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On a Generalization of Tripled Fixed or Best Proximity Points for a Class of Cyclic Contractive Maps

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Abstract. We enrich the known results about tripled fixed points and tripled best proximity points. We generalize the notion of ordered pairs of cyclic contraction maps and we obtain sufficient conditions for the existence and uniqueness of fixed (or best proximity) points. We get a priori and a posteriori error estimates for the tripled fixed points and for the tripled best proximity points, provided that the underlying Banach space has modulus of convexity of power type in the case of best proximity points, obtained by sequences of successive iterations. We illustrate the main result with an example.

1. Introduction and Preliminaries

The Banach contraction principle states that in a complete metric space (X,ρ) any contraction map $T : X \to X$ has a fixed point, i.e. $\min\{\rho(x,Tx) : x \in X\} = 0$. A lot of results in modeling real world processes in applied mathematics lead to the problems, where T depends on two or more variables, e.g. $T : X \times X \times X \to X$. The theory of coupled [13], tripled [1, 8, 19], qudripled [18] and n–order (n–tuples) [19] fixed points. In many kinds of modeling, a crucial point, is the finding of $\min\{\rho(x,Tx) : x \in X\}$. It may happen that the above minimum is greater than zero. One approach for solving the above mentioned problems uses the notion of a best proximity point is introduced in [11], where a sufficient condition for the existence and the uniqueness of best proximity points in uniformly convex Banach spaces is obtained. Combining both mentioned cases we got to the idea of tripled fixed (or best proximity points).

Coupled fixed points defined in [13] and coupled best proximity points defined in [14, 20] are widely investigated recently. Deep results in the theory of coupled fixed points can be found for example in [4–6]. The notion of coupled fixed is as follows: $F : X \times X \to X$ and we are searching for a couple (x,y), such that $x = F(x,y)$ and $y = F(y,x)$ and the notion of coupled best proximity points – (F,G), $F : A\times A \to B$, $G : B\times B \to A$, where $A, B \subset X$ we are searching for a couple (x,y), such that $\rho(x,F(x,y)) = \rho(y,F(y,x)) = \operatorname{dist}(A,B)$. In the mentioned above classical definition of coupled fixed (or best proximity) points the solutions reduce to the point (x,x), provided that the map F or the ordered pair (F,G) be a cyclic contractions [22]. The same observation is made in [4, 7] for coupled fixed points for maps with the mixed monotone property in partially ordered metric spaces. The draw back of the classical technique about coupled fixed (or best proximity) points is that it can be applied only for solving of symmetric linear or nonlinear systems of

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equations. This gap has been filled in [22] by generalizing the defined maps F or the ordered pair of maps (F, G).

There are many problems about fixed points and best proximity points that are not easy to be solved or can not be solved exactly. One of the advantages of the Banach fixed point theorem is the error estimates of the successive iterations and the rate of convergence. That is why an estimation of the error, when an iterative process, is used is of interest, when fixed points or best proximity points are investigated. An extensive study about approximations of fixed points can be found in [3]. A first result in the approximation of the sequence of successive iterations, which converges to the best proximity point for cyclic contractions, is obtained in [21]. This result was expanded for a coupled fixed (or best proximity) points in [16, 22].

The generalization made in [22], was to replace the ordered pair (F, G), $F : A \times A \to B$, $G : B \times B \to A$ an ordered pair (F, G) of ordered couples of maps $F = (F_1, F_2)$ and $G = (G_1, G_2)$, such that $F_1 : A_1 \times A_2 \to B_1$, $F_2 : A_1 \times A_2 \to B_2$, $G_1 : B_1 \times B_2 \to A_1$, $G_2 : B_1 \times B_2 \to A_2$ is considered. Examples are presented in [22] to show that it is possible to apply the coupled fixed (or best proximity) points for solving of non–symmetric systems of equations. If $F_2(x, y) = F_1(y, x)$, $G_2(x, y) = G_1(y, x)$, $A_1 = A_2$, $B_1 = B_2$, the investigated maps in [22] reduce to the known examples [4–6, 13, 14, 20].

The same observations can be made and for tripled fixed (or best proximity) points investigated in [1, 19]. We will try to generalize the notion of tripled fixed (or best proximity) points following the technique from [22]. The notion of tripled fixed points proposed in [8] is a little bit different. Our main result will cover just some of the cases of the considered maps in [8]. We will discus in the last section a possible generalization that will cover and the case investigated in [8].

Let (X, ρ) be a metric space. If $(X, \|\cdot\|)$ be a norm space we will consider the metric to be endowed by the norm, i.e. $\rho(\cdot, \cdot) = \|\cdot - \cdot\|$. Let every where in the text A_i, B_i for $i = 1, 2, 3$ be subset of X. Let us denote, just to simplify the notations, $A^3 = A_1 \times A_2 \times A_3$ and $B^3 = B_1 \times B_2 \times B_3$.

Definition 1.1. *Let A_i, B_i, $i = 1, 2, 3$ be six sets. We say that the ordered pair (F, G) of triples of maps $F = (F_1, F_2, F_3)$ and $G = (G_1, G_2, G_3)$ be a cyclic ordered pair of triple of maps if $F_i : A^3 \to B_i$, $B_i : B^3 \to A_i$ for $i = 1, 2, 3$.*

Just for the the sake of simplicity we will assume for the rest of the article that the pair of maps (F, G) be a cyclic ordered pair of triple of maps.

Following [1, 19] we will give the following definition.

Definition 1.2. *We say that $(\xi_1, \xi_2, \xi_3) \in A^3$ is a tripled fixed point for the ordered pair of triple of maps (F, G) if there holds $\xi_i = F_i(\xi_1, \xi_2, \xi_3)$, for $i = 1, 2, 3$.*

If $A_1 = A_2 = A_3 = B_1 = B_2 = B_3$ and $F_2(x, y, z) = F_1(y, z, x)$, $F_3(x, y, z) = F_1(z, x, y)$ and $G_i(x, y, z) = F_i(x, y, x)$ for $i = 1, 2, 3$ we get the definition of tripled fixed points from [1, 19]. If $A_1 = A_2 = A_3 = B_1 = B_2 = B_3$ and $F_2(x, y, z) = F_1(y, x, y)$, $F_3(x, y, z) = F_1(z, y, x)$ and $G_i(x, y, z) = F_i(x, y, x)$ for $i = 1, 2, 3$ we get the definition of tripled fixed points from [8].

The concept of coupled best proximity points is introduced in [20]. We will generalize this concept from [20, 22] for tripled best proximity points.

Define a distance between two subsets $A, B \subset X$ by $\operatorname{dist}(A, B) = \inf\{\rho(x, y) : x \in A, y \in B\}$. Let A_i, B_i, $i = 1, 2, 3$ be subsets of a metric space (X, ρ). Let us denote $d_i = \operatorname{dist}(A_i, B_i)$ for $i = 1, 2, 3$.

Definition 1.3. *Let A_i, B_i, $i = 1, 2, 3$ be subsets of a metric space (X, ρ) and let (F, G) be a cyclic ordered pair of triple of maps. We say that $(\xi_1, \xi_2, \xi_3) \in A^3$ a tripled best proximity point of F if $\rho(\xi_i, F_i(\xi_1, \xi_2, \xi_3)) = d_i$, for $i = 1, 2, 3$.*

If $A_1 = A_2$, $B_1 = B_2$, $A_3 = B_3 = \emptyset$, $F_2(x, y) = F_1(y, x)$ we get the definition for a coupled fixed point from [20].

We will use the classical technique of successive iterations for the investigations of tripled fixed (or best proximity) points [8, 9].

Definition 1.4. *Let A_i, B_i, $i = 1,2,3$ be six sets and let (F, G) be a cyclic ordered pair of triple of maps. For any triple $\left(x^{(1)}, x^{(2)}, x^{(3)}\right) \in A^3$ we define the sequences $\left\{x_n^{(i)}\right\}_{n=0}^{\infty}$, for $i = 1,2,3$ by $x_0^{(i)} = x^{(i)}$ for $i = 1,2,3$ and*

$$x_{2n+1}^{(i)} = F_i\left(x_{2n}^{(1)}, x_{2n}^{(2)}, x_{2n}^{(3)}\right),\ \ x_{2n+2}^{(i)} = G_i\left(x_{2n+1}^{(1)}, x_{2n+1}^{(2)}, x_{2n+1}^{(3)}\right), \text{ for } i = 1,2,3$$

and for all $n \geq 0$.

If we consider $A_1 = A_2 = A_3 = B_1 = B_2 = B_3$ and $F_2(x, y, z) = F_1(y, z, x)$, $F_3(x, y, z) = F_1(z, x, y)$ and $G_i(x, y, z) = F_i(x, y, x)$ for $i = 1,2,3$ we get the sequence defined in [1]. If we consider $A_1 = A_2 = A_3 = B_1 = B_2 = B_3$ and $F_2(x, y, z) = F_1(y, x, y)$, $F_3(x, y, z) = F_1(z, y, x)$ and $G_i(x, y, z) = F_i(x, y, x)$ for $i = 1,2,3$ we get the sequence defined in [8].

Whenever we consider the sequences $\left\{x_n^{(i)}\right\}_{n=0}^{\infty}$, for $i = 1,2,3$ we assume that they are the sequences defined in Definition 1.4.

The best proximity results need norm-structure of the space X. We will denote the unit sphere and the unit ball of a Banach space $(X, \|\cdot\|)$ by S_X and B_X respectively.

The assumption that the Banach space $(X, \|\cdot\|)$ is uniformly convex plays a crucial role in the investigation of best proximity points.

Definition 1.5. *Let $(X, \|\cdot\|)$ be a Banach space. For every $\varepsilon \in (0,2]$ we define the modulus of convexity of $\|\cdot\|$ by*

$$\delta_{\|\cdot\|}(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_X, \|x - y\| \geq \varepsilon\right\}.$$

The norm is called uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. The space $(X, \|\cdot\|)$ is then called a uniformly convex space.

Indeed, there are results about existence and uniqueness of best proximity points, stated in complete metric space X, where the ordered pair (A, B) of sets $A, B \subset X$ satisfies the properties UC and UC^* [14, 20], but these assumptions are just a replacement of the next two lemmas.

Lemma 1.6. *([11]) Let A be a nonempty closed, convex subset, and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in A and $\{y_n\}_{n=1}^{\infty}$ be a sequence in B satisfying:*
1) $\lim_{n\to\infty} \|x_n - y_n\| = \operatorname{dist}(A, B)$;
2) $\lim_{n\to\infty} \|z_n - y_n\| = \operatorname{dist}(A, B)$;
then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Lemma 1.7. *([11]) Let A be a nonempty closed, convex subset, and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in A and $\{y_n\}_{n=1}^{\infty}$ be a sequence in B satisfying:*
1) $\lim_{n\to\infty} \|z_n - y_n\| = \operatorname{dist}(A, B)$;
2) for every $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$, such that for all $m > n \geq N_0$, $\|x_n - y_n\| \leq \operatorname{dist}(A, B) + \varepsilon$,
then for every $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$, such that for all $m > n > N_1$, holds $\|x_m - z_n\| \leq \varepsilon$.

For obtaining error estimates for the sequence of successive iterations that approximates the best proximity point, which is generated by a cyclic contraction, the modulus of convexity $\delta_{(X,\|\cdot\|)}$ is used [21]. That is why we could not replace the uniform convexity of the Banach space with a metric space and the properties UC and UC^*.

For any uniformly convex Banach space X there holds the inequality

$$\left\|\frac{x+y}{2} - z\right\| \leq \left(1 - \delta_X\left(\frac{r}{R}\right)\right)R \tag{1}$$

for any $x, y, z \in X$, $R > 0$, $r \in [0, 2R]$, $\|x - z\| \leq R$, $\|y - z\| \leq R$ and $\|x - y\| \geq r$ [11].

If $(X, \|\cdot\|)$ is a uniformly convex Banach space, then $\delta_X(\varepsilon)$ is a strictly increasing function. Therefore if $(X, \|\cdot\|)$ is a uniformly convex Banach space, then there exists the inverse function δ^{-1} of the modulus of convexity. If there exist constants $C > 0$ and $q > 0$, such that the inequality $\delta_{\|\cdot\|}(\varepsilon) \ge C\varepsilon^q$ holds for every $\varepsilon \in (0,2]$, we say that the modulus of convexity is of power type q. It is well known that for any Banach space and for any norm there holds the inequality $\delta(\varepsilon) \le K\varepsilon^2$. The modulus of convexity with respect to the canonical norm $\|\cdot\|_p$ in ℓ_p or L_p is $\delta_{\|\cdot\|_p}(\varepsilon) = 1 - \sqrt[p]{1 - \left(\frac{\varepsilon}{2}\right)^p}$ for $p \ge 2$ and for $1 < p < 2$ the modulus of convexity $\delta_{\|\cdot\|_p}(\varepsilon)$ is the solution of the equation $\left(1 - \delta + \frac{\varepsilon}{2}\right)^p + \left|1 - \delta - \frac{\varepsilon}{2}\right|^p = 2$. It is well known that the modulus of convexity with respect to the canonical norm in ℓ_p or L_p is of power type and there hold the inequalities $\delta_{\|\cdot\|_p}(\varepsilon) \ge \frac{\varepsilon^p}{p2^p}$ for $p \ge 2$ and $\delta_{\|\cdot\|_p}(\varepsilon) \ge \frac{(p-1)\varepsilon^2}{8}$ for $p \in (1,2)$ [17].

An extensive study of the Geometry of Banach spaces can be found in [2, 10, 12].

2. Triple fixed (or best proximity) points for cyclic ordered pair of triple of maps

Definition 2.1. *Let (X,ρ) be a metric space. We say that the cyclic ordered pair of triple of maps (F,G) be a generalized cyclic contraction of type 1 if there holds*

$$\sum_{i=1}^{3} \rho\left(F_i\left(x_i^{(1)}, x_i^{(2)}, x_i^{(3)}\right), G_i\left(y_i^{(1)}, y_i^{(2)}, y_i^{(3)}\right)\right) \le \sum_{j=1}^{3}\sum_{i=1}^{3} \alpha_i^{(j)} \rho\left(x_i^{(j)}, y_i^{(j)}\right)$$

for some constants $\alpha_i^{(j)} \in [0,1)$ for $i,j = 1,2,3$, such that $k = \max\left\{\sum_{i=1}^{3}\alpha_i^{(1)}, \sum_{i=1}^{3}\alpha_i^{(2)}, \sum_{i=1}^{3}\alpha_i^{(3)}\right\} < 1$ and any $\left(x_i^{(1)}, x_i^{(2)}, x_i^{(3)}\right) \in A^3$, $\left(y_i^{(1)}, y_i^{(2)}, y_i^{(3)}\right) \in B^3$ for $i = 1,2,3$.

Theorem 2.2. *Let A_i, B_i, $i = 1,2,3$ be nonempty, closed subsets of a complete metric space (X,ρ). Let the ordered pair (F,G) be a generalized cyclic contraction of type 1. Then*

(I) *There exists a unique triple (ξ_1, ξ_2, ξ_3) in $(A_1 \cap B_1) \times (A_2 \cap B_2) \times (A_3 \cap B_3)$, which is a common triple fixed point for the maps F and G. Moreover the iteration sequences $\left\{x_i^{(n)}\right\}_{n=0}^{\infty}$, for $i = 1,2,3$, defined in Definition 1.4 converge to ξ_i for $i = 1,2,3$ respectively for any initial guess triple $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$;*

(II) *a priori error estimates hold*

$$\max\left\{\rho(x_i^{(n)}, \xi_i) : i = 1,2,3\right\} \le \frac{k^n}{1-k}\sum_{i=1}^{3}\rho\left(x_i^{(1)}, x_i^{(0)}\right); \tag{2}$$

(III) *a posteriori error estimates hold*

$$\max\left\{\rho(x_i^{(n)}, \xi) : i = 1,2,3\right\} \le \frac{k}{1-k}\sum_{i=1}^{3}\rho\left(x_i^{(n-1)}, x_i^{(n)}\right); \tag{3}$$

(IV) *The rate of convergence for the sequences of successive iterations is given by*

$$\sum_{i=1}^{3}\rho\left(x_i^{(n)}, \xi_i\right) \le k\sum_{i=1}^{3}\rho\left(x_i^{(n-1)}, \xi_i\right). \tag{4}$$

In what follows we will use the notation $d = \sum_{j=1}^{3} d_j = \sum_{j=1}^{3} \mathrm{dist}(A_j, B_j)$.

Just to fit some of the formulas in the text field we will denote $P_{n,m}(x,y,z) = \|x_n - x_m\| + \|y_n - y_m\| + \|z_n - z_m\|$ and $W_{n,m}(x,y,z) = P_{n,m}(x,y,z) - d = \|x_n - x_m\| + \|y_n - y_m\| + \|z_n - z_m\| - d$, where $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ be three sequences.

Just to simplify some of the statements let us agree that every where (F, G) be an ordered pair of ordered triples of maps, such that $F = (F_1, F_2, F_3)$, $F_i : A_1 \times A_2 \times A_3 \to B_i$ and $G = (G_1, G_2, G_3)$, $G_i : B_1 \times B_2 \times B_3 \to A_i$, for $i = 1, 2, 3$, where A_i, B_i for $i = 1, 2, 3$ be nonempty subsets of the underlying space X, metric or normed space.

Definition 2.3. *We say that the ordered pair (F, G) of ordered triples of maps is a generalized cyclic contraction of type 2 if there holds*

$$\sum_{i=1}^{3} \rho\left(F_i\left(x_i^{(1)}, x_i^{(2)}, x_i^{(3)}\right), G_i\left(y_i^{(1)}, y_i^{(2)}, y_i^{(3)}\right)\right) \leq \sum_{j=1}^{3}\sum_{i=1}^{3} \alpha_i^{(j)} \rho\left(x_i^{(j)}, y_i^{(j)}\right) + \sum_{j=1}^{3}\left(1 - \sum_{i=1}^{3} \alpha_i^{(j)}\right) d_j$$

for some constants $\alpha_i^{(j)} \in [0, 1)$, $i, j = 1, 2, 3$ such that $k = \max\left\{\sum_{i=1}^{3} \alpha_i^{(j)} : j = 1, 2, 3\right\} < 1$ and any $\left(x_i^{(1)}, x_i^{(2)}, x_i^{(3)}\right) \in A^3$, $(y_i^{(1)}, y_i^{(2)}, y_i^{(3)}) \in B^3$ for $i = 1, 2, 3$.

Theorem 2.4. *Let A_i, B_i, for $i = 1, 2, 3$ be nonempty, closed and convex subsets of a uniformly convex Banach space. Let the ordered pair (F, G) of ordered triples of maps be a generalized cyclic contraction of type 2. Then F has a unique tripled best proximity point $(\xi_1, \xi_2, \xi_3) \in A^3$ and G has a unique tripled best proximity point $(\upsilon_1, \upsilon_2, \upsilon_3) \in B^3$, i.e. ($\|\xi_i - F_i(\xi_1, \xi_2, \xi_3)\| = d_i$, $\|\upsilon_i - G_i(\upsilon_1, \upsilon_2, \upsilon_3)\| = d_i$, $i = 1, 2, 3$) and*

$$\begin{aligned} &G_i(F_1(\xi_1, \xi_2, \xi_3), F_2(\xi_1, \xi_2, \xi_3), F_3(\xi_1, \xi_2, \xi_3)) = \xi_i, \\ &F_i(G_1(\upsilon_1, \upsilon_2, \upsilon_3), G_2(\upsilon_1, \upsilon_2, \upsilon_3), G_3(\upsilon_1, \upsilon_2, \upsilon_3)) = \upsilon_i \end{aligned} \tag{5}$$

for $i = 1, 2, 3$. Moreover $\upsilon_i = F_i(\xi_1, \xi_2, \xi_3)$ and $\xi_i = G_i(\upsilon_1, \upsilon_2, \upsilon_3)$ for $i = 1, 2, 3$. For any arbitrary point $\left(x_1^{(0)}, x_2^{(0)}, x_2^{(0)}\right)$ there hold $\lim_{n\to\infty} x_i^{(2n)} = \xi_i$, $\lim_{n\to\infty} x_i^{(2n+1)} = \upsilon_i$ for $i = 1, 2, 3$ and $\sum_{i=1}^{3} \|\xi_i - \upsilon_i\| = d$.

If in addition the modulus of convexity δ is of power type with constants $C > 0$ and $q > 1$ then there holds the error estimates:

(i) *a priori error estimates hold*

$$\max\left\{\left\|\xi_i - x_i^{(2m)}\right\| : i = 1, 2, 3\right\} \leq P_{0,1}(x_1, x_2, x_3) \sqrt[q]{\frac{W_{0,1}(x_1, x_2, x_3)}{Cd}} \cdot \frac{\left(\sqrt[q]{k^2}\right)^m}{1 - \sqrt[q]{k^2}}; \tag{6}$$

(ii) *a posteriori error estimates hold*

$$\max\left\{\left\|\xi_i - x_i^{(2n)}\right\| : i = 1, 2, 3\right\} \leq P_{2n,2n-1}(x_1, x_2, x_3) \sqrt[q]{\frac{W_{2n,2n-1}(x_1, x_2, x_3)}{Cd}} \cdot \frac{\sqrt[q]{k}}{1 - \sqrt[q]{k^2}}, \tag{7}$$

where $x_i = \{x_i^{(n)}\}_{n=0}^{\infty}$ for $i = 1, 2, 3$ are the sequences defined in Definition 1.4.

3. Auxiliary results

Lemma 3.1. *Let (X, ρ) be a metric space and the ordered pair (F, G) of ordered triples of maps be a generalized cyclic contraction of type 2. Then there holds $\lim\limits_{n\to\infty} \rho\left(x_i^{(n)}, x_i^{(n+1)}\right) = d_i$, $i = 1, 2, 3$ for an arbitrary chosen $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) \in A^3$.*

Proof. Let us choose arbitrary $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) \in A^3$ and let us define the sequences $\left\{x_i^{(n)}\right\}_{n=0}^{\infty}$, $i = 1, 2, 3$.

Using the cyclic contraction condition we get that for all $n \in \mathbb{N}$ there holds

$$\begin{aligned} \sum_{i=1}^{3} \rho\left(x_i^{(2n+1)}, x_i^{(2n+2)}\right) &= \sum_{i=1}^{3} \rho\left(F_i\left(x_1^{(2n)}, x_2^{(2n)}, x_3^{(2n)}\right), G_i\left(x_1^{(2n+1)}, x_2^{(2n+1)}, x_3^{(2n+1)}\right)\right) \\ &\leq \sum_{j=1}^{3}\sum_{i} \alpha_i^{(j)} \rho\left(x_j^{(2n)}, x_j^{(2n+1)}\right) + \sum_{j=1}^{3}\left(1 - \sum_{i}^{(j)} \alpha_i^{(j)}\right) d_j \end{aligned}$$

Thus we get

$$\begin{aligned}\sum_{i=1}^{3}\left(\rho\left(x_i^{(2n+1)},x_i^{(2n+2)}\right)-d_i\right) &\leq \sum_{j=1}^{3}\sum_{i}^{3}\alpha_i^{(j)}\left(\rho\left(x_j^{(2n)},x_j^{(2n+1)}\right)-d_j\right)\\ &\leq k\sum_{j=1}^{3}\left(\rho\left(x_j^{(2n)},x_j^{(2n+1)}\right)-d_j\right)\\ &\leq k^2\sum_{j=1}^{3}\left(\rho\left(x_j^{(2n-1)},x_j^{(2n)}\right)-d_j\right)\\ &\quad\cdots\cdots\cdots\cdots\cdots\cdots\\ &\leq k^{2n+1}\sum_{j=1}^{3}\left(\rho\left(x_j^{(0)},x_j^{(1)}\right)-d_j\right)\end{aligned} \tag{8}$$

After taking limit in (8), when $n\to\infty$ we get

$$\lim_{n\to\infty}\left(\sum_{i=1}^{3}\left(\rho\left(x_i^{(2n+1)},x_i^{(2n+2)}\right)-d_i\right)\right)=0$$

and thus from $\rho\left(x_i^{(2n+1)},x_i^{(2n+2)}\right)\geq d_i$ for $i=1,2,3$ we obtain $\lim_{n\to\infty}\rho\left(x_i^{(2n+1)},x_i^{(2n+2)}\right)=d_i$ for $i=1,2,3$. □

It is easy to see that inequality (8) holds also for indexes $m>n$, such that $n+m$ is an odd number

$$\sum_{i=1}^{3}\left(\rho\left(x_i^{(n)},x_i^{(m)}\right)-d_i\right)\leq k^n\sum_{i=1}^{3}\left(\rho\left(x_i^{(0)},x_i^{(1)}\right)-d_i\right). \tag{9}$$

Lemma 3.2. *Let (X,ρ) be a metric space and the ordered pair (F,G) of ordered triples of maps be a generalized cyclic contraction of type 2. For any arbitrary chosen $\left(x_1^{(0)},x_2^{(0)},x_3^{(0)}\right)\in A^3$ the sequences $\left\{x_i^{(2n)}\right\}_{n=0}^{\infty}$, $\left\{x_i^{(2n+1)}\right\}_{n=0}^{\infty}$, $i=1,2,3$ are bounded.*

Proof. Let $\left(x_1^{(0)},x_2^{(0)},x_3^{(0)}\right)\in A^3$ be arbitrary chosen and fixed. From Lemma 3.1 we have that there hold $\lim_{n\to\infty}\rho\left(x_i^{(2n+1)},x_i^{(2n+2)}\right)=d_i$, for $i=1,2,3$ and thus it will be enough to prove that only the sequences $\left\{x_i^{(2n+1)}\right\}_{n=0}^{\infty}$, $i=1,2,3$ are bounded.

Let as choose

$$M>\frac{d+(1+k)k^2\sum_{i=1}^{3}\rho\left(x_i^{(0)},F_i\left(x_1^{(0)},x_2^{(0)},x_3^{(0)}\right)\right)}{1-k^2}.$$

Let us suppose that at least one of the sequences $\left\{x_i^{(2n+1)}\right\}_{n=0}^{\infty}$, $i=1,2,3$ is not bounded. Then there exists $n_0\in\mathbb{N}$, such that there hold

$$\sum_{i=1}^{3}\rho\left(x_i^{(2)},x_i^{(2n_0-1)}\right)=\sum_{i=1}^{3}\rho\left(x_i^{(2)},F_i\left(x_1^{(2n_0-2)},x_2^{(2n_0-2)},x_3^{(2n_0-2)}\right)\right)\leq M$$

and

$$\sum_{i=1}^{3}\rho\left(x_i^{(2)},x_i^{(2n_0+1)}\right)=\sum_{i=1}^{3}\rho\left(x_i^{(2)},F_i\left(x_1^{(2n_0)},x_2^{(2n_0)},x_3^{(2n_0)}\right)\right)>M. \tag{10}$$

From inequality (10) after using (9) with $n = 2$ and $m = 2n_0 + 1$ we get

$$\begin{aligned}
\frac{M-d}{k^2} &< \frac{\sum_{i=1}^{3}\rho\left(x_i^{(2)}, x_i^{(2n_0+1)}\right) - d}{k^2} = \frac{\sum_{i=1}^{3}\rho\left(F_i\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}\right), F_i\left(x_1^{(2n_0)}, x_2^{(2n_0)}, x_3^{(2n_0)}\right)\right) - d}{k^2} \\
&\le \sum_{i=1}^{3}\rho\left(x_i^{(0)}, F_i\left(x_1^{(2n_0-2)}, x_2^{(2n_0-2)}, x_3^{(2n_0-2)}\right)\right) - d \\
&\le \sum_{i=1}^{3}\rho\left(x_i^{(0)}, x_i^{(2)}\right) - d + \sum_{i=1}^{3}\rho\left(x_i^{(2)}, F_i\left(x_1^{(2n_0-2)}, x_2^{(2n_0-2)}, x_3^{(2n_0-2)}\right)\right) \\
&\le \sum_{i=1}^{3}\rho\left(x_i^{(0)}, x_i^{(1)}\right) + \sum_{i=1}^{3}\rho\left(x_i^{(1)}, x_i^{(2)}\right) - d + M \\
&\le \sum_{i=1}^{3}\rho\left(x_i^{(0)}, x_i^{(1)}\right) + k\left(\sum_{i=1}^{3}\rho\left(x_i^{(0)}, x_i^{(1)}\right) - d\right) + M \\
&= (1+k)\sum_{i=1}^{3}\rho\left(x_i^{(0)}, x_i^{(1)}\right) - kd + M < (1+k)\sum_{i=1}^{3}\rho\left(x_i^{(0)}, x_i^{(1)}\right) + M
\end{aligned}$$

which inequality can hold true only if the inequality

$$M \le \frac{d + (1+k)k^2\sum_{i=1}^{3}\rho\left(x_i^{(0)}, x_i^{(1)}\right)}{1-k^2},$$

holds, which contradicts with the choice of M. □

Lemma 3.3. *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space and the ordered pair (F, G) be a generalized cyclic contraction of type 2. Then for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ so that the inequality*

$$\sum_{i=1}^{3}\left\|x_i^{(m)} - x_i^{(n+1)}\right\| < d + \varepsilon$$

holds for any $m > n > n_0$ and $m + n + 1$ be an odd number.

Proof. From Lemma 3.1 we get

$$\lim_{n\to\infty}\left\|x_i^{(n)} - x_i^{(n+1)}\right\| = \lim_{n\to\infty}\left\|x_i^{(n+2)} - x_i^{(n+1)}\right\| = d_i$$

for $i = 1, 2, 3$.

By Lemma 1.6 after using the uniform convexity of $(X, \|\cdot\|)$ it follows that $\lim_{n\to\infty}\left\|x_i^{(n)} - x_i^{(n+2)}\right\| = 0$ for $i = 1, 2, 3$.

Let us suppose that there exists $\varepsilon > 0$ such that for every $j \in \mathbb{N}$ there are $m_j > n_j + 1 \ge j$ so that $\sum_{i=1}^{3}\left\|x_i^{(m_j)} - x_i^{(n_j+1)}\right\| \ge d + \varepsilon$. Let us choose m_j to be the smallest integer so that the above inequality is satisfied, i.e.

$$\sum_{i=1}^{3}\left\|x_i^{(m_j-2)} - x_i^{(n_j+1)}\right\| < d + \varepsilon.$$

Thus we get

$$\begin{aligned}
d + \varepsilon &\le \sum_{i=1}^{3}\left\|x_i^{(m_j)} - x_i^{(n_j+1)}\right\| \le \sum_{i=1}^{3}\left(\left\|x_i^{(m_j)} - x_i^{(m_j-2)}\right\| + \left\|x_i^{(m_j-2)} - x_i^{(n_j+1)}\right\|\right) \\
&< \sum_{i=1}^{3}\left\|x_i^{(m_j)} - x_i^{(m_j-2)}\right\| + d + \varepsilon.
\end{aligned} \tag{11}$$

Letting $j \to \infty$ in (11) we get $\lim_{j\to\infty}\sum_{i=1}^{3}\left\|x_i^{(m_j)} - x_i^{(m_j-2)}\right\| = d + \varepsilon$. Using the boundedness of the sequences $\left\{x_i^{(n)}\right\}_{n=0}^{\infty}$ for $i = 1, 2, 3$ it follows that there exists $M \ge d = \sum_{i=1}^{3} d_i$, such that the inequality

$M \geq \sum_{i=1}^{3} \left\| x_i^{(0)} - x_i^{(n_j - m_j + 1)} \right\|$ holds for every $j \in \mathbb{N}$. The inequality

$$\sum_{i=1}^{3} \|x_i^{(m_j)} - x_i^{(n_j+1)}\| - d \leq k^{m_j} \left(\sum_{i=1}^{3} \|x_i^{(0)} - x_i^{(n_j - m_j + 1)}\| - d \right) \leq k^{m_j}(M - d)$$

holds. For any $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$, such that $k^j(M - d) < \varepsilon$ for every $j \geq j_0$. Therefore for any $m_j > n_j + 1 \geq j_0$ there holds $\sum_{i=1}^{3} \left\| x_i^{(m_j)} - x_i^{(n_j+1)} \right\| < d + \varepsilon$, which is a contradiction. □

Lemma 3.4. *Let* $(X, \|\cdot\|)$ *be a uniformly convex Banach space and the ordered pair* (F, G) *be a generalized cyclic contraction of type* 2*. For an arbitrary chosen* $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) \in A^3$ *the sequences* $\left\{x_i^{(2n)}\right\}_{n=0}^{\infty}$, $\left\{x_i^{(2n+1)}\right\}_{n=0}^{\infty}$ *for* $i = 1, 2, 3$ *are Cauchy sequences.*

Proof. We will prove that $\left\{x_i^{(2n)}\right\}_{n=0}^{\infty}$ is a Cauchy sequence. The proofs for the other five cases are similar. By Lemma 3.3 we have that for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$, so that for all $2m > 2n + 1 \geq n_0$ holds the inequality

$$\sum_{i=1}^{3} \left\| x_i^{(2m)} - x_i^{(2n+1)} \right\| < d + \varepsilon.$$

From the inequalities $d_i \leq \left\| x_i^{(2m)} - x_i^{(2n+1)} \right\|$ for $i = 1, 2, 3$ it follows that the inequality $\left\| x_1^{(2m)} - x_i^{(2n+1)} \right\| < d_1 + \varepsilon$ holds for all $2m > 2n + 1 \geq n_0$. From Lemma 3.1 it follows that $\lim_{n\to\infty} \left\| x_1^{(2n)} - x_1^{(2n+1)} \right\| = d_1$. According to Lemma 1.7 it follows that for every $\varepsilon > 0$ there is $N_0 \in \mathbb{N}$, so that for all $m > n \geq N_0$ holds the inequality $\left\| x_1^{2m} - x_1^{2n} \right\| < \varepsilon$ and consequently $\left\{x_1^{(2n)}\right\}_{n=0}^{\infty}$ is a Cauchy sequence. □

Lemma 3.5. *Let* $(X, \|\cdot\|)$ *be a uniformly convex Banach space and the ordered pair* (F, G) *be a generalized cyclic contraction of type* 2*. Then for any* $1 \leq l \leq 2n$ *there hold the inequalities*

$$\left\| x_i^{(2n+1)} - x_i^{(2n)} \right\| \leq k^l W_{2n+1-l, 2n-l}\left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\right) + d_i$$

for $i = 1, 2, 3$.

Proof. By Lemma 3.1 we have the inequality

$$W_{2n+1,2n}\left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\right) \leq k W_{2n,2n-1}\left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\right).$$

and therefore $W_{2n+1,2n}\left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\right) \leq k^l W_{2n+1-l,2n-l}\left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\right)$.

Consequently using the inequalities $d_i \leq \left\| x_i^{(2n+1)} - x_i^{(2n)} \right\|$ for $i = 1, 2, 3$ and for any $n \in \mathbb{N}$ we get

$$\begin{aligned} \left\| x_i^{(2n+1)} - x_i^{(2n)} \right\| &\leq k^l W_{2n+1-l,2n-l}\left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\right) + d_i + \sum_{\substack{j=1 \\ j \neq i}}^{3} \left(d_j - \left\| x_j^{(2n+1)} - x_j^{(2n)} \right\| \right) \\ &\leq k^l W_{2n+1-l,2n-l}\left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\right) + d_i. \end{aligned}$$

□

Lemma 3.6. *Let* $(X, \|\cdot\|)$ *be a uniformly convex Banach space and the ordered pair* (F, G) *be a generalized cyclic contraction of type* 2*. Then there holds the inequalities*

$$\delta_{\|\cdot\|}\left(\frac{\left\| x_i^{(2n+2)} - x_i^{(2n)} \right\|}{d_i + k^l W_{2n+1-l,2n-l}\left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\right)} \right) \leq \frac{k^l W_{2n+1-l,2n-l}\left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\right)}{d_i + k^l W_{2n+1-l,2n-l}\left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\right)}$$

for $i = 1, 2, 3$.

Proof. From Lemma 3.5 we have the inequalities

$$\left\|x_i^{(2n+1)}-x_i^{(2n)}\right\| \le d_i + k^l W_{2n+1-l,2n-l}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right),$$
$$\left\|x_i^{(2n+2)}-x_i^{(2n+1)}\right\| \le d_i + k^{l+1} W_{2n+1-l,2n-l}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right) \le d_i + k^l W_{2n+1-l,2n-l}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)$$

and

$$\left\|x_i^{(2n+2)}-x_i^{(2n)}\right\| \le \left\|x_i^{(2n+2)}-x_i(2n+1)\right\| + \left\|x_i^{(2n+1)}-x_i^{(2n)}\right\| \le 2\left(d_i + k^l W_{2n+1-l,2n-l}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)\right)$$

for $i=1,2,3$.

After substituting in (1) with $x = x_i^{(2n)}$, $y = x_i^{(2n+2)}$, $z = x_i^{(2n+1)}$, $R = d_i + k^l W_{2n+1-l,2n-l}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)$ and $r = \left\|x_i^{(2n+2)}-x_i^{(2n)}\right\|$ and using the convexity of the set A_i we get the chain of inequalities

$$\begin{array}{rcl} d_i & \le & \left\|\frac{x_i^{(2n)}+x_i^{(2n+2)}}{2} - x_i^{(2n+1)}\right\| \\ & \le & \left(1-\delta_{\|\cdot\|}\left(\frac{\left\|x_i^{(2n+2)}-x_i^{(2n)}\right\|}{d_i+k^lW_{2n+1-l,2n-l}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)}\right)\right)\left(d_i + k^l W_{2n+1-l,2n-l}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)\right) \end{array} \tag{12}$$

for every $i=1,2,3$. Thereafter we obtain the inequality

$$\delta_{\|\cdot\|}\left(\frac{\left\|x_i^{(2n+2)}-x_i^{(2n)}\right\|}{d_i + k^l W_{2n+1-l,2n-l}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)}\right) \le \frac{k^l W_{2n+1-l,2n-l}(x,y)}{d_i + k^l W_{2n+1-l,2n-l}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)}$$

for $i=1,2,3$. □

4. Proof of the Main Results

Proof of Theorem 2.2. It is easy to observe that for any $n\in\mathbb{N}$ there hold the inequalities

$$\begin{array}{rcl} \sum_{i=1}^3 \rho\left(x_i^{(2n+1)},x_i^{(2n)}\right) & = & \sum_{i=1}^3 \rho\left(F_i(x_1^{(2n)},x_2^{(2n)},x_3^{(2n)}),G_i\left(x_i^{(2n-1)},x_i^{(2n-1)},x_i^{(2n-1)}\right)\right. \\ & \le & \sum_{i=1}^3\sum_{j=1}^3 \alpha_j^{(i)}\rho\left(x_i^{(2n)},x_i^{(2n-1)}\right) \le k\sum_{i=1}^3 \rho\left(x_i^{(2n)},x_i^{(2n-1)}\right). \end{array}$$

Consequently

$$\sum_{i=1}^3 \rho\left(x_i^{(2n+1)},x_i^{(2n)}\right) \le k^l \sum_{i=1}^3 \rho\left(x_i^{(2n+1-l)},x_i^{(2n-l)}\right). \tag{13}$$

(I) Let $\left(x_1^{(0)},x_2^{(0)},x_3^{(0)}\right)\in A^3$ be arbitrary chosen. Let $\left\{x_i^{(n)}\right\}_{n=0}^\infty$ for $i=1,2,3$ be the sequences defined in Definition 1.4. Then from (13), because one of $n+1$ or n is even and the other is an odd number, applied for $l=n$ we have

$$\max\left\{\rho\left(x_i^{(n+1)},x_i^{(n)}\right) : i=1,2,3\right\} \le k^n \sum_{i=1}^3 \rho\left(x_i^{(0)},x_i^{(1)}\right).$$

Thus

$$\begin{array}{rcl} \rho\left(x_i^{(n)},x_i^{(n+m)}\right) & \le & \sum_{j=n}^{n+m-1} \rho\left(x_i^{(j)},x_i^{(j+1)}\right) \le \sum_{j=n}^{n+m-1} k^j \sum_{i=1}^3 \rho\left(x_i^{(0)},x_i^{(1)}\right) \\ & \le & k^n \frac{1-k^{n+m}}{1-k}\sum_{i=1}^3 \rho\left(x_i^{(0)},x_i^{(1)}\right). \end{array} \tag{14}$$

Since $k \in (0,1)$ it follows that the sequences $\left\{x_i^{(n)}\right\}_{n=0}^{\infty}$, $i = 1,2,3$ are Cauchy sequences in $A_i \cup B_i$, respectively. Consequently any of the sequences $\left\{x_i^{(n)}\right\}_{n=0}^{\infty}$, $i = 1,2,3$ converges to some $\xi_i \in A_i \cap B_i$. However any of the sequences $\left\{x_i^{(n)}\right\}_{n=0}^{\infty}$ has an infinite number of terms in A_i and in B_i for $i = 1,2,3$, respectively and therefore $\xi_i \in A_i \cap B_i$, $i = 1,2,3$. So $A_i \cap B_i \neq \emptyset$ for $i = 1,2,3$.

Now, we will prove that the pair (ξ_1, ξ_2, ξ_3) is a tripled fixed point of F. Let $\xi_i \in A_i \cap B_i$ for $i = 1,2,3$. WLOG we can assume that $(\xi_1, \xi_2, \xi_3) \in A^3$. It follows that $(F_i\xi_1, \xi_2, \xi_3), F_2(\xi_1, \xi_2, \xi_3), F_3(\xi_1, \xi_2, \xi_3)) \in B^3$. Then by the triangle inequality we get the chain of inequalities

$$\begin{aligned}
\sum_{i=1}^{3} \rho(\xi_i, F_i(\xi_1, \xi_2, \xi_3)) &\leq \sum_{i=1}^{3} \left(\rho\left(\xi_i, x_i^{(2n)}\right) + \rho\left(x_i^{(2n)}, F_i(\xi_1, \xi_2, \xi_3)\right)\right) \\
&\leq \sum_{i=1}^{3} \rho\left(\xi_i, x_i^{(2n)}\right) + \sum_{i=1}^{3} \rho\left(G_i\left(x_1^{(2n-1)}, x_2^{(2n-1)}, x_3^{(2n-1)}\right), F_i(\xi_1, \xi_2, \xi_3)\right) \\
&\leq \sum_{i=1}^{3} \rho\left(\xi_i, x_i^{(2n)}\right) + \sum_{j=1}^{3} \sum_{i=1}^{3} \alpha_i^{(j)} \rho\left(x_j^{(2n-1)}, \xi_j\right).
\end{aligned}$$

Taking the limit when $n \to \infty$, we obtain $\sum_{i=1}^{3} \rho(\xi_i, F_i(\xi_1, \xi_2, \xi_3)) = 0$, i.e $\rho(\xi_i, F_i(\xi_1, \xi_2, \xi_3)) = 0$ for $i = 1,2,3$. Thus the triple (ξ_1, ξ_2, ξ_3) is a tripled fixed point of $F = (F_1, F_2, F_3)$.

The proof that the pair (ξ_1, ξ_2, ξ_3) is a tripled fixed point of $G = (G_1, G_2, G_3)$ can be done in a similar fashion by assuming that $(\xi_1, \xi_2, \xi_3) \in B^3$.

We still have to prove that (ξ_1, ξ_2, ξ_3) is the unique tripled fixed point of F. Arguing by contradiction, suppose there exists $\left(\xi_1^*, \xi_2^*, \xi_3^*\right) \in (A_1 \cup B_1) \times (A_2 \cup B_2) \times (A_3 \cup B_3)$ such that $\left(\xi_1^*, \xi_2^*, \xi_3^*\right) \neq (\xi_1, \xi_2, \xi_3)$ and $\xi_i^* = F_i\left(\xi_1^*, \xi_2^*, \xi_3^*\right)$ for $i = 1,2,3$. If we suppose that $\xi_1^* \in A_1$ then by the definition of a tripled fixed point it follows that $\xi_i^* \in A_i$ for $i = 2,3$ and therefore $\xi_i^* = F_i\left(\xi_1^*, \xi_2^*, \xi_3^*\right) \in B_i$ for $i = 1,2,3$. A similar argument holds if we assume that $\xi_1^* \in B_1$. Thus we can assume that if the pair $\left(\xi_1^*, \xi_2^*, \xi_3^*\right)$ is a tripled fixed point of F then $\left(\xi_1^*, \xi_2^*, \xi_3^*\right) \in (A_1 \cap B_1) \times (A_2 \cap B_2) \times (A_3 \cap B_3)$. Using the observation that (ξ_1, ξ_2, ξ_3) and $\left(\xi_1^*, \xi_2^*, \xi_3^*\right)$ are tripled fixed points and for G, we have the inequalities

$$\sum_{i=1}^{3} \rho\left(\xi_i^*, \xi_i\right) = \sum_{i=1}^{3} \rho\left(F_i\left(\xi_1^*, \xi_2^*, \xi_3^*\right), G_i(\xi_1, \xi_2, \xi_3)\right) \leq \sum_{i=1}^{3} \sum_{i=1}^{3} \alpha_i^{(j)} \rho\left(\xi_j^*, \xi_j\right) < \sum_{i=1}^{3} \rho\left(\xi_i^*, \xi_i\right).$$

It results that $\rho\left(\xi_i^*, \xi_i\right)$ for $i = 1,2,3$, which is a contradiction and therefore (ξ_1, ξ_2, ξ_3) is the unique tripled fixed point of F.

The proof that the pair (ξ_1, ξ_2, ξ_3) is the unique tripled fixed point of G can be done in a similar way.

(II) Letting $m \to \infty$ in (14) we obtain the a priori estimate

$$\rho\left(x_i^{(n)}, \xi_i\right) \leq \frac{k^n}{1-k} \sum_{i=1}^{3} \rho\left(x_i^{(0)}, x_i^{(1)}\right).$$

Therefore

$$\max\{\rho\left(x_i^n, \xi_i\right) : i = 1,2,3\} \leq \frac{k^n}{1-k} \sum_{i=1}^{3} \rho\left(x_i^{(0)}, x_i^{(1)}\right).$$

(III) From the inequality (13) applied for $l = k+1$ we get

$$\begin{aligned}
\rho\left(x_i^{(n)}, x_i^{(n+m)}\right) &\leq \sum_{j=0}^{m-1} \rho\left(x_i^{(n+j)}, x_i^{(n+j+1)}\right) \leq \sum_{j=0}^{m-1} k^{j+1} \sum_{i=1}^{3} \rho\left(x_i^{(n-1)}, x_i^{(n)}\right) \\
&\leq \frac{k}{1-k}\left(1 - k^{m+1}\right) \sum_{i=1}^{3} \rho\left(x_i^{(n-1)}, x_i^{(n)}\right).
\end{aligned}$$

Letting $m \to \infty$ we obtain the a posteriori estimate

$$\rho\left(x_i^{(n)}, \xi_i\right) \le \frac{k}{1-k} \sum_{i=1}^{3} \rho\left(x_i^{(n-1)}, x_i^{(n)}\right)$$

and thus

$$\max\left\{\rho\left(x_i^{(n)}, \xi_i\right) : i = 1,2,3\right\} \le \frac{k}{1-k} \sum_{i=1}^{3} \rho\left(x_i^{(n-1)}, x_i^{(n)}\right).$$

(IV) Considering that the triple (ξ_1, ξ_2, ξ_3) is a tripled fixed point for F we have the inequalities

$$\begin{aligned}\sum_{i=1}^{3} \rho\left(x_1^{(2n)}, \xi_i\right) &= \sum_{i=1}^{3} \rho\left(G_i\left(x_1^{(2n-1)}, x_2^{(2n-1)}, x_3^{(2n-1)}\right), F_i(\xi_1, \xi_2, \xi_3)\right) \le \sum_{j=1}^{3}\sum_{i=1}^{3} \alpha_i^{(j)} \rho(x_j^{(2n-1)}, \xi_j) \\ &\le k \sum_{j=1}^{3} \rho\left(x_j^{(2n-1)}, \xi_j\right).\end{aligned}$$

By similar arguments we get

$$\begin{aligned}\textstyle\sum_{i=1}^{3} \rho\left(x_i^{(2n+1)}, \xi_i\right) &= \textstyle\sum_{i=1}^{3} \rho\left(F_i\left(x_i^{(2n)}, y_i^{(2n)}, z_i^{(2n)}\right), G_i\left(\xi_1, \xi_2, \xi_i\right)\right) \le \sum_{j=1}^{3}\sum_{i=1}^{3} \alpha_i^{(j)} \rho\left(x_i^{(2n)}, \xi_i\right) \\ &\le \textstyle k\sum_{i=1}^{3} \rho\left(x_i^{(2n)}, \xi_i\right).\end{aligned}$$

Consequently $\sum_{i=1}^{3} \rho\left(x_i^{(n)}, \xi_i\right) \le k\sum_{i=1}^{3} \rho\left(x_i^{(n-1)}, \xi_i\right)$. □

Proof of Theorem 2.4. For any initial guess $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) \in A^3$ it follows from Lemma 3.4 that the sequences $\left\{x_i^{(2n)}\right\}_{n=0}^{\infty}$, $\left\{x_i^{(2n+1)}\right\}_{n=0}^{\infty}$ for $i = 1,2,3$ are Cauchy sequences. From the assumptions that $(X, \|\cdot\|)$ is a Banach space and A_i, B_i for $i = 1,2,3$ are closed it follows that there are $(\xi_1, \xi_2, \xi_3) \in A^3$, so that $\lim_{n\to\infty} x_i^{(2n)} = \xi_i \in A_i$ for $i = 1,2,3$.

From the inequalities by using the continuity of the norm function $\|\cdot - \cdot\|$ and Lemma 3.1 we get

$$\begin{aligned}\sum_{i=1}^{3} \|\xi_i - F_i(\xi_1, \xi_2, \xi_3)\| - d &= \lim_{n\to\infty} \sum_{i=1}^{3} \left\|x_i^{(2n)} - F_i(\xi_1, \xi_2, \xi_3)\right\| - d \\ &= \lim_{n\to\infty} \sum_{i=1}^{3} \left\|G_i\left(x_i^{(2n-1)}, x_i^{(2n-1)}, x_i^{(2n-1)}\right) - F_i(\xi_1, \xi_2, \xi_3)\right\| - d \\ &\le \lim_{n\to\infty} \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_j^{(i)} \left(\left\|x_i^{(2n-1)} - \xi_i\right\| - d_i\right) = 0.\end{aligned}$$

Thus $\|\xi_i - F_i(\xi_1, \xi_2, \xi_3)\| = d_i$ for $i = 1,2,3$. It can be proven in a similar fashion that $\sum_{i=1}^{3} \|v_i - G_i(v_1, v_2, v_3)\| \le d$ and consequently $\|v_i - G_i(v_1, v_2, v_3)\| = d_i$ for $i = 1,2,3$.

It has remained to prove that there holds (5). Indeed from the inequalities

$$\begin{aligned}S_5 &= \sum_{i=1}^{3} \|G_i(F_1(\xi_1, \xi_2, \xi_3), F_2(\xi_1, \xi_2, \xi_3), F_3(\xi_1, \xi_2, \xi_3)) - F_i(\xi_1, \xi_2, \xi_3)\| \\ &\le \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_j^{(i)} \|\xi_i - F_i(\xi_1, \xi_2, \xi_3)\| + \sum_{i=1}^{3}\left(1 - \sum_{j=1}^{3} \alpha_j^{(i)}\right) d_i = d\end{aligned}$$

it follows that

$$\|G_i(F_1(\xi_1, \xi_2, \xi_3), F_2(\xi_1, \xi_2, \xi_3), F_3(\xi_1, \xi_2, \xi_3)) - F_i(\xi_1, \xi_2, \xi_3)\| = d_i$$

for $i = 1,2,3$. From the assumption that (ξ_1, ξ_2, ξ_3) is a tripled best proximity point for F i.e. $\|\xi_i - F_x(\xi_1, \xi_2, \xi_3)\| = d_i$ for $i = 1,2,3$ and the uniform convexity of $(X, \|\cdot\|)$ it follows that

$$G_i(F_1(\xi_1, \xi_2, \xi_3), F_2(\xi_1, \xi_2, \xi_3), F_3(\xi_1, \xi_2, \xi_3)) = \xi_i \text{ for } i = 1,2,3.$$

By similar arguments it can be proven that

$$F_i(G_1(v_1, v_2, v_3), G_2(v_1, v_2, v_3), G_3(v_1, v_2, v_3)) = v_i$$

for $i = 1,2,3$.

We will prove the uniqueness of the tripled best proximity points. Let us suppose that the tripled best proximity point (ξ_1, ξ_2, ξ_3) of F is not unique, i.e. there exists $(\xi_1^*, \xi_2^*, \xi_3^*)$, such that $\|\xi_i^* - F_x(\xi_1^*, \xi_2^*, \xi_3^*)\| = d_i$ for $i = 1,2,3$. By similar arguments it can be proven that

$$G_i(F_1(\xi_1, \xi_2, \xi_3), F_2(\xi_1, \xi_2, \xi_3), F_3(\xi_1, \xi_2, \xi_3)) = \xi_i$$

and

$$G_i(F_1(\xi_1^*, \xi_2^*, \xi_3^*), F_2(\xi_1^*, \xi_2^*, \xi_3^*), F_3(\xi_1^*, \xi_2^*, \xi_3^*)) = \xi_i^*$$

for $i = 1,2,3$. Thus we get the inequalities

$$\begin{aligned} \sum_{i=1}^{3} \left\|\xi_i - F_i\left(\xi_1^*, \xi_2^*, \xi_3^*\right)\right\| &= \textstyle\sum_{i=1}^{3} \left\|G_i(F_1(\xi_1, \xi_2, \xi_3), F_2(\xi_1, \xi_2, \xi_3), F_3(\xi_1, \xi_2, \xi_3)) - F_i\left(\xi_1^*, \xi_2^*, \xi_3^*\right)\right\| \\ &\le \sum_{j=1}^{3}\sum_{i=1}^{3} \alpha_i^{(j)} \|\xi_j^* - F_j(\xi_1, \xi_2, \xi_3)\| + \sum_{j=1}^{3}\left(1 - \sum_{i=1}^{3}\alpha_i^{(j)}\right) d_j \\ &\le \sum_{i=1}^{3} \|\xi_i^* - F_i(\xi_1, \xi_2, \xi_3)\|. \end{aligned} \tag{15}$$

By similar calculations we obtain

$$\sum_{i=1}^{3} \left\|\xi_i^* - F_i(\xi_1, \xi_2, \xi_3)\right\| \le \sum_{i=1}^{3} \left\|\xi_i - F_i\left(\xi_1^*, \xi_2^*, \xi_3^*\right)\right\|. \tag{16}$$

Consequently there holds

$$\sum_{i=1}^{3} \left\|\xi_i^* - F_i(\xi_1, \xi_2, \xi_3)\right\| = \sum_{i=1}^{3} \left\|\xi_i - F_i\left(\xi_1^*, \xi_2^*, \xi_3^*\right)\right\|. \tag{17}$$

We will show that $\displaystyle\sum_{i=1}^{3} \left\|\xi_i^* - F_i(\xi_1, \xi_2, \xi_3)\right\| = d$. Let us assume the contrary, i.e. $\displaystyle\sum_{i=1}^{3} \left\|\xi_i^* - F_i(\xi_1, \xi_2, \xi_3)\right\| > d$. Then there holds at least on of the inequalities $\left\|\xi_i^* - F_i(\xi_1, \xi_2, \xi_3)\right\| > d_i$ for some $i = 1,2,3$. Then from the chain of inequalities

$$\begin{aligned} \sum_{i=1}^{3} \|\xi_i - F_i(\xi_1^*, \xi_2^*, \xi_3^*)\| &= \sum_{i=1}^{3} \|G_i(F_1(\xi_1, \xi_2, \xi_3), F_2(\xi_1, \xi_2, \xi_3), F_3(\xi_1, \xi_2, \xi_3\xi_1, \xi_2, \xi_3)) - F_i(\xi_1^*, \xi_2^*, \xi_3^*)\| \\ &\le \sum_{j=1}^{3}\sum_{i=1}^{3} \alpha_i^{(j)} \|\xi_j^* - F_j(\xi_1, \xi_2, \xi_3)\| + \sum_{j=1}^{3}\left(1 - \sum_{i=1}^{3}\alpha_i^{(j)}\right) d_j \\ &< \sum_{i=1}^{3} \|\xi_i^* - F_i(\xi_1, \xi_2, \xi_3)\|. \end{aligned}$$

we get a contradiction with (17). Therefore $\sum_{i=1}^{3}\|\xi_i^*-F_i(\xi_1,\xi_2,\xi_3)\|=d$. From the above equalities, the uniform convexity of $(X,\|\cdot\|)$ and $\|\xi_i-F_i(\xi_1,\xi_2,\xi_3)\|=d_i$ for $i=1,2,3$ it follows that $(\xi_1^*,\xi_2^*,\xi_3^*)=(\xi_1,\xi_2,\xi_3)$.

The proof that $(\upsilon_1,\upsilon_2,\upsilon_3)\in B_1\times B_2\times B_3$ is a unique triples best proximity point of G can be done in a similar fashion.

(i) From the uniform convexity of X is follows that $\delta_{\|\cdot\|}$ is strictly increasing and therefore there exists its inverse function $\delta_{\|\cdot\|}^{-1}$, which is strictly increasing too. From Lemma 3.6 we get

$$\left\|x_i^{(2n)}-x_i(2n+2)\right\|\le\left(d_i+k^lW_{2n+1-l,2n-l}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)\right)\delta_{\|\cdot\|}^{-1}\left(\frac{k^lW_{2n+1-l,2n-l}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)}{d_i+k^lW_{2n+1-l,2n-l}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)}\right)\tag{18}$$

for $i=1,2,3$. By the inequality $\delta_{\|\cdot\|}(t)\ge Ct^q$ it follows that $\delta_{\|\cdot\|}^{-1}(t)\le\left(\frac{t}{C}\right)^{1/q}$. From (18) and the inequalities

$$d_i\le d_i+k^lW_{2n+1-l,2n-l}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)\le P_{2n-l,2n+1-l}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right),\ \ i=1,2,3$$

we obtain

$$\begin{array}{rcl}S_6&=&\left\|x_i^{(2n)}-x_i^{(2n+2)}\right\|\\ &\le&\left(d_i+k^lW_{2n+1-l,2n-l}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)\right)\sqrt[q]{\frac{k^lW_{2n+1-l,2n-l}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)}{C\left(d_x+k^lW_{2n+1-l,2n-l}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)\right)}}\\ &\le&P_{2n-l,2n+1-l}(x_1^{(0)},x_2^{(0)},x_3^{(0)})\sqrt[q]{\frac{W_{2n+1-l,2n-l}(x_1^{(0)},x_2^{(0)},x_3^{(0)})}{Cd_i}}\sqrt[q]{k^l}\end{array}\tag{19}$$

for $i=1,2,3$.

We have proven that there exists a unique pair $(\xi_1,\xi_2,\xi_3)\in A^3$, such that $\|\xi_i-F(\xi_1,\xi_2,\xi_3)\|=d_i$ and ξ_i is a limit of the sequence $\left\{x_i^{(2n)}\right\}_{n=1}^{\infty}$ for any $\left(x_1^{(0)},x_2^{(0)},x_3^{(0)}\right)\in A^3$.

After a substitution with $l=2n$ in (19) we get the inequality

$$\begin{array}{rcl}\displaystyle\sum_{n=1}^{\infty}\left\|x_i^{(2n)}-x_i^{(2n+2)}\right\|&\le&\displaystyle\sum_{i=1}^{3}\left\|x_i^{(0)}-x_i^{(1)}\right\|\sqrt[q]{\frac{\sum_{i=1}^{3}\left\|x_i^{(0)}-x_i^{(1)}\right\|-d}{Cd_i}}\sum_{n=1}^{\infty}\sqrt[q]{k^{2n}}\\ &=&\displaystyle\sum_{i=1}^{3}\left\|x_i^{(0)}-x_i^{(1)}\right\|\sqrt[q]{\frac{\sum_{i=1}^{3}\left\|x_i^{(0)}-x_i^{(1)}\right\|-d}{Cd_i}}\cdot\frac{\sqrt[q]{k^2}}{1-\sqrt[q]{k^2}}\end{array}$$

and consequently the series $\sum_{n=1}^{\infty}\left(x_i^{(2n)}-x_i^{(2n+2)}\right)$ for $i=1,2,3$ are absolutely convergent. Thus for any $m\in\mathbb{N}$ there holds

$$\xi_i=x_i^{(2m)}-\sum_{n=m}^{\infty}\left(x_i^{(2n)}-x_i^{(2n+2)}\right)$$

and therefore we get the inequality

$$\left\|\xi_i-x_i^{(2m)}\right\|\le\sum_{n=m}^{\infty}\left\|x_i^{(2n)}-x_i^{(2n+2)}\right\|\le\sum_{i=1}^{3}\left\|x_i^{(0)}-x_i^{(1)}\right\|\sqrt[q]{\frac{\sum_{i=1}^{3}\left\|x_i^{(0)}-x_i^{(1)}\right\|-d}{Cd_i}}\cdot\frac{\sqrt[q]{k^{2m}}}{1-\sqrt[q]{k^2}}.$$

(ii) After a substitution with $l=1+2j$ in (19) we obtain

$$\left\|x_i^{(2n+2j)}-x_i^{(2n+2(j+1))}\right\|\le P_{2n-1,2n}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)\sqrt[q]{\frac{W_{2n-1,2n}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)}{Cd_i}}\left(\sqrt[q]{k}\right)^{1+2j}.\tag{20}$$

From (20) we get that there holds the inequality

$$\begin{array}{rcl}\left\|x_i^{(2n)}-x_i^{(2(n+m))}\right\| & \leq & \displaystyle\sum_{j=0}^{m-1}\left\|x_i^{(2n+2j)}-x_i^{(2n+2(j+1))}\right\| \\ & \leq & \sum_{j=0}^{m-1}P_{2n-1,2n}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)\sqrt[q]{\frac{W_{2n-1,2n}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)}{Cd_i}}\sqrt[q]{k^{1+2j}} \\ & = & P_{2n-1,2n}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)\sqrt[q]{\dfrac{W_{2n-1,2n}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)}{Cd_x}}\displaystyle\sum_{i=0}^{m-1}\sqrt[q]{k^{1+2j}} \\ & = & P_{2n-1,2n}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)\sqrt[q]{\frac{W_{2n-1,2n}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)}{Cd_i}}\cdot\frac{1-\sqrt[q]{k^{2m}}}{1-\sqrt[q]{k^2}}\sqrt[q]{k}\end{array} \tag{21}$$

and after letting $m\to\infty$ in (21) we obtain the inequality

$$\left\|x_i^{(2n)}-\xi_i\right\|\leq P_{2n,2n-1}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)\sqrt[q]{\frac{W_{2n,2n-1}\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)}{Cd_x i}}\frac{\sqrt[q]{k}}{1-\sqrt[q]{k^2}}$$

for $i=1,2,3$. □

5. Applications

Let us consider some particular cases.

The next theorem is a variant of results from [1, 19].

Theorem 5.1. *Let A and B be a closed subsets of a complete metric space (X,ρ), $F: A\times A\times A\to B$ and $G: B\times B\times B\to B$ be such that the inequality*

$$\rho(F(x,y,z),G(u,v,w))\leq\alpha_1\rho(x,u)+\alpha_2\rho(y,v)+\alpha_3\rho(z,w)$$

holds for any $(x,y,z)\in A\times A\times A$, $(u,v,w)\in B\times B\times B$ and for some $\alpha_i\in[0,1)$ for $i=1,2,3$ satisfying the inequality $\sum_{i=1}^3\alpha_i<1$. Then there exist $\xi\in A\cap B$, such that (ξ,ξ,ξ) is a unique tripled fixed point of F.

If we put $A_x=A_y=A_z=A$, $B_x=B_y=B_z=B$, $F_1(x,y,z)=F(x,y,z)$, $F_2(x,y,z)=F(y,z,x)$, $F_3(z,x,y)=F(x,y,z)$, $G_1(x,y,z)=G(x,y,z)$, $G_2(x,y,z)=G(y,z,x)$, $G_3(z,x,y)=G(x,y,z)$ in Theorem 2.2 we get

$$\sum_{i=1}^3\rho(F_i(x_i^{(1)},x_i^{(2)},x_i^{(3)}),G_i(u_i^{(1)},u_i^{(2)},u_i^{(3)})\leq\sum_{j=1}^3\sum_{i=1}^3\alpha_i^{(j)}\rho(x_i^{(j)},u_i^{(j)}).$$

Thus the map satisfies Theorem 2.2 with $\alpha_1^{(1)}=\alpha_2^{(2)}=\alpha_3^{(3)}=\alpha_1$, $\alpha_2^{(1)}=\alpha_1^{(2)}=\alpha_1^{(3)}=\alpha_3$, $\alpha_3^{(1)}=\alpha_3^{(2)}=\alpha_2^{(3)}=\alpha_2$ for $j=1,2,3$ because $\sum_{i=1}^3\alpha_i\in[0,1)$.

By Theorem 2.2 there are $x,y,z\in A\cap B$, which are the unique tripled (x,y,z) fixed point for the map F and the unique tripled fixed point for the map G too. Then

$$\begin{array}{rcl}\rho(x,y)+\rho(y,z)+\rho(z,x) & = & \rho(F(x,y,z),G(y,z,x)+\rho(F(y,z,x),G(z,x,y)+\rho(F(z,x,y),G(x,y,z) \\ & \leq & \displaystyle\sum_{i=1}^3\alpha_i(\rho(x,y)+\rho(y,z)+\rho(z,x))<\rho(x,y)+\rho(y,z)+\rho(z,x).\end{array}$$

Because of $\sum_{i=1}^3\alpha_i\in[0,1)$ it follows that $\rho(x,y)+\rho(y,z)+\rho(z,x)=0$, i.e. $x=y=z$.

The next result is a variant of [8].

Theorem 5.2. *Let A be a closed subset of a complete metric space and $F : A\times A\times A \to A$ be such that the inequality*

$$\rho(F(x,y,z),F(u,v,w)) \le \alpha\rho(x,u)+\beta\rho(y,v)+\gamma\rho(z,w)$$

holds for any $(x,y,z),(u,v,w)\in A\times A\times A$ and for some $\alpha,\beta,\gamma\in[0,1)$ satisfying the inequality $2\alpha+\beta+\gamma<1$. Then there exist a unique tripled fixed point $(x,y,z)\in A\times A\times A$, such that $x=F(x,y,z)$, $y=F(y,x,y)$ and $z=F(z,y,x)$.

If put $A_x=A_y=A_z=B_x=B_y=B_z=A$, $F_1(x,y,z)=F(x,y,z)$, $F_2(x,y,z)=F(y,x,y)$, $F_3(x,y,z)=F(z,y,x)$ in Theorem 2.2 we get

$$\begin{array}{rcl} S_7 &=& \sum_{i=1}^{3}\rho\left(F_i\left(x_i^{(1)},x_i^{(2)},x_i^{(3)}\right),G_i\left(y_i^{(1)},y_i^{(2)},y_i^{(3)}\right)\right)\\ &\le& \alpha\rho\left(x_1^{(1)},u_1^{(1)}\right)+\beta\rho\left(x_1^{(2)},u_1^{(2)}\right)+\gamma\rho\left(x_1^{(3)},u_1^{(3)}\right)+\beta\rho\left(x_2^{(1)},y_2^{(1)}\right)+(\alpha+\gamma)\rho\left(x_2^{(2)},y_2^{(2)}\right)\\ && +\alpha\rho\left(x_3^{(3)},y_3^{(3)}\right)+\beta\rho\left(x_3^{(2)},y_3^{(2)}\right)+\gamma\rho\left(x_3^{(1)},y_3^{(1)}\right)\\ &=& \alpha\rho\left(x_1^{(1)},u_1^{(1)}\right)+\beta\rho\left(x_2^{(1)},y_2^{(1)}\right)+\gamma\rho\left(x_3^{(1)},y_3^{(1)}\right)\\ && +\beta\left(\rho\left(x_1^{(2)},u_1^{(2)}\right)+\rho\left(x_3^{(2)},y_3^{(2)}\right)\right)+(\alpha+\gamma)\rho\left(x_2^{(2)},y_2^{(2)}\right)+\gamma\rho\left(x_1^{(3)},u_1^{(3)}\right)+\alpha\rho\left(x_3^{(3)},y_3^{(3)}\right) \end{array}$$

Thus the map satisfies Theorem 2.2 with $\alpha_1^{(1)}=\alpha_3^{(3)}=\alpha$, $\alpha_2^{(1)}=\alpha_1^{(2)}=\alpha_3^{(2)}=\beta$, $\alpha_3^{(1)}=\alpha_1^{(3)}=\gamma$, $\alpha_2^{(2)}=\alpha+\gamma$ and $\alpha_2^{(3)}=0$.

Following [15] we may assume that $(\mathbb{R},|\cdot|)$ is a uniformly convex and $\delta_{(\mathbb{R},|\cdot|)}(\varepsilon)=\frac{\varepsilon}{2}$. In this case the inverse function δ_X^{-1} exists ant is equal to 2ε. Thus we get the following corollary of Theorem 2.2:

Corollary 5.3. *Let A_i,B_i, $i=1,2,3$ be nonempty, closed subsets of a complete metric space $(\mathbb{R},|\cdot|)$. Let the ordered pair (F,G) be a generalized cyclic contraction of type 1. Then*

(I) *There exists a unique triple (ξ_1,ξ_2,ξ_3), such that $\xi_i\in(A_i\cap B_i)$ for $i=1,2,3$, which is a common triple fixed point for the maps F and G. Moreover the iteration sequences $\{x_i^{(n)}\}_{n=0}^{\infty}$ for $i=1,2,3$, defined in Definition 1.4 converge to ξ_i, respectively;*

(II) *a priori error estimates hold*

$$\max\left\{\left|x_i^{(n)}-\xi_i\right| : i=1,2,3\right\}\le\frac{k^n}{1-k}\sum_{i=1}^{3}\left|x_i^{(0)}-x_i^{(1)}\right|; \tag{22}$$

(III) *a posteriori error estimates hold*

$$\max\left\{\left|x_i^{(n)}-\xi_i\right| : i=1,2,3\right\}\le\frac{k}{1-k}\sum_{i=1}^{3}\left|x_i^{(n-1)}-x_i^{(n)}\right|; \tag{23}$$

(IV) *The rate of convergence for the sequences of successive iterations is given by*

$$\sum_{i=1}^{3}\left|x_i^{(n)}-\xi_i\right|\le k\sum_{i=1}^{3}\left|x_i^{(n-1)}-\xi_i\right|. \tag{24}$$

We will illustrate Corollary 5.3 by solving the next system:

Example 5.4. *Let us consider the system of nonlinear equations:*

$$\left|\begin{array}{rcccccl} -9x &+& e^{y-1} &+& 3\mathrm{arctg}(z-2) &=& 0\\ -24x+3x^2 &+& e^{(y-1)^2} &+& 3\mathrm{arctg}((z-2)^2) &=& -36\\ -36x+3x^3 &+& e^{(y-1)^3} &+& 3\mathrm{arctg}((z-2)^3) &=& -90. \end{array}\right. \tag{25}$$

Solution. Let us consider the functions

$$F_1(x,y,z)=\frac{x}{4}+\frac{e^{y-1}}{12}+\frac{\text{arctg}(z-2)}{4},\ \ F_2(x,y,z)=\frac{x^2}{8}+\frac{e^{(y-1)^2}}{24}+\frac{\text{arctg}((z-2)^2)}{8}+1.5,$$

$$F_3(x,y,z)=\frac{x^3}{12}+\frac{e^{(y-1)^3}}{36}+\frac{\text{arctg}((z-2)^3)}{12}+2.5.$$

It is easy to check that $F_1:[0,1]\times[1,2]\times[2,3]\to[0,1]$, $F_2:[0,1]\times[1,2]\times[2,3]\to[1,2]$, $F_3:[0,1]\times[1,2]\times[2,3]\to[2,3]$ ($F_1(0,1,2)>0$, $F_1(1,2,3)<1$, $F_2(0,1,2)>1$, $F_2(1,2,3)<2$, $F_3(0,1,2)>2$, $F_3(1,2,3)<3$) and the system

$$\left|\begin{array}{l} x=F_1(x,y,z)\\ y=F_2(x,y,z)\\ y=F_3(x,y,z)\end{array}\right. \tag{26}$$

is equivalent to (25).

Let us put $B_1=A_1=[0,1]$, $B_2=A_y2=[1,2]$, $B_3=A_3=[2,3]$, $G_i=F_i$ for $i=1,2,3$.

Using the inequalities $|e^{(y-1)^n}-e^{(v-1)^n}|\le ne|y-v|$ for $y,v\in[1,2]$; $|\arctan((z-2)^n)-\arctan((w-2)^n)|\le n|z-w|$ for $z,t\in[2,+\infty)$; $|x^n-u^n|\le n|x-u|$ for $x,y\in[0,1]$ it is easy to obtain that

$$\left|F_i(x,y,z)-G_i(u,v,w)\right|\le\frac{|x-u|}{4}+\frac{e|y-v|}{12}+\frac{|z-w|}{4}$$

for $i=1,2,3$. Thus we get

$$\sum_{i=1}^3\left|F_i\left(x_i^{(1)},x_i^{(2)},x_i^{(3)}\right)-G_i\left(y_i^{(1)},y_i^{(0)},y_i^{(3)}\right)\right|\le\frac{1}{4}\sum_{i=1}^3\left|x_i^{(1)}-y_i^{(1)}\right|+\frac{e}{12}\sum_{i=1}^3\left|x_i^{(2)}-y_i^{(2)}\right|+\frac{1}{4}\sum_{i=1}^3\left|x_i^{(3)}-y_i^{(3)}\right|$$

and thus the pair of maps (F,G) satisfies Corollary 5.3 with $\alpha_1^{(1)}=\alpha_2^{(1)}=\alpha_3^{(1)}=1/4$, $\alpha_1^{(2)}=\alpha_2^{(2)}=\alpha_3^{(2)}=e/12$, $\alpha_1^{(3)}=\alpha_2^{(3)}=\alpha_3^{(3)}=1/4$. Thus $k=\max\left\{\sum_{i=1}^3\alpha_i^{(j)}:j=1,2,3\right\}=\max\{3/4,e/4\}=3/4$. Therefore the ordered triple $F=(F_x1,F_2,F_3)$ has a unique tripled fixed points in $[0,1]\times[1,2]\times[2,3]$ and thus the system of equations (25) has a unique solution in $[0,1]\times[1,2]\times[2,3]$. □

Table 1: Number m of iterations needed by the a priori estimate starting from the initial guess $(0,1,2)$

ε	0.1	0.01	0.001	0.0001	0.00001	0.000001
m	14	22	30	38	56	54

Table 2: Number m of iterations needed by the a posteriori estimate $(0,1,2)$

ε	0.1	0.01	0.001	0.0001	0.00001	0.000001
m	5	7	10	12	14	16

We get an approximation of the solution $(0.3741116328, 1.615504553, 2.553428358)$.

References

[1] A. Amini-Harandi. Coupled and tripled fixed point theory in partially ordered metric spaces with application to initial value problem. *Mathematical and Computer Modelling*, 57:2343—-2348, 2013.

[2] B. Beauzamy. *Introduction to Banach Spaces and their Geometry*. North–Holland Publishing Company, Amsterdam, 1979.

[3] V. Berinde. *Iterative Approximation of Fixed Points*. Springer, Berlin, 2007.

[4] V. Berinde. Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces. *Nonlinear Anal.*, 74(18):7347–7355, 2011.

[5] V. Berinde. Coupled fixed point theorems for ϕ–contractive mixed monotone mappings in partially ordered metric spaces. *Nonlinear Anal.*, 75(6):3218–322, 2012.
[6] V. Berinde and M. Păcurar. A constructive approach to coupled fixed point theorems in metric spaces. *Carpathian J. Math*, 31(3):277 – 287, 2015.
[7] T.G. Bhaskar and V. Lakshmikantham. Fixed point theorems in partially ordered metric spaces and applications. *Nonlinear Anal.*, 65(7):1379—-1393, 2006.
[8] M. Borcut and V. Berinde. Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. *Nonlinear Anal.*, 74(15):4889—-4897, 2011.
[9] M. Borcut, M. Păcurar, and V. Berinde. Tripled fixed point theorems for mixed monotone kannan type contractive mappings. *Journal of Applied Mathematics*, 2014/Article ID 120203, 2014.
[10] R. Deville, G. Godefroy, and V. Zizler. *Smothness and renormings in Banach spaces*. Pitman Monographs and Surveys in Pure and Applied Mathematics, 1993.
[11] A. Eldred and P. Veeramani. Existence and convergence of best proximity points. *J. Math. Anal. Appl.*, 323(2):1001–1006, 2006.
[12] M. Fabian, P. Habala, P. Hájek, V. Montesinos, J. Pelant, and V. Zizler. *Functional Analysis and Infinite-Dimensional Geometry*. Springer New York, 2011.
[13] D. Guo and V. Lakshmikantham. Coupled fixed points of nonlinear operators with applications. *Nonlinear Anal.*, 11(5):623–632, 1987.
[14] A. Gupta, S.S. Rajput, and P.S. Kaurav. Coupled best proximity point theorem in metric spaces. *International Journal of Analysis and Applications*, 4(2):201–215, 2014.
[15] A. Ilchev. On an application of coupled best proximity points theorems for solving systems of linear equations. *AIP Conference Proceedings*, 2048, 2018.
[16] A. Ilchev and B. Zlatanov. Error estimates for approximation of coupled best proximity points for cyclic contractive maps. *Applied Mathematics and Computation*, 290:412—-425, 2016.
[17] A. Meir. On the uniform convexity of l_p spaces, $1 < p < 2$s. *Illinois J. Math.*, 28(3):420–424, 1984.
[18] V.O. Olisama, J.O. Olaleru, and H. Olaoluwa. Quadruple best proximity points of q–cyclic contraction pair in abstract metric spaces. *Asian Journal of Mathematics and Applications*, Article ID ama0206, 21 pages, 2015.
[19] B. Samet and C. Vetro. Coupled fixed point, f–invariant set and fixed point of n-order. *Ann.Func.Anal.*, 1(2):46–56, 2010.
[20] W. Sintunavarat and P. Kumam. Coupled best proxitmity point theorem in metric spaces. *Fixed Point Theory Appl.*, 2012/1/93, 2012.
[21] B. Zlatanov. Error estimates for approximating of best proximity points for cyclic contractive maps. *Carpathian J. Math.*, 32(2):241–246, 2016.
[22] B. Zlatanov. Coupled best proximity points for cyclic contractive maps and their applications. *Fixed Point Theory*, 22(1):431–452, 2021.