



Minkowski-Type Inequalities Using Generalized Proportional Hadamard Fractional Integral Operators

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Abstract. The main objective of present investigation is to obtain some Minkowski-type fractional integral inequalities using generalised proportional Hadamard fractional integral operators which is introduced by Rahman et al. in the paper (Certain inequalities via generalized proportional Hadamard fractional integral operators), *Advances in Differential Equations*, 2019, 454(2019). In addition, we establish some other fractional integral inequalities for positive and continuous functions.

1. Introduction

Fractional calculus is generalization of traditional calculus into non-integer differential and integral order. Fractional calculus is very important due to its various application in field of science and technology. Fractional integral inequalities play big role in obtaining uniqueness of solution of fractional ordinary differential equations, fractional partial differential equations and fractional boundary value problems. Recently, a number of researchers in the fields of fractional integral inequalities have established different integral inequalities about fractional integral operators such as Riemann-Liouville, Hadamard, Saigo, generalized Katugampola, Erdélyi-Kober, Riemann-Liouville k -fractional, Hadamard k -fractional, (k,s) -Riemann-Liouville and k -generalized (in terms hypergeometric function) fractional integral operators, see [1, 6–9, 12–16, 19, 20, 23, 28–30, 36, 38, 40]. In [11], Dahmani investigated reverse Minkowski fractional integral inequality by employing Riemann-Liouville fractional integral. Ahmed Anber and et al. [2] presented some fractional integral inequalities which is similar to Minkowski fractional integral inequality using Riemann-Liouville fractional integral. V. L. Chinchane et al. [5] proposed fractional inequalities similar to Minkowski type via Saigo fractional integral operator. In [27], S. K. Panchal et al. investigated weighted fractional integral inequalities using generalized Katugampola fractional integral operator. G. Rahman et al. [31–33] established Minkowski inequality and some other fractional inequalities for convex functions by employing fractional proportional integral operators. In [10], V. L. Chinchane studied the reverse Minkowski fractional integral inequality by considering generalized k -fractional integral operator in terms of the Gauss hypergeometric function. S. Mubeen and et al. [22] have introduced Minkowski inequality involving generalized k -fractional conformable integrals. Moreover, the several kind of fractional

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integral and derivative operators have been introduced. Atangana and Baleanu proposed a new fractional derivative operator with the non-local and non-singular kernel, see [3]. In [18], F. Jarad et al. gave the fractional conformable integral and derivative operators. In [17, 34, 35], F. Jarad et al. and G. Rahman presented concepts of non-local fractional proportional and generalized Hadamard proportional integrals involving exponential functions in their kernels. In [24–26, 39], authors investigated various integral inequalities by employing conformable and generalized conformable fractional integrals. M. Caputo and M. Fabrizio [4] introduced new fractional derivative and integral without singular kernel. Later on, Lasada and Niteto proposed certain properties of fraction derivative without a singular kernel, see [21].

Motivated from [2, 10, 22, 32–35], our purpose in this paper is to propose some new results using generalized Hadamard proportional integrals. The paper has been organized as follows, in Section 2, we recall basic definitions, remarks and lemma related to generalized Hadamard proportional integrals. In Section 3, we obtain reverse Minkowski fractional integral inequality using generalized Hadamard proportional integrals, in Section 4, we present some other inequalities using generalized Hadamard proportional integrals. In Section 5, we give the concluding remarks.

2. Preliminary

Here, we devoted some basic definitions, remarks and lemma about generalized Hadamard proportional integrals which will be used later.

Definition 2.1. *The left and right sided generalized proportional fractional integrals are respectively defined by*

$$({}_a\mathfrak{J}^{\alpha,\beta}z)(x) = \frac{1}{\beta^\alpha\Gamma(\alpha)} \int_a^x e^{[\frac{\beta-1}{\beta}(x-t)]} (x-t)^{\alpha-1} z(t) dt, a < x \tag{1}$$

and

$$(\mathfrak{J}_b^{\alpha,\beta}z)(x) = \frac{1}{\beta^\alpha\Gamma(\alpha)} \int_x^b e^{[\frac{\beta-1}{\beta}(t-x)]} (t-x)^{\alpha-1} z(t) dt, x < b, \tag{2}$$

where the proportionality index $\beta \in (0, 1]$ and $\alpha \in \mathbb{C}$ and $\Re(\alpha) > 0$.

Remark 2.2. *If we consider $\beta = 1$ in (1) and (2), then we get the well known left and right Riemann-Liouville integrals which are respectively defined by*

$$({}_a\mathfrak{J}^\alpha z)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} z(t) dt, a < x \tag{3}$$

and

$$(\mathfrak{J}_b^\alpha z)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} z(t) dt, x < b, \tag{4}$$

where $\alpha \in \mathbb{C}$ and $\Re(\alpha) > 0$.

Recently, Rahman et al. [37] proposed the following generalized Hadamard proportional fractional integrals.

Definition 2.3. *The left sided generalized Hadamard proportional fractional integral of order $\alpha > 0$ and proportional index $\beta \in (0, 1]$ is defined by*

$$({}_a\mathcal{H}^{\alpha,\beta}z)(x) = \frac{1}{\beta^\alpha\Gamma(\alpha)} \int_a^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln t)]} (\ln x - \ln t)^{\alpha-1} \frac{z(t)}{t} dt, a < x. \tag{5}$$

Definition 2.4. The right sided generalized Hadamard proportional fractional integral of order $\alpha > 0$ and proportional index $\beta \in (0, 1]$ is defined by

$$(\mathcal{H}_b^{\alpha,\beta} z)(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_x^b e^{[\frac{\beta-1}{\beta}(\ln t - \ln x)]} (\ln t - \ln x)^{\alpha-1} \frac{z(t)}{t} dt, x < b. \tag{6}$$

Definition 2.5. The one sided generalized Hadamard proportional fractional integral of order $\alpha > 0$ and proportional index $\beta \in (0, 1]$ is defined by

$$(\mathcal{H}_{1,x}^{\alpha,\beta} z)(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln t)]} (\ln x - \ln t)^{\alpha-1} \frac{z(t)}{t} dt, t > 1, \tag{7}$$

where $\Gamma(\alpha)$ is the classical well known gamma function.

Remark 2.6. If we consider $\beta = 1$, then (5)-(7) will led to the following well known Hadamard fractional integrals

$$({}_a \mathcal{H}^\alpha z)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln t)^{\alpha-1} \frac{z(t)}{t} dt, a < x, \tag{8}$$

$$(\mathcal{H}_b^\alpha z)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\ln t - \ln x)^{\alpha-1} \frac{z(t)}{t} dt, x < b \tag{9}$$

and

$$(\mathcal{H}_{1,x}^\alpha z)(x) = \frac{1}{\Gamma(\alpha)} \int_1^x (\ln x - \ln t)^{\alpha-1} \frac{z(t)}{t} dt, x > 1. \tag{10}$$

One can easily prove the following results:

Lemma 2.7. If $\alpha, \lambda > 0$ and proportional index $\beta \in (0, 1]$, then following relation hold:

$$(\mathcal{H}_{1,x}^{\alpha,\beta} e^{[\frac{\beta-1}{\beta}(\ln x)]} (\ln x)^{\lambda-1})(x) = \frac{\Gamma(\lambda)}{\beta^\alpha \Gamma(\alpha + \lambda)} e^{[\frac{\beta-1}{\beta}(\ln x)]} (\ln x)^{\alpha+\lambda-1}, \tag{11}$$

and the semigroup property

$$(\mathcal{H}_{1,x}^{\alpha,\beta})(\mathcal{H}_{1,x}^{\lambda,\beta})z(x) = (\mathcal{H}_{1,x}^{\alpha+\lambda,\beta})z(x). \tag{12}$$

Remark 2.8. If $\beta = 1$, then (11) will reduce to the result of [37] as defined by

$$(\mathcal{H}_{1,x}^\alpha (\ln x)^{\lambda-1})(x) = \frac{\Gamma(\lambda)}{\Gamma(\alpha + \lambda)} (\ln x)^{\alpha+\lambda-1}. \tag{13}$$

3. Reverse Minkowski fractional integral inequality

In this section, we establish reverse Minkowski fractional integral inequality involving generalized proportional Hadamard fractional integral operators.

Theorem 3.1. Let $p \geq 1$ and let f, g be two positive function on $[1, \infty)$, such that for all $x > 0$, $\mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)] < \infty$, $\mathcal{H}_{1,x}^{\alpha,\beta}[g^q(x)] < \infty$. If $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in (1, x)$, we have

$$\left[\mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)] \right]^{\frac{1}{p}} + \left[\mathcal{H}_{1,x}^{\alpha,\beta}[g^q(x)] \right]^{\frac{1}{q}} \leq \frac{1 + M(m + 2)}{(m + 1)(M + 1)} \left[\mathcal{H}_{1,x}^{\alpha,\beta}[(f + g)^p(x)] \right]^{\frac{1}{p}}, \tag{14}$$

where $\alpha > 0$, $\beta \in (0, 1]$ $\alpha \in \mathbb{C}$ and $\Re(\alpha) > 0$.

Proof: Using the condition $\frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in (1, x)$, $x > 1$, we can write

$$(M + 1)^p f(\tau) \leq M^p (f + g)^p(\tau). \quad (15)$$

Consider

$$\psi(x, \tau) = \frac{1}{\beta^\alpha \Gamma(\alpha) \tau} e^{[\frac{\beta-1}{\beta}(\ln x - \ln \tau)]} (\ln x - \ln \tau)^{\alpha-1}. \quad (16)$$

Clearly, we can say that the function $\psi(x, \tau)$ remain positive because for all $\tau \in (1, x)$, ($x > 1$), $\alpha, \beta > 0$. Multiplying both side of (15) by $\psi(x, \tau)$, then integrating resulting identity with respect to τ from 1 to x , we get

$$\begin{aligned} & \frac{(M + 1)^p}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln \tau)]} (\ln x - \ln \tau)^{\alpha-1} f^p(\tau) \frac{d\tau}{\tau} \\ & \leq \frac{M^p}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln \tau)]} (\ln x - \ln \tau)^{\alpha-1} (f + g)^p(\tau) \frac{d\tau}{\tau}, \end{aligned} \quad (17)$$

which is equivalent to

$$\mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)] \leq \frac{M^p}{(M + 1)^p} [\mathcal{H}_{1,x}^{\alpha,\beta}[(f + g)^p(x)]], \quad (18)$$

hence, we can write

$$[\mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)]]^{\frac{1}{p}} \leq \frac{M}{(M + 1)} [\mathcal{H}_{1,x}^{\alpha,\beta}[(f + g)^p(x)]]^{\frac{1}{p}}. \quad (19)$$

On other hand, using condition $m \leq \frac{f(\tau)}{g(\tau)}$, we obtain

$$\left(1 + \frac{1}{m}\right)g(\tau) \leq \frac{1}{m}(f(\tau) + g(\tau)), \quad (20)$$

therefore,

$$\left(1 + \frac{1}{m}\right)^p g^p(\tau) \leq \left(\frac{1}{m}\right)^p (f(\tau) + g(\tau))^p. \quad (21)$$

Now, multiplying both side of (21) by $\psi(x, \tau)$, ($\tau \in (1, x)$, $x > 1$), where $\psi(x, \tau)$ is defined by (16). Then integrating resulting identity with respect to τ from 1 to x , we have

$$[\mathcal{H}_{1,x}^{\alpha,\beta}[g^p(x)]]^{\frac{1}{p}} \leq \left(\frac{1}{m + 1}\right) [\mathcal{H}_{1,x}^{\alpha,\beta}[(f + g)^p(x)]]^{\frac{1}{p}}. \quad (22)$$

The inequalities (14) follows on adding the inequalities (19) and (22).

Our second result is as follows.

Theorem 3.2. Let $p \geq 1$ and f, g be two positive function on $[1, \infty)$, such that for all $x > 0$, $\mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)] < \infty$, $\mathcal{H}_{1,x}^{\alpha,\beta}[g^q(x)] < \infty$. If $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M$, we have

$$\begin{aligned} & [\mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)]]^{\frac{2}{p}} + [\mathcal{H}_{1,x}^{\alpha,\beta}[g^q(x)]]^{\frac{2}{p}} \geq \left(\frac{(M + 1)(m + 1)}{M} - 2\right) [\mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)]]^{\frac{1}{p}} + \\ & \quad [\mathcal{I}_{0,x}^{\alpha,\beta,\eta}[g^q(x)]]^{\frac{1}{p}}, \end{aligned} \quad (23)$$

where $\alpha > 0$, $\beta \in (0, 1]$, $\alpha \in \mathbb{C}$ and $\Re(\alpha) > 0$.

Proof: Multiplying the inequalities (19) and (22), we obtain

$$\left(\frac{(M+1)(m+1)}{M}\right) \left[\mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)]\right]^{\frac{1}{p}} \left[\mathcal{H}_{1,x}^{\alpha,\beta}[g^q(x)]\right]^{\frac{1}{q}} \leq \left(\mathcal{H}_{1,x}^{\alpha,\beta}[(f(x)+g(x))^p]\right)^{\frac{1}{p}}. \tag{24}$$

Applying Minkowski inequalities to the right hand side of (24), we have

$$\left(\mathcal{H}_{1,x}^{\alpha,\beta}[(f(x)+g(x))^p]\right)^{\frac{1}{p}} \leq \left(\mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)]\right)^{\frac{1}{p}} + \left(\mathcal{H}_{1,x}^{\alpha,\beta}[g^q(x)]\right)^{\frac{1}{q}}, \tag{25}$$

which implies that

$$\begin{aligned} \left(\mathcal{H}_{1,x}^{\alpha,\beta}[(f(x)+g(x))^p]\right)^{\frac{2}{p}} &\leq \left(\mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)]\right)^{\frac{2}{p}} + \left(\mathcal{H}_{1,x}^{\alpha,\beta}[g^q(x)]\right)^{\frac{2}{q}} \\ &\quad + 2\left[\mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)]\right]^{\frac{1}{p}} \left[\mathcal{H}_{1,x}^{\alpha,\beta}[g^q(x)]\right]^{\frac{1}{q}}. \end{aligned} \tag{26}$$

using (24) and (26) we obtain (23). Theorem 3.2 is thus proved.

4. Some other fractional integral inequalities

Here, we establish some new integral inequalities using generalized proportional Hadamard fractional integral operators.

Theorem 4.1. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and f, g be two positive function on $[1, \infty)$, such that $\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)] < \infty$, $\mathcal{H}_{1,x}^{\alpha,\beta}[g(x)] < \infty$. If $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M < \infty$, $\tau \in (1, x)$, we have

$$\left[\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]\right]^{\frac{1}{p}} \left[\mathcal{H}_{1,x}^{\alpha,\beta}[g(x)]\right]^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \left[\mathcal{H}_{1,x}^{\alpha,\beta}[[f(x)]^{\frac{1}{p}}[g(x)]^{\frac{1}{q}}]\right], \tag{27}$$

hold, where $\alpha > 0$, $\beta \in (0, 1]$ $\alpha \in \mathbb{C}$ and $\Re(\alpha) > 0$.

Proof:- Since $\frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in [1, x]$ $x > 1$, therefore

$$[g(\tau)]^{\frac{1}{q}} \geq M^{-\frac{1}{q}} [f(\tau)]^{\frac{1}{q}}, \tag{28}$$

and also,

$$\begin{aligned} [f(\tau)]^{\frac{1}{p}} [g(\tau)]^{\frac{1}{q}} &\geq M^{-\frac{1}{q}} [f(\tau)]^{\frac{1}{p}} [f(\tau)]^{\frac{1}{q}} \\ &\geq M^{-\frac{1}{q}} [f(\tau)]^{\frac{1}{p} + \frac{1}{q}} \\ &\geq M^{-\frac{1}{q}} [f(\tau)]. \end{aligned} \tag{29}$$

Multiplying both side of (29) by $\psi(x, \tau)$, ($\tau \in (1, x)$, $x > 1$), where $\psi(x, \tau)$ is defined by (16). Then integrating resulting identity with respect to τ from 1 to x , we have

$$\begin{aligned} \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{l^{\frac{\beta-1}{\beta}}(\ln x - \ln \tau)} (\ln x - \ln \tau)^{\alpha-1} f(\tau)^{\frac{1}{p}} g(\tau)^{\frac{1}{q}}(\tau) \frac{d\tau}{\tau} \\ \geq \frac{M^{-\frac{1}{q}}}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{l^{\frac{\beta-1}{\beta}}(\ln x - \ln \tau)} (\ln x - \ln \tau)^{\alpha-1} f(\tau) \frac{d\tau}{\tau}, \end{aligned} \tag{30}$$

which implies that,

$$\mathcal{H}_{1,x}^{\alpha,\beta} \left[[f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} \right] \geq M^{-\frac{1}{q}} \left[\mathcal{H}_{1,x}^{\alpha,\beta} f(x) \right]. \tag{31}$$

Consequently,

$$\left(\mathcal{H}_{1,x}^{\alpha,\beta} \left[[f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} \right]\right)^{\frac{1}{p}} \geq M^{\frac{-1}{pq}} \left[\mathcal{H}_{1,x}^{\alpha,\beta} f(x) \right]^{\frac{1}{p}}. \tag{32}$$

On other hand, since $mg(\tau) \leq f(\tau)$, $\tau \in [1, x]$, $x > 1$, then we have

$$[f(\tau)]^{\frac{1}{p}} \geq m^{\frac{1}{p}} [g(\tau)]^{\frac{1}{p}}, \tag{33}$$

multiplying equation (33) by $[g(\tau)]^{\frac{1}{q}}$, we have

$$[f(\tau)]^{\frac{1}{p}} [g(\tau)]^{\frac{1}{q}} \geq m^{\frac{1}{p}} [g(\tau)]^{\frac{1}{q}} [g(\tau)]^{\frac{1}{p}} = m^{\frac{1}{p}} [g(\tau)]. \tag{34}$$

Multiplying both side of (34) by $\psi(x, \tau)$, ($\tau \in (1, x)$, $x > 1$), where $\psi(x, \tau)$ is defined by (16). Then integrating resulting identity with respect to τ from 1 to x , we have

$$\begin{aligned} & \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{l^{\frac{\beta-1}{\beta}}(\ln x - \ln \tau)} (\ln x - \ln \tau)^{\alpha-1} f(\tau)^{\frac{1}{p}} g(\tau)^{\frac{1}{q}} (\tau) \frac{d\tau}{\tau} \\ & \geq \frac{m^{\frac{1}{p}}}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{l^{\frac{\beta-1}{\beta}}(\ln x - \ln \tau)} (\ln x - \ln \tau)^{\alpha-1} [g(\tau)] \frac{d\tau}{\tau}. \end{aligned} \tag{35}$$

that is,

$$\mathcal{H}_{1,x}^{\alpha,\beta} \left[[f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} \right] \geq m^{\frac{1}{p}} \left[\mathcal{H}_{1,x}^{\alpha,\beta} g(x) \right]. \tag{36}$$

Hence we can write,

$$\left(\mathcal{H}_{1,x}^{\alpha,\beta} \left[[f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} \right]\right)^{\frac{1}{q}} \geq m^{\frac{1}{pq}} \left[\mathcal{H}_{1,x}^{\alpha,\beta} g(x) \right]^{\frac{1}{q}}, \tag{37}$$

multiplying equation (32) and (37) we get the result (27).

Theorem 4.2. Let f and g be two positive function on $[1, \infty[$, such that $\mathcal{H}_{1,x}^{\alpha,\beta} [f^p(x)] < \infty$, $\mathcal{H}_{1,x}^{\alpha,\beta} [g^q(x)] < \infty$, $x > 0$. If $0 < m \leq \frac{f(\tau)^p}{g(\tau)^q} \leq M < \infty$, $\tau \in [1, x]$. Then we have

$$\left[\mathcal{H}_{1,x}^{\alpha,\beta} f^p(x) \right]^{\frac{1}{p}} \left[\mathcal{H}_{1,x}^{\alpha,\beta} g^q(x) \right]^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \left[\mathcal{H}_{1,x}^{\alpha,\beta} (f(x)g(x)) \right], \tag{38}$$

hold. Where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 0$, $\beta \in (0, 1]$ $\alpha \in \mathbb{C}$ and $\Re(\alpha) > 0$.

Proof:- Replacing $f(\tau)$ and $g(\tau)$ by $f(\tau)^p$ and $g(\tau)^q$, $\tau \in [1, x]$, $x > 1$ in Theorem 4.1, we obtain (38).

. Now, here we present fractional integral inequality related to Minkowsky inequality as follows

Theorem 4.3. Let f and g be two integrable functions on $[1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, and $0 < m < \frac{f(\tau)}{g(\tau)} < M$, $\tau \in (1, x)$, $x > 1$. Then we have

$$\mathcal{H}_{1,x}^{\alpha,\beta} [f(x)g(x)] \leq \frac{2^{p-1}M^p}{p(M+1)^p} \left(\mathcal{H}_{1,x}^{\alpha,\beta} [f^p + g^p](x) \right) + \frac{2^{q-1}}{q(m+1)^q} \left(\mathcal{H}_{1,x}^{\alpha,\beta} [f^q + g^q](x) \right), \tag{39}$$

where $\alpha > 0$, $\beta \in (0, 1]$ $\alpha \in \mathbb{C}$ and $\Re(\alpha) > 0$.

Proof:- Since, $\frac{f(\tau)}{g(\tau)} < M, \tau \in (1, x), x > 1$, we have

$$(M + 1)f(\tau) \leq M(f + g)(\tau). \tag{40}$$

Taking p^{th} power on both side of (40) and multiplying resulting identity by $\psi(x, \tau), (\tau \in (1, x), x > 1)$, where $\psi(x, \tau)$ is defined by (16), then integrate the resulting identity with respect to τ from 1 to x , we have

$$\begin{aligned} & \frac{(M + 1)^p}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln \tau)]} (\ln x - \ln \tau)^{\alpha-1} f(\tau)^p \frac{d\tau}{\tau} \\ & \leq \frac{M^p}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln \tau)]} (\ln x - \ln \tau)^{\alpha-1} [(f + g)^p(\tau)] \frac{d\tau}{\tau}, \end{aligned} \tag{41}$$

therefore,

$$\mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)] \leq \frac{M^p}{(M + 1)^p} \mathcal{H}_{1,x}^{\alpha,\beta}[(f + g)^p(x)]. \tag{42}$$

On other hand, $0 < m < \frac{f(\tau)}{g(\tau)}, \tau \in (1, x), x > 1$, we can write

$$(m + 1)g(\tau) \leq (f + g)(\tau). \tag{43}$$

Taking q^{th} power on both side (43) and multiplying resulting identity by $\psi(x, \tau), (\tau \in (1, x), x > 1)$, where $\psi(x, \tau)$ is defined by (16), then integrate the resulting identity with respect to τ from 1 to x , we have

$$\begin{aligned} & \frac{(m + 1)^q}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln \tau)]} (\ln x - \ln \tau)^{\alpha-1} g^q(\tau) \frac{d\tau}{\tau} \\ & \leq \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln \tau)]} (\ln x - \ln \tau)^{\alpha-1} [(f + g)^q(\tau)] \frac{d\tau}{\tau}, \end{aligned} \tag{44}$$

consequently, we have

$$\mathcal{H}_{1,x}^{\alpha,\beta}[g^q(x)] \leq \frac{1}{(m + 1)^q} \mathcal{H}_{1,x}^{\alpha,\beta}[(f + g)^q(x)]. \tag{45}$$

Now, using Young inequality

$$[f(\tau)g(\tau)] \leq \frac{f^p(\tau)}{p} + \frac{g^q(\tau)}{q}. \tag{46}$$

Multiplying both side of (46) by $\psi(x, \tau), (\tau \in (1, x), x > 1)$, where $\psi(x, \tau)$ is defined by (16), then integrate the resulting identity with respect to τ from 1 to x , we get

$$\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)g(x)] \leq \frac{1}{p} \mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)] + \frac{1}{q} \mathcal{H}_{1,x}^{\alpha,\beta}[g^q(x)], \tag{47}$$

from equation (42), (45) and (47), we have

$$\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)g(x)] \leq \frac{M^p}{p(M + 1)^p} \mathcal{H}_{1,x}^{\alpha,\beta}[(f + g)^p(x)] + \frac{1}{q(m + 1)^q} \mathcal{H}_{1,x}^{\alpha,\beta}[(f + g)^q(x)], \tag{48}$$

now using the inequality $(a + b)^r \leq 2^{r-1}(a^r + b^r), r > 1, a, b \geq 0$, we have

$$\mathcal{H}_{1,x}^{\alpha,\beta}[(f + g)^p(x)] \leq 2^{p-1} \mathcal{H}_{1,x}^{\alpha,\beta}[f^p + g^p](x) \tag{49}$$

and

$$\mathcal{H}_{1,x}^{\alpha,\beta}[(f + g)^q(x)] \leq 2^{q-1} \mathcal{H}_{1,x}^{\alpha,\beta}[f^q + g^q](x). \tag{50}$$

Injecting (49), (50) in (48), we get required inequality (39).

This complete the proof.

Theorem 4.4. Let f, g be two positive functions defined on $[1, \infty)$, such that g is non-decreasing. If

$$\mathcal{H}_{1,x}^{\alpha,\beta} f(x) \geq \mathcal{H}_{1,x}^{\alpha,\beta} g(x), x > 1, \quad (51)$$

then for all $\alpha > 0, \beta \in (0, 1], \alpha \in \mathbb{C}$ and $\Re(\alpha) > 0$,

$$\mathcal{H}_{1,x}^{\alpha,\beta} f^{\gamma-\delta}(x) \leq \mathcal{H}_{1,x}^{\alpha,\beta} f^\gamma(x) g^{-\delta}(x), \quad (52)$$

hold.

Proof:- We use arithmetic-geometric inequality, for $\gamma > 0, \delta > 0$, we have

$$\frac{\gamma}{\gamma - \delta} f^{\gamma-\delta}(\tau) - \frac{\delta}{\gamma - \delta} g^{\gamma-\delta}(\tau) \leq f^\gamma(\tau) g^{-\delta}(\tau), \tau \in (1, x), x > 1. \quad (53)$$

Now, multiplying both side of (53) by $\psi(x, \tau)$, ($\tau \in (1, x), x > 1$), where $\psi(x, \tau)$ is defined by (16), then integrate the resulting identity with respect to τ from 1 to x , we get

$$\begin{aligned} & \frac{\gamma}{\gamma - \delta \beta^\alpha \Gamma(\alpha)} \int_1^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln \tau)]} (\ln x - \ln \tau)^{\alpha-1} f^{\gamma-\delta}(\tau) \frac{d\tau}{\tau} \\ & - \frac{\delta}{\gamma - \delta \beta^\alpha \Gamma(\alpha)} \int_1^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln \tau)]} (\ln x - \ln \tau)^{\alpha-1} g^{\gamma-\delta}(\tau) \frac{d\tau}{\tau} \\ & \leq \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln \tau)]} (\ln x - \ln \tau)^{\alpha-1} f^\gamma(\tau) g^{-\delta}(\tau) \frac{d\tau}{\tau}, \end{aligned} \quad (54)$$

consequently,

$$\frac{\gamma}{\gamma - \delta} \mathcal{H}_{1,x}^{\alpha,\beta} [f^{\gamma-\delta}(x)] - \frac{\delta}{\gamma - \delta} \mathcal{H}_{1,x}^{\alpha,\beta} [g^{\gamma-\delta}(x)] \leq \mathcal{H}_{1,x}^{\alpha,\beta} [f^\gamma(x) g^{-\delta}(x)], \quad (55)$$

which implies that,

$$\frac{\gamma}{\gamma - \delta} \mathcal{H}_{1,x}^{\alpha,\beta} [f^{\gamma-\delta}(x)] \leq \mathcal{H}_{1,x}^{\alpha,\beta} [f^\gamma(x) g^{-\delta}(x)] + \frac{\delta}{\gamma - \delta} \mathcal{H}_{1,x}^{\alpha,\beta} [g^{\gamma-\delta}(x)], \quad (56)$$

that is

$$\mathcal{H}_{1,x}^{\alpha,\beta} [f^{\gamma-\delta}(x)] \leq \frac{\gamma - \delta}{\gamma} \mathcal{H}_{1,x}^{\alpha,\beta} [f^\gamma(x) g^{-\delta}(x)] + \frac{\delta}{\gamma} \mathcal{H}_{1,x}^{\alpha,\beta} [f^{\gamma-\delta}(x)], \quad (57)$$

thus we get the result (52).

Theorem 4.5. Suppose that f, g and h be positive and continuous functions on $[1, \infty)$, such that

$$(g(\tau) - g(\sigma)) \left(\frac{f(\sigma)}{h(\sigma)} - \frac{f(\tau)}{h(\tau)} \right) \geq 0; \tau, \sigma \in [1, x) x > 1, \quad (58)$$

then for all $\alpha > 0, \beta \in (0, 1], \alpha \in \mathbb{C}$ and $\Re(\alpha) > 0$,

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta} [f(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta} [h(x)]} \geq \frac{\mathcal{H}_{1,x}^{\alpha,\beta} [(gf)(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta} [(gh)(x)]}, \quad (59)$$

hold.

Proof: Since f, g and h be three positive and continuous functions on $[1, \infty)$ by (58), we can write

$$g(\tau)\frac{f(\sigma)}{h(\sigma)} + g(\sigma)\frac{f(\tau)}{h(\tau)} - g(\sigma)\frac{f(\sigma)}{h(\sigma)} - g(\tau)\frac{f(\tau)}{h(\tau)} \geq 0; \quad \tau, \sigma \in (1, x), \quad x > 1. \tag{60}$$

Now, multiplying equation (60) by $h(\sigma)h(\tau)$, on both side, we have

$$g(\tau)f(\sigma)h(\tau) - g(\tau)f(\tau)h(\sigma) - g(\sigma)f(\sigma)h(\tau) + g(\sigma)f(\tau)h(\sigma) \geq 0. \tag{61}$$

Now multiplying equation (61) by $\psi(x, \tau)$, ($\tau \in (1, x), x > 1$), where $\psi(x, \tau)$ is defined by (16), then integrate the resulting identity with respect to τ from 1 to x , we get

$$\begin{aligned} & \frac{f(\sigma)}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln \tau)]} (\ln x - \ln \tau)^{\alpha-1} [g(\tau)h(\tau)] \frac{d\tau}{\tau} \\ & - \frac{h(\sigma)}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln \tau)]} (\ln x - \ln \tau)^{\alpha-1} [f(\tau)g(\tau)] \frac{d\tau}{\tau} \\ & - \frac{f(\sigma)g(\sigma)}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln \tau)]} (\ln x - \ln \tau)^{\alpha-1} [h(\tau)] \frac{d\tau}{\tau} \\ & + \frac{g(\sigma)h(\sigma)}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln \tau)]} (\ln x - \ln \tau)^{\alpha-1} [f(\tau)] \frac{d\tau}{\tau} \geq 0, \end{aligned} \tag{62}$$

we get

$$\begin{aligned} & f(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[(gh)(x)] + g(\sigma)h(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)] \\ & - g(\sigma)f(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[h(x)] - h(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[(gf)(x)] \geq 0. \end{aligned} \tag{63}$$

Again multiplying (63) by $\psi(x, \sigma)$, in view of equation (16) and which is positive because $\sigma \in (1, x), x > 1$, then integrate the resulting identity with respect to σ from 1 to x , we have

$$\begin{aligned} & \mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[(gh)(x)] - \mathcal{H}_{1,x}^{\alpha,\beta}[h(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[(gf)(x)] \\ & - \mathcal{H}_{1,x}^{\alpha,\beta}[(gf)(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[h(x)] \\ & + \mathcal{H}_{1,x}^{\alpha,\beta}[(gh)(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)] \geq 0, \end{aligned} \tag{64}$$

which completes the proof.

Theorem 4.6. Suppose that f, g and h be positive and continuous functions on $[1, \infty)$, such that

$$(g(\tau) - g(\sigma))\left(\frac{f(\sigma)}{h(\sigma)} - \frac{f(\tau)}{h(\tau)}\right) \geq 0, \quad \tau, \sigma \in (1, x), \quad x > 0, \tag{65}$$

then for all $\alpha > 0, \beta \in (0, 1], \alpha, \phi \in \mathbb{C}$ and $\Re(\alpha) > 0, \Re(\phi) > 0$,

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]\mathcal{H}_{1,x}^{\phi,\phi}[(gh)(x)] + \mathcal{H}_{1,x}^{\phi,\phi}[f(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[(gh)(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta}[h(x)]\mathcal{H}_{1,x}^{\phi,\phi}[(gf)(x)] + \mathcal{H}_{1,x}^{\phi,\phi}[h(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[(gf)(x)]} \geq 1, \tag{66}$$

hold.

Proof: Multiplying equation (63) by $\frac{1}{\phi^\phi \Gamma(\phi)\sigma} e^{[\frac{\phi-1}{\phi}(\ln x - \ln \sigma)]} (\ln x - \ln \sigma)^{\phi-1}$, which remain positive because $\sigma \in (1, x), (x > 1), \phi, \phi > 0$. Then integrating resulting identity with respect to σ from 1 to x , we have

$$\begin{aligned} & \mathcal{H}_{1,x}^{\phi,\phi}[f(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[(gh)(x)] - \mathcal{H}_{1,x}^{\phi,\phi}[h(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[(gf)(x)] \\ & - \mathcal{H}_{1,x}^{\phi,\phi}[(gf)(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[h(x)] \\ & + \mathcal{H}_{1,x}^{\phi,\phi}[(gh)(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)] \geq 0, \end{aligned} \tag{67}$$

this gives the required inequality (66).

Theorem 4.7. Suppose that f and h are two positive continuous functions such that $f \leq h$ on $[1, \infty)$. If $\frac{f}{h}$ is decreasing and f is increasing on $[1, \infty)$, then for any $p \geq 0$, for all $x > 1$, $\alpha > 0$, $\beta \in (0, 1]$, $\alpha \in \mathbb{C}$ and $\Re(\alpha) > 0$, we have

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta}[h(x)]} \geq \frac{\mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta}[h^p(x)]}. \tag{68}$$

Proof: We take $g = f^{p-1}$ in Theorem 4.5,

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta}[h(x)]} \geq \frac{\mathcal{H}_{1,x}^{\alpha,\beta}[(f f^{p-1})(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta}[(h f^{p-1})(x)]}. \tag{69}$$

Since $f \leq h$ on $[1, \infty)$, then we can write

$$h f^{p-1} \leq h^p. \tag{70}$$

Multiplying equation (70) by $\psi(x, \tau)$, ($\tau \in (1, x)$, $x > 1$), where $\psi(x, \tau)$ is defined by (16), then integrate the resulting identity with respect to τ from 1 to x , we get

$$\mathcal{H}_{1,x}^{\alpha,\beta}[h f^{p-1}(x)] \leq \mathcal{H}_{1,x}^{\alpha,\beta}[h^p(x)], \tag{71}$$

and so, we have

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta}[(f f^{p-1})(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta}[(h f^{p-1})(x)]} \geq \frac{\mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta}[h^p(x)]}, \tag{72}$$

then from equation (69) and (72), we obtain (68).

Theorem 4.8. Suppose that f and h are two positive continuous functions such that $f \leq h$ on $[1, \infty)$. If $\frac{f}{h}$ is decreasing and f is increasing on $[1, \infty)$, then for any $p \geq 1$, for all $x > 1$, $\alpha, \phi > 0$, $\beta, \varphi \in (0, 1]$, $\alpha, \phi \in \mathbb{C}$ and $\Re(\alpha) > 0$, $\Re(\varphi) > 0$,

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[h^p(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[f(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[h^p(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta}[h(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[f^p(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[h(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)]} \geq 1. \tag{73}$$

Proof: We take $g = f^{p-1}$ in Theorem 4.6, then we obtain

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[h f^{p-1}(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[f(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[h f^{p-1}(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta}[h(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[f^p(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[h(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)]} \geq 1, \tag{74}$$

then by hypothesis, $f \leq h$ on $[1, \infty)$, which implies that

$$h f^{p-1} \leq h^p. \tag{75}$$

Now, multiplying both side of (75) by $\frac{1}{\varphi^\phi \Gamma(\phi)\sigma} e^{[\frac{\varphi-1}{\varphi}(\ln x - \ln \sigma)]} (\ln x - \ln \sigma)^{\phi-1}$, which remain positive because $\sigma \in (1, x)$, ($x > 1$), $\phi > 0$, $\varphi \in (0, 1]$. Then integrating resulting identity with respect to σ from 1 to x , we have

$$\mathcal{H}_{1,x}^{\phi,\varphi}[h f^{p-1}(x)] \leq \mathcal{H}_{1,x}^{\phi,\varphi}[h^p(x)], \tag{76}$$

multiplying on both side of (76) by $\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]$, we obtain

$$\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[h f^{p-1}(x)] \leq \mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[h^p(x)], \tag{77}$$

hence by (71) and (77), we obtain

$$\begin{aligned} & \mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]\mathcal{H}_{1,x}^{\phi,\rho}[hf^{p-1}(x)] + \mathcal{H}_{1,x}^{\phi,\rho}[f(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[hf^{p-1}(x)] \\ & \leq \mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]\mathcal{H}_{1,x}^{\phi,\rho}[h^p(x)] + \mathcal{H}_{1,x}^{\phi,\rho}[f(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[h^p(x)]. \end{aligned} \quad (78)$$

By (74) and (78), we complete the proof of this Theorem.

5. Concluding Remarks

Rahaman et al. [35] investigated some integral inequalities by considering generalized proportional Hadamard fractional integral operator. In [34] Rahaman et al. studied certain new weighted fractional integral inequalities involving a family of n ($n \in \mathbb{N}$) positive function by using generalized proportional Hadamard fractional integral operators. Motivated by the above work, here we studied reverse Minkowski fractional integral inequalities and other fractional inequalities by employing generalized proportional Hadamard fractional integral operator. The inequalities investigated in this paper give some contribution in the fields of fractional calculus and generalized proportional Hadamard fractional integral operators. By the help of this study we establish more general inequalities than in the classical cases. Moreover, they are expected to led to some application for finding uniqueness of solutions in fractional differential equations.

References

- [1] G. A. Anastassiou, *Fractional Differentiation Inequalities*, Springer Publishing Company, New York, 2009.
- [2] A. Anber, Z. Dahmani and B. Bendoukha, *New integral inequalities of Feng Qi type via Riemann-Liouville fractional integration*, Facta Universitatis (NIS) Ser. Math. Inform., 27(2), (2012), 13-22.
- [3] A. Atangana and D. Baleanu, *New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model*, Thermal Sci., 20, (2016), 763–769.
- [4] M. Caputo, and M. Fabrizio, *A new Definition of Fractional Derivative without Singular Kernel*, Progr. Fract. Differ. Appl., 1, (2015), 73–85.
- [5] V. L. Chinchane and D. B. Pachpatte *New fractional inequalities involving Saigo fractional integral operator*, Math. Sci. Lett., 3(3), (2014), 1-9.
- [6] V. L. Chinchane and D. B. Pachpatte, *A note on some integral inequalities via Hadamard integral*, J. Fractional Calculus Appl., 4(11), (2013), 1-5.
- [7] V. L. Chinchane and D. B. Pachpatte, *A note on fractional integral inequalities involving convex functions using Saigo fractional integral*, Indian Journal of Mathematics., 61(1), (2019), 27-39.
- [8] V.L. Chinchane *Certain inequalities using Saigo fractional integral operator*, Facta Universitatis. Ser. Math. Inform., 29(14), (2014), 343-350.
- [9] V. L. Chinchane, *On Chebyshev type inequalities using generalized K-fractional integral operator*, Progr. Fract. Differ. Appl., 3(3), (2017), 1-8.
- [10] V. L. Chinchane, *New approach to Minkowski fractional inequalities using generalized K-fractional integral operator*, Journal of the Indian. Math. Soc., 1-2(85), (2018), 32-41.
- [11] Z. Dahmani, *On Minkowski and Hermit-Hadamard integral inequalities via fractional integration*, Ann. Funct. Anal., 1(1), (2010), 51-58.
- [12] Z. Dahmani, *New inequalities in fractional integrals*, Int. J. Nonlinear Sci., 9(4), (2010), 493-497.
- [13] Z. Dahmani, *The Riemann-Liouville operator to generate some new inequalities*, Int. J. Nonlinear Sci., 12(4), (2011), 452-455.
- [14] Z. Dahmani, *Some results associate with fractional integrals involving the extended Chebyshev*, Acta Univ. Apulensis Math. Inform., 27, (2011), 217-224.
- [15] Z. Dahmani, L. Tabharit and S. Taf, *New generalisations of Gruss inequality using Riemann-Liouville fractional integrals*, Bull. Math. Anal. Appl., 2(3), (2010), 93-99.
- [16] Z. Denton and A. S. Vatsala, *Fractional integral inequalities and application*, Comput. Math. Appl., 59, (2010), 1087-1094.
- [17] F. Jarad, T. Abdejawad and J. Alzabut, *Generalized fractional derivatives generated by a class of local proportional derivatives*, Eur. Phys. J. Spec. Top., 226, (2017), 3457-3471.
- [18] F. Jarad, E. Ugurlu, T. Abdeljawad, and D. Baleanu, *On a new class of fractional operators*, Adv. Differ. Equ., 247, 2017, (2017), 1-16.
- [19] U. N. Katugampola, *A new approach to generalized fractional derivatives*, Bull. Math. Anal. Appl., 6(4), (2014), 1-15.
- [20] V. Kiryakova, *On two Saigo's fractional integral operator in the class of univalent functions*, Fract. Calc. Appl. Anal., 9(2), (2006), 159-176.
- [21] J. Losada, and J. J. Nieto, *Properties of a New Fractional Derivative without Singular Kernel*, Progr. Fract. Differ. Appl., 1 (2015), 87–92.
- [22] S. Mubeen, S. Habib and M. N. Naeem, *The Minkowski inequality involving generalized k-fractional conformable integral*, J. Inequal. Appl., 81, 2019, (2019), 1-18.
- [23] A. B. Nale, S. K. Panchal and V. L. Chinchane, *Certain fractional integral inequalities using generalized Katugampola fractional integral operator*, Malaya J. Math., 3(8), (2020), 791-797.

- [24] K. S. Nisar, G. Rahman, K. Mehrez, *Chebyshev type inequalities via generalized fractional conformable integrals*, J. Inequal. Appl., 245, 2019, (2019), 1-9.
- [25] K. S. Nisar, A. Tassadiq, G. Rahman, and A. Khan, *Some inequalities via fractional conformable integral operators*, J. Inequal. Appl., 217, 2019, (2019), 1-8.
- [26] K. S. Nisar, G. Rahman and A. Khan, *Some new inequalities for generalized fractional conformable integral operators*, Adv. Differ. Equ., 427, 2019, (2019), 1-10.
- [27] S. K. Panchal and V. L. Chinchane, A. B. Nale, *On weighted fractional inequalities using generalized Katugampola fractional integral operator*, Fractional Differ. Calc., 2(10), (2020), 255–266.
- [28] S. D. Purohit and R. K. Raina, *Chebyshev type inequalities for the Saigo fractional integral and their q - analogues*, J. Math. Inequal., 7(2), (2013), 239-249.
- [29] S. D. Purohit, R. K. Yadav, *A remark on integral operators involving the Gauss hypergeometric functions*, Math. Rep. Kyushu Univ., 11 (1978), 135-143.
- [30] A. R. Prabhakaran and K. S. Rao, *Saigo operator of fractional integration of Hypergeometric functions*, International Journal of Pure and Applied Mathematics 81(5), (2012), 755-763.
- [31] G. Rahman, T. Abdejawad, A. Khan and K.S Nisar, *Some fractional proportional integral inequalities*, J. Inequal. Appl., 244, 2019, (2019), 1-13.
- [32] G. Rahman, A. Khan, T. Abdejawad and K.S Nisar, *The Minkowski inequalities via generalized proportional fractional integral operators*, Adv. Differ. Equ., 287, 2019, (2019), 1-14.
- [33] G. Rahman, K. S Nisar, T. Abdejawad and S. Ullah, *Certain fractional proportional integral inequalities via convex functions*, Mathematics 8(2), 222, (2020), 1-11.
- [34] G. Rahman, K. S Nisar and T. Abdejawad, *Certain Hadamard proportional fractional integral inequalities*, Mathematics 8(4), 504, (2020), 1-14.
- [35] G. Rahman, T. Abdejawad, F. Jarad, A. Khan and K. S Nisar, *Certain inequalities via generalized proportional Hadamard fractional integral operators*, Adv. Differ. Equ., 454, 2019, (2019), 1-10.
- [36] M. Saigo, *A remark on integral operators involving the Gauss hypergeometric functions*, Math. Rep. Kyushu Univ, 11 (1978), 135-143.
- [37] S. G. Somko, A. A. Kilbas and O. I. Marichev, *Fractional Integral and Derivative Theory and Application*, Gordon and Breach, Yverdon, Switzerland, 1993.
- [38] E. Set, J. Choi and L. Mumcu, *Chebyshev type inequalities involving generalized Katugampola fractional integral operators*, Tamkang Journal of Mathematics, 4(50), (2019), 381-390.
- [39] A. Tassadiq, G. Rahman, K.S. Nisar and M. Samraiz, *Certain fractional conformable inequalities for the weighted and the extended Chebyshev functionals*, Adv. Differ. Equ., 96, 2020, (2020), 1-9.
- [40] N. Virchenko and O. Lisetska, *On some fractional integral operators involving generalized Gauss hypergeometric functions*, Appl. Appl. Math., 5(10), (2010), 1418-1427.
- [41] W. Rudin, *Real and Complex Analysis*, (3rd edition), McGraw-Hill, New York, 1986.
- [42] J. A. Goguen, *L-fuzzy sets*, Journal of Mathematical Analysis and Applications 18 (1967) 145–174.
- [43] P. Erdős, S. Shelah, *Separability properties of almost-disjoint families of sets*, Israel Journal of Mathematics 12 (1972) 207–214.