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On Expansions of Graded 2-Absorbing Hyperideals in Graded Multiplicative Hyperrings

Peyman Ghiasvand[a], Farkhonde Farzalipour[a], Saeed Mirvakili[a]

[a] *Department of Mathematics, Payame Noor University (PNU), P.O.BOX 19395-3697 Tehran, Iran*

Abstract. Let G be an abelian group with identity e. Let R be a graded multiplicative hyperring and $\delta : \mathcal{I}^{gr}(R) \to \mathcal{I}^{gr}(R)$ be an expansion function of $\mathcal{I}^{gr}(R)$, where $\mathcal{I}^{gr}(R)$ is the set of all graded hyperideals of R. In this paper, we introduce and study the concepts of graded δ-primary hyperideals of R and graded 2-absorbing δ-primary hyperideals of R which are the extended classes of graded prime and graded 2-absorbing hyperideals of R, respectively. Moreover, we give the basic properties of these new types of graded hyperideals and investigate the relations among these structures.

1. Introduction

Hyperstructure theory was first introduced by the French mathematician F. Marty in 1934 [27]. He, at the 8th Congress of Scandinavian Mathematicians, defined hypergroups, as a natural generalization of groups, based on the notion of hyperoperation, and has since then been studied by many authors (see for example [16, 17, 32]). In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Algebraic hyperstructures are a suitable generalization of classical algebraic structures, with broad applications in the mathematical foundations of geometry, lattices, cryptography, automata, graphs and hypergraphs, fuzzy set, probability and rough set theory, physics, chemistry and so on (see [16, 17]). The notion of hyperrings was introduced by M. Krasner in 1983, where the addition is a hyperoperation, while the multiplication is an operation [24]. Prime, primary, and maximal subhypermodules of a hypermodule in the sense of krasner hyperring R were discussed by M. M. Zahedi and R. Ameri in [38]. The concept of 2-absorbing hyperideals on Krasner hyperrings was introduced in [7] as a generalization of the notion of prime hyperideals in Krasner hyperrings. H. Bordbar and I. Cristea in [13, 14] introduced and studied height of hyperideals in Krasner hyperrings. The concept of δ-primary hyperideals on Krasner hyperrings was introduced in [9]. R. Ameri *et al.* in [3] introduced Krasner (m, n)-hyperrings and in [4] studied prime and primary subhypermodules of (m, n)- hypermodules. Also, K. Hila *et al.* in [23] introduced and studied (k, n)-absorbing hyperideals in Krasner (m, n)-hyperrings. The notion of multiplicative hyperrings are an important class of algebraic hyperstructures which generalize rings, initiated the study by Rota in 1982, where the multiplication is a hyperoperation, while the addition is an operation [33]. Procesi and Rota introduced and studied in brief the prime hyperideals of multiplicative

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Communicated by Dijana Mosić
Email addresses: p_ghiasvand@pnu.ac.ir (Peyman Ghiasvand), f_farzalipour@pnu.ac.ir (Farkhonde Farzalipour), saeed_mirvakili@pnu.ac.ir (Saeed Mirvakili)

hyperrings [29–31] and this idea is further generalized in a paper by Dasgupta [18]. R. Ameri *et al.* in [2] described multiplicative hyperring of fractions and coprime hyperideals. Later on, many researches have observed that generalizations of prime hyperideals in multiplicative hyperrings [5, 6, 8, 34]. Recently, The concept of graded multiplicative hyperrings and graded hyperideals was introduced in [22]. Furthermore, the study of graded rings arises naturally out of the study of affine schemes and allows them to formalize and unify arguments by induction [35]. However, this is not just an algebraic trick. The concept of grading in algebra, in particular graded modules is essential in the study of homological aspect of rings. Much of the modern development of the commutative algebra emphasizes graded rings. Graded rings play a central role in algebraic geometry and commutative algebra. Gradings appear in many circumstances, both in elementary and advanced level. In recent years, rings with a group-graded structure have become increasingly important and consequently, the graded analogues of different concepts are widely studied (see [19–21, 25, 28]). Theory of graded hyperrings can be considered as an extension theory of hyperrings. The notion of 2-absorbing ideals over commutative rings which is a generalization of prime ideals has been introduced and investigated by A. Badawi in [10]. D. Zhao in [39] introduced the concept of δ-primary ideals of commutative rings. This concept was studied extensively in [11] and [12]. Many results in this paper are inspired by the work of the authors in [11] and [12]. After that in [1, 36], the authors extended the notion of 2-absorbing ideals to graded rings. Recently, G. Ulucak [37] introduced and study the concept of δ-primary hyperideals and 2-absorbing δ-primary hyperideals in multiplicative hyperring which are the extended classes of prime and 2-absorbing hyperideals, respectively. In this paper, we introduce and study the notions of graded δ-primary hyperideals of R and graded 2-absorbing δ-primary hyperideals of R which are the extended classes of graded prime and graded 2-absorbing hyperideals of R, respectively. Moreover, we give a number of main results and the basic properties concerning these classes of graded hyperideals and their homogeneous components.

2. Basic definitions and results

In this section we give some definitions and results of hyperstructures which we need to develop our paper. We refer to [17, 18] for these basic properties and information on hyperstructures.

Definition 2.1. *[33] Let R be a nonempty set. By $P^*(R)$, we mean the set of all nonempty subset of R. Let $\circ$ be a hyperoperation from $R \times R$ to $P^*(R)$. Rota called $(R, +, \circ)$ a multiplicative hyperring, if it has the following properties:*

(i) $(R, +)$ *is an abelian group;*

(ii) $(R, \circ)$ *is a hypersemigroup;*

(iii) For all $a, b, c \in R$, $a \circ (b + c) \subseteq a \circ b + a \circ c$ *and* $(b + c) \circ a \subseteq b \circ a + c \circ a$;

(iv) $a \circ (-b) = (-a) \circ b = -(a \circ b)$.

If in (iii) we have equalities instead of inclusions, then we say that the multiplicative hyperring is strongly distributive.

Here, we mean a hypersemigroup by a nonempty set R with an associative hyperoperation $\circ$, i.e.,

$$a \circ (b \circ c) = \bigcup_{t \in (b \circ c)} a \circ t = \bigcup_{s \in (a \circ b)} s \circ c = (a \circ b) \circ c$$

for all $a, b, c \in R$.

Further, if R is a multiplicative hyperring with $a \circ b = b \circ a$ for all $a, b \in R$, then R is called a commutative multiplicative hyperring.

Definition 2.2. *[30] (a) Let $(R, +, \circ)$ be a multiplicative hyperring and S be a nonempty subset of R. Then S is said to be a subhyperring of R if $(S, +, \circ)$ is itself a multiplicative hyperring.*

(b) A subhyperring I of a multiplicative hyperring R is a hyperideal of $(R, +, \circ)$ if $I - I \subseteq I$ and for all $x \in I$, $r \in R$; $x \circ r \cup r \circ x \subseteq I$.

Definition 2.3. *[17] (a) A proper hyperideal M of a multiplicative hyperring R is maximal in R, if for any hyperideal I of R, $M \subset I \subseteq R$, then $I = R$.*

(b) A proper hyperideal P of a multiplicative hyperring R is said to be a prime hyperideal of R, if for any $a, b \in R$, $a \circ b \subseteq P$, then $a \in P$ or $b \in P$.

(c) A proper hyperideal Q of a multiplicative hyperring R is said to be a primary hyperideal of R, if for any $a, b \in R$, $a \circ b \subseteq Q$, then $a \in Q$ or $b^n \subseteq Q$ for some $n \in \mathbb{N}$.

Definition 2.4. *[22] Let G be a group with identity element e. A multiplicative hyperring (R, G) is called a G-graded multiplicative hyperring, if there exists a family $\{R_g\}_{g\in G}$ of additive subgroups of R indexed by the elements $g \in G$ such that $R = \bigoplus_{g\in G} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$ where $R_gR_h = \bigcup\{r_g \circ r_h : r_g \in R_g, r_h \in R_h\}$. For simplicity, we will denote the graded multiplicative hyperring (R, G) by R.*

An element of a graded hyperring R is called homogeneous if it belongs to $\bigcup_{g\in G} R_g$ and this set of homogeneous elements is denoted by $h(R)$. If $x \in R_g$ for some $g \in G$, then we say that x is of degree g, and it is denoted by x_g. If $x \in R$, then there exist unique elements $x_g \in h(R)$ such that $x = \sum_{g\in G} x_g$.

In fact, every hyperring is trivially a G-graded hyperring by letting $R_e = R$ and $R_g = 0$ for all $g \neq e$. If $R = \bigoplus_{g\in G} R_g$ is a graded multiplicative hyperring, then R_e is a subhyperring of R where e is the identity element of group G. Also, if R has an identity element 1 such that $x \circ 1 = 1 \circ x = \{x\}$, then $1 \in R_e$.

Example 2.5. *Let $G = (\mathbb{Z}_2, +)$ be the cyclic group of order 2 and $R = \{a, b, c, d\}$. Consider the multiplicative hyperring $(R, +, \circ)$, where operation $+$ and hyperoperation $\circ$ defined on R as follow:*

$+$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a, d\}$	$\{a, c\}$	$\{a, b\}$
c	$\{a\}$	$\{a, c\}$	$\{a\}$	$\{a, c\}$
d	$\{a\}$	$\{a, b\}$	$\{a, c\}$	$\{a, d\}$

Let $R_0 = \{a, d\}$ and $R_1 = \{a, b\}$. Then it is easy to verify that R_0 and R_1 are subgroups of $(R, +)$ and we can write $a = a + a$, $b = a + b$, $c = d + b$ and $d = d + a$ uniquely. Hence, $R = R_0 \bigoplus R_1$. We have $R_0R_0 \subseteq R_0$, $R_0R_1 \subseteq R_1$, $R_1R_0 \subseteq R_1$ and $R_1R_1 \subseteq R_0$. Therefore (R, G) is a graded hyperring and $h(R) = \{a, b, d\}$.

Definition 2.6. *Let $R = \bigoplus_{g\in G} R_g$ be a graded multiplicative hyperring. A subhyperring S of R is called a graded subhyperring of R, if $S = \bigoplus_{g\in G}(S \cap R_g)$. Equivalently, S is graded if for every element $f \in S$, all the homogeneous components of f (as an element of R) are in S.*

Definition 2.7. *Let I be a hyperideal of a graded multiplicative hyperring R. Then I is a graded hyperideal, if $I = \bigoplus_{g\in G}(I \cap R_g)$. For any $a \in I$ and for some $r_g \in h(R)$ that $a = \sum_{g\in G} r_g$, then $r_g \in I \cap R_g$ for all $g \in G$.*

Lemma 2.8. *[22] Let I and J be graded hyperideals of a graded multiplicative hyperring R. Then*

(i) $I \cap J$ is a graded hyperideal of R.

(ii) IJ is a graded hyperideal of R.

(iii) $I \cup J$ is a graded hyperideal of R if and only if $I \subseteq J$ or $J \subseteq I$.

(iv) $I + J$ is a graded hyperideal of R.

Definition 2.9. *Let I be a graded hyperideal of a graded multiplicative hyperring $(R, +, \circ)$. The intersection of all graded prime hyperideals of R containing I is called the graded radical of I, denoted by $Grad(I)$. If the graded multiplicative hyperring R does not have any graded prime hyperideal containing I, we define $Grad(I) = R$.*

Let I be a graded hyperideal of a graded multiplicative hyperring R. We define $D(I) = \{r \in R :$ for any $g \in G$, $r_g^{n_g} \subseteq I$ for some $n_g \in \mathbb{N}\}$. It is clear that $D(I)$ is a graded hyperideal of R.

Definition 2.10. *Let R be a graded multiplicative hyperring and C be the class of all finite products of homogeneous elements of R i. e. $C = \{r_1 \circ r_2 \circ \cdots \circ r_n : r_i \in h(R), n \in \mathbb{N}\} \subseteq P^*(h(R))$. A graded hyperideal I of R is said to be a C^{gr}-ideal of R if for any $A \in C$, $A \cap I \neq \emptyset$, then $A \subseteq I$.*

Theorem 2.11. *[22] Let $I = \bigoplus_{g\in G} I_g = \bigoplus_{g\in G}(I \cap R_g)$ be a graded hyperideal of a commutative graded multiplicative hyperring $R = \bigoplus_{g\in G} R_g$. Then $D(I) \subseteq Grad(I)$. The equality holds when I is a C^{gr}-ideal of R.*

Definition 2.12. *[22] Let $R = \bigoplus_{g\in G} R_g$ and $S = \bigoplus_{g\in G} S_g$ be two graded multiplicative hyperrings. The function $f : R \to S$ is called a graded homomorphism, if*

(i) for any $a, b \in R$, $f(a + b) = f(a) + f(b)$,

(ii) for any $a, b \in R$, $f(a \circ b) \subseteq f(a) \circ f(b)$, and

(iii) $f(R_g) \subseteq S_g$ for any $g \in G$.

In particular, f is called a graded good homomorphism in case $f(a \circ b) = f(a) \circ f(b)$. The kernel of a graded homomorphism is defined as $Ker(f) = f^{-1}(\langle 0\rangle) = \{r \in R : f(r) \in \langle 0\rangle\}$ and note that $f(r)$ may not be a zero element.

If Q is a graded hyperideal of S and $f : R \to S$ is a graded good homomorphism, then $f^{-1}(Q)$ is a graded hyperideal of R. If I is a graded hyperideal of R and $f : R \to S$ is an onto graded good homomorphism, then $f(I)$ is a graded hyperideal of S.

Throughout this paper, we assume that all graded hyperrings are commutative graded multiplicative hyperrings with absorbing zero, i. e. $0 \in R$ such that $x = 0 + x$ and $0 \in x \circ 0 = 0 \circ x$ for all $x \in R$.

3. On expansion of graded prime hyperideals

Definition 3.1. (*a*) *A proper graded hyperideal I of a graded multiplicative hyperring R is called a graded prime hyperideal of R if, for any $a_g, b_h \in h(R)$, $a_g \circ b_h \subseteq I$, then $a_g \in I$ or $b_h \in I$.*
(*b*) *A proper graded hyperideal I of a graded multiplicative hyperring R is called a graded primary hyperideal of R if, for any $a_g, b_h \in h(R)$, $a_g \circ b_h \subseteq I$, then $a_g \in I$ or $b_h^n \subseteq I$ for some $n \in \mathbb{N}$.*

Let R be a multiplicative hyperring. By $\mathcal{I}^{gr}(R)$ and $\mathcal{I}^{gr}_*(R)$, we mean all graded hyperideals of R and proper graded hyperideals of R, respectively.

Definition 3.2. *The function $\delta : \mathcal{I}^{gr}(R) \to \mathcal{I}^{gr}(R)$ is said to be an expansion function of $\mathcal{I}^{gr}(R)$ if it satisfies the following two conditions:* (1) *$I \subseteq \delta(I)$,* (2) *If $I \subseteq J$, then $\delta(I) \subseteq \delta(J)$ for all graded hyperideals I, J of R.*

In the following examples, we explain the definition of expansion functions over commutative graded multiplicative hyperrings.

Example 3.3.

1. *The function δ_0 is an expansion function of $\mathcal{I}^{gr}(R)$ with $\delta_0(I) = I$ for every graded hyperideal $I \in \mathcal{I}^{gr}(R)$.*
2. *The function δ_1 is an expansion function of $\mathcal{I}^{gr}(R)$ with $\delta_1(I) = D(I)$ for every graded hyperideal $I \in \mathcal{I}^{gr}(R)$.*
3. *The function δ_2 is an expansion function of $\mathcal{I}^{gr}(R)$ with $\delta_2(I) = Grad(I)$ for every graded hyperideal $I \in \mathcal{I}^{gr}(R)$.*
4. *The function δ_r is an expansion function of $\mathcal{I}^{gr}(R)$ with $\delta_r(I) = R$ for every graded hyperideal $I \in \mathcal{I}^{gr}(R)$.*
5. *Let δ_i and δ_j be expansion functions of graded hyperideals of R. δ is defined by $\delta(I) = \delta_i(I) \cap \delta_j(I)$ for each graded hyperideal I of R. Notice that δ is an expansion function of $\mathcal{I}^{gr}(R)$.*
6. *Let $\delta_{\mathcal{I}^{gr}(R)}$ be defined by $\delta_{\mathcal{I}^{gr}(R)}(J) = \bigcap\{I \in \mathcal{I}^{gr}(R) | J \subseteq I\}$. Then $\delta_{\mathcal{I}^{gr}(R)}$ is an expansion function of graded hyperideals of R.*

7. *The compound function $\delta \circ \gamma$ of two expansion functions δ and γ of $\mathcal{I}^{gr}(R)$ is an expansion of $\mathcal{I}^{gr}(R)$ with $\delta \circ \gamma(I) = \delta(\gamma(I))$ for every graded hyperideal I of R.*

8. *Let δ_+ be defined by $\delta_+(I) = I + J$ for every graded hyperideal I of R where J is a graded hyperideal of R. It is easy to see that δ_+ is an expansion function of $\mathcal{I}^{gr}(R)$.*

Definition 3.4. *Let δ be an expansion function of $\mathcal{I}^{gr}(R)$. $I \in \mathcal{I}^{gr}_*(R)$ is called a graded δ-primary hyperideal of R, if $a_g, b_h \in h(R)$ and $a_g \circ b_h \subseteq I$ imply either $a_g \in I$ or $b_h \in \delta(I)$.*

Example 3.5. *Consider the expansion function δ_r of a graded multiplicative hyperring R (see Example 3.3 (4)). Then every proper graded hyperideal of R is a graded δ_r-primary hyperideal.*

Example 3.6. 1. *It is clear that a graded hyperideal is graded δ_0-primary if and only if it is a graded prime hyperideal.*

2. *If a graded hyperideal of R is graded δ_1-primary, then it is graded primary.*

3. *Let every graded hyperideal of R is a C^{gr}-hyperideal. Then a graded hyperideal of R is graded δ_2-primary if and only if it is a graded primary hyperideal.*

Example 3.7. *Let $(R, +, \cdot)$ be a ring. Then corresponding to every subset $A \in P^*(R)(|A| \geq 2)$, there exists a multiplicative hyperring with absorbing zero $(R_A, +, \circ)$, where $R_A = R$ and for any $\alpha, \beta \in R_A$, $\alpha \circ \beta = \{\alpha \cdot a \cdot \beta : a \in A\}$. If $(R_A, +, \circ)$ be a commutative multiplicative hyperring and element x indeterminate over R_A. Consider the polynomial multiplicative hyperring $S = (R_A[x], +, *)$, where operation $+$ and hyperoperation $*$ defined on S as follows: for all $f(x) = \sum_{k=0}^{n} a_k x^k$ and $g(x) = \sum_{k=0}^{m} b_k x^k$ of S, we consider*

$$f(x) + g(x) = \sum_{k=0} (a_k + b_k) x^k, \;\; f(x) * g(x) = \left\{ \sum_{k=0}^{n+m} c_k x^k \mid c_k \in \sum_{i+j=k} a_i b_j \right\}.$$

Let $R_A = (\mathbb{R}, +, \circ)$ with $A = \{3, 4, -6\}$ be the multiplicative hyperring and $G = (\mathbb{Z}, +)$ be the integers group. Consider the multiplicative polynomial hyperring $S = (R_A[x, y], +, *)$. Then $S = \bigoplus S_n$ is a G- graded multiplicative hyperring such that for $m = (m_1, m_2) \in \mathbb{N}^2$ and $X^m = x^{m_1} y^{m_2}$ we have $S_n = \{\sum_{m \in \mathbb{N}^2} r_m X^m \mid r_m \in \mathbb{R}, m_1 + m_2 = n\}$. Let $I = \langle x \rangle$. Then I is a graded hyperideal of S such that it is a graded δ-primary hyperideal for any expansion function δ of R.

Example 3.8. *Let $R = (\mathbb{Z}[i], +, \cdot)$ be the Gaussian integers ring and $G = (\mathbb{Z}_2, +)$ be the cyclic group of order 2. Consider the multiplicative hyperring $(R_A, +, \circ) = (\mathbb{Z}[i], +, \circ) = \{a + bi \mid a, b \in \mathbb{Z}\}$ with $A = \{-1, 3\}$, where $R_A = R$ and for any $x, y \in R_A$, $x \circ y = \{x \cdot a \cdot y : a \in A\}$. Then, $(R_A, +, \circ)$ is a G-graded multiplicative hyperring with $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$ and $R_A = R_0 \bigoplus R_1$. Consider the graded hyperideal $J = 2R = \{-2a - 2bi, 6a + 6bi : a, b \in \mathbb{Z}\}$. Then it is clear that J is a graded prime hyperideal of R. Hence J is a graded δ_2-primary hyperideal of R. But J is not a δ_2-primary hyperideal because 2 is not irreducible in R_A.*

Proposition 3.9. *Let δ and γ be expansion functions of $\mathcal{I}^{gr}(R)$ and $\delta(I) \subseteq \gamma(I)$ for each graded hyperideal I of R. Every graded δ-primary hyperideal of R is a graded γ-primary hyperideal of R.*

Proof. It is straightforward. □

By Proposition 3.9, every graded prime hyperideal is graded δ-primary hyperideal for any expansion function δ of a graded multiplicative hyperring. However, the next example shows that the converse is not true, in general.

Example 3.10. *Consider the $\mathbb{Z}_2$-graded multiplicative hyperring $(R_A, +, \circ) = (\mathbb{Z}_A[i], +, \circ)$ with $A = \{-1, 2, 3, 5\}$ and $\mathbb{Z}_A[i] = \mathbb{Z}[i]$. Then $Q = \langle 4 \rangle \oplus \langle 0 \rangle$ is a graded δ_1-primary hyperideal of R. But it is not a graded prime hyperideal of R, because*

$$(2, 0) \circ (2, 0) = \bigcup_{a \in A} (2, 0) \cdot a \cdot (2, 0) = \{(-4, 0), (8, 0), (12, 0), (20, 0)\} \subseteq Q$$

but $(2, 0) \notin Q$.

Theorem 3.11. *If I is a graded primary hyperideal of R and $Grad(\delta(I)) = \delta(I)$, then I is a graded δ-primary hyperideal of R.*

Proof. Let $a_g \circ b_h \subseteq I$ where $a_g, b_h \in h(R)$. Since I is a graded primary hyperideal of R, we have $a_g \in I$ or $b_h \in D(I) \subseteq Grad(I) \subseteq Grad(\delta(I))$, and so $a_g \in I$ or $b_h \in \delta(I)$ because $Grad(\delta(I)) = \delta(I)$. Hence I is a graded δ-primary hyperideal of R. □

Proposition 3.12. *Let $I \in \mathcal{I}^{gr}_*(R)$. Then I is a graded δ-primary hyperideal of R if and only if $L \circ K \subseteq I$ for each $L, K \in \mathcal{I}^{gr}(R)$ implies $L \subseteq I$ or $K \subseteq \delta(I)$.*

Proof. ($\Rightarrow$) Suppose that $L \circ K \subseteq I$ and $L \nsubseteq I$ and $K \nsubseteq \delta(I)$ for some $L, K \in \mathcal{I}^{gr}(R)$. Hence there exist $a_g, b_h \in h(R)$ such that $a_g \in L - I$ and $b_h \in K - \delta(I)$. Then $a_g \circ b_h \subseteq L \circ K \subseteq I$, which is a contradiction.
($\Leftarrow$) Let $a_g \circ b_h \subseteq I$ where $a_g, b_h \in h(R)$. By [22], it is obtained that $\langle a_g \rangle \circ \langle b_h \rangle \subseteq \langle a_g \circ b_h \rangle \subseteq I$. Consequently, $\langle a_g \rangle \subseteq I$ or $\langle b_h \rangle \subseteq \delta(I)$ by assumption. Therefore, $a_g \in I$ or $b_h \in \delta(I)$, as needed. □

Lemma 3.13. *If I, J be graded hyperideals of a strongly distributive graded multiplicative hyperring R, then $(I :_R J) = \{r \in R : r \circ J \subseteq I\}$ is a graded hyperideal of R, and also, if $a_g \in h(R)$, then $(I :_R a_g) = \{r \in R : r \circ a_g \subseteq I\}$ is a graded hyperideal of R.*

Proof. Let $r \in (I : J)$. Then we can write $r = \sum_{i=1}^{n} r_{g_i}$ where $0 \neq r_{g_i} \in R_{g_i}$. It is enough to show that $r_{g_i} \in (I : J)$ for any $i \in \{1, 2, \cdots, n\}$. We have $\sum_{i=1}^{n} r_{g_i} \circ J \subseteq I$. Let $x \in J$, and hence $x = \sum_{i=1}^{m} x_{g_i}$ where $0 \neq x_{g_i} \in J \cap R_{g_i}$ since J is a graded hyperideal of R. Thus $\sum_{i=1}^{n} r_{g_i} \circ \sum_{i=1}^{m} x_{g_i} = (r_{g_1} + \cdots + r_{g_n}) \circ (x_{h_1} + \cdots + x_{h_m}) = r_{g_1} \circ x_{h_1} + \cdots + r_{g_n} \circ x_{h_m} \subseteq I$. Now we show that $r_{g_1} \circ x_{h_1} \subseteq I$. Suppose that $t_{g_1 h_1} \in r_{g_1} \circ x_{h_1}$. Since $r_{g_i} \circ x_{h_j} \neq \emptyset$, for any $1 \leq i \leq n$ and $1 \leq j \leq m$, then there exist $t_{g_2 h_2} \in r_{g_2} \circ x_{h_2}, \cdots, t_{g_n h_m} \in r_{g_n} \circ x_{h_m}$ such that $t_{g_1 h_1} + t_{g_2 h_2} + \cdots + t_{g_n h_m} \in r_{g_1} \circ x_{h_1} + \cdots + r_{g_n} \circ x_{h_m} \subseteq I$, and so $t_{g_i h_j} \in I$ for any i, j, because I is a graded hyperideal of R. Therefore $r_{g_1} \circ x_{h_1} \subseteq I$. In order to, $r_{g_1} \circ x_{h_j} \subseteq I$ for any $j \in \{1, 2, \cdots, m\}$. Thus $r_{g_1} \circ x = r_{g_1} \circ (x_{h_1} + \cdots + x_{h_m}) = r_{g_1} \circ x_{h_1} + \cdots + r_{g_1} \circ x_{h_m} \subseteq I$, so $r_{g_1} \in (I : J)$. Consequently, we get $r_{g_i} \in (I : J)$ for every $i \in \{1, \cdots, n\}$. Thus $(I : J)$ is a graded hyperideal of R. □

Theorem 3.14. *Let I be a graded δ-primary hyperideal of a strongly distributive graded multiplicative hyperring R. Then*

(i) $(I : K) = I$ *for each graded hyperideal K of $\mathcal{I}^{gr}(R)$ with $K \nsubseteq \delta(I)$.*

(ii) $(I : H)$ *is a graded δ-primary hyperideal of R for each graded hyperideal H of R.*

Proof. (*i*) Let $r \in I$. Then $r \circ K \subseteq I$ since I is a hyperideal, so $I \subseteq (I : K)$. Conversely, consider $(I : K) \circ K$. Then $(I : K) \circ K = \bigcup_{r \in (I:K), x \in K} (r \circ x) \subseteq I$. Since I is a graded δ-primary hyperideal of R and $K \nsubseteq \delta(I)$, we get $(I : K) \subseteq I$ by Proposition 3.12.
(*ii*) By Lemma 3.13, $(I : H)$ is a graded hyperideal of R. Let $a_g \circ b_h \subseteq (I : H)$ and $a_g \notin (I : H)$ for some $a_g, b_h \in h(R)$. Hence there exists $h_k \in H \cap h(R)$ such that $a_g \circ h_k \nsubseteq I$. Thus $a_g \circ b_h \circ h_k = a_g \circ h_k \circ b_h \subseteq I$ and $a_a \circ h_k \nsubseteq I$, that is, we get $\langle a_g \circ h_k \rangle \circ \langle b_h \rangle \subseteq I$ and $\langle a_a \circ h_k \rangle \nsubseteq I$. Hence $\langle b_h \rangle \subseteq \delta(I) \subseteq \delta(I : H)$. Therefore $b_h \in \delta(I : H)$. □

Theorem 3.15. *If I is a graded δ-primary C^{gr}-hyperideal of a graded multiplicative hyperring R with $Grad(\delta(I)) = \delta(I)$, then $Grad(I)$ is a graded δ-primary C^{gr}-hyperideal of R.*

Proof. Notice that $D(I) = Grad(I)$ because I is a C^{gr}-hyperideal of R ([22]). Let $a_g \circ b_h \subseteq Grad(I)$ and $a \notin Grad(I)$ where $a_g, b_h \in h(R)$. Hence $a_g^n \circ b_h^n = (a_g \circ b_h)^n \subseteq I$ for some positive integer n and $a_g^m \nsubseteq I$ for each positive integer m. By assumption and $a_g^{mn} \circ b_h^{mn} \subseteq I$, we obtain $b_h^{mn} \subseteq \delta(I)$. Hence $b_h \in Grad(\delta(I)) \subseteq \delta(Grad(I))$, and so $Grad(I)$ is a graded δ-primary hyperideal of R. □

Definition 3.16. *If δ holds $\delta(I \cap J) = \delta(I) \cap \delta(J)$ for every $I, J \in \mathcal{I}^{gr}(R)$, we say that δ has the property of intersection preserving.*

Theorem 3.17. *Let δ has the property of intersection preserving. If I_i is a graded δ-primary hyperideal of a graded multiplicative hyperring R and $\delta(I_i) = P$ for all $i \in \{1, 2, \cdots, n\}$. Then $I = \bigcap_{i=1}^{n} I_i$ is so.*

Proof. Let $a_g \circ b_h \subseteq I$ and $a_g \notin I$ for some $a_g, b_h \in h(R)$. Hence $a_g \notin I_j$ for some $j \in \{1, 2, \cdots, n\}$. Thus $b_h \in \delta(I_j) = P$ and $\delta(I) = \delta(\bigcap_{i=1}^{n} I_i) = \delta(I_1) \cap \cdots \delta(I_n) = P$. Therefore $b_h \in \delta(I)$, so $I = \bigcap_{i=1}^{n} I_i$ is a graded δ-primary hyperideal of R. □

Definition 3.18. *Let R, S be graded multiplicative hyperrings and $f : R \to S$ be a graded good homomorphism. Let δ and γ be expansion functions of $\mathcal{I}^{gr}(R)$ and $\mathcal{I}^{gr}(S)$, respectively. Then f is called a $\delta\gamma$-homomorphism if $\delta(f^{-1}(J)) = f^{-1}(\gamma(J))$ for each graded hyperideal J of S.*

Consider the expansion function γ_1 of $\mathcal{I}^{gr}(S)$ and δ_1 of $\mathcal{I}^{gr}(R)$ defined in a similar manner of Example 3.3 (2). It is seen that each graded homomorphism from R to S is an example of $\delta_1\gamma_1$-homomorphism. If every graded hyperideal of R is a C^{gr}-hyperideal, any graded homomorphism from R to S is a $\delta_2\gamma_2$-homomorphism where the graded radical operations γ_2 of $\mathcal{I}^{gr}(S)$ and δ_2 of $\mathcal{I}^{gr}(R)$ (see Example 3.3 (3)). Also, note that $\gamma(f(I)) = f(\delta(I))$ where f is a $\delta\gamma$-epimorphism and $I \in \mathcal{I}^{gr}(R)$ with $Ker(f) \subseteq I$.

Theorem 3.19. *Let R, S be graded multiplicative hyperrings and $f : R \to S$ be a $\delta\gamma$-homomorphism. Then the followings hold:*

(i) If J is a graded γ-primary hyperideal of S, then $f^{-1}(J)$ is a graded δ-primary hyperideal of R.

(ii) Let f be a graded epimorphism and $I \in \mathcal{I}^{gr}(R)$ with $Ker(f) \subseteq I$. Then I is a graded δ-primary hyperideal of R if and only if $f(I)$ is a graded γ-primary hyperideal of S.

Proof. (*i*) By [22], $f^{-1}(J)$ is a proper graded hyperideal of R. Let $a_g \circ b_h \subseteq f^{-1}(J)$ for each $a_g, b_h \in h(R)$. We have $f(a_g \circ b_h) = f(a_g) \circ f(b_h) \subseteq J$. Since J is a graded γ-primary hyperideal of S, we obtain that $f(a_g) \in J$ or $f(b_h) \in \gamma(J)$. Hence $a_g \in f^{-1}(J)$ or $b_h \in f^{-1}(\gamma(J))$, so by assumption, $a_g \in f^{-1}(J)$ or $b_h \in \delta(f^{-1}(J))$. Thus $f^{-1}(J)$ is a graded δ-primary hyperideal of R.
(*ii*) Let I be a graded δ-primary hyperideal of R. Assume that $x_g \circ y_h \subseteq f(I)$ with $x_g, y_h \in h(S)$. Since f is a graded epimorphism, $x_g = f(a_g)$ and $y_h = f(b_h)$ for some $a_g, b_h \in h(R)$. Hence $f(a_g \circ b_h) = f(a_g) \circ f(b_h) \subseteq f(I)$. We show that $a_g \circ b_h \subseteq I$. Let $t \in a_g \circ b_h$. Then $f(t) \in f(a_g \circ b_h) \subseteq f(I)$ and so $f(t) = f(x)$ for some $x \in I$. This implies that $f(t - x) = f(t) - f(x) = 0 \in \langle 0 \rangle$, that is, $t - x \in Ker(f) \subseteq I$ and so $t \in I$. Thus $a_g \circ b_h \subseteq I$. Since I is a graded δ-primary hyperideal of R, we have $a_g \in I$ or $b_h \in \delta(I)$ and so $f(a_g) \in f(I)$ or $f(b_h) \in f(\delta(I)) = \gamma(f(I))$ by assumption. Consequently, $f(I)$ is a graded γ-primary hyperideal of S. Conversely, Let $a_g \circ b_h \subseteq I$ where $a_g, b_h \in h(R)$. Hence $f(a_g \circ b_h) = f(a_g) \circ f(b_h) \subseteq f(I)$. Since $f(I)$ is a graded γ-primary hyperideal of S, then $f(a_g) \in f(I)$ or $f(b_h) \in \gamma(f(I)) = f(\delta(I))$. Hence $a_g \in f^{-1}(f(I)) \subseteq I$ or $b_h \in f^{-1}(f(\delta(I))) \subseteq \delta(I)$, as needed. □

Suppose that I is a graded hyperideal of a graded multiplicative hyperring $R = \bigoplus_{g \in G} R_g$. Then quotient group $R/I = \{a + I : a \in R\}$ becomes a multiplicative hyperring with the multiplication $(a + I) \circ (b + I) = \{r + I : r \in a \circ b\}$. One can easily prove that R/I is a graded hyperring with $R/I = \bigoplus_{g \in G}(R/I)_g$ where for all $g \in G$, $(R/I)_g = (R_g + I)/I$. Also, all graded hyperideals of R/I are of the form J/I, where J is a graded hyperideal of R containing I since the natural graded homomorphism $\phi : R \to R/I$ is a graded good epimorphism ([22]).

Let δ be an expansion function of $\mathcal{I}^{gr}(R)$ and $I \in \mathcal{I}^{gr}(R)$. Let the function $\delta_q : R/I \to R/I$ be defined by $\delta_q(J/I) = \delta(J)/I$ for all graded hyperideals J ($I \subseteq J$) of R. Note that δ_q is an expansion function of $\mathcal{I}^{gr}(R/I)$.

Proposition 3.20. *Let I and J be graded hyperideals of a graded multiplicative hyperring R with $I \subseteq J$. Then J is a graded δ-primary hyperideal of R if and only if J/I is a graded δ_q-primary hyperideal of the graded quotient hyperring R/I.*

Proof. Let $(a_g + I) \circ (b_h + I) \subseteq J/I$ for each $a_g + I, b_h + I \in h(R/I)$. Thus $a_g \circ b_h \subseteq J$, because if $r \in a_g \circ b_h$, then $r + I \in (a_g + I) \circ (b_h + I)$, and so $r \in J$. Hence $a_g \in J$ or $b_h \in \delta(J)$ since J is a graded δ-primary hyperideal of R. Therefore, $a_g + I \in J/I$ or $b_h + I \in \delta(J)/I = \delta_q(J/I)$. Conversely, Let $a_g \circ b_h \subseteq J$. Then we can see $(a_g + I) \circ (b_h + I) \subseteq J/I$. Hence $a_g + I \in J/I$ or $b_h + I \in \delta_q(J/I) = \delta(J)/I$. Therefore, $a_g \in J$ or $b_h \in \delta(J)$, as required. □

Definition 3.21. *Let $(R, +, \circ)$ be a graded multiplicative hyperring.*

(i) A homogeneous element $r_g \in h(R)$ is defined as zero divisor if there is a homogeneous element $0 \neq r'_h \in h(R)$ such that $r_g \circ r'_h = \{0\}$.

(ii) A homogeneous element $r_g \in h(R)$ is a δ-nilpotent if $r_g \in \delta(0)$.

Theorem 3.22. *A graded hyperideal I of a graded multiplicative hyperring R is a graded δ-primary hyperideal of R if and only if every zero divisor of the graded quotient hyperring R/I is a δ_q-nilpotent.*

Proof. ($\Rightarrow$) Let $I \in \mathcal{I}^{gr}(R)$ is a graded δ-primary hyperideal. Assume that $r_g + I \in h(R/I)$ is a zero divisor element of R/I. Then there exists $I \neq r'_h + I \in h(R/I)$ such that $I = (r_g + I) \circ (r'_h + I)$. As the result of $I = r_g \circ r'_h + I$, we have $r_g \circ r'_h \subseteq I$. Since I is a graded δ-primary, $r_g \circ r'_h \subseteq I$ and $r'_h \notin I$, we conclude $r_g \in I$. Consider the expansion function δ_q of $\mathcal{I}^{gr}(R/I)$ and the natural homomorphism $\pi : R \to R/I$. We obtain that π is a $\delta\delta_q$-epimorphism. Thus we have $\delta(I) = \delta(\pi^{-1}(0_{R/I})) = \pi^{-1}(\delta_q(I))$. Note that $r_g + I \in \delta(I)/I = \pi(\delta(I)) = \delta_q(0_{R/I})$. Hence $r_g + I \in \delta_q(0_{R/I})$.
($\Leftarrow$) Let every zero divisor of R/I be a δ_q-nilpotent. Let $r_g \circ r'_h \subseteq I$ and $r_g \notin I$ for $r_g, r'_h \in h(R)$. Then $r'_h + I$ is a zero divisor element of R/I as $r_g \circ r'_h + I = (r_g + I) \circ (r'_h + I) = I$ and $r_g + I \neq I$. By assumption, we get $r'_h + I \in \delta_q(0_{R/I}) = \delta(I)/I$. Consequently, $r'_h \in \delta(I)$. □

Theorem 3.23. *Let I be a graded δ-primary hyperideal of a graded multiplicative hyperring R and $I_1, I_2, \cdots, I_n \in \mathcal{I}^{gr}_*(R)$ with $\bigcap_{i=1}^n I_i \subseteq I$. Then $I_i \subseteq \delta(I)$ for some $i \in \{1, \cdots, n\}$. If $\bigcap_{i=1}^n I_i = I$ and $\delta(\delta(J)) = \delta(J)$ for each $J \in \mathcal{I}^{gr}(R)$, then $\delta(I_i) = \delta(I)$ for some $i \in \{1, \cdots, n\}$.*

Proof. Suppose that $I_i \nsubseteq \delta(I)$ for every $i \in \{1, \cdots, n\}$. Then there exist elements $x_1, \cdots, x_n \in h(R)$ with $x_i \in I_i - \delta(I)$. We get $x_1 \circ \cdots \circ x_n \subseteq I_i$ for every i and so $x_1 \circ \cdots \circ x_n \subseteq \bigcap_{i=1}^n I_i \subseteq I$. Since I is δ-primary and $x_1, \cdots, x_n \notin \delta(I)$, then $x_i \in I \subseteq \delta(I)$ for each $i \in \{1, \cdots, n\}$, which is a contradiction. Let $\bigcap_{i=1}^n I_i = I$. Then $\delta(I_i) = \delta(I)$ since $I \subseteq I_i$, and $\delta(I) \subseteq \delta(I_i)$. □

Let G be an abelian group. Let $(R_1, +_1, \circ_1)$ and $(R_2, +_2, \circ_2)$ be two graded multiplicative hyperrings where $R_1 = \bigoplus_{g \in G}(R_1)_g$ and $R_2 = \bigoplus_{g \in G}(R_2)_g$. Then $(R = R_1 \times R_2, +, \circ)$ is a multiplicative hyperring with operation $+$ and the hyperoperation $\circ$ are defined respectively as $(x, y) + (z, t) = (x +_1 z, y +_2 t)$ and $(x, y) \circ (z, t) = \{(a, b) \in R \mid a \in x \circ_1 z\ ,\ b \in y \circ t\}$ for all $(x, y), (z, t) \in R$. Also, $(R = R_1 \times R_2, +, \circ)$ becomes a G-graded hyperring with homogeneous elements $h(R) = \bigcup_{g \in G} R_g$, where $R_g = (R_1)_g \times (R_2)_g$ for all $g \in G$. Note that each graded hyperideal of R is the Cartesian product of graded hyperideals of R_1 and R_2. Suppose that δ_1 and δ_2 are expansion functions of graded hyperideals of R_1 and R_2, respectively. Let δ_R be a function of graded hyperideals of R with $\delta(I_1 \times I_2) = \delta_1(I_1) \times \delta_2(I_2)$ for every graded hyperideal I_i of R_i for $i \in \{1, 2\}$. It is seen that the function δ_R is an expansion function of graded hyperideals of R.

Theorem 3.24. *Let $(R_1, +_1, \circ_1)$ and $(R_2, +_2, \circ_2)$ be two graded multiplicative hyperrings and δ_1 and δ_2 be expansion functions of hyperideals of R_1 and R_2, respectively. Let $I_1 \in \mathcal{I}^{gr}_*(R_1)$, $I_2 \in \mathcal{I}^{gr}_*(R_2)$ and $R = (R_1 \times R_2, \circ, +)$. Then the followings hold:*

(i) I_1 is a graded δ_1-primary hyperideal of R_1 if and only if $I_1 \times R_2$ is a graded δ_R-primary hyperideal of R.

(ii) I_2 is a graded δ_2-primary hyperideal of R_2 if and only if $R_1 \times I_2$ is a graded δ_R-primary hyperideal of R.

Proof. (*i*) ($\Rightarrow$) Let $(x_g, y_g), (z_h, t_h) \in h(R)$ with $(x_g, y_g) \circ (z_h, t_h) \subseteq I_1 \times R_2$. Hence we get $x_g \circ_1 z_h \subseteq I_1$. Thus $x_g \in I_1$ or $z_g \in \delta_1(I_1)$ and so $(x_g, y_g) \in I_1 \times R_2$ or $(z_h, t_h) \in \delta_R(I_1 \times R_2)$.
($\Leftarrow$) Let I_1 be not a graded δ_1-primary hyperideal of R_1. So we have $x_g, y_h \in h(R_1)$ with $x_g \circ_1 y_h \subseteq I_1$, $x_g \notin I_1$ and $y_h \notin \delta_1(I_1)$. Note that $(x_g, 0_{R_2}) \circ (y_h, 0_{R_2}) \subseteq I_1 \times R_2$. By assumption, $(x_g, 0_{R_2}) \in I_1 \times R_2$ or $(h_h, 0_{R_2}) \in \delta_R(I_1 \times R_2)$. It means $x_g \in I_1$ or $y_h \in \delta_1(I_1)$, a contradiction. Therefore, I_1 is a graded δ_1-primary hyperideal of R_1.
(*ii*) The proof is similar to (*i*). □

4. On expansion of graded 2-absorbing hyperideals

Definition 4.1. (a) *A proper graded hyperideal I of a graded multiplicative hyperring R is called a graded 2-absorbing hyperideal, if $a_g, b_h, c_k \in h(R)$ and $a_g \circ b_h \circ c_k \subseteq I$, then $a_g \circ b_h \subseteq I$ or $a_g \circ c_k \subseteq I$ or $b_h \circ c_k \subseteq I$.*
(b) *A proper graded hyperideal I of a graded multiplicative hyperring R is called a graded 2-absorbing primary hyperideal, if $a_g, b_h, c_k \in h(R)$ and $a_g \circ b_h \circ c_k \subseteq I$, then $a_g \circ b_h \subseteq I$ or $a_g \circ c_k \subseteq Grad(I)$ or $b_h \circ c_k \subseteq Grad(I)$.*

Definition 4.2. *Let δ be an expansion function of $\mathcal{I}^{gr}(R)$ and $I \in \mathcal{I}^{gr}_*(R)$. I is called a graded 2-absorbing δ-primary hyperideal of R if $a_g, b_h, c_k \in h(R)$ and $a_g \circ b_h \circ c_k \subseteq I$, then $a_g \circ b_h \subseteq I$ or $a_g \circ c_k \subseteq \delta(I)$ or $b_h \circ c_k \subseteq \delta(I)$.*

Remark 4.3.
1. *Every graded δ-primary hyperideal of a graded multiplicative hyperring R is a graded 2-absorbing δ-primary hyperideal of R.*
2. *I is a graded 2-absorbing δ_0-primary hyperideal if and only if I is a graded 2-absorbing hyperideal.*
3. *I is a graded 2-absorbing δ_2-primary hyperideal if and only if I is a graded 2-absorbing primary hyperideal.*

Example 4.4. *Let $R_A = (\mathbb{R}, +, \circ)$ where $A = \{-3, 5, 6\}$ and $G = (\mathbb{Z}, +)$ be the integers group. Consider the $\mathbb{Z}$-graded multiplicative polynomial hyperring $S = (R_A[x, y, z], +, *)$. Let $I = \langle xy \rangle = \langle x \rangle \bigcap \langle y \rangle$, which is intersection of two graded prime hyperideals, is a graded 2-absorbing δ-primary hyperideal for any expansion function δ of R.*

Example 4.5. *In the graded multiplicative hyperring $R_A = \mathbb{Z}[i]$ with $A = \{2, 3\}$, the graded hyperideal $J = \langle 6 \rangle \oplus \langle 0 \rangle$ of R is a graded 2-absorbing δ_2-primary hyperideal, but it is not a graded δ_2-primary hyperideal. Since, for all $\alpha \in A$ we have $(2, 0) \circ (3, 0) = (2, 0) \cdot \alpha \cdot (3, 0) = \{(12, 0), (18, 0)\} \subseteq J$ but $(2, 0) \notin J$ and $(3, 0) \notin \delta(J)$. This example shows that a graded 2-absorbing δ_2-primary hyperideal of a graded multiplicative hyperring R is not necessarily a graded δ_2-primary hyperideal of R.*

Theorem 4.6. *Let R be a graded multiplicative hyperring. Then the following statements hold:*

(i) Let γ be an expansion function of $\mathcal{I}^{gr}(R)$ satisfied $\delta(I) \subseteq \gamma(I)$ for each $I \in \mathcal{I}^{gr}(R)$. Then every graded 2-absorbing δ-primary hyperideal of R is a graded 2-absorbing γ-primary. Additionally, every graded 2-absorbing hyperideal is a graded 2-absorbing δ-primary hyperideal since $I \subseteq \delta(I)$ for each expansion function δ of $\mathcal{I}^{gr}(R)$.

(ii) Let I be a graded 2-absorbing primary hyperideal of R and $\delta(I)$ be a graded radical hyperideal (i. e. $Grad(\delta(I)) = \delta(I)$). Then I is a graded 2-absorbing δ-primary hyperideal of R.

Proof. (i) It is clear by assumption.
(ii) Let $a_g \circ b_h \circ c_k \subseteq I$ where $a_g, b_h, c_k \in h(R)$. Hence $a_g \circ b_h \subseteq I$ or $a_g \circ c_k \subseteq Grad(I)$ or $b_h \circ c_k \subseteq Grad(I)$ by assumption. We have $Grad(I) \subseteq Grad(\delta(I))$ because $I \subseteq \delta(I)$. Thus $a_g \circ b_h \subseteq I$ or $a_g \circ c_k \subseteq Grad(\delta(I))$ or $b_h \circ c_k \subseteq Grad(\delta(I))$. Since $Grad(\delta(I)) = \delta(I)$, we have $a_g \circ b_h \subseteq I$ or $a_g \circ c_k \subseteq \delta(I)$ or $b_h \circ c_k \subseteq \delta(I)$. □

Theorem 4.7. *If $\delta(I)$ be a graded prime hyperideal of a graded multiplicative hyperring R, then I is a graded 2-absorbing δ-primary hyperideal of R.*

Proof. Let $a_g \circ b_h \circ c_k \subseteq I$ and $a_g \circ b_h \nsubseteq I$ where $a_g, b_h, c_k \in h(R)$. Let us consider two situations. Firstly, let $a_g \circ b_h \nsubseteq \delta(I)$. Thus $c_k \in \delta(I)$ since $\delta(I)$ is a graded prime hyperideal. Therefore, $a_g \circ c_k \subseteq \delta(I)$ and $b_h \circ c_k \subseteq \delta(I)$. Secondary, take $a_g \circ b_h \subseteq \delta(I)$. By assumption, we get $a_g \in \delta(I)$ or $b_h \in \delta(I)$. Hence $a_g \circ c_k \subseteq \delta(I)$ or $b_h \circ c_k \subseteq \delta(I)$, as needed. □

Theorem 4.8. *Let I be a graded 2-absorbing δ-primary C^{gr}-hyperideal of a graded multiplicative hyperring R with $Grad(\delta(I)) \subseteq \delta(Grad(I))$. Then $Grad(I)$ is a graded 2-absorbing δ-primary C^{gr}-hyperideal of R.*

Proof. It can be proved in a similar manner to Theorem 3.15. □

Theorem 4.9. *Let I, K and L be proper graded hyperideals of a graded multiplicative hyperring R with $L \subseteq K \subseteq I$. If I is a graded δ-primary hyperideal of R such that $\delta(I) = \delta(L)$, then K is a graded 2-absorbing δ-primary hyperideal of R.*

Proof. Let $a_g \circ b_h \circ c_k \subseteq K$ and $a_g \circ b_h \nsubseteq K$ where $a_g, b_h, c_k \in h(R)$. We get two cases as $K \subseteq I$. The first case: Let $a_g \circ b_h \nsubseteq I$. Then $c_k \in \delta(I) = \delta(L) \subseteq \delta(K)$ with our assumption. Thus $a_g \circ c_k \subseteq \delta(K)$ and $b_h \circ c_k \subseteq \delta(K)$. The second case: Let $a_g \circ b_h \subseteq I$. It means $a_g \in I \subseteq \delta(K)$ or $b_h \in \delta(I) = \delta(L) \subseteq \delta(K)$ by assumption. Hence $a_g \circ c_k \subseteq \delta(K)$ and $b_h \circ c_k \subseteq \delta(K)$. In the both cases, we obtain that K is a graded 2-absorbing δ-primary hyperideal of R. □

Corollary 4.10. *Let I be a graded δ-primary hyperideal of R and $K \in \mathcal{I}^{gr}(R)$ with $K \subseteq I$ and $\delta(I) = \delta(K)$. Then K is a graded 2-absorbing δ-primary hyperideal of R.*

Proof. The proof holds by Theorem 4.9. □

Theorem 4.11. *Let δ and η be two expansion functions of $\mathcal{I}^{gr}(R)$ and $I \in \mathcal{I}^{gr}(R)$. If $\eta(I)$ is a graded prime hyperideal of R, then I is a graded 2-absorbing $\delta \circ \eta$-primary hyperideal of R.*

Proof. Let $a_g \circ b_h \circ c_k \subseteq I$ and $a_g \circ b_h \nsubseteq I$ where $a_g, b_h, c_k \in h(R)$. We consider two cases. Case 1: $a_g \circ b_h \nsubseteq \eta(I)$. Then $c_k \in \eta(I)$, and so $c_k \in \delta(\eta(I))$ since $\eta(I) \subseteq \delta(\eta(I))$ ($\eta(I) \in \mathcal{I}^{gr}(R)$). Thus $a_g \circ c_k \subseteq \delta \circ \eta(I)$ and $b_h \circ c_k \subseteq \delta \circ \eta(I)$. Case 2: $a_g \circ b_h \subseteq \eta(I)$. Hence $a_g \in \eta(I)$ or $b_h \in \eta(I)$ since $\eta(I)$ is graded prime. Therefore, $a_g \circ c_k \subseteq \eta(I) \subseteq \delta(\eta(I))$ or $b_h \circ c_k \subseteq \delta(\eta(I))$. Therefore, I is a graded 2-absorbing $\delta \circ \eta$-primary hyperideal of R. □

Theorem 4.12. *Let δ be an expansion function of $\mathcal{I}^{gr}(R)$ and I, J be graded δ-primary hyperideals of R with $\delta(I \cap J) = \delta(I) \cap \delta(J)$. Then $I \cap J$ is a graded 2-absorbing δ-primary hyperideal of R.*

Proof. Let $a_g \circ b_h \circ c_k \subseteq I \cap J$ and $a_g \circ b_h \nsubseteq I \cap J$ where $a_g, b_h, c_k \in h(R)$. Thus it means $a_g \circ b_h \nsubseteq I$ or $a_g \circ b_h \nsubseteq J$. Hence we consider the following cases:

Case 1: $a_g \circ b_h \subseteq I$ and $a_g \circ b_h \nsubseteq J$. Since $a_g \circ b_h \nsubseteq J$, there exists $r_{gh} \in a_g \circ b_h$ such that $r_{gh} \notin J$. Since $r_{gh} \circ c_k \subseteq J$ and $r_{gh} \notin J$, then $c_k \in \delta(I)$. Hence $a_g \circ c_k \subseteq \delta(J)$ and $b_h \circ c_k \subseteq \delta(J)$ since $\delta(I)$ is a hyperideal of R. Also, $a_g \in I \subseteq \delta(I)$ or $b_h \in \delta(I)$ as $a_g \circ b_h \subseteq I$ and I is graded δ-primary. Hence $a_g \circ c_k \subseteq \delta(I)$ or $b_h \circ c_k \subseteq \delta(I)$. Then we obtain $a_g \circ c_k \subseteq \delta(I) \cap \delta(J) = \delta(I \cap J)$ or $b_h \circ c_k \subseteq \delta(I) \cap \delta(J) = \delta(I \cap J)$.

Case 2: Let $a_g \circ b_h \nsubseteq I$ and $a_g \circ b_h \subseteq J$. Then the proof holds by a similar way to the proof of Case 1.

Case 3: Let $a_g \circ b_h \nsubseteq I$ and $a_g \circ b_h \nsubseteq J$. We have homogeneous elements $r_{gh}, s_{gh} \in a_g \circ b_h$ with $r_{gh} \notin I$ and $s_{gh} \notin J$. Thus we have $r_{gh} \circ c_k \subseteq I$ and $s_{gh} \circ c_k \subseteq J$. Hence $c_k \in \delta(I)$ and $c_k \in \delta(J)$ by our assumption. Consequently, $a_g \circ c_k \subseteq \delta(I) \cap \delta(J) = \delta(I \cap J)$ or $b_h \circ c_k \subseteq \delta(I) \cap \delta(J) = \delta(I \cap J)$. □

Theorem 4.13. *Let δ has the property of intersection preserving and $K = I \cap J$ for some graded δ-primary hyperideals I and J of R. Then K is a graded 2-absorbing δ-primary hyperideal of R.*

Proof. It is clear by Theorem 4.12. □

Theorem 4.14. *Let R, S be graded multiplicative hyperrings and $f : R \to S$ be a $\delta\gamma$-homomorphism. Then the followings hold:*

(i) If J is a graded 2-absorbing γ-primary hyperideal of S, then $f^{-1}(J)$ is a graded 2-absorbing δ-primary hyperideal of R.

(ii) Let f be a graded epimorphism and $I \in \mathcal{I}^{gr}(R)$ with $Ker(f) \subseteq I$. Then I is a graded 2-absorbing δ-primary hyperideal of R if and only if $f(I)$ is a graded 2-absorbing γ-primary hyperideal of S.

Proof. (i) Let $a_g \circ b_h \circ c_k \subseteq f^{-1}(J)$ for each $a_g, b_h, c_k \in h(R)$. We have $f(a_g \circ b_h \circ c_k) = f(a_g) \circ f(b_h) \circ f(c_k) \subseteq J$. Since J is a graded 2-absorbing γ-primary hyperideal of S, we obtain that $f(a_g) \circ f(b_h) = f(a_g \circ b_h) \subseteq J$ or $f(a_g) \circ f(c_k) = f(a_g \circ c_k) \subseteq \gamma(J)$ or $f(b_h) \circ f(c_k) = f(b_h \circ c_k) \subseteq \gamma(J)$. Hence $a_g \circ b_h \subseteq f^{-1}(J)$ or $a_g \circ c_k \subseteq f^{-1}(\gamma(J))$ or $b_h \circ c_k \subseteq f^{-1}(\gamma(J))$, so by assumption, $a_g \circ b_h \subseteq f^{-1}(J)$ or $a_g \circ c_k \subseteq \delta(f^{-1}(J))$ or $b_h \circ c_k \subseteq \delta(f^{-1}(J))$. Thus $f^{-1}(J)$ is a graded 2-absorbing δ-primary hyperideal of R.

(ii) Let I be a graded δ-primary hyperideal of R. Assume that $x_g \circ y_h \circ z_k \subseteq f(I)$ with $x_g, y_h, z_k \in h(S)$. Since f is a graded epimorphism, $x_g = f(a_g)$, $y_h = f(b_h)$ and $z_k = f(c_k)$ for some $a_g, b_h, c_k \in h(R)$. Hence $f(a_g) \circ f(b_h) \circ f(c_k) = f(a_g) \circ b_h \circ c_k) \subseteq f(I)$. We show that $a_g \circ b_h \circ c_k \subseteq I$. Let $t \in a_g \circ b_h \circ c_k$. Then

$f(t) \in f(a_g \circ b_h \circ c_k) \subseteq f(I)$ and so $f(t) = f(x)$ for some $x \in I$. This implies that $f(t - x) = f(t) - f(x) = 0 \in \langle 0 \rangle$, that is, $t - x \in Ker(f) \subseteq I$ and so $t \in I$. Thus $a_g \circ b_h \circ c_k \subseteq I$. Since I is a graded 2-absorbing δ-primary hyperideal of R, we have $a_g \circ b_h \subseteq I$ or $a_g \circ c_k \subseteq \delta(I)$ or $b_h \circ c_k \subseteq \delta(I)$ and so $f(a_g \circ b_h) \subseteq f(I)$ or $f(a_g \circ c_k) \subseteq f(\delta(I)) = \gamma(f(I))$ or $f(b_h \circ c_k) \subseteq f(\delta(I)) = \gamma(f(I))$ by assumption. Consequently, $f(I)$ is a graded 2-absorbing γ-primary hyperideal of S. The converse part is verified from (i). □

Corollary 4.15. *Let I and J be graded hyperideals of a graded multiplicative hyperring R with $I \subseteq J$. Then J is a graded 2-absorbing δ-primary hyperideal of R if and only if J/I is a graded 2-absorbing δ_q-primary hyperideal of the graded quotient hyperring R/I.*

Proof. The proof is completely straightforward. □

Proposition 4.16. *Let R be a graded multiplicative hyperring and $I, J, K \in \mathcal{I}^{gr}_*(R)$. If $I \subseteq J \cup K$, then $I \subseteq J$ or $I \subseteq K$.*

Proof. Let $I \subseteq J \cup K$, $I \nsubseteq J$ and $I \nsubseteq K$. There exist $a, b \in R$ such that $a \in I - J$ and $b \in I - K$. Then $a - b \in I$. Thus $a \in J$ or $b \in K$, which is a contradiction. □

Theorem 4.17. *Let R be a graded multiplicative hyperring and $I = \bigoplus_{g \in G} I_g = \bigoplus_{g \in G}(I \cap R_g)$ a graded hyperideal of R. The following statements are equivalent:*

(i) I is a graded 2-absorbing δ-primary hyperideal of R.

(ii) $(I_k :_R a_g \circ b_h) \subseteq (\delta(I) \cap R_{kh^{-1}} :_R a_g) \cup (I_{kg^{-1}} :_R b_h)$ for $a_g, b_h \in h(R)$ such that $a_g \circ b_h \nsubseteq \delta(I)$.

(iii) $(I_k :_R a_g \circ b_h) \subseteq (\delta(I) \cap R_{kh^{-1}} :_R a_g)$ or $(I_k :_R a_g \circ b_h) = (I_{kg^{-1}} :_R b_h)$ for $a_g, b_h \in h(R)$ such that $a_g \circ b_h \nsubseteq \delta(I)$.

Proof. $(i) \Rightarrow (ii)$ Let $x_{g^{-1}h^{-1}k} \in (I_k : a_g \circ b_h)$. Then $a_g \circ b_h \circ x_{g^{-1}h^{-1}k} \subseteq I_k \subseteq I$. Thus $b_h \circ x_{g^{-1}h^{-1}k} \subseteq I$ or $a_g \circ x_{g^{-1}h^{-1}k} \subseteq \delta(I)$ since $a_g \circ b_h \nsubseteq \delta(I)$ and I is a graded 2-absorbing δ-primary hyperideal. Therefore, $x_{g^{-1}h^{-1}k} \in (I_{kg^{-1}} : b_h)$ or $x_{g^{-1}h^{-1}k} \in (\delta(I) \cap R_{kh^{-1}} : a_g)$, that is, $x_{g^{-1}h^{-1}k} \in (I_{kg^{-1}} : b_h) \cup (\delta(I) \cap R_{kh^{-1}} : a_g)$.
$(ii) \Rightarrow (i)$ Assume that $a_g \circ b_h \circ c_k \subseteq I$, $a_g \circ b_h \nsubseteq \delta(I)$ and $b_h \circ c_k \nsubseteq I$ for each $a_g, b_h, c_k \in h(R)$. Then we have $c_k \in (I_{kgh} : a_g \circ b_h)$. Hence $c_k \in (I_{kh} : b_h) \cup (\delta(I) \cap R_{kg} : a_g)$. Since $b_h \circ c_k \nsubseteq I$, then we obtain $a_g \circ c_k \subseteq \delta(I)$. Thus I is a graded 2-absorbing δ-primary hyperideal of R.
$(ii) \Leftrightarrow (iii)$ It is clear from Proposition 4.16 and $(I_{kg^{-1}} : b_h) \subseteq (I_k : a_g \circ b_h)$. □

Lemma 4.18. *Let I be a graded 2-absorbing δ-primary hyperideal of a graded multiplicative hyperring $R = \bigoplus_{g \in G} R_g$. Let $k \in G$ and J_k be a subgroup of R_k. If $a_g \circ b_h \circ J_k \subseteq I$ and $a_g \circ b_h \nsubseteq I$ for $a_g, b_h \in h(R)$, then $a_g \circ J_k \subseteq \delta(I)$ or $b_h \circ J_k \subseteq \delta(I)$.*

Proof. Suppose that $a_g \circ J_k \nsubseteq \delta(I)$ and $b_h \circ J_k \nsubseteq \delta(I)$. Since $a_g \circ J_k = \bigcup_{j_k \in J_k} a_g \circ j_k \nsubseteq \delta(I)$ and $b_h \circ J_k = \bigcup_{j_k \in J_k} b_h \circ j_k \nsubseteq \delta(I)$. Hence there exist $c_k, d_k \in J_k$ such that $a_g \circ c_k \nsubseteq \delta(I)$ and $b_h \circ d_k \nsubseteq \delta(I)$. Since $a_g \circ b_h \circ c_k \subseteq I$, $a_g \circ b_h \nsubseteq I$, $a_g \circ c_k \nsubseteq \delta(I)$ and I is a graded 2-absorbing δ-primary hyperideal of R, then $b_h \circ c_k \subseteq \delta(I)$. Similarly, Since $a_g \circ b_h \circ d_k \subseteq I$, $a_g \circ b_h \nsubseteq I$, $b_h \circ d_k \nsubseteq \delta(I)$ and I is a graded 2-absorbing δ-primary hyperideal of R, then $a_g \circ d_k \subseteq \delta(I)$. Now since $a_g \circ b_h \circ (c_k + d_k) \subseteq I$, $a_g \circ b_h \nsubseteq I$ and I is a graded 2-absorbing δ-primary hyperideal of R, then $a_g \circ (c_k + d_k) \subseteq \delta(I)$ or $b_h \circ (c_k + d_k) \subseteq \delta(I)$. If $a_g \circ (c_k + d_k) \subseteq \delta(I)$, then $a_g \circ c_k = a_g \circ (c_k + d_k - d_k) \subseteq a_g \circ (c_k + d_k) - a_g \circ d_k \subseteq \delta(I)$ since $a_g \circ d_k \subseteq \delta(I)$, which is a contradiction. Similarly, let $b_h \circ (c_k + d_k) \subseteq \delta(I)$. Then $b_h \circ d_k = b_h \circ (c_k + d_k - c_k) \subseteq b_h \circ (c_k + d_k) - b_h \circ c_k \subseteq \delta(I)$ as $b_h \circ c_k \subseteq \delta(I)$, which is a contradiction. Thus $a_g \circ J_k \subseteq \delta(I)$ or $b_h \circ J_k \subseteq \delta(I)$. □

Theorem 4.19. *Let I be a graded hyperideal of a graded multiplicative hyrperring R. I is a graded 2-absorbing δ-primary hyperideal of R if and only if for any subgroups J_g, K_h, L_k of R_g, R_h, R_k, respectively, $J_g \circ K_h \circ L_k \subseteq I$, then $J_g \circ K_h \subseteq I$ or $J_g \circ L_k \subseteq \delta(I)$ or $K_h \circ L_k \subseteq \delta(I)$.*

Proof. Let I be a graded 2-absorbing δ-primary hyperideal of R and $J_g \circ K_h \circ L_k \subseteq I$ and $J_g \circ K_h \nsubseteq I$. We show that $J_g \circ L_k \subseteq \delta(I)$ or $K_h \circ L_k \subseteq \delta(I)$. Suppose that $J_g \circ L_k \nsubseteq \delta(I)$ and $K_h \circ L_k \nsubseteq \delta(I)$. Hence $j_g \circ L_k \nsubseteq \delta(I)$ and $k_h \circ L_k \nsubseteq \delta(I)$ for some $j_g \in J_g$ and $k_h \in K_h$. By Lemma 4.18, since $j_g \circ k_h \circ L_k \subseteq I$ but $j_g \circ L_k \nsubseteq \delta(I)$ and $k_h \circ L_k \nsubseteq \delta(I)$, we get $j_g \circ k_h \subseteq I$. Since $J_g \circ K_h \nsubseteq I$, so there exist $a_g \in J_g$ and $b_h \in K_h$ such that $a_g \circ b_h \nsubseteq I$. Since $(a_g \circ b_h) \circ L_k \subseteq J_g \circ K_h \circ L_k \subseteq I$ and $a_g \circ b_h \nsubseteq I$, by Lemma 4.18, $a_g \circ L_k \subseteq \delta(I)$ or $b_h \circ L_k \subseteq \delta(I)$.

Case 1: Suppose that $a_g \circ L_k \subseteq \delta(I)$ and $b_h \circ L_k \nsubseteq \delta(I)$. Since $(j_g \circ b_h) \circ L_k \subseteq J_g \circ K_h \circ L_k \subseteq I$, $b_h \circ L_k \nsubseteq \delta(I)$ and $j_g \circ L_k \nsubseteq \delta(I)$, we have $j_g \circ b_h \subseteq I$ by Lemma 4.18. As $a_g \circ L_k \subseteq \delta(I)$ and $j_g \circ L_k \nsubseteq \delta(I)$, it means that $(a_g + j_g) \circ L_k \nsubseteq \delta(I)$. Indeed, if $(a_g + j_g) \circ L_k \subseteq \delta(I)$, then we get $(a_g + j_g) \circ x_k \subseteq \delta(I)$ for every $x_k \in L_k$ and it is obtained $j_g \circ x_k \subseteq (a_g + j_g - a_g) \circ x_k \subseteq (a_g + j_g) \circ x_k - a_g \circ x_k \subseteq \delta(I)$, a contradiction. By Lemma 4.18, we have $(a_g + j_g) \circ b_h \subseteq I$ as $(a_g + j_g) \circ b_h \circ L_k \subseteq I$, $(a_g + j_g) \circ L_k \nsubseteq \delta(I)$ and $b_h \circ L_k \nsubseteq \delta(I)$. Then $a_g \circ b_h = (a_g + j_g - j_g) \circ b_h \subseteq (a_g + j_g) \circ b_h - (j_g \circ b_h) \subseteq I$, that is, we get $a_g \circ b_h \subseteq I$, which is a contradiction.

Case 2: Suppose that $a_g \circ L_k \nsubseteq \delta(I)$ and $b_h \circ L_k \subseteq \delta(I)$. Then $a_g \circ k_h \subseteq I$ by Lemma 4.18. As $b_h \circ L_k \subseteq \delta(I)$ and $k_h \circ L_k \nsubseteq \delta(I)$, we get $(b_h + k_h) \circ L_k \nsubseteq \delta(I)$. Indeed, if $(b_h + k_h) \circ L_k \subseteq \delta(I)$, then we get $(b_h + k_h) \circ x_k \subseteq \delta(I)$ for every $x_k \in L_k$ and it is obtained $k_h \circ x_k \subseteq (b_h + k_k - b_h) \circ x_k \subseteq (b_h + k_h) \circ x_k - b_h \circ x_k \subseteq \delta(I)$, a contradiction. By Lemma 4.18, we have $a_g \circ (b_h + k_h) \subseteq I$ as $a_g \circ (b_h + k_h) \circ L_k \subseteq I$, $(b_h + k_h) \circ L_k \nsubseteq \delta(I)$ and $a_g \circ L_k \nsubseteq \delta(I)$. Then $a_g \circ b_h = (b_h + k_h - k_h) \circ a_g \subseteq (b_h + k_h) \circ a_g - (k_h \circ a_g) \subseteq I$, that is, we get $a_g \circ b_h \subseteq I$, which is a contradiction.

Case 3: Suppose that $a_g \circ L_k \subseteq \delta(I)$ and $b_h \circ L_k \subseteq \delta(I)$. Since $b_h \circ L_k \subseteq \delta(I)$ and $k_h \circ L_k \nsubseteq \delta(I)$, we have $(b_h + k_h) \circ L_k \nsubseteq \delta(I)$. If $(b_h + k_k) \circ L_k \subseteq \delta(I)$, then we get $(b_h + k_h) \circ x_k \subseteq \delta(I)$ for every $x_k \in L_k$. Then $k_h \circ x_k = (b_h + k_h - b_h) \circ x_k \subseteq (b_h + k_h) \circ x_k - b_h \circ x_k \subseteq \delta(I)$, a contradiction. By Lemma 4.18, we get $j_g \circ (b_h + k_h) \subseteq I$ as $j_g \circ (b_h + k_h) \circ L_k \subseteq I$, $(b_h + k_h) \circ L_k \nsubseteq \delta(I)$ and $j_g \circ L_k \nsubseteq \delta(I)$. Since $j_g \circ k_h \circ L_k \subseteq I$, $j_g \circ L_k \nsubseteq \delta(I)$ and $k_h \circ L_k \nsubseteq \delta(I)$, then by Lemma 4.18, we have $j_g \circ k_h \subseteq I$. Also, it is obtained $(a_g + j_g) \circ L_k \nsubseteq \delta(I)$ as $a_g \circ L_k \subseteq \delta(I)$ and $j_g \circ L_k \nsubseteq \delta(I)$ by a similar way to the explanation in above. By Lemma 4.18, we obtain $(a_g + j_g) \circ k_h \subseteq I$ as $(a_g + j_g) \circ k_h \circ L_k \subseteq I$, $(a_g + j_g) \circ L_k \nsubseteq \delta(I)$ and $k_h \circ L_k \nsubseteq \delta(I)$. Then it is clear that $(a_g + j_g) \circ (b_h + k_h) \subseteq I$ since $(a_g + j_g) \circ (b_h + k_h) \circ L_k \subseteq I$, $(a_g + j_g) \circ L_k \nsubseteq \delta(I)$ and $(b_h + k_h) \circ L_k \nsubseteq \delta(I)$. Thus $a_g \circ b_h = (a_g + j_g - j_g) \circ (b_h + k_h - k_h) \subseteq (a_g + j_g) \circ (b_h + k_h) - (a_g + j_g) \circ k_h - j_g \circ (b_h + k_h) - j_g \circ k_h \subseteq I$ since $(a_g + j_g) \circ (b_h + k_h) \subseteq I$, $(a_g + j_g) \circ k_h \subseteq I$, $j_g \circ (b_h + k_h) \subseteq I$ and $j_g \circ k_h \subseteq I$. Hence $a_g \circ b_h \subseteq I$, a contradiction. Consequently, $J_g \circ L_k \subseteq \delta(I)$ or $K_h \circ L_k \subseteq \delta(I)$.

Conversely, Let $a_g \circ b_h \circ c_k \subseteq I$ where $a_g, b_h, c_k \in h(R)$. Consider $\langle a_g \rangle$, $\langle b_h \rangle$ and $\langle c_k \rangle$ the subgroups of generated by a_g, b_h, c_k, respectively. Hence we get $\langle a_g \rangle \circ \langle b_h \rangle \circ \langle c_k \rangle \subseteq I$. So by assumption, we have $\langle a_g \rangle \circ \langle b_h \rangle \subseteq I$ or $\langle b_h \rangle \circ \langle c_k \rangle \subseteq \delta(I)$ or $\langle b_h \rangle \circ \langle c_k \rangle \subseteq \delta(I)$. Therefore, $a_g \circ b_h \subseteq \langle a_g \rangle \circ \langle b_h \rangle \subseteq I$ or $b_h \circ c_k \subseteq \langle b_h \rangle \circ \langle c_k \rangle \subseteq \delta(I)$ or $a_g \circ c_k \subseteq \langle b_h \rangle \circ \langle c_k \rangle \subseteq \delta(I)$. □

Theorem 4.20. *Let $(R_1, +_1, \circ_1)$ and $(R_2, +_2, \circ_2)$ be two graded multiplicative hyperrings and δ_1 and δ_2 be expansion functions of hyperideals of R_1 and R_2, respectively. Let $I_1 \in \mathcal{I}^{gr}_*(R_1)$, $I_2 \in \mathcal{I}^{gr}_*(R_2)$ and $R = (R_1 \times R_2, \circ, +)$. The following statements hold:*

(i) I_1 is a graded 2-absorbing δ_1-primary hyperideal of R_1 if and only if $I_1 \times R_2$ is a graded 2-absorbing δ_R-primary hyperideal of R.

(ii) I_2 is a graded 2-absorbing δ_2-primary hyperideal of R_2 if and only if $R_1 \times I_2$ is a graded 2-absorbing δ_R-primary hyperideal of R.

Proof. (i) $(\Rightarrow)$ Let $(x_g, y_g), (z_h, t_h), (u_k, v_k) \in h(R)$ with $(x_g, y_g) \circ (z_h, t_h) \circ (u_k, v_k) \subseteq I_1 \times R_2$. Hence $x_g \circ_1 z_h \circ_1 u_k \subseteq I_1$. Thus $x_g \circ_1 z_h \subseteq I_1$ or $z_h \circ_1 u_k \subseteq \delta_1(I_1)$ or $x_g \circ_1 u_k \subseteq \delta_1(I_1)$ and so $(x_g, y_g) \circ (z_h, t_h) \subseteq I_1 \times R_2$ or $(z_h, t_h) \circ (u_k, v_k) \subseteq \delta_R(I_1 \times R_2)$ or $(x_g, y_g) \circ (u_k, v_k) \subseteq \delta(I_1 \times R_2)$.

$(\Leftarrow)$ Let I_1 be not a graded 2-absorbing δ_1-primary hyperideal of R_1. So we have $x_g, y_h, z_k \in h(R_1)$ with $x_g \circ_1 y_h \circ_1 z_k \subseteq I_1$, $x_g \circ_1 z_h \nsubseteq I_1$, $z_h \circ_1 u_k \nsubseteq \delta_1(I_1)$ and $x_g \circ_1 u_k \nsubseteq \delta_1(I_1)$. Note that $(x_g, 0_{R_2}) \circ (z_h, 0_{R_2}) \circ (u_k, 0_{R_2}) \subseteq I_1 \times R_2$. By assumption, $(x_g, 0_{R_2}) \circ (z_h, 0_{R_2}) \subseteq I_1 \times R_2$ or $(z_h, 0_{R_2}) \circ (u_k, 0_{R_2}) \subseteq \delta_R(I_1 \times R_2)$ or $(x_g, 0_{R_2}) \circ (u_k, 0_{R_2}) \subseteq \delta_R(I_1 \times R_2)$. It means $x_g \circ_1 z_h \subseteq I_1$ or $z_h \circ_1 u_k \subseteq \delta_1(I_1)$ or $x_g \circ_1 u_k \subseteq \delta_1(I_1)$, a contradiction. Therefore, I_1 is a graded 2-absorbing δ_1-primary hyperideal of R_1.
(ii) The proof is similar to (i). □

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