcal{G}_1, ..., \mathcal{G}_n)$ was arbitrary elements of $\Delta_1 \times ... \times \Delta_n$ in the above inequality, we have

$$\begin{aligned}
&\omega^T(\Upsilon(\Delta_1 \times ... \times \Delta_n), \varepsilon) \\
&\leq \Gamma^{-1}(\psi(\Gamma(2|t_2 - t_1| + \max\{\omega^T(\mathcal{G}_1, \omega^T(\alpha, \varepsilon)), ..., \omega^T(\mathcal{G}_n, \omega^T(\alpha, \varepsilon))\} + (\zeta(t_2) - \zeta(t_1))U_{r_0}(\pi, \varepsilon) + \zeta(T)\omega^T_{r_0}(\pi, \varepsilon)))).
\end{aligned} \tag{16}$$

Moreover, in the light of the uniform continuity of the functions π and ζ on

$$[0, T] \times [0, \zeta(T)] \times [-r_0, r_0]^n,$$

and

$$[0, T]$$

respectively, $\omega_{r_0}^T(\pi,\varepsilon) \longrightarrow 0$ and $(\zeta(t_2)-\zeta(t_1))U_{r_0}(\pi,\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

Also, because of the uniform continuity of α and ζ on $[0,T]$, $\omega^T(\alpha,\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

Now, this remarks and the inequality 16 via tending $\varepsilon \to 0$ imply that

$$\omega_0^T(\Upsilon(\Delta_1\times\ldots\times\Delta_n)) \leq \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\max\Big\{\omega_0^T(\Delta_1),\ldots,\omega_0^T(\Delta_n)\Big\}\Big)\Big)\Big). \tag{17}$$

Furthermore, taking $T \to \infty$, we obtain that

$$\omega_0(\Upsilon(\Delta_1\times\ldots\times\Delta_n)) \leq \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\max\Big\{\omega_0(\Delta_1),\ldots,\omega_0(\Delta_n)\Big\}\Big)\Big)\Big). \tag{18}$$

Now, for arbitrary elements $(\mathcal{G}_1,\ldots,\mathcal{G}_n),(\mathcal{U}_1,\ldots,\mathcal{U}_n) \in \Delta_1\times\ldots\times\Delta_n$ and for all $t \in R_+$, we have

$$\begin{aligned}&\Big|\Upsilon(\mathcal{G}_1,\ldots,\mathcal{G}_n)(t)-\Upsilon(\mathcal{U}_1,\ldots,\mathcal{U}_n)(t)\Big|\\ &\leq \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\max\Big\{\max\Big\{\mathcal{G}_1(\alpha(t))-\mathcal{U}_1(\alpha(t))|,\ldots,|\mathcal{G}_n(\alpha(t))-\mathcal{U}_n(\alpha(t))|\Big\},2a(t)\int_0^{\zeta(t)}b(s)ds\Big\}\Big)\Big)\Big).\end{aligned} \tag{19}$$

Now, using the above inequality and the notion of diameter of a set, we have

$$\begin{aligned}&diam(\Upsilon(\Delta_1\times\ldots\times\Delta_n)(t))\\ &\leq \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\max\Big\{\max\Big\{diam\Delta_1(\alpha(t)),\ldots,diam\Delta_n(\alpha(t))\Big\},2a(t)\int_0^{\zeta(t)}b(s)ds\Big\}\Big)\Big)\Big)\end{aligned} \tag{20}$$

and hence

$$\begin{aligned}&\limsup_{t\to\infty} diam(\Upsilon(\Delta_1\times\ldots\times\Delta_n)(t))\\ &\leq \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\max\Big\{\limsup_{t\to\infty} diam\Delta_1(t),\ldots,\limsup_{t\to\infty} diam\Delta_n(t)\Big\}\Big)\Big)\Big).\end{aligned} \tag{21}$$

Adding inequalities 18 and 21 and by assuming that the combination function $\Gamma^{-1}\Big(\psi\Big(\Gamma\Big)\Big)$ is super-additive, we obtain that

$$\begin{aligned}&m(\Upsilon(\Delta_1\times\ldots\times\Delta_n))\\ &= \omega_0(\Upsilon(\Delta_1\times\ldots\times\Delta_n)) + \limsup_{t\to\infty} diam(\Upsilon(\Delta_1\times\ldots\times\Delta_n)(t))\\ &\leq \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\max\Big\{\omega_0(\Delta_1),\ldots,\omega_0(\Delta_n)\Big\}\Big)\Big)\Big)\\ &+\Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\max\Big\{\limsup_{t\to\infty} diam\Delta_1(t),\ldots,\limsup_{t\to\infty} diam\Delta_n(t)\Big\}\Big)\Big)\Big)\\ &\leq \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\max\Big\{m(\Delta_1),\ldots,m(\Delta_n\Big\}\Big)\Big)\Big).\end{aligned} \tag{22}$$

So, we get

$$\Gamma(m(\Upsilon(\Delta_1\times\ldots\times\Delta_n))) \leq \psi\Big(\Gamma\Big(\max\Big\{m(\Delta_1),\ldots,m(\Delta_n)\Big\}\Big)\Big). \tag{23}$$

Thus, from Theorem 3.6 we obtain that the operator Υ has a Presić type fixed point. Thus, the system of functional integral equations (13) has at least one solution in $(BC(\mathbb{R}_+))^n$. □

5. Example

Example 5.1. *Let:*

$$\begin{cases} \mathcal{G}_1(\iota) = \frac{1}{7}e^{-\iota^2} + \frac{1}{2}\dfrac{\frac{e^{-\iota}}{2} + \frac{(\sum_{i=1}^{n}\mathcal{G}_i(\iota))}{2+\iota}}{1 + \frac{e^{-\iota}}{2} + \frac{(\sum_{i=1}^{n}\mathcal{G}_i(\iota))}{2+\iota}} + \dfrac{1}{2}\dfrac{\displaystyle\int_0^{\iota^2} \frac{e^{-(\iota+\kappa)}\sin^2(\sum_{i=1}^{n}\mathcal{G}_i^i(\kappa))}{\cosh(2\sum_{i=1}^{n}\mathcal{G}_i^i(\kappa))}d\kappa}{1 + \displaystyle\int_0^{\iota^2} \frac{e^{-(\iota+\kappa)}\sin^2(\sum_{i=1}^{n}\mathcal{G}_i^i(\kappa))}{\cosh(2\sum_{i=1}^{n}\mathcal{G}_i^i(\kappa))}d\kappa} \\ \qquad\qquad\qquad\qquad \dots \\ \mathcal{G}_n(\iota) = \frac{1}{7}e^{-\iota^2} + \frac{1}{2}\dfrac{\frac{e^{-\iota}}{2} + \frac{(\sum_{i=1}^{n}\mathcal{G}_i(\iota))}{2+\iota}}{1 + \frac{e^{-\iota}}{2} + \frac{(\sum_{i=1}^{n}\mathcal{G}_i(\iota))}{2+\iota}} + \dfrac{1}{2}\dfrac{\displaystyle\int_0^{\iota^2} \frac{e^{-(\iota+\kappa)}\sin^2(\sum_{i=1}^{n}\mathcal{G}_i^i(\kappa))}{\cosh(2\sum_{i=1}^{n}\mathcal{G}_i^i(\kappa))}d\kappa}{1 + \displaystyle\int_0^{\iota^2} \frac{e^{-(\iota+\kappa)}\sin^2(\sum_{i=1}^{n}\mathcal{G}_i^i(\kappa))}{\cosh(2\sum_{i=1}^{n}\mathcal{G}_i^i(\kappa))}d\kappa}. \end{cases} \tag{24}$$

The above system is a special case of (13) *with*

$$\alpha(\iota) = \gamma(\iota) = \iota, \ \zeta(\iota) = \iota^2, \qquad \iota \in [0, \infty),$$

$$\varpi(\iota, \mathcal{G}_1, \mathcal{G}_2) = \frac{1}{7}e^{-\iota^2} + \frac{1}{2}\frac{\mathcal{G}_1}{1+\mathcal{G}_1} + \frac{1}{2}\frac{\mathcal{G}_2}{1+\mathcal{G}_2},$$

$$\pi(\iota, \kappa, \mathcal{G}_1, ..., \mathcal{G}_n) = \frac{e^{-(\iota+\kappa)}\sin^2(\sum_{i=1}^{n}\mathcal{G}_i^i(\kappa))}{\cosh(2\sum_{i=1}^{n}\mathcal{G}_i^i(\kappa))},$$

and

$$h(\iota, \mathcal{G}_1, ..., \mathcal{G}_n) = \frac{e^{-\iota}}{2} + \frac{(\sum_{i=1}^{n}\mathcal{G}_i(\iota))}{2+\iota}.$$

Also, take $\mathcal{D}(\iota) = \frac{2}{3}$. To prove the existence of a solution for this system, we should interrogate the conditions (i)-(v) *of 1heorem 4.1.*
Condition (i) is clearly evident. Now

$$\begin{aligned} |h(\iota_1, \mathcal{G}_1, ..., \mathcal{G}_n) - h(\iota_2, \mathcal{U}_1, ..., \mathcal{U}_n)| &\le |\frac{e^{-\iota_1}}{2} - \frac{e^{-\iota_2}}{2}| + |\frac{(\sum_{i=1}^{n}\mathcal{G}_i(\iota_1))}{n+\iota_1} - \frac{(\sum_{i=1}^{n}\mathcal{U}_i(\iota_2))}{n+\iota_2}| \\ &\le |\iota_1 - \iota_2| + \frac{1}{n}(|\mathcal{G}_1 - \mathcal{U}_1| + ... + |\mathcal{G}_n - \mathcal{U}_n|) \\ &\le |\iota_1 - \iota_2| + \max\{|\mathcal{G}_1 - \mathcal{U}_1|, ..., |\mathcal{G}_n - \mathcal{U}_n|\}, \end{aligned} \tag{25}$$

and

$$\begin{aligned} \left|\varpi(\iota_1, \mathcal{G}_1, \mathcal{G}_2) - \varpi(\iota_2, \mathcal{U}_1, \mathcal{U}_2)\right| &\le \ln \frac{[e^{\frac{1}{7}|\iota_1-\iota_2| + \frac{1}{2}|\frac{\mathcal{G}_1}{1+\mathcal{G}_1} + \frac{\mathcal{G}_2}{1+\mathcal{G}_2} - \frac{\mathcal{U}_1}{1+\mathcal{U}_1} - \frac{\mathcal{U}_2}{1+\mathcal{U}_2}|}]^{\frac{1}{3}}}{2} \\ &\le \ln \frac{[e^{\frac{1}{7}|\iota_1-\iota_2| + \frac{1}{2}(\frac{|\mathcal{G}_1-\mathcal{U}_1|}{(1+\mathcal{G}_1)(1+\mathcal{U}_1)} + \frac{|\mathcal{G}_2-\mathcal{U}_2|}{(1+\mathcal{G}_2)(1+\mathcal{U}_2)})}]^{\frac{1}{3}}}{2} \\ &\le \ln \frac{[e^{\frac{1}{7}|\iota_1-\iota_2| + \frac{1}{2}(|\mathcal{G}_1-\mathcal{U}_1| + |\mathcal{G}_2-\mathcal{U}_2|)}]^{\frac{1}{3}}}{2} \\ &\le \ln \frac{[e^{1+|\iota_1-\iota_2| + \max\{|\mathcal{G}_1-\mathcal{U}_1|, |\mathcal{G}_2-\mathcal{U}_2|\}}]^{\frac{1}{3}}}{2}. \end{aligned}$$

Therefore, we have

$$\Big|\varpi(\iota_1,\mathcal{G}_1,\mathcal{G}_2)-\varpi(\iota_2,\mathcal{U}_1,\mathcal{U}_2)\Big| \leq \Gamma^{-1}\Big(\psi\Big(\Gamma\Big(|t_1-t_2|+\max\{|\mathcal{G}_1-\mathcal{U}_1|,|\mathcal{G}_2-\mathcal{U}_2|\}\Big)\Big)\Big)$$

where $\Gamma(x)=e^x$ *and* $\psi(x)=\frac{x^{\frac{1}{3}}}{2}$.

We can find that ϖ *satisfies condition (ii) of Theorem 4.1. Also,*

$$M=\sup\{|\varpi(\iota,0,0)|:\iota\in[0,\infty)\}=\sup\{\frac{1}{7}e^{-\iota^2}:\iota\in[0,\infty)\}\simeq 0.14285714285$$

and

$$N=\sup\{|h(\iota,0,...,0)|:\iota\in[0,\infty)\}=\sup\{\frac{1}{2}e^{-\iota}:\iota\in[0,\infty)\}=\frac{1}{2}.$$

Moreover, π *is continuous on* $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^2$ *and*

$$|\pi(\iota,\kappa,\mathcal{G}_1,...,\mathcal{G}_n)|=|\frac{e^{-(\iota+\kappa)}\sin^2(\sum_{i=1}^n\mathcal{G}_i^i(\kappa))}{\cosh(2\sum_{i=1}^n\mathcal{G}_i^i(\kappa))}|\leq e^{-(\iota+\kappa)}.$$

Therefore, $a(\iota)=e^{-\iota}, b(\kappa)=e^{-\kappa}$.

On the other hand,

$$\begin{aligned}Q&=\sup\{|\int_0^{\iota^2}a(\iota)b(\kappa)d\kappa|:\iota,\kappa\in[0,\infty)\}\\&=\sup\{|\int_0^{\iota^2}e^{-(\iota+\kappa)}d\kappa|:\iota,\kappa\in[0,\infty)\}\\&=\sup\{|e^{-\iota}(1-e^{-\iota^2})|:\iota,\kappa\in[0,\infty)\}\simeq 0.23384.\end{aligned}$$

Furthermore, for every $r\geq 0$,

$$\begin{aligned}&\Gamma^{-1}\Big(\psi\Big(\Gamma\Big(\max\Big\{r+N,Q\Big\}\Big)\Big)\Big)+M\\&=\ln(\frac{(e^{\max\left\{r+\frac{1}{2},0.23384\right\}})^{\frac{1}{3}}}{2})+1/7\\&=\ln(\frac{e^{\frac{r}{3}+\frac{1}{6}}}{2})+1/7\leq r.\end{aligned}$$

Consequently, all the reservations of Theorem 4.1 are fulfilled. Hence, the system of integral equations (24) *has at least one solution which belongs to the space* $(BC(\mathbb{R}_+))^n$.

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