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Strong Super Convergence of the Balanced Euler Method for a Class of Stochastic Volterra Integro-Differential Equations With Non-Globally Lipschitz Continuous Coefficients

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Abstract. In this paper, we propose the balanced Euler method of a class of stochastic Volterra integro-differential equations with non-globally Lipschitz continuous coefficients. The moment boundedness is studied and the strong convergence is shown to be 1. Moreover, the theoretical results are illustrated by a numerical example.

1. Introduction

Volterra integral equation (VIEs) has been widely used in many fields. Due to the effects of random noise and uncertain factors, such problems are modeled by stochastic Volterra integral equations (SVIEs). Recently many researchers have paid great interest in the theoretical analysis of SVIEs (see [13] and the references cited therein). However, most of SVIEs can not obtain the theoretical solutions because of the complexity of such equations, numerical methods become an important tool. The numerical solutions of SVIEs have been studied by many authors (see [1],[2], [8], and more references cited in it).

Especially recently, some researchers have paid more attention to stochastic Volterra integro-differential equations (SVIDEs). In 2000, Mao [9] considered the stability of a SVIDE as follows:

$$dX(t) = f(X(t), t)dt + g\left(\int_0^t G(t,s)X(s)ds, t\right)dw(t).$$

Mao and Riedle [10] later studied the stability of a more generalized type of equation as follows:

$$dX(t) = \left[f(X(t), t) + g\left(\int_0^t G(t,s)X(s)ds, t\right)\right]dt + h\left(\int_0^t H(t,s)X(s)ds, t\right)dw(t).$$

SVIDEs can be regarded as the more generalized type of SDEs. The theoretical and numerical analysis of SDEs have been well investigated (see, for example, [5–7, 11, 12, 17]). Up to now, there are some numerical

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results about SVIDEs (see [3, 4, 15] and references cited therein). In 2020, we (see [18]) considered the Euler-Maruyama method for generalized SVIDEs

$$\begin{aligned} dX(t) =& f\left(X(t), \int_0^t k_1(t,s)X(s)\,ds, \int_0^t \sigma_1(t,s)X(s)\,dw(s)\right)dt \\ &+ g\left(Y(t), \int_0^t k_2(t,s)X(s)\,ds, \int_0^t \sigma_2(t,s)X(s)\,dw(s)\right)dw(t), \end{aligned}$$

under global Lipschitz condition and showed that the strong convergence order is 1/2. In the same year, the convergence of the truncated Euler-Maruyama method for a class of SVIDEs is studied (see [19]) under non-globally Lipschitz condition and its strong convergence is close to 1/2. In 2019, Yang et al. (see [16]) studied the semi-implicit Euler method for the following nonlinear SVIDEs:

$$\frac{dX(t)}{dt} = f(X(t)) + \int_0^t \sigma(t,s)X(s)dw(s). \tag{1}$$

for $t \in [0,T]$ with initial data $X(0) = X_0 \in \mathbb{R}$. Here $f : \mathbb{R} \to \mathbb{R}$. The kernel σ: $D \to \mathbb{R}$ are continuous on $D := \{(t,s) : 0 \le s \le t \le T\}$. Set $\|\sigma\|_\infty = \max_{(t,s)\in D} |\sigma(t,s)|$.

In this paper, we will further consider SVIDE (1), the classical Euler methods of SDEs can not converge when the coefficients do not satisfy the linear growth conditions (see [5]), due to the cheap computation costs and acceptable convergence orders of explicit methods and motivated by [14] and [20], the balanced Euler method of SVIDE (1) is proposed in Section 3. Its boundedness is considered in Section 4 and its strong convergence rate is shown to be 1 in Section 5. Finally, we will give an example in Section 6 to illustrate the theoretical results of SVIDE (1).

2. Preliminary

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge 0},\mathbb{P})$ denote a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (i.e, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and let $\mathbb{E}$ be the expectation corresponding to $\mathbb{P}$. Let $w(t)$ be a 1-dimensional Brownian motion defined on the probability space. The family of $\mathbb{R}$-valued $\mathcal{F}_t$-adapted processes $\{x(t)\}_{t\in[0,T]}$ such that $\mathbb{E}|x(t)|^p < \infty$ $(p \ge 1)$ is denoted by $\mathcal{L}^p([0,T];\mathbb{R})$. Let $\mathcal{M}^2([0,T];\mathbb{R})$ be the family of processes $\{x(t)\}_{t\in[0,T]}$ in $\mathcal{L}^2([0,T];\mathbb{R})$ such that $\mathbb{E}\int_0^T |x(t)|^2 dt < \infty$. For $a, b \in \mathbb{R}$, we use $a \vee b$ and $a \wedge b$ for $\max\{a,b\}$ and $\min\{a,b\}$, respectively. If G is a subset of Ω, let χ_G denote its indicator function. $\lfloor\cdot\rfloor$ denotes the greatest-integer function.

Define

$$Z_{x_0}(z) := \int_0^z \sigma(z,s)X_{x_0}(s)dw(s).$$

The integration of SVIDE (1) is written as the following form

$$\begin{aligned} X_{x_0}(t) =& x_0 + \int_0^t f\left(X_{x_0}(z)\right)dz + \int_0^t \left(\int_0^z \sigma(z,s)X_{x_0}(s)dw(s)\right)dz \\ =& x_0 + \int_0^t f\left(X_{x_0}(z)\right)dz + \int_0^t Z_{x_0}(z)dz, \quad t \in [0,T]. \end{aligned} \tag{2}$$

We impose three assumptions as standing hypotheses:

Assumption 2.1. *There is a pair of constants $\gamma \ge 1$ and $K_1 > 0$ such that*

$$|f(x) - f(y)|^2 \le K_1\left(1 + |x|^{2\gamma-2} + |y|^{2\gamma-2}\right)|x-y|^2 \tag{3}$$

for all x, $y \in \mathbb{R}$.

Remark 2.2. *Due to* (3)*, we have*

$$|f(x)|^2 \le \bar{K}_1 \left(1 + |x|^{2\gamma}\right)$$

for all $x \in \mathbb{R}$*, where* $\bar{K}_1 = \left(2|f(0)|^2\right) \vee \frac{2K_1(\gamma+1)}{\gamma}$.

Assumption 2.3. *There is a constant* $K_2 > 0$ *such that*

$$(x - y)[f(x) - f(y)] \le K_2|x - y|^2$$

for all $x, y \in \mathbb{R}$.

Remark 2.4. *See from Assumption 2.3, we have*

$$xf(x) \le \bar{K}_2 \left(1 + |x|^2\right)$$

for all $x \in \mathbb{R}$*, where* $\bar{K}_2 = \left(\frac{1}{2}|f(0)|^2\right) \vee \left(K_2 + \frac{1}{2}\right)$.

Assumption 2.5. *There exists a constant* $K_3 > 0$ *such that*

$$|\sigma(t, s) - \sigma(\hat{t}, \hat{s})| \le K_3(|t - \hat{t}| + |s - \hat{s}|)$$

for all (t, s)*,* $(\hat{t}, \hat{s}) \in D$*. Moreover, set* $\sigma(t, t) = 0$*, for* $(t, t) \in D$*.*

A known result (see [16]) is stated as the following lemma.

Lemma 2.6. *Let Assumption 2.1 and Assumption 2.3 hold. If* $X_{x_0}(t)$ *is a solution of SVIDE* (1)*, for* $p \ge 1$*, then*

$$\mathbb{E}|X_{x_0}(t)|^{2p} \le K\mathbb{E}|x_0|^{2p}, \quad t \in [0, T],$$

where K *depends on* σ*,* T*,* K_1*,* K_2 *(but independent of* h *later) and its value may change between occurrences.*

Let $X_x(t + h)$ denote the solution of (1). Then

$$X_x(t + h) = x + \int_t^{t+h} f(X_x(s))ds + \int_t^{t+h} Z_x(s)ds,$$

where $x = X_{x_0}(t)$ and

$$Z_x(s) := \int_0^s \sigma(s, z)X_x(z)dw(z). \tag{4}$$

Remark 2.7. *Due to the flow property of SVIDE* (1)*, under Assumption 2.1 and Assumption 2.3, for* $p \ge 1$*, SVIDE* (1) *has a unique global solution* $X_x(s) = X_{x_0}(s)$ *and, moreover,*

$$\mathbb{E}|X_x(s)|^{2p} \le K\left(1 + \mathbb{E}|x|^{2p}\right), \quad \forall t \le s \le T.$$

3. The balanced Euler method

Let the step size $h \in (0, 1)$, $T = Nh$, $t_n = nh$, $n = 1, 2, \cdots, N$ and $N \in \mathbb{N}$. Motivated by [20], we introduce the one-step approximation $\bar{Y}_x(t + h)$, for the solution $X_x(t + h)$, which is defined as follows

$$\bar{Y}_x(t + h) = X_x(t) + \sin(hf(X_x(t))) + \sin\left(h\bar{Z}_x(\underline{t})\right),$$

for $0 \leq n \leq N-1$, with the initial point x, a time step h, where

$$\underline{s} := t_n, \quad \text{for} \quad s \in [t_n, t_{n+1}),$$

$$X_x(\underline{s}) := Y_n, \quad \text{for} \quad s \in [t_n, t_{n+1}),$$

$$\bar{Z}_x(\underline{t}) := \int_0^{\underline{t}} \sigma(\underline{t}, \underline{s}) X_x(\underline{s}) dw(s). \tag{5}$$

We propose the following balanced Euler scheme $\bar{Y}_{x_0}(t_n) = Y_n \approx X_{x_0}(t_n)$ for $t_n = nh$ by setting $Y_0 = x_0$ and computing

$$Y_{n+1} = \bar{Y}_{x_0}(t_{n+1}) = Y_n + \sin(hf(Y_n)) + \sin(hZ_n), \tag{6}$$

where

$$Z_n := \sum_{l=0}^{n-1} \sigma(t_n, t_l) Y_l \Delta w_l, \tag{7}$$

where $\Delta w_l = w(t_{l+1}) - w(t_l)$ are Gaussian $\mathcal{N}(0, \sqrt{h})$ i.i.d. random variable.

4. Moment boundedness of the balanced Euler method

Define

$$\tilde{\Omega}_{R,n} := \{\omega : |Y_k| \leq R(h),\ k = 0, 1, 2, \cdots, n\},$$

where $\gamma > 1$ and $R^\gamma(h) < 1/h$.

In order to obtain the moment boundedness of the balanced Euler method, we present the following lemma.

Lemma 4.1. *Let Assumption 2.1 and Assumption 2.3 hold. For $p \geq 1$, then we have*

$$h^{p+1}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Z_n|^{2p}\right) \leq K\|\sigma\|_\infty^{2p} T^{2p-1} h^2 \sum_{l=0}^{n-1} \mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_l|^{2p}\right)$$

and

$$\sum_{l=0}^{n-1} h^{p+1}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Z_n|^{2p}\right) \leq K\|\sigma\|_\infty^{2p} T^{2p} h \sum_{l=0}^{n-1} \mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_l|^{2p}\right).$$

Proof. By the definition of σ and (7), we get

$$h^{p+1}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Z_n|^{2p}\right) \leq \|\sigma\|_\infty^{2p} h^{p+1}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)\left|\sum_{l=0}^{n-1} Y_l \Delta w_l\right|^{2p}\right).$$

It is known that $\mathbb{E}(\Delta w_k) = 0$,

$$\mathbb{E}\left(\Delta w_i \Delta w_j\right) = \begin{cases} 0, & if \quad i \neq j, \\ h, & if \quad i = j \end{cases}$$

and $\mathbb{E}\left(|\Delta w_k|^{2n}\right) = (2n-1)!!h^n$.

Hence, we have

$$\begin{aligned}
h^{p+1}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Z_n|^{2p}\right) \leq & \|\sigma\|_\infty^{2p} h^{p+1}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)\left|\sum_{l=0}^{n-1} Y_l\Delta w_l\right|^{2p}\right)\\
\leq & \|\sigma\|_\infty^{2p} n^{2p-1} h^{p+1}\sum_{l=0}^{n-1}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)\mathbb{E}\,|Y_l\Delta w_l|^{2p}\right)\\
\leq & K\|\sigma\|_\infty^{2p} n^{2p-1} h^{2p+1}\sum_{l=0}^{n-1}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)\mathbb{E}\,|Y_l|^{2p}\right)\\
\leq & K\|\sigma\|_\infty^{2p} T^{2p-1} h^{2}\sum_{l=0}^{n-1}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)\mathbb{E}\,|Y_l|^{2p}\right).
\end{aligned}$$

Therefore, we get

$$h^{p+1}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Z_n|^{2p}\right) \leq \|\sigma\|_\infty^{2p} T^{2p-1} h^{2}\sum_{l=0}^{n-1}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_l|^{2p}\right) \tag{8}$$

and

$$\begin{aligned}
\sum_{l=0}^{n-1} h^{p+1}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Z_n|^{2p}\right) \leq & K\|\sigma\|_\infty^{2p} T^{2p-1} h^{2}\sum_{l=0}^{n-1}\sum_{r=0}^{l-1}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_r|^{2p}\right)\\
\leq & K\|\sigma\|_\infty^{2p} T^{2p-1} h^{2} n\sum_{l=0}^{n-1}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_l|^{2p}\right)\\
\leq & K\|\sigma\|_\infty^{2p} T^{2p} h\sum_{l=0}^{n-1}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_l|^{2p}\right).
\end{aligned}$$

□

In the similar way of [20], we obtain the following lemma.

Lemma 4.2. *Let Assumption 2.1 and Assumption 2.3 hold. Define* $G(\gamma) = \max\left\{2(\gamma-1), \chi_{p>1}\frac{3(\gamma-1)}{2}\right\}$, *for* $p \geq 1$, *we have*

$$\mathbb{E}|Y_n|^{2p} \leq K\left(1 + \mathbb{E}|x_0|^{2p\beta}\right),$$

where $\beta \geq 2 + \frac{(2p+1)G(\gamma)}{p}$.

Proof. We divide the proof into two cases.

Case (I) If $\gamma > 1$, let $\tilde{\Lambda}_{R,n}$ denote the compliments of $\tilde{\Omega}_{R,n}$.

By (6), we have

$$|Y_{n+1}| \leq |Y_n| + 2 \leq |x_0| + 2(n+1). \tag{9}$$

For any integer $p \geq 1$, we get

$$\begin{aligned}
\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n+1}}(\omega)|Y_{n+1}|^{2p}\right) \leq & \mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_{n+1}|^{2p}\right)\\
= & \mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|\left(Y_{n+1}-Y_n\right)+Y_n|^{2p}\right)\\
= & \mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p}\right) + \mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p-2}B\right)
\end{aligned}$$

$$+ K\sum_{l=3}^{2p}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p-l}|Y_{n+1}-Y_n|^l\right), \tag{10}$$

where $B := \chi_{\tilde{\Omega}_{R,n}}E\left[2pY_n(Y_{n+1}-Y_n)+p(2p-1)|Y_{n+1}-Y_n|^2|\mathcal{F}_{t_n}\right]$.

Since Δw_n are independent of $\mathcal{F}_{t_n}$, we have

$$\chi_{\tilde{\Omega}_{R,n}}(\omega)\mathbb{E}\left(hZ_n\middle|\mathcal{F}_{t_n}\right)=0,$$
$$\chi_{\tilde{\Omega}_{R,n}}(\omega)\mathbb{E}\left(|hZ_n|^2\middle|\mathcal{F}_{t_n}\right)=\chi_{\tilde{\Omega}_{R,n}}(\omega)\mathbb{E}\,|Z_n|^2\,h^2.$$

Using the asymmetry of the sine function, we get

$$\chi_{\tilde{\Omega}_{R,n}}(\omega)\mathbb{E}\left(\sin(hZ_n)\middle|\mathcal{F}_{t_n}\right)=0.$$

Noting that $|\sin x|\le|x|$ and for some $\theta\in[0,1]$,

$$|x-\sin x| = |(1-\cos(\theta x))x| \le 2|x|\left|\sin\left(\frac{\theta x}{2}\right)\right|^2, \tag{11}$$

together with the elementary inequality, we have

$$\begin{aligned} B &= 2p\chi_{\tilde{\Omega}_{R,n}}(\omega)\mathbb{E}\left[\left(Y_n(Y_{n+1}-Y_n)+\frac{2p-1}{2}|Y_{n+1}-Y_n|^2\right)\middle|\mathcal{F}_{t_n}\right] \\ &= 2p\chi_{\tilde{\Omega}_{R,n}}(\omega)\mathbb{E}\left[\left(Y_nf(Y_n)h+(2p-1)\left|\sin(f(Y_n)h)\right|^2+(2p-1)\left|\sin(hZ_n)\right|^2\right)\middle|\mathcal{F}_{t_n}\right] \\ &\quad +2p\chi_{\tilde{\Omega}_{R,n}}(\omega)\mathbb{E}\left[\left(Y_n\left(-f(Y_n)h+\sin(f(Y_n)h)\right)\right)\middle|\mathcal{F}_{t_n}\right] \\ &\le 2p\chi_{\tilde{\Omega}_{R,n}}(\omega)\mathbb{E}\left[\left(Y_nf(Y_n)h+(2p-1)\left|hZ_n\right|^2+\left|f(Y_n)\right|^2h^2\right)\middle|\mathcal{F}_{t_n}\right] \\ &\quad +4p\chi_{\tilde{\Omega}_{R,n}}(\omega)\mathbb{E}\left(\left|Y_nf(Y_n)\right|h\middle|\mathcal{F}_{t_n}\right). \end{aligned}$$

Applying Assumption 2.3 and Remark 2.2, we get

$$B \le K\chi_{\tilde{\Omega}_{R,n}}(\omega)h\left(1+|Y_n|^2+|Y_n|^{2\gamma}h+h\,|Z_n|^2\right). \tag{12}$$

Consequently, we have

$$\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p-2}B\right) \le Kh\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p-2}\left(1+|Y_n|^2+|Y_n|^{2\gamma}h+h\,|Z_n|^2\right)\right].$$

Using (6) and Remark 2.2, we obtain

$$\begin{aligned} \mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p-l}|Y_{n+1}-Y_n|^l\right) &\le K\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p-l}\left(|hf(Y_n)|^l+|hZ_n|^l\right)\right] \\ &\le K\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p-l}h^l\left(1+|Y_n|^{l\gamma}+|Z_n|^l\right)\right]. \end{aligned} \tag{13}$$

Substituting (12), (13) into (10), by the Young inequality, we have

$$\begin{aligned} \mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n+1}}(\omega)|Y_{n+1}|^{2p}\right) \le& \mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p}\right)+K\sum_{l=3}^{2p}\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p-l}h^l\left(1+|Y_n|^{l\gamma}+|Z_n|^l\right)\right] \\ &+Kh\mathbb{E}\left[\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p-2}\left(1+|Y_n|^2+|Y_n|^{2\gamma}h+h\,|Z_n|^2\right)\right] \end{aligned}$$

$$
\begin{aligned}
&\le \mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p}\right) + Kh\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p}\right) + Kh \\
&\quad + Kh^{p+1}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Z_n|^{2p}\right) + Kh\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p+2\gamma-2}h\right) \\
&\quad + K\sum_{l=3}^{2p}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p-l}h^l\right) + Kh^{p+1}\sum_{l=3}^{2p}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Z_n|^{2p}\right) \\
&\quad + Kh\sum_{l=3}^{2p}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p+l(\gamma-1)}h^{l-1}\right) \\
&\le \mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p}\right) + Kh\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p}\right) + Kh \\
&\quad + Kh^{p+1}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Z_n|^{2p}\right) + Kh\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p+2\gamma-2}h\right) \\
&\quad + Kh\sum_{l=3}^{2p}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p+l(\gamma-1)}h^{l-1}\right).
\end{aligned}
$$

Choose $R = R(h) = h^{-1/G(\gamma)}$, where $G(\gamma) = \max\left\{2(\gamma-1), \chi_{p>1}\frac{3(\gamma-1)}{2}\right\}$, we get,

$$
\begin{aligned}
\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p+2\gamma-2}h &\le \chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p}, \\
\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p+l(\gamma-1)}h^{l-1} &\le \chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p}.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n+1}}(\omega)|Y_{n+1}|^{2p}\right) &\le \mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p}\right) + Kh\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p}\right) \\
&\quad + Kh^{p+1}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Z_n|^{2p}\right) + Kh.
\end{aligned}
$$

Using Lemma 4.1, we obtain

$$
\begin{aligned}
\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n+1}}(\omega)|Y_{n+1}|^{2p}\right) &\le \mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p}\right) + Kh\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_n|^{2p}\right) + Kh \\
&\quad + K\|\sigma\|_\infty^{2p}T^{2p-1}h^2\sum_{l=0}^{n-1}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_l|^{2p}\right) \\
&\le |x_0|^{2p} + Kh\sum_{l=0}^{n}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_l|^{2p}\right) + Kh \\
&\quad + K\|\sigma\|_\infty^{2p}T^{2p-1}h^2\sum_{r=0}^{n}\sum_{l=0}^{r-1}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_l|^{2p}\right) \\
&\le |x_0|^{2p} + Kh\sum_{l=0}^{n}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_l|^{2p}\right) + Kh \\
&\quad + K\|\sigma\|_\infty^{2p}T^{2p-1}h^2 n\sum_{l=0}^{n}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_l|^{2p}\right) \\
&\le |x_0|^{2p} + Kh\sum_{l=0}^{n}\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,n}}(\omega)|Y_l|^{2p}\right) + Kh.
\end{aligned}
$$

The well-known Gronwall inequality yields that

$$
\mathbb{E}\left(\chi_{\tilde{\Omega}_{R,u}}(\omega)|Y_n|^{2p}\right) \le K(1 + \mathbb{E}|x_0|^{2p}).
$$

Noting that (see [14])

$$\begin{aligned}\chi_{\tilde{\Lambda}_{R,n}} =& 1-\chi_{\tilde{\Omega}_{R,n}} = 1-\chi_{\tilde{\Omega}_{R,n-1}}\chi_{|Y_n|\le R}\\ =&\chi_{\tilde{\Lambda}_{R,n-1}}+\chi_{\tilde{\Omega}_{R,n-1}}\chi_{|Y_n|>R}=\cdots=\sum_{k=0}^{n}\chi_{\tilde{\Omega}_{R,k-1}}\chi_{|Y_k|>R},\end{aligned}$$

where we put $\chi_{\tilde{\Omega}_{R,-1}}=1$, and using (9), the Hölder inequality and the Markov's inequality, we get

$$\begin{aligned}\mathbb{E}[\chi_{\tilde{\Lambda}_{R,n}}(\omega)|Y_n|^{2p}] \le& \left(E\left(|x_0|+2n\right)^{4p}\right)^{1/2}\sum_{k=0}^{n}\frac{\left(\mathbb{E}\left(\chi_{\tilde{\Omega}_{R(h),k-1}}(\omega)|Y_k|^{2(2p+1)G(\gamma)}\right)\right)^{1/2}}{R(h)^{(2p+1)G(\gamma)}}\\ \le& K\left(\mathbb{E}\left(|x_0|+2n\right)^{4p}\right)^{1/2}\left(\mathbb{E}(1+|x_0|)^{2(2p+1)G(\gamma)}\right)^{1/2}nh^{2p+1}\\ \le& K\left(1+\mathbb{E}|x_0|^{2p\beta}\right),\end{aligned}$$

where $\beta \ge 2+\frac{(2p+1)G(\gamma)}{p}$.

Case (II): If $\gamma = 1$, it can be proved in the similar way of the case (I).

The proof is complete. □

Remark 4.3. *Due to the flow property of the SVIDE* (1), *under Assumption 2.1 and Assumption 2.3, for* $p \ge 1$, *we have*

$$\mathbb{E}|\bar{Y}_{x_0}(t_u)|^{2p} \le K\left(1+\mathbb{E}|x_0|^{2p\beta}\right), \quad \forall \left\lfloor \frac{t}{\Delta}\right\rfloor \le u \le N,$$

where $x_0 \in \mathbb{R}$ *and* $\beta \ge 2+\frac{(2p+1)G(\gamma)}{p}$.

5. Convergence order of the balanced Euler method

Lemma 5.1. *Let Assumption 2.1 and Assumption 2.3 hold. Then for all* $1 \le l$ *and* $0 \le t \le s \le T$, *we have*

$$\mathbb{E}\left|f(X_x(s))-f(X_x(t))\right|^l \le K\left(1+|x|^{l(2\gamma-1)}\right)(s-t)^l, \tag{14}$$

where $x \in \mathbb{R}$.

Proof. For all $1 \le l$ and $t \le s$, by Assumption 2.1, we have

$$\begin{aligned}\mathbb{E}|f(X_x(s))-f(X_x(t))|^l \le& K\mathbb{E}\left[\left(1+|X_x(s)|^{\gamma-1}+|X_x(t)|^{\gamma-1}\right)|X_x(s)-X_x(t)|\right]^l\\ \le& K\mathbb{E}\left[\left(1+|X_x(s)|^{\gamma-1}+|X_x(t)|^{\gamma-1}\right)\left(\left|\int_t^s f(X_x(z))dz\right|^l+\left|\int_t^s Z_x(z)dz\right|^l\right)\right].\end{aligned}$$

Applying Hölder inequality and Remark 2.7, we get

$$\begin{aligned}\mathbb{E}\left|f(X_x(s))-f(X_x(t))\right|^l \le& K\left(1+\mathbb{E}|X_x(s)|^{2l(\gamma-1)}+\mathbb{E}|X_x(t)|^{2l(\gamma-1)}\right)^{1/2}\\ &\cdot\left(\mathbb{E}\left|\int_t^s f(X_x(z))dz\right|^{2l}+\mathbb{E}\left|\int_t^s Z_x(z)dz\right|^{2l}\right)^{1/2}.\end{aligned} \tag{15}$$

Due to the Hölder inequality and Remark 2.7, we obtain

$$\left(\mathbb{E}\left|\int_t^s f(X_x(z))dz\right|^{2l}\right)^{1/2} \le \left((s-t)^{2l-1}\mathbb{E}\int_t^s\left|f(X_x(z))\right|^{2l}dz\right)^{1/2}$$

$$\begin{aligned} &\leq K\left((s-t)^{2l-1}\int_t^s \left(1+\mathbb{E}|X_x(z)|^{2l\gamma}\right)dz\right)^{1/2} \\ &\leq K(s-t)^l\left(1+|x|^{2l\gamma}\right)^{1/2}. \end{aligned} \tag{16}$$

By (4), we get

$$Z_x(z) := \int_0^z \sigma(z,s)X_{x_0}(s)dw(s).$$

Using the Itô isometry, we have

$$\mathbb{E}|Z_x(z)|^{2l} \leq \|\sigma\|_\infty^{2l} T \sup_{0\leq u\leq s}\mathbb{E}|X_x(u)|^{2l} \leq K\sup_{0\leq u\leq s}\mathbb{E}|X_x(u)|^{2l} \leq K|x|^{2l}. \tag{17}$$

Hence, we get

$$\begin{aligned} \left(\mathbb{E}\left|\int_t^s Z_x(z)dz\right|^{2l}\right)^{1/2} &\leq \left(K(s-t)^{2l-1}\int_t^s \mathbb{E}|Z_x(z)|^{2l}dz\right)^{1/2} \\ &\leq K(s-t)^l|x|^l. \end{aligned} \tag{18}$$

Substituting (16), (18) to (15), we have (20). The proof is complete. □

Lemma 5.2. *Let Assumption 2.1, Assumption 2.3 and Assumption 2.5 hold. Then for $p \geq 1$ and $s \geq t$, we have*

$$\mathbb{E}\left|Z_x(s) - \bar{Z}_x(\underline{t})\right|^{2p} \leq K\left(1+|x|^{2p\gamma}\right)h^{2p}, \tag{19}$$

where $x \in \mathbb{R}$.

Proof. Using $\sigma(t,t) = 0$, we have

$$\mathbb{E}\left|\int_{\underline{t}}^s \sigma(\underline{t},\underline{u})X_x(\underline{u})dw(u)\right|^{2p} = 0.$$

By (2), the elementary inequality, Remark 2.2 and (17), we get

$$\begin{aligned} \int_0^s \mathbb{E}|X_x(u) - X_x(\underline{u})|^{2p}du &= \int_0^s \mathbb{E}\left|\int_{\underline{u}}^u f(X_x(z))dz + \int_{\underline{u}}^u Z_x(z)dz\right|^{2p}du \\ &\leq Kh^{2p-1}\int_0^s\int_{\underline{u}}^u \mathbb{E}\left(|f(X_x(z))|^{2p} + |Z_x(z)|^{2p}\right)dzdu \\ &\leq Kh^{2p-1}\int_0^s\int_{\underline{u}}^u \mathbb{E}\left(1+|X_x(z)|^{2p\gamma} + |Z_x(z)|^{2p}\right)dzdu \\ &\leq K\left(1+|x|^{2p\gamma}\right)h^{2p}. \end{aligned}$$

Applying (4), (5), the elementary inequality and Assumption 2.5, we get

$$\begin{aligned} \mathbb{E}\left|Z_x(s) - \bar{Z}_x(\underline{t})\right|^{2p} &= \mathbb{E}\left|\int_0^s \sigma(s,u)X_x(u)dw(u) - \int_0^{\underline{t}} \sigma(\underline{t},\underline{u})X_x(\underline{u})dw(u)\right|^{2p} \\ &\leq 5^{2p-1}\mathbb{E}\left|\int_0^s \sigma(s,u)X_x(u)dw(u) - \int_0^s \sigma(s,\underline{u})X_x(u)dw(u)\right|^{2p} \\ &\quad +5^{2p-1}\mathbb{E}\left|\int_0^s \sigma(s,\underline{u})X_x(u)dw(u) - \int_0^s \sigma(s,\underline{u})X_x(\underline{u})dw(u)\right|^{2p} \end{aligned}$$

$$
\begin{aligned}
&+5^{2p-1}\mathbb{E}\left|\int_0^s \sigma(s,\underline{u})X_x(\underline{u})dw(u) - \int_0^s \sigma(\underline{s},\underline{u})X_x(\underline{u})dw(u)\right|^{2p} \\
&+5^{2p-1}\mathbb{E}\left|\int_0^s \sigma(\underline{s},\underline{u})X_x(\underline{u})dw(u) - \int_0^s \sigma(\underline{t},\underline{u})X_x(\underline{u})dw(u)\right|^{2p} \\
&+5^{2p-1}\mathbb{E}\left|\int_{\underline{t}}^s \sigma(\underline{t},\underline{u})X_x(\underline{u})dw(u)\right|^{2p} \\
\leq\;& 5^{2p-1}\mathbb{E}\left|\int_0^s [\sigma(s,u)-\sigma(s,\underline{u})]X_x(u)dw(u)\right|^{2p} \\
&+5^{2p-1}\mathbb{E}\left|\int_0^s \sigma(s,\underline{u})[X_x(u)-X_x(\underline{u})]dw(u)\right|^{2p} \\
&+5^{2p-1}\mathbb{E}\left|\int_0^s [\sigma(s,\underline{u})-\sigma(\underline{s},\underline{u})]X_x(\underline{u})dw(u)\right|^{2p} \\
&+5^{2p-1}\mathbb{E}\left|\int_0^s [\sigma(\underline{s},\underline{u})-\sigma(\underline{t},\underline{u})]X_x(\underline{u})dw(u)\right|^{2p} \\
&+5^{2p-1}\mathbb{E}\left|\int_{\underline{t}}^s \sigma(\underline{t},\underline{u})X_x(\underline{u})dw(u)\right|^{2p} \\
\leq\;& 5^{2p-1}\mathbb{E}\left|\int_0^s K_3(u-\underline{u})X_x(u)dw(u)\right|^{2p} \\
&+5^{2p-1}\|\sigma\|_\infty^{2p}h^p\int_0^s \mathbb{E}|X_x(u)-X_x(\underline{u})|^{2p}du \\
&+5^{2p-1}\mathbb{E}\left|\int_0^s K_3(s-\underline{s})X_x(\underline{u})dw(u)\right|^{2p} \\
&+5^{2p-1}\mathbb{E}\left|\int_0^s K_3(\underline{s}-\underline{t})X_x(\underline{u})dw(u)\right|^{2p} \\
&+5^{2p-1}\mathbb{E}\left|\int_{\underline{t}}^s \sigma(\underline{t},\underline{u})X_x(\underline{u})dw(u)\right|^{2p} \\
\leq\;& K\left(1+|x|^{2p}\right)h^{2p} + 5^{2p-1}\|\sigma\|_\infty^{2p}h^p\int_0^s \mathbb{E}|X_x(u)-X_x(\underline{u})|^{2p}du \\
\leq\;& K\left(1+|x|^{2p\gamma}\right)h^{2p}.
\end{aligned}
$$

The proof is complete. □

Lemma 5.3. *Let Assumption 2.1, Assumption 2.3 and Assumption 2.5 hold. Then we have*

$$
\left|\mathbb{E}\int_t^{t+h}[Z_x(s)-Z_x(t)]\,ds\right| = 0, \tag{20}
$$

where $x \in \mathbb{R}$.

Proof. Applying (4), (5), the elementary inequality, Assumption 2.5 and the properties of the Paley-Wiener-Zygmund integral, we get

$$
\left|\mathbb{E}\int_t^{t+h}[Z_x(s)-Z_x(t)]\,ds\right| = \left|\mathbb{E}\int_t^{t+h}\left(\int_0^s \sigma(s,u)X_x(u)dw(u) - \int_0^t \sigma(t,u)X_x(u)dw(u)\right)ds\right|
$$

$$\begin{aligned}
&\leq 2\|\sigma\|_\infty \left|\mathbb{E}\int_t^{t+h}\int_0^s X_x(u)dw(u)ds\right| \\
&\leq 2\|\sigma\|_\infty \sup_{0\leq u\leq s}\mathbb{E}|X_x(u)|\left|\mathbb{E}\int_t^{t+h}\int_0^s dw(u)ds\right| \\
&= 0.
\end{aligned}$$

The proof is complete. □

Theorem 5.4. *Let Assumption 2.1, Assumption 2.3 and Assumption 2.5 hold. For $p \geq 1$, we have*

$$\left|\mathbb{E}\left[X_x(t+h) - \bar{Y}_x(t+h)\right]\right| \leq Kh^2\left(1+|x|^{3\gamma}\right) \tag{21}$$

and

$$\mathbb{E}\left|X_x(t+h) - \bar{Y}_x(t+h)\right|^{2p} \leq Kh^{4p}\left(1+|x|^{6p\gamma}\right). \tag{22}$$

where $\bar{Y}(t) = X(t) = x \in \mathbb{R}$.

Proof. We divided the proof into to three steps.

Step 1: We consider the one-step approximation to the explicit Euler scheme:

$$\tilde{X}_x(t+h) = X_x(t) + hf(X_x(t)) + h\bar{Z}_x(\underline{t}).$$

Define

$$\tilde{\rho}(t+h) := X_x(t+h) - \tilde{X}_x(t+h).$$

By Lemma 5.1 and Lemma 5.3, we have

$$\begin{aligned}
\left|\mathbb{E}\tilde{\rho}(t+h)\right| &\leq \left|\mathbb{E}\int_t^{t+h}\left[f(X_x(s)) - f(X_x(t))\right]ds\right| + \left|\mathbb{E}\int_t^{t+h}\left[Z_x(s) - Z_x(t)\right]ds\right| \\
&\leq \mathbb{E}\int_t^{t+h}|f(X_x(s)) - f(X_x(t))|ds \\
&\leq Kh^2\left(1+|x|^{2\gamma}\right).
\end{aligned} \tag{23}$$

Using Lemma 5.1 and Lemma 5.2, we obtain

$$\begin{aligned}
\mathbb{E}\left|\tilde{\rho}(t+h)\right|^{2p} &\leq \mathbb{E}\left|\int_t^{t+h}[f(X_x(s)) - f(X_x(t))]ds\right|^{2p} + K\mathbb{E}\left|\int_t^{t+h}[Z_x(s) - \bar{Z}_x(\underline{t})]ds\right|^{2p} \\
&\leq Kh^{2p-1}\int_t^{t+h}\mathbb{E}|f(X_x(s)) - f(X_x(t))|^{2p}ds + Kh^{2p-1}\int_t^{t+h}\mathbb{E}|Z_x(s) - \bar{Z}_x(\underline{t})|^{2p}ds \\
&\leq Kh^{4p}\left(1+|x|^{2p(2\gamma-1)}\right).
\end{aligned} \tag{24}$$

Step 2: We give the one-step approximation to the balanced Euler scheme

$$\bar{Y}_x(t+h) = X_x(t) + \sin(hf(X_x(t))) + \sin\left(h\bar{Z}_x(\underline{t})\right).$$

Define

$$\begin{aligned}
\bar{\rho}(t+h) :=& \tilde{X}_x(t+h) - \bar{Y}_x(t+h) \\
=& hf(X_x(t)) - \sin(hf(X_x(t))) + h\bar{Z}_x(\underline{t}) - \sin\left(h\bar{Z}_x(\underline{t})\right).
\end{aligned}$$

Using (11), we get

$$
\begin{aligned}
|\mathbb{E}\bar{\rho}(t+h)| &= \left|\mathbb{E}\left[f(X_x(t))h - \sin(hf(X_x(t)))\right]\right| \\
&\leq K\mathbb{E}\left|f(X_x(t))h\right|^3 \\
&\leq Kh^3\left(1+|x|^{3\gamma}\right) \qquad (25)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}|\bar{\rho}(t+h)|^{2p} &\leq 2^{2p-1}\mathbb{E}\left|f(X_x(t))h - \sin(hf(X_x(t)))\right|^{2p} + 2^{2p-1}\mathbb{E}\left|h\bar{Z}_x(\underline{t}) - \sin\left(h\bar{Z}_x(\underline{t})\right)\right|^{2p} \\
&\leq K\mathbb{E}\left|f(X_x(t))h\right|^{6p} + K\mathbb{E}|h\bar{Z}_x(\underline{t})|^{6p} \\
&\leq Kh^{6p}\left(1+|x|^{6p\gamma}\right). \qquad (26)
\end{aligned}
$$

Step 3. Define

$$\rho(t+h) := X_x(t+h) - \bar{Y}_x(t+h) = \tilde{\rho}(t+h) - \bar{\rho}(t+h).$$

Applying (23) and (25), we get

$$
\begin{aligned}
\left|\mathbb{E}\rho(t+h)\right| &\leq \left|\mathbb{E}\tilde{\rho}(t+h)\right| + \left|\mathbb{E}\bar{\rho}(t+h)\right| \\
&\leq Kh^2\left(1+|x|^{3\gamma}\right).
\end{aligned}
$$

Using (24) and (26), we have

$$
\begin{aligned}
\mathbb{E}\left|\rho(t+h)\right|^{2p} &\leq 2^{2p-1}\mathbb{E}\left|\tilde{\rho}(t+h)\right|^{2p} + 2^{2p-1}\mathbb{E}\left|\bar{\rho}(t+h)\right|^{2p} \\
&\leq Kh^{4p}\left(1+|x|^{6p\gamma}\right).
\end{aligned}
$$

The proof is complete. □

Lemma 5.5. *Let Assumption 2.1, Assumption 2.3 and Assumption 2.5 hold. Define*

$$X_x(t+\theta) - X_y(t+\theta) = x - y + U_{x,y}(t+\theta) \qquad (27)$$

for $\theta \in [0,h]$*, where* $X(t) = x \in \mathbb{R}$*,* $X(t) = y \in \mathbb{R}$*. For* $p \geq 1$*, we have*

$$\mathbb{E}\left|X_x(t+h) - X_y(t+h)\right|^{2p} \leq |x-y|^{2p}(1+Kh), \qquad (28)$$

$$\mathbb{E}\left|U_{x,y}(t+h)\right|^{2p} \leq K\left(1+|x|^{2\gamma-2}+|y|^{2\gamma-2}\right)^{p/2}|x-y|^{2p}h^p. \qquad (29)$$

Proof. Define

$$S_{x,y}(s) := S(s) := X_x(s) - X_y(s). \qquad (30)$$

Hence , Using the Hölder inequality, (27) and (30), we have $U_{x,y}(s) := U(s) := S(s) - (x-y)$.

$$
\begin{aligned}
\int_t^{t+\theta} \mathbb{E}\left|Z_x(s) - Z_y(s)\right|^{2p} ds &= \int_t^{t+\theta} \mathbb{E}\left|\int_0^s \sigma(s,z)X_x(z)dw(z) - \int_0^s \sigma(s,z)X_y(z)dw(z)\right|^{2p} ds \\
&\leq \int_t^{t+\theta} \mathbb{E}\left|\int_0^s \sigma(s,z)(X_x(z) - X_y(z))dw(z)\right|^{2p} ds \\
&\leq \|\sigma\|_\infty^{2p-1} s^{(2p-2)/2} \int_t^{t+\theta}\int_0^s \mathbb{E}\left|S(z)\right|^{2p} dzds
\end{aligned}
$$

$$\leq \quad \|\sigma\|_\infty^{2p-1} s^{(2p-2)/2} T \int_t^{t+\theta} \mathbb{E}\,|S(s)|^{2p}\, ds.$$

Applying the Itô formula, Assumption 2.3 and the Young inequality, for $\theta \geq 0$, we get

$$\begin{aligned}
& \mathbb{E}\,|S(t+\theta)|^{2p} \\
\leq \quad & |x-y|^{2p} + 2p\mathbb{E} \int_t^{t+\theta} |S(s)|^{2p-2} \Big(S(s) \left[f(X_x(s)) - f(X_y(s)) \right] \\
& + \frac{2p-1}{2} \left| Z_x(s) - Z_y(s) \right|^2 \Big) ds \\
\leq \quad & |x-y|^{2p} + 2p\mathbb{E} \int_t^{t+\theta} |S(s)|^{2p-2} \left(K_2 \left| X_x(s) - X_y(s) \right|^2 + \frac{2p-1}{2} \left| Z_x(s) - Z_y(s) \right|^2 \right) ds \\
\leq \quad & |x-y|^{2p} + (4p-2)K \int_t^{t+\theta} \mathbb{E}|S(s)|^{2p} ds + 2K \int_t^{t+\theta} \mathbb{E} \left| Z_x(s) - Z_y(s) \right|^{2p} ds \\
\leq \quad & |x-y|^{2p} + K \int_t^{t+\theta} \mathbb{E}|S(s)|^{2p} ds.
\end{aligned}$$

By the Gronwall inequality, we have (28).

Applying the Itô formula and Assumption 2.3, for $\theta \geq 0$, we get

$$\begin{aligned}
\mathbb{E}\,|U(t+\theta)|^{2p} \leq & 2p\mathbb{E} \int_t^{t+\theta} |U(s)|^{2p-2} \Big(U(s) \left[f(X_x(s)) - f(X_y(s)) \right] \\
& + \frac{2p-1}{2} |Z_x(s) - Z_y(s)|^2 \Big) ds \\
\leq & 2p\bar{K}_1 \mathbb{E} \int_t^{t+\theta} |U(s)|^{2p-2} \Big(S(s) \left[f(X_x(s)) - f(X_y(s)) \right] \\
& + \frac{2p-1}{2} |Z_x(s) - Z_y(s)|^2 \Big) ds \\
& - 2p\mathbb{E} \int_t^{t+\theta} |U(s)|^{2p-2} (x-y) \left[f(X_x(s)) - f(X_y(s)) \right] ds \\
\leq & 2p\bar{K}_1 \mathbb{E} \int_t^{t+\theta} |U(s)|^{2p-2} \left(|S(s)|^2 + \frac{2p-1}{2} \left| Z_x(s) - Z_y(s) \right|^2 \right) ds \\
& - 2p\mathbb{E} \int_t^{t+\theta} |U(s)|^{2p-2} (x-y) \left[f(X_x(s)) - f(X_y(s)) \right] ds.
\end{aligned}$$

Using the Young inequality, we obtain

$$\begin{aligned}
& 2p\bar{K}_1 \mathbb{E} \int_t^{t+\theta} |U(s)|^{2p-2} \left(|S(s)|^2 + |Z_x(s) - Z_y(s)|^2 \right) ds \\
\leq & K \int_t^{t+\theta} \mathbb{E}|U(s)|^{2p} ds + K \int_t^{t+\theta} \mathbb{E}|S(s)|^{2p} ds \\
\leq & K \int_t^{t+\theta} \mathbb{E}|U(s)|^{2p} ds + K|x-y|^{2p}.
\end{aligned}$$

Applying the Hölder inequality, Assumption 2.1 and Lemma 5.1, we get

$$- 2p\mathbb{E} \int_t^{t+\theta} |U(s)|^{2p-2} (x-y) [f(X_x(s)) - f(X_y(s))] ds$$

$$\begin{aligned}
&\le 2p\mathbb{E}\int_t^{t+\theta}|U(s)|^{2p-2}|x-y||f(X_x(s))-f(X_y(s))|ds\\
&\le K|x-y|\int_t^{t+\theta}\left(\mathbb{E}|U(s)|^{2p}\right)^{(p-1)/p}\left(\mathbb{E}|f(X_x(s))-f(X_y(s))|^p\right)^{1/p}ds\\
&\le K|x-y|\int_t^{t+\theta}\left(\mathbb{E}|U(s)|^{2p}\right)^{(p-1)/p}\Big(\mathbb{E}\Big[(1+|X_x(s)|^{2\gamma-2}\\
&\quad +|X_y(s)|^{2\gamma-2})^{p/2}\left(|X_x(s)-X_y(s)|^2\right)^{p/2}\Big]\Big)^{1/p}ds\\
&\le K|x-y|\int_t^{t+\theta}\left(\mathbb{E}|U(s)|^{2p}\right)^{(p-1)/p}\left(\mathbb{E}|X_x(s)-X_y(s)|^{2p}+\mathbb{E}|Z_x(s)-Z_y(s)|^{2p}\right)^{1/2p}\\
&\quad\times\left[\mathbb{E}\left(1+|X_x(s)|^{2p(\gamma-1)}+|X_y(s)|^{2p(\gamma-1)}\right)\right]^{1/2p}ds\\
&\le K|x-y|^2\left(1+|x|^{2\gamma-2}+|y|^{2\gamma-2}\right)^{1/2}\int_t^{t+\theta}\left(\mathbb{E}|U(s)|^{2p}\right)^{(p-1)/p}ds.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\mathbb{E}|U(t+\theta)|^{2p} \quad\le\quad & K|x-y|^2\left(1+|x|^{2\gamma-2}+|y|^{2\gamma-2}\right)^{1/2}\int_t^{t+\theta}\left(\mathbb{E}|U(s)|^{2p}\right)^{(p-1)/p}ds\\
&+K\int_t^{t+\theta}\mathbb{E}|U(s)|^{2p}ds. \qquad (31)
\end{aligned}$$

By the Young inequality and the Gronwall inequality, we get (29).

The proof is complete. □

Theorem 5.6. *Let Assumption 2.1, Assumption 2.3 and Assumption 2.5 hold. For some $p \ge 1$, we have*

$$\mathbb{E}\left|X_{x_0}(t_n)-\bar{Y}_{x_0}(t_n)\right|^{2p} \le K\left(1+\mathbb{E}|x_0|^{12p\beta\gamma}\right)h^{2p+2}. \qquad (32)$$

Proof. Define

$$\begin{aligned}
\rho_{n+1} \quad:=\quad & X_{x_0}(t_{n+1})-\bar{Y}_{x_0}(t_{n+1}) = X_{X_{x_0}(t_n)}(t_{n+1})-\bar{Y}_{\bar{Y}_{x_0}(t_n)}(t_{n+1})\\
=\quad & \left(X_{X_{x_0}(t_n)}(t_{n+1})-X_{Y_n}(t_{n+1})\right)+\left(X_{Y_n}(t_{n+1})-\bar{Y}_{Y_n}(t_{n+1})\right). \qquad (33)
\end{aligned}$$

Recalling (33), we get

$$\begin{aligned}
S_{n+1} \quad:=\quad & S_{X_{x_0}(t_n),\bar{Y}_{x_0}(t_n)}(t_{n+1}) = X_{X_{x_0}(t_n)}(t_{n+1})-X_{\bar{Y}_{x_0}(t_n)}(t_{n+1})\\
=\quad & (X_{x_0}(t_n)-\bar{Y}_{x_0}(t_n))+U_{X_{x_0}(t_n),\bar{Y}_{x_0}(t_n)}(t_{n+1})\\
=\quad & (X_{x_0}(t_n)-Y_n)+U_{X_{x_0}(t_n),Y_n}(t_{n+1})\\
:=\quad & \rho_n+U_{n+1}.
\end{aligned}$$

Define

$$r_{n+1} := X_{Y_n}(t_{n+1})-\bar{Y}_{Y_n}(t_{n+1}).$$

See from Theorem 5.4, for $l \ge 2$, we have

$$\begin{aligned}
\mathbb{E}|r_{n+1}|^l \quad\le\quad & Kh^{2l}\left(1+\mathbb{E}|Y_n|^{3l\gamma}\right)\\
\le\quad & Kh^{2l}\left(1+\mathbb{E}|x_0|^{3l\beta\gamma}\right) \qquad (34)
\end{aligned}$$

and

$$\mathbb{E}|r_{n+1}| \quad\le\quad Kh^2\left(1+\mathbb{E}|Y_n|^{3\gamma}\right)$$

$$\leq \quad Kh^2\left(1+\mathbb{E}|x_0|^{3\beta\gamma}\right). \tag{35}$$

Hence, we get

$$\begin{aligned}
\mathbb{E}\left|\rho_{n+1}\right|^{2p} &= \mathbb{E}|S_{n+1}+r_{n+1}|^{2p} \\
&= \mathbb{E}\left(|S_{n+1}|^2+2S_{n+1}r_{n+1}+|r_{n+1}|^2\right)^p \\
&\leq \mathbb{E}|S_{n+1}|^{2p}+2p\mathbb{E}\left[|S_{n+1}|^{2p-2}(\rho_n+U_{n+1})r_{n+1}\right] \\
&\quad +K\sum_{l=2}^{2p}\mathbb{E}\left(|S_{n+1}|^{2p-l}|r_{n+1}|^l\right).
\end{aligned} \tag{36}$$

Applying Lemma 5.5, we have

$$\mathbb{E}\,|S_{n+1}|^{2p} \quad \leq \quad \mathbb{E}|\rho_n|^{2p}(1+Kh) \tag{37}$$

and

$$\begin{aligned}
\mathbb{E}\left(|S_{n+1}|^{2p-2}(\rho_n+U_{n+1})r_{n+1}\right) &= \mathbb{E}\left(|\rho_n|^{2p-2}\rho_n r_{n+1}\right)+\mathbb{E}\left[\left(|S_{n+1}|^{2p-2}-|\rho_n|^{2p-2}\right)\rho_n r_{n+1}\right] \\
&\quad +\mathbb{E}\left(|S_{n+1}|^{2p-2}U_{n+1}r_{n+1}\right).
\end{aligned} \tag{38}$$

Due to $\mathcal{F}_{t_n}$-measurability of ρ_n, by the elementary inequality, the Young inequality, (34) and (35), we get

$$\begin{aligned}
\mathbb{E}\left(|\rho_n|^{2p-2}\rho_n r_{n+1}\right) &\leq \mathbb{E}\left[|\rho_n|^{2p-1}\mathbb{E}\left(|r_{n+1}||\mathcal{F}_{t_n}\right)\right] \\
&\leq \mathbb{E}\Big[|\rho_n|^{2p-1}\left(1+\mathbb{E}|x_0|^{3\beta\gamma}\right)\Big]h^2 \\
&\leq Kh\mathbb{E}|\rho_n|^{2p}+K\left(1+\mathbb{E}|x_0|^{6p\beta\gamma}\right)h^{2p+1}.
\end{aligned} \tag{39}$$

Noting that

$$\begin{aligned}
|S_{n+1}|^{2p-2}-|\rho_n|^{2p-2} &= |\rho_n+U_{n+1}|^{2p-2}-|\rho_n|^{2p-2} \\
&= |\rho_n|^{2p-2}+\sum_{l=1}^{2p-2}|U_{n+1}|^{2p-2-l}|\rho_n|^l-|\rho_n|^{2p-2} \\
&= \sum_{l=1}^{2p-2}|U_{n+1}|^{2p-2-l}|\rho_n|^l,
\end{aligned}$$

together with the elementary inequality and the Young inequality, we have

$$\begin{aligned}
&\mathbb{E}\left(\left(|S_{n+1}|^{2p-2}-|\rho_n|^{2p-2}\right)\rho_n r_{n+1}\right) \\
\leq\ & \mathbb{E}\left(|U_{n+1}||\rho_n||r_{n+1}|\sum_{l=0}^{2p-3}|U_{n+1}|^{2p-3-l}|\rho_n|^l\right) \\
=\ & \mathbb{E}\left(|r_{n+1}|\sum_{l=0}^{2p-3}|U_{n+1}|^{2p-2-l}|\rho_n|^{l+1}\right) \\
\leq\ & \sum_{l=0}^{2p-3}\mathbb{E}\left[\mathbb{E}\left(|r_{n+1}|)^2|\mathcal{F}_{t_n}\right)^{1/2}\mathbb{E}\left(|U_{n+1}|^{4(2p-2-l)}|\mathcal{F}_{t_n}\right)^{1/4}\mathbb{E}\left(|\rho_n|^{4l+4}|\mathcal{F}_{t_n}\right)^{1/4}\right] \\
\leq\ & K\sum_{l=0}^{2p-3}\mathbb{E}\left[|\rho_n|^{2p-1}\left(1+|X_{Y_n}(t_{n+1})|^{(\gamma-1)/2}+|X_{Y_n}([t_{n+1}])|^{(\gamma-1)/2}\right)(1+\mathbb{E}|x_0|^{3\beta\gamma})h^{p+1-l/2}\right] \\
\leq\ & K\mathbb{E}\left[|\rho_n|^{2p-1}\left(1+|X_{Y_n}(t_{n+1})|^{(\gamma-1)/2}+|X_{Y_n}([t_{n+1}])|^{(\gamma-1)/2}\right)(1+\mathbb{E}|x_0|^{3\beta\gamma})h^{5/2}\right]
\end{aligned}$$

$$\begin{aligned} &\leq K\mathbb{E}\left[\frac{2p-1}{2p}\left(|\rho_n|h^{(2p-1)/2p}\right)^{2p}\right] \\ &\quad + K\mathbb{E}\left[\frac{1}{2p}\left(\left(1+|X_{Y_n}(t_{n+1})|^{(\gamma-1)/2}+|X_{Y_n}([t_{n+1}])|^{(\gamma-1)/2}\right)(1+\mathbb{E}|x_0|^{3\beta\gamma})h^{5/2-(2p-1)/2p}\right)^{2p}\right] \\ &\leq Kh\mathbb{E}|\rho_n|^{2p}+K\left(1+\mathbb{E}|x_0|^{p\beta(7\gamma-1)}\right)h^{3p+1}, \end{aligned} \tag{40}$$

$$\begin{aligned} &\mathbb{E}\left(|S_{n+1}|^{2p-2}U_{n+1}r_{n+1}\right) \\ &\leq \mathbb{E}\left[\mathbb{E}\left(|S_{n+1}|^{4p-4}|\mathcal{F}_{t_n}\right)^{1/2}\mathbb{E}\left(|U_{n+1}|^4|\mathcal{F}_{t_n}\right)^{1/4}\mathbb{E}\left(|r_{n+1}|^4|\mathcal{F}_{t_n}\right)^{1/4}\right] \\ &\leq \mathbb{E}\left[|\rho_n|^{2p-1}\left(1+\mathbb{E}|X_{x_0}([t_{n+1}])|^{2(\gamma-1)}+\mathbb{E}|U_n|^{2(\gamma-1)}\right)^{1/4}\left(1+\mathbb{E}|x_0|^{12\beta\gamma}\right)^{1/4}h^3\right] \\ &\leq Kh\mathbb{E}|\rho_n|^{2p}+K\left(1+\mathbb{E}|x_0|^{p\beta(7\gamma-1)}\right)h^{4p+1} \end{aligned} \tag{41}$$

and

$$\begin{aligned} K\sum_{l=2}^{2p}\mathbb{E}\left(|S_{n+1}|^{2p-l}|r_{n+1}|^l\right) &\leq \sum_{l=2}^{2p}\mathbb{E}\left[\mathbb{E}\left(|S_{n+1}|^{4p-2l}|\mathcal{F}_{t_n}\right)^{1/2}\mathbb{E}\left(|r_{n+1}|^{2l}|\mathcal{F}_{t_n}\right)^{1/2}\right] \\ &\leq K\sum_{l=2}^{2p}\mathbb{E}\left[|\rho_n|^{2p-l}(1+Kh)\left(1+\mathbb{E}|x_0|^{6l\beta\gamma}\right)^{1/2}h^{2l}\right] \\ &\leq K\mathbb{E}h|\rho_n|^{2p}+K\left(1+\mathbb{E}|x_0|^{12p\beta\gamma}\right)h^{6p+1}. \end{aligned} \tag{42}$$

Substituting (37) − (42) into (36), we obtain

$$\mathbb{E}|\rho_{n+1}|^{2p} \leq \mathbb{E}|\rho_n|^{2p}+Kh\mathbb{E}|\rho_n|^{2p}+K\left(1+\mathbb{E}|x_0|^{12p\beta\gamma}\right)h^{2p+2}.$$

By the Gronwall inequality, we get (32).

The proof is complete. □

6. A numerical example

In this section, a numerical example is given to illustrate the result of Theorem 5.6. We use discrete Brownian paths over $[0,1]$ with $\Delta = 2^{-15}$. Let $X_h^i(T)$ be the numerical solution of the balanced Euler method along the ith sample path at $t = T$ with step size h. Let $X_\Delta^i(T)$ be the numerical solution to be an approximation of the analytic solution and compare this with the numerical approximation using $h = 2^3\Delta$, $h = 2^4\Delta$, $h = 2^5\Delta$ and $h = 2^6\Delta$ over $N = 500$ sample paths. Here the mean-square error is denoted as follows:

$$Error_h := \left(\frac{1}{N}\sum_{i=1}^{N}\left|X_h^i(T)-X_\Delta^i(T)\right|^2\right)^{1/2}.$$

The strong convergence order is defined numerically by

$$Order = \log\frac{Error_h}{Error_{h/2}}/\log(2).$$

Consider the following example:

Example 6.1. *Consider the following SVIDE*

$$\frac{dX(t)}{dt} = -X^3(t)+\int_0^t \sin(t-s)X(s)dw(t) \tag{43}$$

on $t \geq 0$ with initial data $X_0 = 1$, where $w(t)$ is a 1-dimension Brownian motion.

Here $\sigma(t,s) = \sin(t-s)$, $f(x) = -x^3$. It is obvious that all the assumptions are fulfilled.
Define the balanced method as follows:

$$Y_{n+1} = Y_n + \sin(f(Y_n)h) + \sin(hZ_n),$$

where

$$Z_n := \sum_{l=0}^{n-1} \sin(t_n - t_l)Y_l \Delta w_l.$$

Table 1: Strong convergence order for Example 6.1.

step size	Error	order
$2^5\Delta t$	0.0518	-
$2^6\Delta t$	0.1110	1.0995
$2^7\Delta t$	0.2295	1.0479
$2^8\Delta t$	0.4667	1.0240

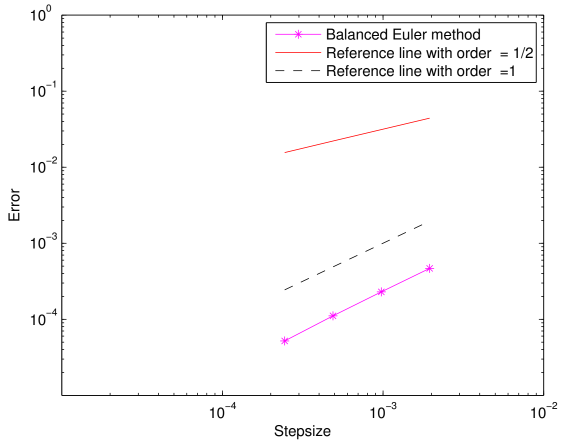

Figure 1: Mean square errors of the balanced Euler method of SVIDE (43)

In Table 1 and Figure 1, we can see that the strong convergence order of balanced method for SVIDE (43) is 1.

7. Conclusion

In this paper, we present the balanced Euler method of the nonlinear SVIDE (1). We give its moment boundedness and show a strong convergence order of 1 under polynomial growth coefficients and one-sided Lipschitz condition. Moreover, the theoretical results are illustrated by a numerical example.

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