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Topobooleans and Boolean Contact Algebras with Interpolation Property

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Abstract. In this paper we study the connections between topobooleans [A. A. Estaji, A. Karimi Feizabadi, and M. Zarghani, Categ. Gen. Algebr. Struct. Appl. 4 (2016), 75–94] and Boolean contact algebras with the interpolation property (briefly, *ICAs*) [G. Dimov and D. Vakarelov, Fund. Inform. 74 (2006), 209–249]. We prove that every complete *ICA* generates a topoboolean and, conversely, if a topoboolean satisfies some natural conditions then it generates a complete *ICA* which, in turn, generates it. We introduce the category **ICA** of *ICAs* and suitable morphisms between them. We show that the category **ICA** has products and every **ICA**-monomorphism is an injective function. We prove as well that if A and B are complete Boolean algebras, $f\colon B_1 \to B_2$ is a complete Boolean homomorphism and (A, C) is an *ICA*, then B possesses a final **ICA**-structure in respect of f.

1. Introduction

Efremovič proximity spaces were introduced by Efremovič in 1951 [21, 22], when he axiomatically characterized the proximity relation "A is near B" for subsets A and B of a given set X. Later, Leader [33, 34] and Lodato [35, 36] worked on structures having weaker axioms than those of Efremovič proximity spaces, what enabled them to equip the underlying sets with arbitrary topologies.

Proximity spaces play a very significant role in the study of many problems that involve topological spaces, such as compactifications, extension problems and so on. Each proximity space determines a natural topology with nice properties, and the theory possesses deep results, and rich machinery and tools; Naimpally and Warrack's book [37] is the most complete exposition of the classical theory of proximity spaces.

A proximity space is a non-empty set X with a special relation on the Boolean algebra $P(X)$. It is natural to look for suitable generalizations of this notion. In [40], a paper written under the direction of V. A. Efremovič, A. S. Šwartz introduced the notion of *proximity distributive lattice* and proved a topological representation theorem. It was mentioned there, that the paper can be considered as an attempt to build a pointfree analogue of the notion of proximity space. Later on, H. de Vries [42] introduced the notion of *compingent algebra* (now known as *de Vries algebra* [3]) and proved that the category **deV** of complete

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compingent algebras and their appropriate morphisms is dually equivalent to the category **KHaus** of compact Hausdorff spaces and continuous maps.This was a convincing demonstration of the usefulness of the point-free analogues of the notion of proximity space. In [28], V. V. Fedorchuk introduced the notion of *Boolean δ-algebra* (now known also as normal contact algebra or *IECA* [16]) as a pair (B, δ) of a Boolean algebra B and a proximity-like relation δ on B. He noticed that this notion is equivalent to the notion of a compingent algebra and proved that the category **Fed** of complete Boolean δ-algebras and complete Boolean homomorphisms which reflect the contact relation is dually equivalent to the category $\mathbf{KHaus}_{qo}$ of compact Hausdorff spaces and quasi-open maps. Some generalizations of the results of de Vries [42] and Fedorchuk [28] were obtained in [8], [9], [5], [13] and [12].

By taking some suitable subsets of the set of the axioms of Boolean δ-algebras, the new notions of *contact algebra* ([16]), *extensional contact algebra* ([20]) and *precontact algebra* [19] (called so in [17] and originally called *proximity algebra*) were introduced and topological (or discrete) representation and duality theorems for these classes were obtained (see [19], [20], [17], [16], [18], [14]). It is interesting that the class of precontact algebras was introduced independently and in a completely different form (as *quasi-modal algebras*) by S. Celani [6]. So that the notion of a precontact algebra arose naturally in the fields of logic, topology and theoretical computer science (especially in *Region-based theory of space* - see [1], [30], [38], [41] for surveys). Finally, the *local proximities* of Leader [22] were generalized by Roeper [39].

Although the contact algebras which satisfy the interpolation property (for short, *ICAs*) were introduced in[16], they were studied very briefly there. In this paper they will be the main object of our investigation."

Instead of being focused on the points of a space, *point-free topology* (*frame theory*) focuses on its open subsets. It deals with the abstractly defined "lattices of open sets", known as *frames*, and their homomorphisms. Frames are complete lattices in which the meet distributes over all joins. For instance, for any topological space X, the power set $\mathcal{P}(X)$ and the family of all open sets, $O(X)$, are frames. Just as Boolean algebras can be seen as models for the classical propositional logic, frames can be seen as models for the geometric propositional logic, which is a logic with finite conjunctions and infinite disjunctions. Let us also recall that Banaschewski [2] and Frith [29] introduced the notion of *proximity frame* and in [15] a category isomorphic to that of proximity frames was constructed. Proximity frames were studied as well in [4] and [7].

The second conceptual ingredient in the framework of point-free topologies is to view topoframes as "point-free spaces". Estaji, Karimi Feizabadi and Zarghani [24, 25] defined topoframes: A *topoframe* is a pair (L, τ), abbreviated as L_τ, consisting of a frame L and a subframe τ whose all elements are complemented in L. For example, with the notation above, the pair $\big(\mathcal{P}(X), O(X)\big)$ is a topoframe. In a topoframe (L, τ), we have both open and closed elements (the members of τ and their complements, respectively). We refer the interested reader to [23, 27, 44] and the references therein for more details. A topoframe (L, τ) in which L is a complete Boolean algebra is called a topoboolean. In this paper we will study the connections between topobooleans and *ICAs*.

The paper is organized as follows. In Section 2, we review some basic notions together with some known properties of Boolean algebras and topobooleans. In Section 3, we prove that every complete *ICA* generates a topoboolean (see Theorem 3.8) and, conversely, if a topoboolean satisfies some natural conditions then it generates a complete *ICA* which, in turn, generates it (see Theorem 3.19). Finally, in Section 4 we introduce the category **ICA** of *ICAs* as a full subcategory of the category **PCA** from [14]. We show that the category **ICA** has products and every **ICA**-monomorphism is an injective function (see Propositions 4.8 and 4.9). We prove as well that if A and B are complete Boolean algebras, $f: B_1 \to B_2$ is a complete Boolean homomorphism and (A, C) is an *ICA*, then B possesses a final **ICA**-structure in respect of f (see Theorem 4.3).

This research has been continued by the authors in [26]. The work contains some results concerning the weight of *ICAs*.

2. Preliminaries

In this section, we recall some definitions and results on Boolean algebras and topobooleans. For further information on the concepts related to Boolean algebras and topobooleans, we refer the reader to [31] and [25], respectively.

Let L be a lattice. We denote the *top element* and the *bottom element* of a bounded lattice by $\top$ and $\bot$, respectively. An element a of a bounded lattice L is called an *atom* if $a \neq \bot$, and $\bot < b \leq a$ implies $b = a$ for $b \in L$. In what follows, the set of all atoms of a bounded lattice L is denoted by $At(L)$.

A bounded lattice L is said to be *complemented* if every $a \in L$ has a complement, that is, for every $a \in L$ there exists an element b of L such that $a \wedge b = \bot$ and $a \vee b = \top$. A distributive complemented lattice is called a *Boolean algebra*. Notice that every element a of a Boolean algebra has a unique complement, which is denoted by a'. See [31].

A *frame* L is a complete lattice in which the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}$$

holds for all $a \in L$ and $S \subseteq L$.

We recall from [32] that every frame is isomorphic to a subframe of a complete Boolean algebra.

In what follows, B denotes a complete Boolean algebra. Also, L denotes a subframe of a complete Boolean algebra B.

Definition 2.1. ([24]) A **topoframe** is a pair (L, τ), abbreviated as L_τ, consisting of a frame $(L\colon \wedge, \vee, \top, \bot)$ and a subset τ of L satisfying the following conditions.

(1) Every element p of τ has a complement p' in L.

(2) If $\{p_i\}_{i\in I}$ is a subfamily of τ, then the supremum of $\{p_i\}_{i\in I}$ belongs to τ.

(3) If $\{p_i\}_{i\in I}$ is a finite subfamily of τ, then the infimum of $\{p_i\}_{i\in I}$ belongs to τ.

If $(B\colon \wedge, \vee, \top, \bot)$ is a complete Boolean algebra, then the topoframe (B, τ) is called a **topoboolean**. Now, let B_τ be a topoboolean. An element $a \in B$ is said to be **compact** if $a \leq \bigvee S$ for $S \subseteq \tau$ implies $a \leq \bigvee F$ for some finite $F \subseteq S$. A topoboolean B_τ is said to be compact whenever its top element is compact. Hereafter, the real line $\mathbb{R}$ is always assumed to be endowed with the natural topology $\mathfrak{O}(\mathbb{R})$.

Definition 2.2. ([24]) Let B_τ be a topoboolean. A function $f\colon \mathcal{P}(\mathbb{R}) \to B$ is called a τ- **real-continuous function** on B (or a **real-continuous function** on B_τ) if $f\colon \mathcal{P}(\mathbb{R}) \to B$ is a complete Boolean homomorphism and $f\big(\mathfrak{O}(\mathbb{R})\big) \subseteq \tau$. The set of all real-continuous functions on B_τ is denoted by $\mathcal{R}(B_\tau)$.

We recall from [24] that the set $\mathcal{R}(B_\tau)$ with the operator $\diamond \in \{+, \cdot, \wedge, \vee\}$ defined by

$$(f \diamond g)(X) = \bigvee \{f(Y) \wedge g(Z) \mid Y \diamond Z \subseteq X\}$$

is an f-ring. Here,

$$Y \diamond Z = \{y \diamond z \mid y \in Y, z \in Z\}$$

or equivalently,

$$(f \diamond g)(X) = \bigvee \{f(\{x\}) \wedge g(\{y\}) \mid x \diamond y \in X\}.$$

Also, we recall from [25] that for every $f \in \mathcal{R}(B_\tau)$, $f(\{0\})$ and $f(-\infty, 0) \vee f(0, +\infty)$ are called a **zero-element** and a **cozero-element** of f and are denoted by $z(f)$ and $coz(f)$, respectively. Obviously, $z(f) = \big(coz(f)\big)'$.

We recall form [43] that for every $f, g \in \mathcal{R}(B_\tau)$, the following statements are equivalent:

(1) $f \leq g$.

(2) For every $p \in \mathbb{Q}$, $f(p, +\infty) \leq g(p, +\infty)$.

(3) For every $q \in \mathbb{Q}$, $f(-\infty, q) \geq g(-\infty, q)$.

Definition 2.3. ([24]) Let B_τ be a topoboolean. For each $c \in \mathbb{R}$, function $\mathbf{c}$ defined by

$$\mathbf{c}(X) = \begin{cases} 1_B & c \in X, \\ 0_B & c \notin X, \end{cases}$$

for every $X \in \mathcal{P}(\mathbb{R})$, is called a **constant** real-valued functions on a topoboolean B_τ.

Proposition 2.4. ([25]) *Let B_τ be a topoboolean. For every $f, g \in \mathcal{R}(B_\tau)$, the following statements hold:*

(1) $z(f \wedge g) = z(f) \vee z(g)$, *while* $f, g \geq \mathbf{0}$.

(2) $z(f) \wedge z(g) = z(|f| + |g|) = z(f^2 + g^2)$.

(3) *An element f is unit if and only if $coz(f) = \top$.*

Let B_τ be a topoboolean. If $p \in B$, then the closure of p is the element

$$\mathrm{cl}_\tau(p) := \bigwedge\{q \in B | q' \in \tau, p \leq q\},$$

and the interior of p is the element

$$p^\circ = \mathrm{int}_\tau(p) := \bigvee\{t \in \tau | t \leq p\}.$$

Lemma 2.5. ([43]) *Let B_τ be a topoboolean. Then, the following hold for every $a \in B$.*

(1) $(\mathrm{cl}_\tau(a))' = \mathrm{int}_\tau(a')$.

(2) $\mathrm{cl}_\tau(a') = \left(\mathrm{int}_\tau(a)\right)'$.

Lemma 2.6. ([43]) *Let B_τ be a topoboolean. Then the following hold for all $a, b \in B$.*

(1) $\mathrm{cl}_\tau(\bot) = \bot$ *and* $\mathrm{cl}_\tau(\top) = \top$.

(2) *If $a \leq b$, then* $\mathrm{cl}_\tau(a) \leq \mathrm{cl}_\tau(b)$.

(3) $a \leq \mathrm{cl}_\tau(a)$.

(4) $\mathrm{cl}_\tau\left(\mathrm{cl}_\tau(a)\right) = \mathrm{cl}_\tau(a)$.

(5) $\mathrm{cl}_\tau(a \vee b) = \mathrm{cl}_\tau(a) \vee \mathrm{cl}_\tau(b)$.

3. Boolean contact algebras with interpolation property

In this section, we consider the topoboolean $B_{\tau(C)}$, which is induced by an contact relation C on B with interpolation property, and study its properties.

Definition 3.1. ([16]) An algebraic system $B = (B, \bot, \top, \vee, \wedge, ', C)$ is called a **contact algebra** (abbreviated as *CA*) if $(B, \bot, \top, \vee, \wedge, ')$ is a non-degenerate Boolean algebra and C is a binary relation on B, called **contact**, satisfying the following axioms.

(C1) aCb implies bCa.

(C2) $(a \vee b)Cc$ if and only if aCc or bCc.

(C3) If aCb, then $a \neq \bot$ and $b \neq \bot$.

(C4) If $a \neq \bot$, then aCa.

We shall simply write (B, C) for a contact algebra. A contact algebra (B, C) is called **complete** (**atomic**) *CA* if the Boolean algebra B is complete (atomic). The negation of C will be denoted by $(-C)$.

Let (B, C) be a contact algebra. The relation $\ll_C$ (or simply $\ll$) of non-tangential inclusion is defined by $a \ll b$ if and only if $a(-C)b'$. The definition of *CA* can be equivalently reformulated using this relation. This can be done using the following axioms.

($\ll$ 1) $\top \ll \top$.

($\ll$ 2) If $a \ll b$, then $a \leq b$.

($\ll$ 3) It follows from $a \leq b \ll c \leq d$ that $a \ll b$.

($\ll$ 4) $a \ll b$ implies $b' \ll a'$.

($\ll$ 5) $a \ll b$ and $a \ll c$ imply $a \ll (b \wedge c)$.

A mapping f between two contact algebras (B_1, C_1) and (B_2, C_2) is called a *CA*-**morphism** ([14]) if $f : B_1 \to B_2$ is a Boolean homomorphism, and $f(a)C_2 f(b)$ implies $aC_1 b$, for any $a, b \in B_1$. Note that $f : (B_1, C_1) \to (B_2, C_2)$ is a *CA*-morphism if and only if $a \ll_{C_1} b$ implies $f(a) \ll_{C_2} f(b)$ for any $a, b \in B_1$. Two contact algebras (B_1, C_1) and (B_2, C_2) are *CA*-**isomorphic** if and only if there exists a bijection $f : B_1 \to B_2$ such that f and f^{-1} are *CA*-morphisms.

Lemma 3.2. ([16]) *Let C be a contact relation on a complete Boolean algebra B. If $a\ C\ b$, $a \leq a_1$ and $b \leq b_1$, then $a_1\ C\ b_1$.*

As it follows easily from Lemma 3.2, the condition ($C4$) is equivalent to the following condition.

(C4') If $a \wedge b \neq \bot$, then aCb.

Example 3.3. Let B_τ be a topoboolean. For every $a, b \in B$, we define aCb if and only if $\mathrm{cl}_\tau(a) \wedge \mathrm{cl}_\tau(b) \neq \bot$. Then, using Lemma 2.6, we can easily show that (B, C) is a contact algebra.

Definition 3.4. ([16]) Let B be a Boolean algebra. A contact relation C on B which satisfies the interpolation property

(C5) $x \ll y \Rightarrow (\exists z)(x \ll z \ll y)$,

will be called, for the sake of brevity, **I-contact relation**. If C is an I-contact relation, then (B, C) is called an **I-contact algebra** (abbreviated as *ICA*).

Let (B, C) be an *ICA*. Hereafter, we set

$$\tau(C) := \Big\{ b \in B \colon \forall x \in At(B)(x \leq b \Rightarrow x \ll_C b) \Big\}.$$

Fact 3.5. $b \in \tau(C) \Leftrightarrow \forall x \in At(B)(xCb' \Rightarrow x \leq b')$.

Remark 3.6. Note that if B is atomless, then $\tau(C) = B$.

Theorem 3.7. *Let (B, C_1) and (B, C_2) be two ICAs. Then, $C_2 \subseteq C_1$ implies $\tau(C_1) \subseteq \tau(C_2)$.*

Proof. Let $b \in \tau(C_1)$ and $x \in At(B)$ be such that $(x, b') \in C_2$. Since $C_2 \subseteq C_1$, we conclude that $(x, b') \in C_1$, and so $x \leq b'$. Then, $b \in \tau(C_2)$. $\square$

Theorem 3.8. *If (B, C) is a complete ICA, then $B_{\tau(C)}$ is a topoboolean.*

Proof. This is straightforward. □

Theorem 3.9. *Let (B, C) be a complete ICA, $b_0 \in B$ and $b_0 \leq \bigvee\{x \in At(B) \colon x \, C \, b_0\}$. Then*

$$\mathrm{cl}_{\tau(C)}(b_0) = \bigvee\{x \in At(B) \colon x \, C \, b_0\}.$$

Proof. Put $S = \{x \in At(B) \colon xCb_0\}$, $T = \{a \in B \colon a' \in \tau(C), b_0 \leq a'\}$ and $s = \bigvee S$, $t = \bigwedge T$, $\ll := \ll_C$. We have to show that $t = s$.
For proving that $s \leq t$, let $a \in T$ and $x \in S$. Since $b_0 \leq a$, we obtain that xCa. Thus $x \not\ll a'$. Then $x \nleq a'$ because $a' \in \tau(C)$. Therefore $x \leq a$. Hence, $x \leq t$. Thus $s \leq t$.

For proving that $t \leq s$, it is enough to show that $s \in T$. Since, by our hypothesis, $b_0 \leq s$, we only need to verify that $s' \in \tau(C)$. We will do this using Fact 3.5. So, let $x \in At(B)$ and xCs. We will first show that $x \in S$. Suppose that $x \notin S$. Then $x(-C)b_0$, i.e., $x \ll (b_0)'$. Hence, there exist $c, d \in B$ such that $x \ll c \ll d \ll (b_0)'$. Then $x(-C)c'$, $d(-C)b_0$ and $c \leq d$. Thus $c(-C)b_0$. For every $y \in S$, we have that yCb_0 and hence $y \leq c'$ (because $c(-C)b_0$). Therefore, $s \leq c'$. Since $x(-C)c'$, we obtain that $x(-C)s$, a contradiction. Hence $x \in S$. Thus $x \leq s$. Now, Fact 3.5 implies that $s' \in \tau(C)$. Hence, the inequality $t \leq s$ is proved. □

Corollary 3.10. *Let (B, C) be a complete atomic ICA. Then, for every $b \in B$,*

$$\mathrm{cl}_{\tau(C)}(b) = \bigvee\{x \in At(B) \colon x \, C \, b\}.$$

Corollary 3.11. *Suppose that (B, C) is a complete ICA, $b \in B$ and $b \leq \bigvee\{x \in At(B) \colon x \, C \, b\}$. Then the following statements are true for every $a \in B$ with $a(-C)b$.*

(1) $\mathrm{cl}_{\tau(C)}(b) \leq a'$.

(2) $\mathrm{cl}_{\tau(C)}(b) \leq \left(\mathrm{int}_{\tau(C)}(a)\right)'$.

Proof. (1) Let $x \in At(B)$, xCb and $x \nleq a'$. Since x is an atom of B, we conclude that $x \leq a$. On the other hand, xCb and so by Definition 3.1 (2), $(x \vee a)Cb$. Then, aCb. Hence, by Theorem 3.9, $\mathrm{cl}_{\tau(C)}(b) \leq a'$.

(2) By (1), Lemma 2.6 (2) and (4), and Lemma 2.5,

$$\mathrm{cl}_{\tau(C)}(b) = \mathrm{cl}_{\tau(C)}\left(\mathrm{cl}_{\tau(C)}(b)\right) \leq \mathrm{cl}_{\tau(C)}(a') = \left(\mathrm{int}_{\tau(C)}(a)\right)'.$$

□

Lemma 3.12. *For a complete ICA (B, C), the following statements are true.*

(1) *For every $a, b \in B$, if aCb, then $\left(\mathrm{cl}_{\tau(C)}(a)\right)C\left(\mathrm{cl}_{\tau(C)}(b)\right)$.*

(2) *Let $a, b \in B \setminus \{\perp\}$ be given. If $b \leq \bigvee\{x \in At(B) \colon x \, C \, b\}$ and $a \leq \bigvee\{x \in At(B) \colon x \, C \, a\}$, then $\left(\mathrm{cl}_{\tau(C)}(a)\right)C\left(\mathrm{cl}_{\tau(C)(b)}\right)$ implies that aCb.*

Proof. (1) By Lemma 2.6, $a \leq \mathrm{cl}_{\tau(C)}(a)$ and $b \leq \mathrm{cl}_{\tau(C)}(b)$. Now, by Lemma 3.2 we conclude that $\left(\mathrm{cl}_{\tau(C)}(a)\right)C\left(\mathrm{cl}_{\tau(C)}(b)\right)$.

(2) Suppose that $a(-C)b$. Then, by Definition 3.4, there exists $c \in B$ such that $a(-C)c$ and $c'(-C)b$. By Corollary 3.11, $\mathrm{cl}_{\tau(C)}(b) \leq c$ and by Lemma 3.2, $a(-C)c$ implies $a(-C)\mathrm{cl}_{\tau(C)}(b)$. A similar reasoning allows us to conclude that $\left(\mathrm{cl}_{\tau(C)}(a)\right)(-C)\left(\mathrm{cl}_{\tau(C)}(b)\right)$, a contradiction. Hence aCb. □

Lemma 3.13. *Let (B, C) be a complete ICA, $a, b \in B$ and $a(-C)b$. If $a \leq \bigvee\{x \in At(B) \colon x \, C \, a\}$, then $b(-C)\mathrm{cl}_{\tau(C)}(a)$. If $b \leq \bigvee\{x \in At(B) \colon x \, C \, b\}$, then $a(-C)\mathrm{cl}_{\tau(C)}(b)$.*

Proof. See the proof of Lemma 3.12 (2). □

Definition 3.14. Let B_τ be a topoboolean. Two elements a and b of B are said to be **completely separated** (from one another) in B_τ if there exists a real-continuous function f in $\mathcal{R}(B_\tau)$ such that

$$a \le f(\{0\}), \quad b \le f(\{1\}) \quad \textit{and} \quad \mathbf{0} \le f \le \mathbf{1}\,.$$

Proposition 3.15. *Let B_τ be a topoboolean. The relation C, defined on B by $a(-C)b$ if and only if a and b are completely separated, is an I-contact relation on B.*

Proof. We check the conditions (C1)–(C5) for C.

(C1). This is clear.

(C2). By Definition 3.14, $(a \vee b)(-C)c$ gives us $f, g \in \mathcal{R}(B_\tau)$ such that $z(f) \wedge z(g) = \bot$, $a \vee b \le z(f)$ and $c \le z(g)$. Now, since $a \le a \vee b$ and $b \le a \vee b$, we conclude that $a(-C)c$ and $b(-C)c$. Conversely, if $a(-C)c$ and $b(-C)c$, then there exist $f_i, g_i \in \mathcal{R}(B_\tau)$ such that $z(f_i) \wedge z(g_i) = \bot$ for $i = 1, 2$. Moreover, $a \le z(f_1)$, $b \le z(f_2)$ and $c \le z(g_i)$. Set $f = |f_1| \wedge |f_2|$ and $g = |g_1| + |g_2|$. By Proposition 2.4, $z(g) = z(g_1) \wedge z(g_2)$ and $z(f) = z(f_1) \vee z(f_2)$. Furthermore,

$$\begin{aligned} z(g) \wedge z(f) &= \big(z(g_1) \wedge z(g_2)\big) \wedge \big(z(f_1) \vee z(f_2)\big) \\ &= \big(z(g_1) \wedge z(g_2) \wedge z(f_1)\big) \vee \big(z(g_1) \wedge z(g_2) \wedge z(f_2)\big) \\ &= \big(\bot \wedge z(g_2)\big) \vee \big(z(g_1) \wedge \bot\big) \\ &= \bot. \end{aligned}$$

Also, $a \vee b \le z(f)$ and $c \le z(g)$. Then, $(a \vee b)(-C)c$.

(C3). If aCb and $a = \bot$, set $f = \mathbf{1}$ and $g = \mathbf{0}$. Then, $z(f) = \bot$ and $z(g) = \top$. Hence, $a \le z(f)$, $b \le z(g)$ and $z(f) \wedge z(g) = \bot$, which mean that $a(-C)b$.

(C′4). If $a(-C)b$, then there exist $f, g \in \mathcal{R}(B_\tau)$ such that $a \le z(f)$, $b \le z(g)$ and $z(f) \wedge z(g) = \bot$. Hence, $a \wedge b = \bot$.

(C5). If $a(-C)b$, then there exists an element f of $\mathcal{R}(B_\tau)$ such that $a \le f(\{0\})$, $b \le f(\{1\})$ and $\mathbf{0} \le f \le \mathbf{1}$. We set $c := f([\frac{1}{2}, 1])$. Since the map $g\colon \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ defined by

$$g(A) = \begin{cases} \frac{1}{2}A \cap (-\infty, \frac{1}{2}], & 1 \notin A, \\ \big(\frac{1}{2}A \cap (-\infty, \frac{1}{2}]\big) \cup [\frac{1}{2}, \infty), & 1 \in A, \end{cases}$$

is a frame map, $fg \in \mathcal{R}(B_\tau)$, so that

$$a \le f(\{0\}) = fg(\{0\})$$

and

$$c \le f\Big([\frac{1}{2}, \infty)\Big) = fg(\{1\}).$$

Hence, $a(-C)c$.
Define $h\colon \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by

$$h(A) = \{\frac{1}{2}(a+1) \colon a \in A\} \cap [\frac{1}{2}, +\infty)$$

when $0 \notin A$, and

$$h(A) = \Big(\{\frac{1}{2}(a+1) \colon 0 \neq a \in A\} \cap [\frac{1}{2}, +\infty)\Big) \cup (-\infty, \frac{1}{2}]$$

otherwise. Since h is a frame map, $fh \in \mathcal{R}(B_\tau)$, so that $b \leq f(\{1\}) = fh(\{1\})$. Also, from $f(1, +\infty) = \perp$ we infer that

$$c' = f(-\infty, \frac{1}{2}) \vee f(1, +\infty) = f(-\infty, \frac{1}{2}) = fh(\{0\}).$$

Therefore, $c'(-C)b$

□

Definition 3.16. A topoboolean B_τ is said to be **completely regular** if for every $t \in \tau$, there exists a subset $\{f_i\}_i$ of $\mathcal{R}(B_\tau)$ such that $t = \bigvee_i coz(f_i)$.

Lemma 3.17. *Let B_τ be a completely regular topoboolean. If $b \in \tau$ and $x \in At(B)$ are such that $x \leq b$, then there exist $f, h \in \mathcal{R}(B_\tau)$ such that*

$$x \leq int\Big(z(h)\Big) \leq z(h) \leq coz(f) \leq b.$$

Proof. By the hypothesis, there exists a subset $\{f_i\}_i$ of $\mathcal{R}(B_\tau)$ such that $b = \bigvee_i coz(f_i)$. This gives us an element f of $\mathcal{R}(B_\tau)$ such that $b' \leq z(f)$ and $x \leq coz(f)$, because $x \nleq b' = \bigwedge_i z(f_i)$. Since

$$\begin{aligned} coz(f) &= \bigvee_{n\in\mathbb{N}} \Big(f(-\infty, -\frac{1}{n}) \vee f(\frac{1}{n}, +\infty)\Big) \\ &= \bigvee_{n\in\mathbb{N}} \Big(f(-\infty, -\frac{1}{n}]^\circ \vee f[\frac{1}{n}, +\infty)^\circ\Big) \\ &= \bigvee_{n\in\mathbb{N}} \Big(f(-\infty, -\frac{1}{n}] \vee f[\frac{1}{n}, +\infty)\Big), \end{aligned}$$

we conclude that there exists an element n of $\mathbb{N}$ such that

$$x \leq f(-\infty, -\frac{1}{n}]^\circ \vee f[\frac{1}{n}, +\infty)^\circ \leq f(-\infty, -\frac{1}{n}] \vee f[\frac{1}{n}, +\infty) \leq coz(f).$$

If $h = \Big((f - \frac{1}{\mathbf{n}}) \vee 0\Big)\Big((f + \frac{1}{\mathbf{n}}) \wedge 0\Big)$, then $z(h) = f(-\infty, -\frac{1}{n}] \vee f[\frac{1}{n}, +\infty)$, which completes the proof. □

Definition 3.18. Let B_τ be a topoboolean, and let (B, C) be a *ICA*. Then τ and C are said to be **compatible** if $\tau = \tau(C)$.

We recall that if g is an invertible element of a ring R, then $\frac{f}{g}$ is defined as $f.g^{-1}$ for every $f \in R$.

Theorem 3.19. *Let B_τ be a completely regular topoboolean, and assume that the contact relation C is defined by $a(-C)b$ if and only if a and b are completely separated. Then, the following statements are true.*

(1) $\tau \subseteq \tau(C)$.

(2) *If B is an atomic Boolean algebra, then C is compatible with τ.*

Proof. (1) Let $b \in \tau$ be given, $x \in At(B)$ and $x \leq b$. Then, by Lemma 3.17, there exist $f, h \in \mathcal{R}(B_\tau)$ such that

$$x \leq int\Big(z(h)\Big) \leq z(h) \leq coz(f) \leq b.$$

Since $coz(|f| + |h|) = \top$, by Proposition 2.4 we conclude that $|f| + |h|$ is unit, and so we set $g := \frac{|f|}{|f|+|h|}$. Then $g \in \mathcal{R}(B_\tau)$, $b \leq g(\{0\})$, $x \leq g(\{1\})$ and $\mathbf{0} \leq g \leq \mathbf{1}$. Hence $x(-C)b'$, which implies that $b \in \tau(C)$.

(2) Let $b \in \tau(C)$ be given, $x \in At(B)$ and $x \leq b$. Then $x(-C)b'$, and so there exists an element f_x of $\mathcal{R}(B_\tau)$ such that

$$x \le f_x(\{0\}) \le f_x(-\infty, \tfrac{1}{2}),\ b' \le f_x(\{1\}) \text{ and } \mathbf{0} \le f_x \le \mathbf{1}.$$

Since B is an atomic Boolean algebra, we conclude that

$$b = \bigvee_{\substack{x \in At(B) \\ x \le b}} x \le \bigvee_{\substack{x \in At(B) \\ x \le b}} f_x(-\infty, \frac{1}{2}) \le b,$$

which implies $b \in \tau$. □

Remark 3.20. We recall from [24] that a topoboolean B_τ is **normal** if for every $a, b \in \tau$ with $a \vee b = \top$, there exist $u, v \in \tau$ such that

$$a' \le u,\ b' \le v \text{ and } u \wedge v = \bot.$$

Also, a topoboolean B_τ is normal if and only if for any $s, t \in \tau$ with $s' \le t$, there exists an element v of τ such that

$$s' \le v \le \mathrm{cl}_\tau(v) \le t.$$

Proposition 3.21. *Let B_τ be a normal topoboolean. Define a relation C on B by writing aCb if and only if $\mathrm{cl}_\tau(a) \wedge \mathrm{cl}_\tau(b) \neq \bot$. Then, C is an I-contact relation on B.*

Proof. By Example 3.3, C is a contact relation on B. Now, let $a, b \in B$ with $a(-C)b$ be given. Then $\mathrm{cl}_\tau(a) \wedge \mathrm{cl}_\tau(b) = \bot$ and by Remark 3.20, there exists an element v of τ such that

$$\mathrm{cl}_\tau(b) \le v \le \mathrm{cl}_\tau(v) \le \big(\mathrm{cl}_\tau(a)\big)'.$$

This implies

$$\mathrm{cl}_\tau(a) \wedge \mathrm{cl}_\tau(v) = \bot$$

and

$$\mathrm{cl}_\tau(v') \wedge \mathrm{cl}_\tau(b) = v' \wedge \mathrm{cl}_\tau(b) = \bot.$$

Hence, $a(-C)v$ and $v'(-C)b$. Therefore, C is an I-contact relation on B. □

Proposition 3.22. *If a completely regular topoboolean B_τ has a compatible I-contact relation C defined by aCb if and only if $\mathrm{cl}_\tau(a) \wedge \mathrm{cl}_\tau(b) \neq \bot$, then B_τ is a normal topoboolean.*

Proof. Let $a, b \in \tau$ be given with $a \vee b = \top$. Then $a'(-C)b'$, which gives us an element c of B such that $a'(-C)c$ and $c'(-C)b'$. So,

$$a' \le \mathrm{int}_\tau(c'),\ b' \le \mathrm{int}_\tau(c) \text{ and } \mathrm{int}_\tau(c') \wedge \mathrm{int}_\tau(c) = \bot.$$

Therefore, B_τ is a normal topoboolean. □

Lemma 3.23. *Let (B, C) be a complete ICA and $a, b \in B$. Then, the following statements are true.*

(1) If $a \le \bigvee\{x \in At(B) \colon x\, C\, a\}$, then $a \ll b$ implies $\mathrm{cl}_{\tau(C)}(a) \ll b$.

(2) If $b' \le \bigvee\{x \in At(B) \colon x\, C\, b'\}$, then $a \ll b$ implies $a \ll \mathrm{int}_{\tau(C)}(b)$.

Proof. (1) By Lemma 3.13, $a(-C)b'$ implies $\mathrm{cl}_{\tau(C)}(a)(-C)b'$. So, $\mathrm{cl}_{\tau(C)}(a) \ll b$.

(2) It follows from $a(-C)b'$ that $a(-C)\mathrm{cl}_{\tau(C)}(b')$. By Lemma 2.5, we conclude that $a(-C)\big(\mathrm{int}_{\tau(C)}(b)\big)'$, that is, $a \ll \mathrm{int}_{\tau(C)}(b)$. □

Corollary 3.24. *Let (B, C) be a complete atomic ICA, $a, b \in B$ and $a(-C)b$. Then there exists an element c of B such that $a \ll \mathrm{int}_{\tau(C)}(c)$ and $\big(\mathrm{int}_{\tau(C)}(c)\big)(-C)b$.*

Proof. Since $a(-C)b$, there exist $c, d \in B$ such that $a \ll c \ll d \ll b'$. Then $a(-C)c'$, $d(-C)b$ and $c \le d$. Hence, by Lemma 3.23, $a \ll \mathrm{int}_{\tau(C)}(c)$ and $\mathrm{int}_{\tau(C)}(c) \le c \le d$. These imply $b(-C)\big(\mathrm{int}_{\tau(C)}(c)\big)$. □

Lemma 3.25. *Let B_τ be a completely regular topoboolean, and let C be a contact relation on B compatible with τ. If B is an atomic Boolean algebra and $a, b \in B$ are such that a is compact, b is closed, and $a \wedge b = \bot$, then $a(-C)b$.*

Proof. If x is an atom of B and $x \le a$, then $x(-C)b$ because b is closed. Hence, given any atom x of B, Corollary 3.24 gives us an element c_x of τ such that $x \ll c_x$ and $c_x(-C)b$. Now, $\{c_x : x \text{ is an atom of } B\}$ is an open cover of the compact element a. So, there exists a finite subcover $\{c_{x_1}, c_{x_2}, \ldots, c_{x_n}\}$ of a such that $(\bigvee_{i=1}^{n} c_{x_i})(-C)b$, by the definition of contact algebra. Therefore, $a(-C)b$. □

Remark 3.26. Let B_τ be a compact topoboolean. Then, each closed element $k \in B$ is compact. In fact, if $k \le \bigvee S$ for some $S \subseteq \tau$, then $k' \vee \bigvee S = 1$ and so, by the compactness of $\top$, there exist $s_1, \ldots, s_n \in S$ such that $k' \vee (s_1 \vee \cdots \vee s_n) = \top$. Consequently, $k \le s_1 \vee \cdots \vee s_n$, as desired.

Theorem 3.27. *Let B_τ be a compact topoboolean which is completely regular. If B is an atomic Boolean algebra, then there exists a unique compatible contact relation with the interpolation property given by*

$$aCb \quad \text{if and only if} \quad \mathrm{cl}_\tau(a) \wedge \mathrm{cl}_\tau(b) \neq \bot.$$

Proof. By Theorem 3.19(2), there exists a compatible contact relation on B. Let C be any compatible contact relation and $\mathrm{cl}_\tau(a) \wedge \mathrm{cl}_\tau(b) \neq \bot$. Then, by Definition 3.1 (4), $\mathrm{cl}_\tau(a)(-C)\mathrm{cl}_\tau(b)$. So, by Lemma 3.12(2), aCb. Conversely, let aCb. Suppose that $\mathrm{cl}_\tau(a) \wedge \mathrm{cl}_\tau(b) = \bot$. Then, by Remark 3.26 and Lemma 3.25, we conclude that $\mathrm{cl}_\tau(a)(-C)\mathrm{cl}_\tau(b)$. Hence, by Lemma 3.2, $a(-C)b$, which is a contradiction. Thus, $\mathrm{cl}_\tau(a) \wedge \mathrm{cl}_\tau(b) \neq \bot$. □

4. *CA*-morphisms between ICAs and the category ICA

A pair (B, C) is said to be a precontact algebra ([6, 17, 19]) if it satisfies the axioms $(C2)$ and $(C3)$ from Definition 3.1. In [14], the category **PCA** of all precontact algebras and all Boolean homomorphisms $f : (B_1, C_1) \to (B_2, C_2)$ between them such that, for all $a, b \in B$, $f(a)C_2f(b)$ implies that aCb was introduced and studied. In this section, we introduce the category of I-contact algebras as a full subcategory of this category, and present some of its properties. These include the existence of products and initial objects.

Recall that if B_1 and B_2 are two complete Boolean algebras, then any complete Boolean homomorphism $f : B_1 \to B_2$ has a right Galois adjoint $f_* : B_2 \to B_1$ and a left Galois adjoint $f^* : B_2 \to B_1$.

Remark 4.1. (see [8]) Let (B_1, C_1) and (B_2, C_2) be two complete *CA*s, and let $f : B_1 \to B_2$ be a complete Boolean homomorphism. Then $f : (B_1, C_1) \to (B_2, C_2)$ is a *CA*-morphism if and only if, for every $a, b \in B_2$, aC_2b implies $f^*(a)C_1f^*(b)$.

Theorem 4.2. *Let $f : (B_1, C_1) \to (B_2, C_2)$ be a CA-morphism such that for every $b \in At(B_2)$, there exists $a_b \in At(B_1)$ with $b \le f(a)$. Then $f(\tau(C_1)) \subseteq \tau(C_2)$.*

Proof. Let $b \in \tau(C_1)$ and $x \in At(B_2)$ be such that $x \le f(b)$. Suppose that $x \not\ll f(b)$. Then $xC_2f(b')$. There exists $a \in At(B_1)$ such that $x \le f(a)$. Then, by Lemma 3.2, $f(a)C_2f(b')$ and hence aC_1b'. Since $a \in At(B_1)$ and $b \in \tau(C_1)$, we obtain that $a \le b'$. Thus, $x \le f(a) \le f(b')$. Therefore, $x \le f(b)$ and $x \le f(b')$, which imply $x = \bot$, a contradiction. Hence $x \ll f(b)$. □

Theorem 4.3. *Suppose that f is a complete Boolean homomorphism from a complete Boolean algebra B_1 to a complete Boolean algebra B_2. Let (B_1, C_1) be an ICA. We define the relation C_2 on B_2 by writing $a(-C_2)b$ if and only if there exists an element c of B_1 such that*

$$f^*(a)(-C_1)c' \text{ and } f(c) \leq b'.$$

Then, the following statements are true.

(1) *The relation C_2 is a I-contact relation on B_2.*

(2) *The mapping $f\colon (B_1, C_1) \to (B_2, C_2)$ is a CA-morphism.*

(3) *If C is a I-contact relation on B_2 such that $f\colon (B_1, C_1) \to (B_2, C)$ is a CA-morphism, then $C \subseteq C_2$.*

(4) *If for each $y \in At(B_2)$ there exists a $x_y \in At(B_1)$ such that $y \leq f(x_y)$, then $f(\tau(C_1)) \subseteq \tau(C_2)$.*

Proof. (1) We check the conditions (C1)–(C5) for C_2.

(C1). If $a(-C_2)b$, then there exists an element c of B_1 such that $f^*(a)(-C_1)c'$ and $f(c) \leq b'$. We set $d := \big(f^*(a)\big)'$. Since $f^*(b) \leq f^*\big(f(c')\big) \leq c'$, we conclude that $f^*(b)(-C_1)d'$ and $f(d) = f\big((f^*(a))'\big) \leq a'$, which imply $b(-C_2)a$.

(C2). If $(a \vee b)(-C_2)c$, then there exists an element d of B_1 such that $f^*(a \vee b)(-C_1)d'$ and $f(d) \leq c'$. Since $f^*(a) \leq f^*(a \vee b)$ and $f^*(b) \leq f^*(a \vee b)$, we conclude that $a(-C_2)c$ and $b(-C_2)c$. Conversely, assume that $a(-C_2)c$ and $b(-C_2)c$. Then there exist $d_1, d_2 \in B_1$ such that $f^*(a) \not C_1\, d_1'$, $f^*(b) \not C_1\, d_2'$, $f(d_1) \leq c'$ and $f(d_2) \leq c'$. Set $d = d_1 \vee d_2$. Then $f^*(a \vee b) \not C_1\, d'$ and $f(d) \leq c'$, which mean that $a \vee b(-C_2)c$.

(C3). Since $(f^*(\bot), \top) = (\bot, \top) \notin C_2$ and $f(\bot) = \bot \leq b'$, $\bot(-C_2)b$ for every $b \in B_2$.

(C4). If $a(-C)b_2$, then there exists an element c of B_1 such that $f^*(a)(-C_1)c'$ and $f(c) \leq b'$. It follows that

$$a \wedge b \leq a \wedge f(c') \leq f f^*(a) \wedge f(c') \leq f\big(f^*(a) \wedge c'\big) = f(\bot) = \bot.$$

(C5). If $a(-C_2)b$, then there exists an element c of B_1 such that $f^*(a)(-C_1)c'$ and $f(c) \leq b'$, which imply the existence of an element d of B_1 such that $f^*(a)(-C_1)d$ and $d'(-C_1)c'$. We set $e := f(d)$. Since $f^*(e') = f^*\big(f(d')\big) \leq d'$, we infer that $f^*(e') \not C_1\, c'$. This implies $e'(-C)_2 b$. Also, from $f^*(a)(-C_1)d$ and $e' = f(d')$ we conclude that $a(-C_2)e$.

(2) In order to show that $f\colon (B_1, C_1) \to (B_2, C_2)$ is a CA-morphism, suppose that $f^*(a)(-C_1)f^*(b)$ for some $a, b \in B_2$. Since $f^*(a) \ll \big(f^*(b)\big)'$, there exists c in B_1 such that $f^*(a) \ll c \ll \big(f^*(b)\big)'$. This implies $f^*(a)(-C_1)c'$ and

$$f(c) \leq f\Big(\big(f^*(b)\big)'\Big) = \Big(f\big(f^*(b)\big)\Big)' \leq b'.$$

It follows that $a(-C_2)b$. Therefore, by Remark 4.1, $f\colon (B_1, C_1) \to (B_2, C_2)$ is a CA-morphism.

(3) If $a(-C_2)b$, then there exists an element c of B_1 such that $f^*(a)(-C_1)c'$ and $f(c) \leq b'$, which imply $f\big(f^*(a)\big)(-C)f(c')$ and $b \leq f(c')$. Therefore, it follows that $a(-C)b$. Hence, $C \subseteq C_2$.

(4) This follows from (2) and Theorem 4.2. □

Remark 4.4. ([11]) If (B_1, C_1) and (B_2, C_2) are two CAs, $f\colon B_1 \to B_2$ is a Boolean homomorphism which preserves the contact relation C_1 (that is, aC_1b implies $f(a)C_2f(b)$ for all $a, b \in B_1$), then f is an injection.

According to the above remark, the following conditions are equivalent.

1. For every $a, b \in B_1$, aCb_1 implies $f(a)C_2f(b)$.
2. For every $c, d \in B_2$, $c(-C)d$ implies $a(-C_1)b$ for every $a \in f^{-1}(c)$ and $b \in f^{-1}(d)$.
3. For every $c, d \in B_2$, $c(-C)d$ implies $\big(\bigvee f^{-1}(c)\big)(-C_1)\big(\bigvee f^{-1}(d)\big)$.

Remark 4.5. Let B be a complete Boolean algebra. We recall from [16] that the largest and smallest contact relations exist on B; the largest one, C_l, is defined by aC_lb iff $a \neq \bot$ and $b \neq \bot$; the smallest one, C_s, being defined by aC_sb iff $a \wedge b \neq \bot$; moreover, (B, C_l) is an ICA and (B, C_s) is a normal contact algebra.

In the sequel, **CBoo** represents the category of complete Boolean algebras and complete homomorphisms.

Proposition 4.6. *The category* **ICA** *has an initial object.*

Proof. Let $B = \{\bot, \top\}$ and C_I be defined on B as in Remark 4.5. Let (M, C) be an *ICA*. Since B is an initial object in the category **CBoo**, there exists exactly one morphism f from B to M in **CBoo**. Now, assume that $a(-C_I)b$. Then $a = \bot$ or $b = \bot$, and so $f(a) = \bot$ or $f(b) = \bot$. By Definition 3.1 (C3), we conclude that $f(a)(-C)f(b)$. Hence, $f: (B, C_I) \to (M, C)$ is a *CA*-morphism. □

Proposition 4.7. *The category* **ICA** *does not have a terminal object.*

Proof. Suppose that (A, C_A) is an *ICA*. Let B denote a complete Boolean algebra with four elements, and let p denote an element of B other than $\bot$ and $\top$. Consider the binary relation C_I on B. Define $g: (B, C_I) \to (A, C_A)$ by $g := \begin{pmatrix} \bot & p & p' & \top \\ \bot & \bot & \top & \top \end{pmatrix}$ and $h: (B, C_I) \to (A, C_A)$ by $h := \begin{pmatrix} \bot & p & p' & \top \\ \bot & \top & \bot & \top \end{pmatrix}$. It is easy to see that g and h are *CA*-morphisms from (B, C_I) to (A, C_A). Therefore, (A, C_A) is not a terminal object. □

Let $\{(B_\alpha, C_\alpha) : \alpha \in I\}$ be a family of *ICA*s. Let $B = \Pi_{\alpha \in I} B_\alpha$ be the product of $\{B_\alpha\}_{\alpha \in I}$ in the category **CBoo**, and let p_α be the projection Boolean homomorphism of B to B_α. In the proposition below, we define an I-contact relation C on B.

Proposition 4.8. *Let $\{(B_\alpha, C_\alpha)\}_{\alpha \in I}$ be a family of ICAs, and let $B = \Pi_{i \in I} B_\alpha$ be the product of $\{B_\alpha\}_{\alpha \in I}$ in the category* **CBoo**. *Define a binary relation C on B by*

$$aCb \text{ if and only if } \big(p_\alpha(a)\big) C_\alpha \big(p_\alpha(b)\big) \text{ for some } \alpha \in I.$$

Then, the following statements are true.

(1) *The relation C is an I-contact relation on B.*

(2) *For every $\alpha \in I$, the projection map $p_\alpha: (B, C) \to (B_\alpha, C_\alpha)$ is a CA-morphism.*

(3) *A complete Boolean homomorphism $f: M \to B$ is a CA-morphism from (M, C_1) to (B, C) if and only if the composition $p_\alpha \circ f: (M, C_1) \to (B_\alpha, C_\alpha)$ is a CA-morphism for each projection p_α.*

(4) *The ICA (B, C) is a product of the family $\{(B_\alpha, C_\alpha)\}_{\alpha \in A}$ of ICAs in the category* **ICA**.

Proof. (1) See [10, Fact 3.2].
(2) This is a direct consequence of the definition of *CA*-morphisms and the definition of C.

(3) *Necessity.* By (2), $p_\alpha \circ f$ is a *CA*-morphism.
Sufficiency. Let $a, b \in M$ be such that $a(-C_1)b$. Since for each $\alpha \in I$, $p_\alpha \circ f: (M, C_1) \to (B_\alpha, C_\alpha)$ is a *CA*-morphism, $\big(p_\alpha \circ f(a)\big)(-C_\alpha)\big(p_\alpha \circ f(b)\big)$. Then, $f(a)(-C)f(b)$ by our definition of C.

(4) Let $\{(B_\alpha, C_\alpha)\}_{\alpha \in A}$ be a family of *ICA*s. Consider the product $B = \Pi_{\alpha \in A} B_\alpha$ in the category **CBoo**. Define a binary relation C on B by

$$aCb \quad \text{if and only if } \big(p_\alpha(a)\big) C_\alpha \big(p_\alpha(b)\big) \text{ for some } \alpha \in A.$$

By Proposition 4.8, (B, C) is an *ICA*. Suppose that (M, σ) is any *ICA*, and that $g_\alpha: (M, \sigma) \to (B_\alpha, C_\alpha)$ is a *CA*-morphism. Then, there exists a unique morphism $g: M \to B$ in **CBoo** such that $p_\alpha \circ g = g_\alpha$ for every $\alpha \in A$. It is sufficient to show that g is a *CA*-morphism. Let $m_1, m_2 \in M$ be given with $m_1(-\sigma)m_2$. Then $g_\alpha(m_1)(-C)_\alpha g_\alpha(m_2)$, which implies $p_\alpha\big(g(m_1)\big)(-C_\alpha)p_\alpha\big(g(m_2)\big)$. Hence, $g(m_1)(-C)g(m_2)$. □

Proposition 4.9. *Every monomorphism in the category* **ICA** *is an injective function.*

Proof. Let $f\colon (B_1, C_1) \to (B_2, C_2)$ be a monomorphism, and let $a, b \in B_1$ be such that $f(a) = f(b)$. Also, let B denote a complete Boolean algebra with four elements, and let p denote an element of B other than $\bot$ and $\top$. Consider the binary relation C_l on B . Define $g\colon (B, C_l) \to (B_1, C_1)$ by $g := \begin{pmatrix} \bot & p & p' & \top \\ \bot & a & a' & \top \end{pmatrix}$ and $h\colon (B, C_l) \to (B_1, C_1)$ by $h := \begin{pmatrix} \bot & p & p' & \top \\ \bot & b & b' & \top \end{pmatrix}$. It is easy to see that g and h are CA-morphisms, and that $f \circ g = f \circ h$. Since $f\colon (B_1, C_1) \to (B_2, C_2)$ is a monomorphism, we conclude that $g = h$, and so $a = b$. □

Let B_τ be a normal topoboolean. By Proposition 3.21, the relation C_τ defined on B by $aC_\tau b$ if and only if $\mathrm{cl}_\tau(a) \wedge \mathrm{cl}_\tau(b) \neq \bot$ is an I-contact relation. Then, (B, C_τ) is an ICA.

Proposition 4.10. *Let B_τ and M_σ be two normal topobooleans, and let $f\colon B \to M$ be a complete Boolean homomorphism such that $f(\tau) \subseteq \sigma$. Then, $f\colon (B, C_\tau) \to (M, C_\sigma)$ is a CA-morphism.*

Proof. Let $a, b \in B$, and let $a(-C_\tau)b$. Then $\mathrm{cl}_\tau(a) \wedge \mathrm{cl}_\tau(b) = \bot$, and so $f\big(\mathrm{cl}_\tau(a)\big) \wedge f\big(\mathrm{cl}_\tau(b)\big) = f\big(\mathrm{cl}_\tau(a) \wedge \mathrm{cl}_\tau(b)\big) = f(\bot) = \bot$. Also,

$$a \leq \mathrm{cl}_\tau(a) \Rightarrow f(a) \leq f\big(\mathrm{cl}_\tau(a)\big) \Rightarrow \mathrm{cl}_\sigma\big(f(a)\big) \leq f\big(\mathrm{cl}_\tau(a)\big).$$

By a similar reasoning,

$$\mathrm{cl}_\sigma\big(f(b)\big) \leq f\big(\mathrm{cl}_\tau(b)\big)$$

and so

$$\mathrm{cl}_\sigma\big(f(a)\big) \wedge \mathrm{cl}_\sigma\big(f(b)\big) = \bot.$$

This means that $f(a)(-C_\sigma)f(b)$. □

A function f like that in Proposition 4.10 will be called a **topoboolean map**.

Now, let **NTboo** be the category of normal topobooleans and topoboolean maps.

Proposition 4.11. *It follows that $G\colon$ **NTboo** $\to$ **ICA**, given by*

$$\begin{array}{ccc} B_\tau & \stackrel{G}{\longmapsto} & (B, C_\tau) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G(f)=f} \\ M_\sigma & \underset{G}{\longmapsto} & (M, C_\sigma) \end{array}$$

is a functor.

Proof. This is a direct consequence of Propositions 3.21 and 4.10. □

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