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Nonlinear Quadratic Volterra-Urysohn Functional-Integral Equations in Orlicz Spaces

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Abstract. The current article discusses the existence of monotonic discontinuous solutions for general nonlinear quadratic Volterra-Urysohn functional-integral equations in Orlicz spaces E_φ when the function φ satisfies the Δ_2-condition. The case of Lebesgue spaces L_p $(p > 1)$ are also examined. We use the arguments of measure of noncompactness with Darbo fixed point theorem to prove our results.

1. Introduction

The current article is devoted to study the following quadratic functional-integral equation

$$x(t) = G_2(x(\eta_2))(t) + G_1(x)(t) \cdot \int_0^t u(t, s, x(\eta_1(s)))\, ds, \quad t \in [0, d] \tag{1}$$

in Orlicz spaces E_φ, when φ satisfies the Δ_2-condition, where $G_i(x)$, $i = 1, 2$ are general operators acting on some Orlicz spaces.

We consider operators with non-polynomial growth (of exponential growth, for instance), then we cannot anticipate the solutions will be continuous (cf. [8, 11, 20]) and it is better to examine the solutions in Orlicz spaces. The considered integral equation with exponential nonlinearities

$$x(t) + \int_I k(t, s) \exp x(s) ds = 0,$$

has application in the thermodynamical problems. The general Chandrasekhar integral equation of the form

$$x(t) = 1 + \lambda \cdot x(t) \int_0^1 \frac{t}{t+s} \psi(s) \log(1 + \sqrt{x(s)})\, ds \tag{2}$$

can be treated as an example of equation (1). Equation (2) describes scattering through a homogeneous semi-infinite plane atmosphere and discontinuous solutions for such equation are useful estimation of non-homogeneous atmosphere (cf. [7, 19]), then solutions in Orlicz spaces are imperious (see also some

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comments in [12]). Such equations have applications in various branches such as in traffic theory, kinetic theory of gases, in the theory of radiative transfer, in the theory of neutron transport, and in mathematical physics (cf. [3, 9, 10]).

Most of the papers were devoted to study quadratic integral equations in Banach algebras (cf. [5, 19], for example). These equations were also discussed in some Banach-Orlicz algebras (cf. [12]).

We use the technique in [17] which is not algebras by fixing a space of (target) solutions, and then we nominate the proper intermediate spaces. In particular, L_1-solutions or L_p-solutions were discussed in [7, 14, 21] with a polynomial growth, this implies many restrictions on the growth of considered functions.

The quadratic Hammerstein integral equations were studied in Orlicz spaces in [13, 22] using L_∞-space as one of the intermediate spaces and this leads to some restriction on an operator. In [15] the authors extended these results and discussed the existence of L_φ-solutions in two separate cases when φ satisfying Δ', Δ_3-conditions separately. The case of φ satisfying Δ_2-condition was discussed in [16].

This article is concerned to generalize and extend the results in the previous studies by skipping several restrictions like in [7, 13, 22]. We discuss the existence of a.e. monotonic solutions of general nonlinear quadratic Volterra-Urysohn functional-integral equations in arbitrary Orlicz spaces by controlling the conditions on the operators G_1, G_2 and on the Urysohn operator, where they act on different spaces under a general set of assumptions. This allows us to cover all these previous studies as particular cases. The case of Lebesgue spaces L_p for $p > 1$ are also investigated.

2. Basic notations.

Let $\mathbb{R}$ be the field of real numbers and I be a compact interval $[0, a] \subset \mathbb{R}$. Denote by $(E, \|\cdot\|)$ an arbitrary Banach space with zero element θ. Denote by $B_r(x)$ the closed ball with center at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$. We will use notation $B_r(E)$ to indicate the space. Let $X \subset E$, then $\bar{X}$ and $convX$ stand for the closure and convex hull of X, respectively.

Next, let M and N be complementary N-functions i.e. $N(x) = \sup_{y\geq 0}(xy - M(x))$, where $N : [0, +\infty) \to [0, +\infty)$ is continuous, even and convex with $\lim_{x\to 0} \frac{N(x)}{x} = 0$, $\lim_{x\to\infty} \frac{N(x)}{x} = \infty$ and $N(x) > 0$ if $x > 0$ ($N(x) = 0 \iff x = 0$). The Orlicz class, denoted by O_P, consists of measurable functions $x : I \to \mathbb{R}$ for which $\rho(x; M) = \int_I M(x(t))dt < \infty$. We shall denote by $L_M(I)$ the Orlicz space of all measurable functions $x : I \to \mathbb{R}$ for which

$$\|x\|_M = \inf_{\lambda>0} \left\{ \int_I M\left(\frac{x(s)}{\lambda}\right) ds \leq 1 \right\}.$$

Let $E_M(I)$ be the closure in $L_M(I)$ of the set of all bounded functions. Note that $E_M(I) \subseteq L_M(I) \subseteq O_M(I)$. The inclusion $L_M(I) \subset L_P(I)$ holds if, and only if, there exists positive constants u_0 and a such that $P(u) \leq aM(u)$ for $u \geq u_0$.

An important property of $E_M(I)$ spaces lies in the fact that this is a class of functions from $L_M(I)$ having absolutely continuous norms.

Definition 2.1. *[20] We say that the N-function $M(t)$ satisfies the Δ_2-condition for large values of u if there exist constants $\omega > 0$, $t_0 \geq 0$ such that*

$$M(2t) \leq \omega M(t), \quad \text{and } t \geq t_0.$$

For example the N-functions $M_1(u) = \frac{u^p}{p}$ and $M_2(u) = |u|^\alpha(|\ln|u|| + 1)$ for $\alpha \geq \frac{3+\sqrt{5}}{2}$ satisfy this condition, while the function $M_3(u) = \exp|u| - |u| - 1$ does not. Moreover, the complement functions to $M_4(u) = \exp u^2 - 1$ and $M_5(u) = \exp|u| - |u| - 1$ satisfy this condition while the original functions M_4 and M_5 do not.

Note that, if M satisfies the Δ_2-condition, then we have $E_M(I) = L_M(I) = O_M(I)$.

Let $S = S(I)$ denotes the set of measurable (in the Lebesgue sense) functions on I and let *meas* denote the Lebesgue measure on $\mathbb{R}$. Identifying the functions equals almost everywhere the set S furnished with the metric

$$\rho(x, y) = \inf_{\epsilon>0}[a + meas\{s : |x(s) - y(s)| \geq \epsilon\}]$$

becomes a complete space. Moreover, the space S with the topology convergence in measure on ρ is a metric space, because the convergence in measure is equivalent to convergence with respect to ρ (cf. Proposition 2.14 in [26]). The compactness in such a topology is called a "compactness in measure".

Lemma 2.2. [12] *Let X be a bounded subset of $L_M(I)$ consisting of functions which are a.e. nondecreasing (or a.e. nonincreasing) on the interval I. Then X is compact in measure in $L_M(I)$.*

Denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact subsets.

Definition 2.3. [4] *A mapping $\mu : \mathcal{M}_E \to [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:*

(i) $\mu(X) = 0 \Rightarrow X \in \mathcal{N}_E$.
(ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
(iii) $\mu(\bar{X}) = \mu(convX) = \mu(X)$.
(iv) $\mu(\lambda X) = |\lambda|\, \mu(X),\ for\ \lambda \in \mathbb{R}$.
(v) $\mu(X + Y) \leq \mu(X) + \mu(Y)$.
(vi) $\mu(X \bigcup Y) = \max\{\mu(X), \mu(Y)\}$.
(vii) If X_n is a sequence of nonempty, bounded, closed subsets of E such that $X_{n+1} \subset X_n$, $n = 1, 2, 3, \cdots$, and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

An example of such mappings is the following:

Definition 2.4. [4] *Let X be a nonempty and bounded subset of E. The Hausdorff measure of noncompactness $\beta_H(X)$ is defined as*
$\beta_H(X) = inf\{r > 0 :$ *there exists a finite subset Y of E such that* $x \subset Y + B_r\}$.

For any $\varepsilon > 0$, let $c(X)$ be a measure of equiintegrability of the set X in $L_M(I)$ (cf. Definition 3.9 in [26] or [18]):

$$c(X) = \lim_{\varepsilon\to 0} \sup_{measD\leq\varepsilon} \sup_{x\in X} \|x \cdot \chi_D\|_{L_M(I)}, \tag{3}$$

where χ_D denotes the characteristic function of D.

Proposition 2.5. *[16] Let X be a nonempty, bounded and compact in measure subset of an Orlicz space $L_\varphi(I)$, where φ satisfies the Δ_2-condition. Then*

$$\beta_H(X) = c(X).$$

As a consequence, we obtain that bounded sets which are additionally compact in measure are compact in $L_M(I)$ iff they are equiintegrable in this space (i.e. have equiabsolutely continuous norms, in particular $X \subset E_M(I)$), which is important to apply the Darbo theorem:

Theorem 2.6. *[4] Let Q be a nonempty, bounded, closed and convex subset of E and let $V : Q \to Q$ be a continuous transformation which is a contraction with respect to the measure of noncompactness μ, i.e. there exists $k \in [0, 1)$ such that*

$$\mu(V(X)) \leq k\mu(X),$$

for any nonempty subset X of E. Then V has at least one fixed point in the set Q.

Definition 2.7. *[2] Assume that a function $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions i.e. it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in I$. Then to every function $x(t)$ being measurable on I we may assign the function*

$$F_f(x)(t) = f(t, x(t)), \quad t \in I.$$

The operator F_f is said to be the superposition (Nemytskii) operator generated by the function f

Lemma 2.8. [20, Theorem 17.5] *Assume that a function $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions. Then*

$$M_2(f(s,x)) \leq a(s) + bM_1(x),$$

where $b \geq 0$ and $a \in L_1(I)$, if and only if the superposition operator F_f acts from $L_{M_1}(I) \to L_{M_2}(I)$.

Lemma 2.9. *[12] Let f be a Carathéodory function. If the superposition operator $F_f : L_{M_1}(I) \to E_{M_2}(I)$, then it is continuous.*

For multiplications of operators, we have:

Lemma 2.10. ([20, Lemma 13.5]) *Let φ_1, φ_2 and φ are arbitrary N-functions. The following conditions are equivalent:*

1. *For every functions $u \in L_{\varphi_1}(I)$, $w \in L_{\varphi_2}$ and $u \cdot w \in L_{\varphi}(I)$.*
2. *There exists a constant $k > 0$ such that for all measurable u, w on I we have $\|uw\|_{\varphi} \leq k\|u\|_{\varphi_1}\|w\|_{\varphi_2}$.*
3. *There exists numbers $C > 0$, $u_0 \geq 0$ such that for all $s, t \geq u_0$, $\varphi\left(\frac{st}{C}\right) \leq \varphi_1(s) + \varphi_2(t)$.*
4. $\limsup_{t\to\infty} \frac{\varphi_1^{-1}(t)\varphi_2^{-1}(t)}{\varphi(t)} < \infty$.

For functions which shall satisfy the above Lemma, we have

Lemma 2.11. ([20, p. 223]) *If there exist complementary N-functions Q_1 and Q_2 such that the inequalities*

$$Q_1(\alpha u) < \varphi^{-1}[\varphi_1(u)]$$

$$Q_2(\alpha u) < \varphi^{-1}[\varphi_2(u)]$$

hold, then for every functions $u \in L_{\varphi_1}(I)$ and $w \in L_{\varphi_2}$, $u \cdot w \in L_{\varphi}(I)$. If moreover φ satisfies the Δ_2-condition, then it is sufficient that the inequalities

$$Q_1(\alpha u) < \varphi_1[\varphi^{-1}(u)]$$

$$Q_2(\alpha u) < \varphi_2[\varphi^{-1}(u)]$$

hold.

For the composition operators $C_\eta(x(t)) = x(\eta(t))$ in Orlicz spaces, we have the following:

Lemma 2.12. *[16] Let $\eta : I \to I$ be a measurable mapping such that there is a constant $K > 1$ with $meas(\eta^{-1}(E)) \leq K \cdot meas(E)$, for all measurable $E \subset I$. Then $C_\eta : E_\varphi(I) \to E_\varphi(I)$.*

3. Main results.

Let $J = [0, d]$ and rewrite equation (1) in operator form as following

$$x = Bx,$$

where $B(x) = G_2(x(\eta_2)) + A(x)$, $A(x) = G_1(x) \cdot U(x(\eta_1))$, $U(x(\eta_1))(t) = \int_0^t u(t, s, x(\eta_1(s)))\, ds$, and G_1, G_2 are general operators acting on some Orlicz spaces.

Now, consider equation (1) with the following assumptions.
Assume that, $\varphi, \varphi_1, \varphi_2$ are N-functions and that M and N are complementary N-functions and that:

(N1) There exists a constant $k_1 > 0$ such that for every $w_1 \in L_{\varphi_1}(J)$ and $w_2 \in L_{\varphi_2}(J)$ we have $\|w_1 w_2\|_{\varphi} \leq k_1\|w_1\|_{\varphi_1}\|w_2\|_{\varphi_2}$.

(C1) $u : J \times J \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions i.e. it is measurable in (t,s) for any $x \in \mathbb{R}$ and continuous in x for almost all (t,s). Assume that the function u is nondecreasing with respect to each variable, separately.

(C2) $|u(t,s,x)| \leq K(t,s)(b(t) + R(|x|))$ for $t,s \in J$ and $x \in \mathbb{R}$, where $b \in E_N(J)$ and R is non-negative, nondecreasing, continuous function defined on $\mathbb{R}^+$ and $K(t,s) \geq 0$ for $t,s \in J$.

(C3) Assume that φ is N-functions and the function N satisfies the Δ_2-condition and suppose that there exist $\gamma \geq 0$ such that

$$R(u) \leq \gamma N^{-1}(\varphi(u)) \quad \text{for} \quad u \geq 0.$$

(G) $G_1 : L_\varphi(J) \to L_{\varphi_1}(J)$, takes continuously $E_\varphi(J)$ into $E_{\varphi_1}(J)$ and the operator $G_2 : L_\varphi(J) \to L_\varphi(J)$, takes continuously $E_\varphi(J)$ into itself. There exist positive functions $g_1 \in L_{\varphi_1}(J)$, $g_2 \in L_\varphi(J)$ such that for $t \in J$, $|G_i(x)(t)| \leq g_i(t)\|x\|_\varphi, i = 1,2$ and each G_i takes the set of all a.e. nondecreasing functions into itself. Moreover, assume that for any $x \in E_\varphi(J)$ we get $G_1(x) \in E_{\varphi_1}(J), G_2(x) \in E_\varphi(J)$.

(K1) $s \to K(t,s) \in L_M(J)$ for a.e. $t \in J$ and $p(t) = \|K(t,\cdot)\|_M \in E_{\varphi_2}(J)$.

(K2) $\eta_i : J \to J$ is increasing absolutely continuous function and there are positive constants Z_i such that $\eta_i' \geq Z_i$ a.e. on $(0,d), i = 1,2$.

Assume that for some $q > 0$ the following inequality holds true on an interval $J_0 = [0,a] \subset J = [0,d]$

$$\int_{J_0} \varphi\left(g_2(t)\left(1 + \frac{q-1}{Z_2}\right) + g_1(t) \cdot q \cdot |p(t)| \cdot \left[\|b\|_N + \left(\gamma + \frac{\gamma}{Z_1}(q-1)\right)\right]\right) dt \leq q.$$

Proposition 3.1. *(a) Note, that by the assumption (K2) each $\eta_i, i = 1,2$ are strictly increasing, it is nonsingular and for all measurable subsets $X \subset J$ with meas$(\eta_i^{-1}(X)) \leq d$ meas(X), $i = 1,2$. This allows us to use Lemma 2.12 and $x(\eta_i(\cdot)) : E_\varphi(J) \to E_\varphi(J)$, $i = 1,2$.*

(b) It should be noted that the assumption (N1) implies, that $p \in L_{\varphi_2}(J)$ implies $p \in L_\varphi(J)$ (by putting $w_1 = const.$ and $w_2 = p$). The same holds true for the function g_1 with values in $L_{\varphi_1}(J)$ and g_2 with values in $L_\varphi(J)$.

(c) Let V denote the closure of the set $\{x \in E_\varphi(J_0) : \int_0^a \varphi(|x(s)|)\, ds \leq r - 1\}$. Take an arbitrary $x \in V$. By using the Hölder inequality and ([20, Theorem 10.5 with $k = 1$]), we obtain that for any $t \in J_0$

$$\begin{aligned}
|U(x(\eta_1))(t)| &\leq \left|\int_0^t K(t,s)[\, b(s) + R(|x(\eta_1(s))|)]\, ds\right| \\
&\leq |p(t)|\Big[\|b\|_N + \|R(|x(\eta_1)\chi_{[0,t]}|)\|_N\Big] \\
&\leq |p(t)|\Big[\|b\|_N + \gamma\left\|N^{-1}\left(\varphi\left(|x(\eta_1)\chi_{[0,t]}|\right)\right)\right\|_N\Big] \\
&\leq |p(t)|\Big[\|b\|_N + \gamma + \gamma\int_0^t \varphi\left(|x(\eta_1(s))|\right)\, ds\Big] \\
&\leq |p(t)|\Big[\|b\|_N + \gamma + \gamma\int_0^t \varphi\left(|x(\eta_1(s))|\right)\frac{\eta_1'(s)}{Z_1}\, ds\Big] \\
&= |p(t)|\Big[\|b\|_N + \gamma + \frac{\gamma}{Z_1}\int_{\eta_1(0)}^{\eta_1(t)} \varphi\left(|x(u)|\right)\, du\Big] \\
&\leq |p(t)|\Big[\|b\|_N + \gamma + \frac{\gamma}{Z_1}\int_{J_0} \varphi\left(|x(u)|\right)\, du\Big].
\end{aligned}$$

(d) *For an arbitrary $x \in X$ and for a set $D \subset J_0$, meas $D \leq \varepsilon$, and by assumption (G) (for more details see [13]), we obtain*

$$\|G_1(x) \cdot \chi_D\|_{\varphi_1} \leq \|G_1(x \cdot \chi_D)\|_{\varphi_1} \leq \|g_1\|_{\varphi_1} \|x \cdot \chi_D\|_{\varphi}$$

and

$$\|G_2(x) \cdot \chi_D\|_{\varphi} \leq \|g_2 \cdot \chi_D \|x\|_{\varphi}\|_{\varphi} \leq \|g_2 \cdot \chi_D\|_{\varphi} \|x\|_{\varphi}.$$

Now, we can introduce our main theorem.

Theorem 3.2. *Let the above mentioned assumptions (N1)–(K2) be satisfied. Moreover, if*

$$\left(k_1 \|g_1\|_{\varphi_1} \|p\|_{\varphi_2} \left[\|b\|_N + \gamma + \frac{\gamma}{Z_1}(r-1) \right]\right) < 1,$$

then there exists an a.e. nondecreasing solution $x \in E_{\varphi}(J_0)$ of (1) *on $J_0 \subset J$.*

Proof. We need to divide the proof into a few steps.

Step I. First of all observe that by the assumptions (C1)-(C3), (cf. [20, Lemma 16.3 and Theorem 16.3] (with $M_1 = N, M_2 = \varphi_2$ and $N_1 = M$) implies that the operator U is continuous mappings from the unit ball $B_1(E_{\varphi}(J))$ into $E_{\varphi_2}(J)$. By our assumption (G) the operator G_1 is continuous from $B_1(E_{\varphi}(J))$ into $E_{\varphi_1}(J)$ and then by (N1) the operator A is a continuous mapping from $B_1(E_{\varphi}(J))$ into the space $E_{\varphi}(J)$. Finally, since G_2 is continuous from $B_1(E_{\varphi}(J))$ into $E_{\varphi}(J)$, we can deduce that the operator B maps $B_1(E_{\varphi}(J))$ into $E_{\varphi}(J)$ continuously.

Step II. We will structure an invariant set $V \subset B_1(E_{\varphi}(J))$ for the operator B is bounded in $L_{\varphi}(J)$.

Denote by Q the set of all positive numbers q for which

$$\int_{J_0} \varphi\left(g_2(t)\left(1 + \frac{q-1}{Z_2}\right) + g_1(t) \cdot q \cdot |p(t)| \cdot \left[\|b\|_N + \left(\gamma + \frac{\gamma}{Z_1}(q-1)\right)\right]\right) dt \leq q.$$

By r we will denote $\sup Q$. Recal, that $J_0 = [0, a] \subset J$.

Let V denote the closure of the set $\{x \in E_{\varphi}(J_0) : \int_0^a \varphi(|x(s)|)\, ds \leq r - 1\}$. Clearly V is not a ball in $E_{\varphi}(J_0)$, but $V \subset B_r(E_{\varphi}(J_0))$ (cf. [20, p. 222]). Notice that $\overline{V}$ is a bounded closed and convex subset of $E_{\varphi}(J_0)$.

Thus for any measurable subset T of J. For arbitrary $x \in V$ and $t \in J_0$ and by using Proposition 3.1(c), we have

$$\begin{aligned}
|B(x)(t)| &\leq |G_2(x(\eta_2))(t)| + |Ax(t)| \\
&\leq |G_2(x(\eta_2))(t)| + |G_1(x)(t)| \cdot |U(x(\eta_1))(t)| \\
&\leq g_2(t) \cdot \|x(\eta_2)\|_{\varphi} + g_1(t) \cdot \|x\|_{\varphi} \cdot |p(t)| \left[\|b\|_N + \|R(|x(\eta_1)\chi_{[0,t]}|)\|_N\right] \\
&\leq g_2(t) \cdot \left(1 + \int_{J_0} \varphi(|x(\eta_2(t))|)\, dt\right) + g_1(t) \cdot \left(1 + \int_{J_0} \varphi(|x(t)|)\, dt\right) |p(t)| \times \\
&\qquad \times \left[\|b\|_N + \left(\gamma + \frac{\gamma}{Z_1} \int_{J_0} \varphi\left(|x(u)|\right)\, du\right)\right] \\
&\leq g_2(t) \cdot \left(1 + \frac{1}{Z_2} \int_{J_0} \varphi(|x(v)|)\, dv\right) + g_1(t) \cdot r \cdot |p(t)| \left[\|b\|_N + \left(\gamma + \frac{\gamma}{Z_1}(r-1)\right)\right] \\
&\leq g_2(t)\left(1 + \frac{r-1}{Z_2}\right) + g_1(t) \cdot r \cdot |p(t)| \cdot \left[\|b\|_N + \left(\gamma + \frac{\gamma}{Z_1}(r-1)\right)\right].
\end{aligned}$$

Therefor,

$$\int_{J_0} \varphi(B(x)(t))\, dt \leq \int_{J_0} \varphi\left(g_2(t)\left(1 + \frac{r-1}{Z_2}\right) + g_1(t) \cdot r \cdot |p(t)| \cdot \left[\|b\|_N + \left(\gamma + \frac{\gamma}{Z_1}(r-1)\right)\right]\right) dt.$$

By the definition of r we get $\int_{J_0} \varphi(B(x)(t))\, dt \le r$ and then $B(V) \subset V$. Consequently $B(\overline{V}) \subset \overline{B(V)} \subset \overline{V} = V$.

Then $B : V \to V$ is continuous on $V \subset B_r(E_\varphi(J_0))$.

Step III. Now, let $Q_r \subset V$ consists of a.e. nondecreasing functions. This set is nonempty, bounded (by r), closed, and convex (cf. [16]). Now, in view of Lemma 2.2 the set Q_r is compact in measure.

Step IV. For the monotonicity properties of B in Q_r. Take $x \in Q_r$, then x and $x(\eta_i), i = 1, 2$ are a.e. nondecreasing on J and consequently $U(x(\eta_i)), i = 1, 2$ are also of the same type in virtue of the assumption (C1). Since the pointwise product of a.e. monotonic functions is still of the same type and by assumption (G), the operator A is a.e. nondecreasing on J_0. Moreover, the assumption (G) permits us to deduce that $B(x) = G_2(x(\eta_2)) + A(x)$ is also a.e. nondecreasing on J_0. This estimate that $B : Q_r \to Q_r$ is continuous.

Step V. Assume that $X \subset Q_r$ is a nonempty and let the fixed constant $\varepsilon > 0$ be arbitrary. Then for an arbitrary $x \in X$ and for a set $D \subset J_0$, meas $D \le \varepsilon$, and by using Prposition 3.1(d), we obtain

$$\begin{aligned}
\|B(x) \cdot \chi_D\|_\varphi &\le \|G_2(x(\eta_2)) \cdot \chi_D\|_\varphi + \|A(x) \cdot \chi_D\|_\varphi \\
&\le \|(g_2 \cdot \|x(\eta_2)\|_\varphi) \cdot \chi_D\|_\varphi + \|(G_1(x) \cdot U(x(\eta_1))) \cdot \chi_D\|_\varphi \\
&\le \|g_2 \cdot \chi_D\|_\varphi \|x(\eta_2)\|_\varphi + k_1 \|G_1(x) \cdot \chi_D\|_{\varphi_1} \cdot \|U(x(\eta_1))\|_{\varphi_2} \\
&\le \|g_2 \cdot \chi_D\|_\varphi \left(1 + \int_{J_0} \varphi(|x(\eta_2(s))|)\, ds\right) \\
&\quad + k_1 \|g_1\|_{\varphi_1} \|x \cdot \chi_D\|_\varphi \left\| \int_{J_0} u(\cdot, s, x(\eta_1(s)))\, ds \right\|_{\varphi_2} \\
&\le \|g_2 \cdot \chi_D\|_\varphi \left(1 + \frac{1}{Z_2} \int_{J_0} \varphi(|x(v)|) dv\right) \\
&\quad + k_1 \|g_1\|_{\varphi_1} \|x \cdot \chi_D\|_\varphi \|p\|_{\varphi_2} \Big[\|b\|_N + \|R(|x(\eta_1)|)\|_N\Big] \\
&\le \|g_2 \cdot \chi_D\|_\varphi \left(1 + \frac{r-1}{Z_2}\right) + k_1 \|g_1\|_{\varphi_1} \|x \cdot \chi_D\|_\varphi \|p\|_{\varphi_2} \Big[\|b\|_N + \gamma + \frac{\gamma}{Z_1}(r-1)\Big].
\end{aligned}$$

Hence, taking into account that $g_2 \in E_\varphi$

$$c(X) = \lim_{\varepsilon \to 0} \sup_{measD \le \varepsilon} \sup_{x \in X} \|g_2 \cdot \chi_D\|_\varphi.$$

Thus by using Definition 3, we get

$$c(B(X)) \le k_1 \|g_1\|_{\varphi_1} \|p\|_{\varphi_2} \Big[\|b\|_N + \gamma + \frac{\gamma}{Z_1}(r-1)\Big] c(X).$$

Since $X \subset Q_r$ is a nonempty, bounded and compact in measure subset of a regular part E_φ of L_φ, we can use Proposition 2.5 and we get

$$\beta_H(B(X)) \le k_1 \|g_1\|_{\varphi_1} \|p\|_{\varphi_2} \Big[\|b\|_N + \gamma + \frac{\gamma}{Z_1}(r-1)\Big] \beta_H(X).$$

The inequality obtained above together with the properties of the operator B and the set Q_r established before and the inequality

$$k_1 \|g_1\|_{\varphi_1} \|p\|_{\varphi_2} \Big[\|b\|_N + \gamma + \frac{\gamma}{Z_1}(r-1)\Big] < 1$$

allow us to apply the Darbo fixed point theorem 2.6, which completes the proof. □

Next, we discuss the case of Lebesgue spaces which can be considered as Orlicz spaces $L_p = L_{M_p}$ for $M_p(x) = \frac{x^p}{p}$, $p > 1$ satisfies Δ_2-condition.

Let us consider a special case of equation (1) with the operators $G_i(x) = h_i(t, x(t)),\ i = 1, 2$, which still represent an interesting and general case.

Then equation (1) takes the form

$$x(t) = h_2(t, x(\eta_2(t))) + h_1(t, x(t)) \cdot \int_0^t u(t, s, x(\eta_1(s)))\, ds, \quad t \in [0, d]. \tag{4}$$

We deem equation (4) with the following set of assumptions. Assume that $p > 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.

(i) Assume that functions $h_1, h_2 : J \times \mathbb{R} \to \mathbb{R}$ satisfy Carathéodory conditions and there are positive constants b_i $(i = 1, 2)$ and positive functions $a_1 \in L_{p_1}, a_2 \in L_p$ such that

$$|h_1(t, x)| \le a_1(t) + b_1|x|^{\frac{p}{p_1}} \text{ and } |h_2(t, x)| \le a_2(t) + b_2|x|$$

for all $t \in J$ and $x \in \mathbb{R}$. Moreover, the functions h_1, h_2 are assumed to be nondecraeasing with respect to both variables t and x separately.

(ii) $u : J \times J \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions. The function u is nondecreasing with respect to each variable, separately.

(iii) Suppose that for arbitrary non-negative $z(t) \in L_{p_2}(J)$

$$\lim_{\delta \to 0} \sup_{|x| \le z} \| \int_D u(t, s, x(s))\, ds \|_{L_{p_2}} = 0$$

and that

$$|u(t, s, x)| \le k(t, s)(a_3(s) + b_3|x|^{\frac{p}{p_1}}), \text{ for all } t, s \in J \text{ and } x \in \mathbb{R},$$

where the function k is measurable in (t, s), $a_3 \in L_{p_1}(J)$ and a constant $b_3 > 0$. Moreover, assume that the function $\|K_0\| = \|t \to \|K(t, \cdot)\|_{p'_1}\|_{p_2}$, where $\frac{1}{p_1} + \frac{1}{p'_1} = 1$

(iv) $\eta_i : J \to J$ is increasing absolutely continuous function and there are positive constant Z_i such that $\eta'_i \ge Z_i$ $(i = 1, 2)$ a.e. on $(0, d)$. In addition, let r be a positive number such that

$$\begin{aligned}\Big(\|a_2\|_p &+ \|a_1\|_{p_1}\|a_3\|_{p_1}\|K_0\|\Big) + \left(\frac{b_2}{Z_2^{\frac{1}{p}}} - 1\right) \cdot r \\ &+ \|K_0\|\left(\frac{b_3\|a_1\|_{p_1}}{Z_1^{\frac{1}{p_1}}} + b_1\|a_3\|_{p_1}\right) \cdot r^{\frac{p}{p_1}} + \frac{b_1 b_3\|K_0\|}{Z_1^{\frac{1}{p_1}}} \cdot r^{\frac{2p}{p_1}} = 0.\end{aligned}$$

Corollary 3.3. *Let assumptions (i)–(iv) be satisfied. If*

$$\left(\frac{b_2}{Z_2^{\frac{1}{p}}} + b_1\|K_0\| r^{\frac{p}{p_1}-1}\Big(\|a_3\|_{p_1} + \frac{b_3}{Z_1^{\frac{1}{p_1}}} r^{\frac{p}{p_1}}\Big)\right) < 1,$$

then equation (4) *has at least one $L_p(J)$-solution a.e nondecreasing on some subinterval* $[0, a] \subset J$.

4. Remarks and Examples

Remark 4.1. *The quadratic integral equation is not in general compact even in the simplest case when* $G_i(x) = f_i(t, x)$, *so we cannot apply the method used in [1] for instance. We use Darbo fixed point theorem and measure of noncompactness to exclude these difficulties.*

Remark 4.2. *The classical Volterra-Urysohn integral equations were studied in Orlicz spaces* L_φ *for* φ *satisfying* Δ_2*-condition (cf. [23, 24] and in generalized Orlicz spaces (Musielak-Orlicz spaces were discussed in [6, 25]). Let me mention that some additional properties of solutions in Orlicz spaces like constant-sign solutions are also examined (see [1], for instance).*

Remark 4.3. *The complete details for acting and continuity conditions for the operator* $G_i(x) = a_i(t)x(t)$ *in Orlicz spaces can be found in [20, Theorem 18.2] (cf. our assumption (G)).*

Example 4.4. *Let* $\tau \geq 0$ *is a constant and* $G_i(x)(t) = q_i(t) \cdot x(t)$, $i = 1, 2$, *then we have the quadratic integral equations*

$$x(t) \; = \; q_2(t) \cdot x(t-\tau) + q_1(t) \cdot x(t) \cdot \int_0^t u(t, s, x(s-\tau))\, ds, \quad t \in [0, d], \tag{5}$$

which is a particular case of equation (1).

Example 4.5. *Let* $G_1(x) = \lambda \cdot x(t)$ *and* $G_2(x) = 1$ *in equation (1), we have*

$$x(t) = 1 + \lambda\, x(t) \int_0^1 \frac{t}{t+s} e^{-s} \cdot \chi_{[0,t]}(\log(1 + |x(\sqrt{s})|))\, ds, \tag{6}$$

which represents a general form of Chandrasekhar equation studied in [3, 10].

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