Filomat 35:9 (2021), 2883–2893
https://doi.org/10.2298/FIL2109883B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Fejér Sort Inequalities for Products Two Harmonically-Convex Functions via Fractional Integrals

Madeeha Baloch[a], Muhammad Amer Latif[b], Sabir Hussain[c], Muhammad Saeed[d]

[a]*Department of mathematics, University of Sialkot, Pakistan*
[b]*Department of Basic Sciences, Deanship of Preparatory Year, King Faisal University, Hofuf 31982, Al-Hasa,Saudi Arabia*
[c]*Department of Mathematics, University of Engineering and Technology Lahore, Pakistan*
[d]*Department of Mathematics, University of Management and Technology Lahore, Pakistan*

Abstract. In this paper, some new estimates on Fejér sort inequalities via Riemann Liouville fractional integral for the products of two harmonically-convex functions are obtained.

1. Introduction

It is certified that a function $\Im : \mathbf{K} \subset \mathbb{R} \to \mathbb{R}$ is convex, if

$$\Im\left(\eta\vartheta + (1-\eta)\theta\right) \leq \eta\Im(\vartheta) + (1-\eta)\Im(\theta),$$

grip for all $\vartheta, \theta \in \mathbf{K}$ and $\eta \in [0,1]$. The Hermite-Hadamard inequality is onoff essential inequalities for convex functions.

This inequalities startest if, for $\ell_1, \ell_2 \in \mathbf{K}$ with $\ell_1 < \ell_2$, the $\Im : \mathbf{K} \subset \mathbb{R} \to \mathbb{R}$ is a convex function spell out on a convex meantime $\mathbf{K}$ of real numbers.Then

$$\Im\left(\frac{\ell_1+\ell_2}{2}\right) \leq \frac{1}{\ell_2-\ell_1}\int_{\ell_1}^{\ell_2}\Im(x)\,dx \leq \frac{\Im(\ell_1)+\Im(\ell_2)}{2}. \tag{1}$$

In [8], Fejér turn over weighted generalization of (1).

Consider the integral $\int_{\ell_1}^{\ell_2}\Im(x)\Re(x)dx$, where $\Re$ is a positive function in the meantime (ℓ_1, ℓ_2) as a consequence $\Im(x)$ is a convex function in the same meantime such that:

$$0 \leq t \leq \frac{1}{2}(\ell_1+\ell_2), \Re(\ell_2+\ell_1-t) = \Re(t),$$

i.e., $\varepsilon = \Re(x)$ is symmetric curve in regard to the straight line which normal to the x-axis and contains the point $\left(\frac{1}{2}(\ell_1+\ell_2), 0\right)$. Then

$$\Im\left(\frac{\ell_1+\ell_2}{2}\right)\int_{\ell_1}^{\ell_2}\Re(x)dx \leq \int_{\ell_1}^{\ell_2}\Im(x)\Re(x)dx \leq \frac{\Im(\ell_1)+\Im(\ell_2)}{2}\int_{\ell_1}^{\ell_2}\Re(x)dx. \tag{2}$$

2020 *Mathematics Subject Classification*. 26A51, 26A33, 26D07, 26D10, 26D15.
Keywords. Fejér sort inequality, Riemann-Liouville fractional integral, harmonically-convex function
Received: 16 July 2020; Accepted: 12 December 2020
Communicated by Miodrag Spalević
Email addresses: madeehabaloch@yahoo.com (Madeeha Baloch), m_amer_latif@hotmail.com (Muhammad Amer Latif), sabirhus@gmail.com (Sabir Hussain), muhammad.saeed@umt.edu.pk (Muhammad Saeed)

A number of version of the inequality (1) and (2) have been secured by researchers from all over the world in the past three decades [1, 7, 13, 17, 22, 24].

Definition 1.1. *[11] Let* $\mathbf{K} \subset \mathbb{R}\setminus\{0\}$ *be a real interval. A function* $\Im : \mathbf{K} \to \mathbb{R}$ *is convex harmonically , then*

$$\Im\left(\frac{\vartheta\theta}{\rho\vartheta + (1-\rho)\theta}\right) \leq \rho\Im(\theta) + (1-\rho)\Im(\vartheta), \tag{3}$$

for all $\vartheta, \theta \in \mathbf{K}$ *and* $\rho \in [0,1]$. *If (3) is reversed, then* $\Im$ *is concave harmonically.*

Theorem 1.2. *[11] Let* $\Im : \mathbf{K} \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ *be a convex harmonically function on* $\mathbf{K}^\circ$ *as a consequence* $\ell_1, \ell_2 \in \mathbf{K}^\circ$ *with* $\ell_1 < \ell_2$ *such that* $\Im' \in \mathrm{L}([\ell_1, \ell_2])$*.Then*

$$\Im\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right) \leq \frac{\ell_1\ell_2}{\ell_2-\ell_1}\int_{\ell_1}^{\ell_2}\frac{\Im(\nu)}{\nu^2}d\nu \leq \frac{\Im(\ell_1)+\Im(\ell_2)}{2}. \tag{4}$$

Theorem 1.3. *[5] Let* $\Im : \mathbf{K} \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ *is convex harmonically function on* $\mathbf{K}^\circ$ *as a consequence* $\ell_1, \ell_2 \in \mathbf{K}$ *with* $\ell_1 < \ell_2$ *such that* $\Im' \in \mathrm{L}([\ell_1, \ell_2])$*.Then*

$$\Im\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right)\int_{\ell_1}^{\ell_2}\frac{\Re(\nu)}{\nu^2}d\nu \leq \frac{\ell_1\ell_2}{\ell_2-\ell_1}\int_{\ell_1}^{\ell_2}\frac{\Im(\nu)\Re(\nu)}{\nu^2}d\nu \leq \frac{\Im(\ell_1)+\Im(\ell_2)}{2}\int_{\ell_1}^{\ell_2}\frac{\Re(\nu)}{\nu^2}d\nu, \tag{5}$$

where $\Re : \mathbf{K} \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ *is integrable, non-negative and satisfies :*

$$\Re\left(\frac{\ell_1\ell_2}{\nu}\right) = \Re\left(\frac{\ell_1\ell_2}{\ell_1+\ell_2-\nu}\right).$$

Here, we also indicate the sequential definition of harmonically symmetric functions to be used in the progression of the paper.

Definition 1.4. *[16] A function* $\Re : [\ell_1, \ell_2] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ *is harmonically symmetric in regard to* $\frac{2\ell_1\ell_2}{\ell_1+\ell_2}$*, then*

$$\Re(\nu) = \Re\left(\frac{1}{\frac{1}{\ell_1}+\frac{1}{\ell_2}-\frac{1}{\nu}}\right),$$

holds for all $\nu \in [\ell_1, \ell_2]$.

Definition 1.5. *[14] Let* $\Im \in \mathrm{L}([\ell_1, \ell_2])$ *, then for order* $\alpha > 0$ *with* $\ell_2 > \ell_1 \geq 0$*, the right handed as a consequence left handed Riemann-Liouville fractional integrals* $J^{\alpha}_{\ell_1^+}\Im$ *as a consequence* $J^{\alpha}_{\ell_2^-}\Im$*, are defined as:*

$$J^{\alpha}_{\ell_1^+}\Im(\nu) = \frac{1}{\Gamma(\alpha)}\int_a^{\nu}(\nu-\rho)^{\alpha-1}\Im(\rho)\,d\rho, \nu > \ell_1$$

and

$$J^{\alpha}_{\ell_2^-}\Im(\nu) = \frac{1}{\Gamma(\alpha)}\int_a^{\nu}(\rho-\nu)^{\alpha-1}\Im(\rho)\,d\rho, \nu < \ell_2$$

where, $\Gamma(\alpha) = \int_0^\infty e^{\rho}\rho^{\alpha-1}d\rho$ *and* $J^0_{\ell_1^+}\Im(\nu) = J^0_{\ell_2^-}\Im(\nu) = \Im(\nu)$.

Theorem 1.6. *[9] Let* $\Im : \mathbf{K} \subseteq (0,\infty) \to \mathbb{R}$ *be a differentiable on* $\mathbf{K}$ *as a consequence* $\ell_1, \ell_2 \in \mathbf{K}$ *with* $\ell_1 < \ell_2$ *as a consequence* $\Im \in \mathrm{L}([\ell_1, \ell_2])$. *If* $\Im$ *is convex harmonically on* $[\ell_1, \ell_2]$*, then*

$$\Im\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right) \leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{\ell_1}-}(\Im o\hbar)\left(\frac{1}{\ell_2}\right) + J^{\alpha}_{\frac{1}{\ell_2}+}(\Im o\hbar)\left(\frac{1}{\ell_1}\right)\right] \leq \frac{\Im(\ell_1)+\Im(\ell_2)}{2}, \tag{6}$$

with $\alpha > 0$ *and* $\hbar(\nu) = \frac{1}{\nu}$, $\nu \in \left[\frac{1}{\ell_2}, \frac{1}{\ell_1}\right]$.

Theorem 1.7. *[10] Let $\Im : [\ell_1, \ell_2] \to \mathcal{R}$ be harmonically convex with $\ell_1 < \ell_2$ as a consequence $\Im \in \mathrm{L}([\ell_1, \ell_2])$. If $\Re : [\ell_1, \ell_2] \to \mathcal{R}$ is integrable, non-negative as a consequence harmonically symmetric in regard to $\frac{2\ell_1\ell_2}{\ell_1+\ell_2}$, then*

$$\Im\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right)\left[J^{\alpha}_{\frac{1}{\ell_1}-}(\Re o\hbar)\left(\frac{1}{\ell_2}\right)+J^{\alpha}_{\frac{1}{\ell_2}+}(\Re o\hbar)\left(\frac{1}{\ell_1}\right)\right] \leq \left[J^{\alpha}_{\frac{1}{\ell_1}-}(\Im\Re o\hbar)\left(\frac{1}{\ell_2}\right)+J^{\alpha}_{\frac{1}{\ell_2}+}(\Im\Re o\hbar)\left(\frac{1}{\ell_1}\right)\right]$$
$$\leq \frac{\Im(\ell_1)+\Im(\ell_2)}{2}\left[J^{\alpha}_{\frac{1}{\ell_1}-}(\Re o\hbar)\left(\frac{1}{\ell_2}\right)+J^{\alpha}_{\frac{1}{\ell_2}+}(\Re o\hbar)\left(\frac{1}{\ell_1}\right)\right], \quad (7)$$

with $\alpha > 0$ and $\hbar(\nu) = \frac{1}{\nu}$, $\nu \in \left[\frac{1}{\ell_2}, \frac{1}{\ell_1}\right]$.

Budak [4], granted some Fejér sort inequalities for products of two convex mappings as a consequence by applying these inequalities for Riemann-Liouville fractional integrals, he rooted some Fejér sort inequalities involving Riemann-Liouville fractional integrals. For well-purposed details on Hermite-Hadamard-Fejér integral inequalities with convexities see, [2, 3, 6, 12, 15, 18–21, 23]. Motivated by this work, we present some Fejér sort inequalities for products of two harmonically convex mappings as a consequence by applying these inequalities for Riemann-Liouville fractional integrals.

2. Fejér sort inequalities for product for harmonically-convex functions

Theorem 2.1. *Suppose that $\beth : [\ell_1, \ell_2] \to [0, \infty)$ is non-negative, integrable as a consequence harmonically symmetric in regard to $\frac{2\ell_1\ell_2}{\ell_1+\ell_2}$. If $\Im, \Re : \mathbf{K} \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ are two non-negative, real-valued as a consequence harmonic convex functions on $\mathbf{K}$, then for any $\ell_1, \ell_2 \in \mathbf{K}$, we attain*

$$\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}} \Im\left(\frac{1}{\varepsilon}\right)\Re\left(\frac{1}{\varepsilon}\right)\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon$$
$$\leq \left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^2\left[M(\ell_1,\ell_2)\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^2\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon + N(\ell_1,\ell_2)\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)\left(\varepsilon-\frac{1}{\ell_2}\right)\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon\right], \quad (8)$$

or equivalently

$$\int_{\ell_1}^{\ell_2} \Im(\varepsilon)\Re(\varepsilon)\frac{\beth(\varepsilon)}{\varepsilon^2}d\varepsilon$$
$$\leq \left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^2\left[M(\ell_1,\ell_2)\int_{\ell_1}^{\ell_2}\left(\frac{1}{\ell_1}-\frac{1}{\varepsilon}\right)^2\frac{\beth(\varepsilon)}{\varepsilon^2}d\varepsilon + N(\ell_1,\ell_2)\int_{\ell_1}^{\ell_2}\left(\frac{1}{\ell_1}-\frac{1}{\varepsilon}\right)\left(\frac{1}{\varepsilon}-\frac{1}{\ell_2}\right)\frac{\beth(\varepsilon)}{\varepsilon^2}d\varepsilon\right], \quad (9)$$

where

$$M(\ell_1,\ell_2) = \Im(\ell_1)\Re(\ell_1) + \Im(\ell_2)\Re(\ell_2)$$

and

$$N(\ell_1,\ell_2) = \Im(\ell_1)\Re(\ell_2) + \Re(\ell_1)\Im(\ell_2).$$

Proof. Since $\Im$ and $\Re$ are harmonic-convex functions on $[\ell_1, \ell_2]$, we attain

$$\Im\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right) \leq \nu\Im(\ell_2) + (1-\nu)\Im(\ell_1). \quad (10)$$

$$\Re\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right) \leq \nu\Re(\ell_2) + (1-\nu)\Re(\ell_1). \quad (11)$$

From (10) and (11), we attain

$$\Im\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)\Re\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)$$
$$\leq \nu^2\Im(\ell_2)\Re(\ell_2)+(1-\nu)^2\Im(\ell_1)\Re(\ell_1)+\nu(1-\nu)\left[\Im(\ell_1)\Re(\ell_2)+\Re(\ell_1)\Im(\ell_2)\right]. \quad (12)$$

Multiplying beside of (12) over $\beth\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)$, soon after integrating in regard to ν from 0 to 1, we attain

$$\int_0^1 \Im\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)\Re\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)\beth\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)d\nu$$
$$\leq \Im(\ell_2)\Re(\ell_2)\int_0^1 \nu^2\beth\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)d\nu+\Im(\ell_1)\Re(\ell_1)\int_0^1(1-\nu)^2\beth\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)d\nu$$
$$+\left[\Im(\ell_1)\Re(\ell_2)+\Re(\ell_1)\Im(\ell_2)\right]\int_0^1\nu(1-\nu)\beth\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)d\nu. \quad (13)$$

By change of variable $\varepsilon=\frac{\nu\ell_1+(1-\nu)\ell_2}{\ell_1\ell_2}$, we attain

$$\int_0^1 \Im\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)\Re\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)\beth\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)d\nu$$
$$=\frac{\ell_1\ell_2}{\ell_2-\ell_1}\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\Im\left(\frac{1}{\varepsilon}\right)\Re\left(\frac{1}{\varepsilon}\right)\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon, \quad (14)$$

and

$$\int_0^1 \nu^2\beth\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)d\nu=\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^3\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^2\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon. \quad (15)$$

Since $\beth(\varepsilon)$ is harmonically symmetric in regard to $\frac{2\ell_1\ell_2}{\ell_1+\ell_2}$, we attain

$$\int_0^1 (1-\nu)^2\beth\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)d\nu=\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^3\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\varepsilon-\frac{1}{\ell_2}\right)^2\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon$$
$$=\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^3\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^2\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon. \quad (16)$$

We also attain

$$\int_0^1 \nu(1-\nu)\beth\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)d\nu=\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^3\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)\left(\varepsilon-\frac{1}{\ell_2}\right)\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon. \quad (17)$$

By substitution the equalities (14)-(17) in (13), we attain

$$\frac{\ell_1\ell_2}{\ell_2-\ell_1}\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\Im\left(\frac{1}{\varepsilon}\right)\Re\left(\frac{1}{\varepsilon}\right)\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon$$
$$\leq\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^3\left[\Im(\ell_1)\Re(\ell_1)+\Im(\ell_2)\Re(\ell_2)\right]\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^2\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon$$
$$+\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^3\left[\Im(\ell_1)\Re(\ell_2)+\Re(\ell_1)\Im(\ell_2)\right]\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)\left(\varepsilon-\frac{1}{\ell_2}\right)\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon. \quad (18)$$

If we multiply beside of (18) by $\frac{\ell_2-\ell_1}{\ell_1\ell_2}$, then we attain the desired result. □

Corollary 2.2. *If we elect* $\Re\left(\frac{1}{\varepsilon}\right) = 1$, *for all* $\varepsilon \in \left[\frac{1}{\ell_2}, \frac{1}{\ell_1}\right]$ *in Theorem 2.1, then*

$$\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}} \Im\left(\frac{1}{\varepsilon}\right) \daleth\left(\frac{1}{\varepsilon}\right) d\varepsilon \leq \frac{\Im(\ell_1) + \Im(\ell_2)}{2} \int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}} \daleth\left(\frac{1}{\varepsilon}\right) d\varepsilon,$$

or equivalently

$$\int_{\ell_1}^{\ell_2} \Im(\varepsilon) \frac{\daleth(\varepsilon)}{\varepsilon^2} d\varepsilon \leq \frac{\Im(\ell_1) + \Im(\ell_2)}{2} \int_{\ell_1}^{\ell_2} \frac{\daleth(\varepsilon)}{\varepsilon^2} d\varepsilon.$$

Proof. For $\Re\left(\frac{1}{\varepsilon}\right) = 1$, for all $\varepsilon \in \left[\frac{1}{\ell_2}, \frac{1}{\ell_1}\right]$, from the inequality (8), we attain

$$\begin{aligned}
&\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}} \Im\left(\frac{1}{\varepsilon}\right) \daleth\left(\frac{1}{\varepsilon}\right) d\varepsilon \\
&\leq [\Im(\ell_1) + \Im(\ell_2)] \left(\frac{\ell_1\ell_2}{\ell_2 - \ell_1}\right)^2 \left[\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}} \left(\frac{1}{\ell_1} - \varepsilon\right)^2 \daleth\left(\frac{1}{\varepsilon}\right) d\varepsilon + \int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}} \left(\frac{1}{\ell_1} - \varepsilon\right)\left(\varepsilon - \frac{1}{\ell_2}\right) \daleth\left(\frac{1}{\varepsilon}\right) d\varepsilon\right] \\
&= [\Im(\ell_1) + \Im(\ell_2)] \left(\frac{\ell_1\ell_2}{\ell_2 - \ell_1}\right) \int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}} \left(\frac{1}{\ell_1} - \varepsilon\right) \daleth\left(\frac{1}{\varepsilon}\right) d\varepsilon. \qquad (19)
\end{aligned}$$

Since $\daleth$ is harmonic symmetric in regard to $\frac{2\ell_1\ell_2}{\ell_1+\ell_2}$, we attain

$$\begin{aligned}
&\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}} \left(\frac{1}{\ell_1} - \varepsilon\right) \daleth\left(\frac{1}{\varepsilon}\right) d\varepsilon = \int_{\frac{1}{\ell_2}}^{\frac{\ell_1+\ell_2}{2\ell_1\ell_2}} \left(\frac{1}{\ell_1} - \varepsilon\right) \daleth\left(\frac{1}{\varepsilon}\right) d\varepsilon + \int_{\frac{\ell_1+\ell_2}{2\ell_1\ell_2}}^{\frac{1}{\ell_1}} \left(\frac{1}{\ell_1} - \varepsilon\right) \daleth\left(\frac{1}{\varepsilon}\right) d\varepsilon \\
&= \int_{\frac{1}{\ell_2}}^{\frac{\ell_1+\ell_2}{2\ell_1\ell_2}} \left(\frac{1}{\ell_1} - \varepsilon\right) \daleth\left(\frac{1}{\varepsilon}\right) d\varepsilon + \int_{\frac{1}{\ell_2}}^{\frac{\ell_1+\ell_2}{2\ell_1\ell_2}} \left(\varepsilon - \frac{1}{\ell_2}\right) \daleth\left(\frac{1}{\frac{1}{\ell_1} + \frac{1}{\ell_2} - \varepsilon}\right) d\varepsilon \\
&= \frac{\ell_2 - \ell_1}{\ell_1\ell_2} \int_{\frac{1}{\ell_2}}^{\frac{\ell_1+\ell_2}{2\ell_1\ell_2}} \daleth\left(\frac{1}{\varepsilon}\right) d\varepsilon = \frac{\ell_2 - \ell_1}{2\ell_1\ell_2} \int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}} \daleth\left(\frac{1}{\varepsilon}\right) d\varepsilon. \qquad (20)
\end{aligned}$$

Using (20) in (19), we attain the required result. □

Theorem 2.3. *Suppose that conditions of Theorem 2.1 holds, then*

$$\begin{aligned}
&2\Im\left(\frac{2\ell_1\ell_2}{\ell_1 + \ell_2}\right) \Re\left(\frac{2\ell_1\ell_2}{\ell_1 + \ell_2}\right) \int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}} \daleth\left(\frac{1}{\varepsilon}\right) d\varepsilon \leq \int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}} \Im\left(\frac{1}{\varepsilon}\right) \Re\left(\frac{1}{\varepsilon}\right) \daleth\left(\frac{1}{\varepsilon}\right) d\varepsilon \\
&+ \left(\frac{\ell_1\ell_2}{\ell_2 - \ell_1}\right)^2 \left[M(\ell_1, \ell_2) \int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}} \left(\frac{1}{\ell_1} - \varepsilon\right)^2 \daleth\left(\frac{1}{\varepsilon}\right) d\varepsilon + N(\ell_1, \ell_2) \int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}} \left(\frac{1}{\ell_1} - \varepsilon\right)\left(\varepsilon - \frac{1}{\ell_2}\right) \daleth\left(\frac{1}{\varepsilon}\right) d\varepsilon\right], \qquad (21)
\end{aligned}$$

or equivalently

$$\begin{aligned}
&2\Im\left(\frac{2\ell_1\ell_2}{\ell_1 + \ell_2}\right) \Re\left(\frac{2\ell_1\ell_2}{\ell_1 + \ell_2}\right) \int_{\ell_1}^{\ell_2} \frac{\daleth(\varepsilon)}{\varepsilon^2} d\varepsilon \leq \int_{\ell_1}^{\ell_2} \Im(\varepsilon) \Re(\varepsilon) \frac{\daleth(\varepsilon)}{\varepsilon^2} d\varepsilon \\
&+ \left(\frac{\ell_1\ell_2}{\ell_2 - \ell_1}\right)^2 \left[M(\ell_1, \ell_2) \int_{\ell_1}^{\ell_2} \left(\frac{1}{\ell_1} - \frac{1}{\varepsilon}\right)^2 \frac{\daleth(\varepsilon)}{\varepsilon^2} d\varepsilon + N(\ell_1, \ell_2) \int_{\ell_1}^{\ell_2} \left(\frac{1}{\ell_1} - \frac{1}{\varepsilon}\right)\left(\frac{1}{\varepsilon} - \frac{1}{\ell_2}\right) \frac{\daleth(\varepsilon)}{\varepsilon^2} d\varepsilon\right], \qquad (22)
\end{aligned}$$

where $M(\ell_1, \ell_2)$ *and* $N(\ell_1, \ell_2)$ *are defined as in Theorem 2.1.*

Proof. For $\nu \in [0,1]$, we can write

$$\frac{2\ell_1\ell_2}{\ell_1+\ell_2} = \frac{2\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)\left(\frac{\ell_1\ell_2}{(1-\nu)\ell_1+\nu\ell_2}\right)}{\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)+\left(\frac{\ell_1\ell_2}{(1-\nu)\ell_1+\nu\ell_2}\right)}.$$

Using the harmonic-convexity of $\mathfrak{I}$ and $\mathfrak{R}$, we attain

$$\begin{aligned}
&\mathfrak{I}\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right)\mathfrak{R}\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right) \\
&= \mathfrak{I}\left(\frac{2\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)\left(\frac{\ell_1\ell_2}{(1-\nu)\ell_1+\nu\ell_2}\right)}{\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)+\left(\frac{\ell_1\ell_2}{(1-\nu)\ell_1+\nu\ell_2}\right)}\right)\mathfrak{R}\left(\frac{2\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)\left(\frac{\ell_1\ell_2}{(1-\nu)\ell_1+\nu\ell_2}\right)}{\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)+\left(\frac{\ell_1\ell_2}{(1-\nu)\ell_1+\nu\ell_2}\right)}\right) \\
&\le \frac{1}{4}\left[\mathfrak{I}\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)+\mathfrak{I}\left(\frac{\ell_1\ell_2}{(1-\nu)\ell_1+\nu\ell_2}\right)\right]\left[\mathfrak{R}\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)+\mathfrak{R}\left(\frac{\ell_1\ell_2}{(1-\nu)\ell_1+\nu\ell_2}\right)\right] \\
&= \frac{1}{4}\left[\mathfrak{I}\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)\mathfrak{R}\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)+\mathfrak{I}\left(\frac{\ell_1\ell_2}{(1-\nu)\ell_1+\nu\ell_2}\right)\mathfrak{R}\left(\frac{\ell_1\ell_2}{(1-\nu)\ell_1+\nu\ell_2}\right)\right] \\
&\quad + \frac{1}{4}\left[\mathfrak{I}\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)\mathfrak{R}\left(\frac{\ell_1\ell_2}{(1-\nu)\ell_1+\nu\ell_2}\right)+\mathfrak{I}\left(\frac{\ell_1\ell_2}{(1-\nu)\ell_1+\nu\ell_2}\right)\mathfrak{R}\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)\right]. \quad (23)
\end{aligned}$$

For the second expression in the last equality, by using again the HA-convexity of $\mathfrak{I}$ and $\mathfrak{R}$, we attain

$$\begin{aligned}
&\mathfrak{I}\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right)\mathfrak{R}\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right) \\
&\le \frac{1}{4}\left[\mathfrak{I}\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)\mathfrak{R}\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)+\mathfrak{I}\left(\frac{\ell_1\ell_2}{(1-\nu)\ell_1+\nu\ell_2}\right)\mathfrak{R}\left(\frac{\ell_1\ell_2}{(1-\nu)\ell_1+\nu\ell_2}\right)\right] \\
&\quad + \frac{1}{4}\left[\nu^2+(1-\nu)^2\right]\left[\mathfrak{I}(\ell_1)\mathfrak{R}(\ell_1)+\mathfrak{I}(\ell_2)\mathfrak{R}(\ell_2)\right]+\frac{1}{2}t(1-t)\left[\mathfrak{I}(\ell_1)\mathfrak{R}(\ell_2)+\mathfrak{R}(\ell_1)\mathfrak{I}(\ell_2)\right]. \quad (24)
\end{aligned}$$

Multiplying beside of (12) over $\beth\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)$, soon after integrating in regard to ν from 0 to 1, we attain

$$\begin{aligned}
&\mathfrak{I}\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right)\mathfrak{R}\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right)\int_0^1 \beth\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)d\nu \\
&\le \frac{1}{4}\int_0^1 \mathfrak{I}\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)\mathfrak{R}\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)\beth\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)d\nu \\
&+\frac{1}{4}\int_0^1 \mathfrak{I}\left(\frac{\ell_1\ell_2}{(1-\nu)\ell_1+\nu\ell_2}\right)\mathfrak{R}\left(\frac{\ell_1\ell_2}{(1-\nu)\ell_1+\nu\ell_2}\right)\beth\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)d\nu \\
&+\frac{M(\ell_1,\ell_2)}{4}\int_0^1\left[\nu^2+(1-\nu)^2\right]\beth\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)d\nu+\frac{N(\ell_1,\ell_2)}{2}\int_0^1 \nu(1-\nu)\beth\left(\frac{\ell_1\ell_2}{\nu\ell_1+(1-\nu)\ell_2}\right)d\nu. \quad (25)
\end{aligned}$$

Using the identities (14)-(17) in (25), we attain

$$\begin{aligned}
&\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)\mathfrak{I}\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right)\mathfrak{R}\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right)\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon \le \frac{1}{2}\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\mathfrak{I}\left(\frac{1}{\varepsilon}\right)\mathfrak{R}\left(\frac{1}{\varepsilon}\right)\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon \\
&\quad +\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^3\left[\frac{M(\ell_1,\ell_2)}{2}\int_{\frac{1}{b}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^2\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon+\frac{N(\ell_1,\ell_2)}{2}\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)\left(\varepsilon-\frac{1}{\ell_2}\right)\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon.\right] \quad (26)
\end{aligned}$$

Multiply the beside of (26) by $\frac{2(\ell_2-\ell_1)}{\ell_1\ell_2}$, then we attain the desired result (21). □

Corollary 2.4. *If we elect* $\mathfrak{R}\left(\frac{1}{\varepsilon}\right)=1$, *for all* $\varepsilon\in\left[\frac{1}{\ell_2},\frac{1}{\ell_1}\right]$ *in Theorem 2.3, then*

$$2\mathfrak{I}\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right)\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon\leq\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\mathfrak{I}\left(\frac{1}{\varepsilon}\right)\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon+\frac{\mathfrak{I}(\ell_1)+\mathfrak{I}(\ell_2)}{2}\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon,$$

or equivalently

$$2\mathfrak{I}\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right)\int_{\ell_1}^{\ell_2}\frac{\beth(\varepsilon)}{\varepsilon^2}d\varepsilon\leq\int_{\ell_1}^{\ell_2}\mathfrak{I}(\varepsilon)\frac{\beth(\varepsilon)}{\varepsilon^2}d\varepsilon+\frac{\mathfrak{I}(\ell_1)+\mathfrak{I}(\ell_2)}{2}\int_{\ell_1}^{\ell_2}\frac{\beth(\varepsilon)}{\varepsilon^2}d\varepsilon.$$

Proof. For $\mathfrak{R}\left(\frac{1}{\varepsilon}\right)=1$, for all $\varepsilon\in\left[\frac{1}{\ell_2},\frac{1}{\ell_1}\right]$, from the inequality (21), we attain

$$\begin{aligned}&2\mathfrak{I}\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right)\int_{\ell_1}^{\ell_2}\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon\leq\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\mathfrak{I}\left(\frac{1}{\varepsilon}\right)\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon\\&+\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^2[\mathfrak{I}(\ell_1)+\mathfrak{I}(\ell_2)]\left[\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^2\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon+\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)\left(\varepsilon-\frac{1}{\ell_2}\right)\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon\right]\\&=\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\mathfrak{I}\left(\frac{1}{\varepsilon}\right)\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon+\frac{\mathfrak{I}(\ell_1)+\mathfrak{I}(\ell_2)}{2}\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon.\end{aligned}\tag{27}$$

By applying the equality (20), we attain the desired result. □

3. Some Results For Riemann-Liouville Fractional Integrals

In this section, we spread the inequalities obtain in Section 2 to Riemann-Liouville fractional integrals. Thus, we establish some Fejér sort inequalities involving Riemann-Liouville fractional integrals.

Theorem 3.1. *Suppose that* $\beth:[\ell_1,\ell_2]\to[0,\infty)$ *is integrable, non-negative as a consequence harmonically symmetric in regard to* $\frac{2\ell_1\ell_2}{\ell_1+\ell_2}$. *If* $\mathfrak{I},\mathfrak{R}:\mathbf{K}\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ *are two non-negative, real-valued as a consequence harmonic convex functions on* $\mathbf{K}$, *then for any* $\ell_1,\ell_2\in\mathbf{K}$, *we attain*

$$\begin{aligned}&J^{\alpha}_{\frac{1}{\ell_2}^+}(\mathfrak{I}\mathfrak{R}\beth)(\ell_1)+J^{\alpha}_{\frac{1}{\ell_1}^-}(\mathfrak{I}\mathfrak{R}\beth)(\ell_2)\\&\leq\frac{1}{\Gamma(\alpha)}\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^2\Bigg\{M(\ell_1,\ell_2)\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha-1}\left[\left(\frac{1}{\ell_1}-\varepsilon\right)^2+\left(\varepsilon-\frac{1}{\ell_2}\right)^2\right]\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon\\&\qquad+2N(\ell_1,\ell_2)\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha}\left(\varepsilon-\frac{1}{\ell_2}\right)\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon\Bigg\},\end{aligned}\tag{28}$$

or equivalently

$$\begin{aligned}&J^{\alpha}_{\frac{1}{\ell_2}^+}(\mathfrak{I}\mathfrak{R}\beth)(\ell_1)+J^{\alpha}_{\frac{1}{\ell_1}^-}(\mathfrak{I}\mathfrak{R}\beth)(\ell_2)\\&\leq\frac{1}{\Gamma(\alpha)}\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^2\Bigg\{M(\ell_1,\ell_2)\int_{\ell_1}^{\ell_2}\left(\frac{1}{\ell_1}-\frac{1}{\varepsilon}\right)^{\alpha-1}\left[\left(\frac{1}{\ell_1}-\frac{1}{\varepsilon}\right)^2+\left(\frac{1}{\varepsilon}-\frac{1}{\ell_2}\right)^2\right]\frac{\beth(\varepsilon)}{\varepsilon^2}d\varepsilon\\&\qquad+2N(\ell_1,\ell_2)\int_{\ell_1}^{\ell_2}\left(\frac{1}{\ell_1}-\frac{1}{\varepsilon}\right)^{\alpha}\left(\frac{1}{\varepsilon}-\frac{1}{\ell_2}\right)\frac{\beth(\varepsilon)}{\varepsilon^2}d\varepsilon\Bigg\},\end{aligned}\tag{29}$$

where Γ is the Gamma function.

Proof. Based on the assumption that $\beth$ is integrable, non-negative and symmetric $\frac{2\ell_1\ell_2}{\ell_1+\ell_2}$, it is obvious that $h\left(\frac{1}{\varepsilon}\right)=\frac{1}{\Gamma(\alpha)}\left[\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha-1}+\left(\varepsilon-\frac{1}{\ell_2}\right)^{\alpha-1}\right]\beth\left(\frac{1}{\varepsilon}\right)$ is integrable, non-negative and symmetric in regard to $\frac{2\ell_1\ell_2}{\ell_1+\ell_2}$. Thus by using Theorem 2.1 we attain

$$\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\mathfrak{I}\left(\frac{1}{\varepsilon}\right)\mathfrak{R}\left(\frac{1}{\varepsilon}\right)h\left(\frac{1}{\varepsilon}\right)d\varepsilon \leq\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^2\left[M(\ell_1,\ell_2)\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^2h\left(\frac{1}{\varepsilon}\right)d\varepsilon+N(\ell_1,\ell_2)\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)\left(\varepsilon-\frac{1}{\ell_2}\right)h\left(\frac{1}{\varepsilon}\right)d\varepsilon\right], \quad (30)$$

as

$$\frac{1}{\Gamma(\alpha)}\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\mathfrak{I}\left(\frac{1}{\varepsilon}\right)\mathfrak{R}\left(\frac{1}{\varepsilon}\right)\left[\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha-1}+\left(\varepsilon-\frac{1}{\ell_2}\right)^{\alpha-1}\right]\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon$$
$$\leq\frac{1}{\Gamma(\alpha)}\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^2\left\{M(\ell_1,\ell_2)\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^2\left[\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha-1}+\left(\varepsilon-\frac{1}{\ell_2}\right)^{\alpha-1}\right]\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon\right.$$
$$\left.+N(\ell_1,\ell_2)\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)\left(\varepsilon-\frac{1}{\ell_2}\right)\times\left[\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha-1}+\left(\varepsilon-\frac{1}{\ell_2}\right)^{\alpha-1}\right]\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon\right\}. \quad (31)$$

From the Definition 1.6, we attain

$$\frac{1}{\Gamma(\alpha)}\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha-1}\mathfrak{I}\left(\frac{1}{\varepsilon}\right)\mathfrak{R}\left(\frac{1}{\varepsilon}\right)\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon+\frac{1}{\Gamma(\alpha)}\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\varepsilon-\frac{1}{\ell_2}\right)^{\alpha-1}\mathfrak{I}\left(\frac{1}{\varepsilon}\right)\mathfrak{R}\left(\frac{1}{\varepsilon}\right)\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon$$
$$=J^{\alpha}_{\frac{1}{\ell_2}^+}(\mathfrak{I}\mathfrak{R}\beth)(\ell_1)+J^{\alpha}_{\frac{1}{\ell_1}^-}(\mathfrak{I}\mathfrak{R}\beth)(\ell_2). \quad (32)$$

Moreover, since $\beth$ is symmetric in regard to $\frac{2\ell_1\ell_2}{\ell_1+\ell_2}$, we attain

$$\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^2\left[\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha-1}+\left(\varepsilon-\frac{1}{\ell_2}\right)^{\alpha-1}\right]\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon$$
$$=\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha+1}\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon+\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^2\left(\varepsilon-\frac{1}{\ell_2}\right)^{\alpha-1}\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon$$
$$=\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha+1}\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon+\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha-1}\left(\varepsilon-\frac{1}{\ell_2}\right)^2\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon$$
$$=\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha-1}\left[\left(\frac{1}{\ell_1}-\varepsilon\right)^2+\left(\varepsilon-\frac{1}{\ell_2}\right)^2\right]\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon, \quad (33)$$

and similarly

$$\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)\left(\varepsilon-\frac{1}{\ell_2}\right)\left[\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha-1}+\left(\varepsilon-\frac{1}{\ell_2}\right)^{\alpha-1}\right]\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon=\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\varepsilon-\frac{1}{\ell_2}\right)\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha}\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon$$
$$+\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)\left(\varepsilon-\frac{1}{\ell_2}\right)^{\alpha}\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon=\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\varepsilon-\frac{1}{\ell_2}\right)\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha}\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon$$
$$+\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\varepsilon-\frac{1}{\ell_2}\right)\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha}\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon=2\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\varepsilon-\frac{1}{\ell_2}\right)\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha}\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon. \quad (34)$$

If we substitute the equalities (32)-(34) in (31), then we attain the desired inequality (28). □

Corollary 3.2. *If we elect $\Re\left(\frac{1}{\varepsilon}\right)=1$, for all $\varepsilon\in\left[\frac{1}{\ell_2},\frac{1}{\ell_1}\right]$, in Theorem 3.1, then we have*

$$J^{\alpha}_{\frac{1}{\ell_2}^{+}}(\Im\beth)(\ell_1)+J^{\alpha}_{\frac{1}{\ell_1}^{-}}(\Im\beth)(\ell_2)\le\frac{\Im(\ell_1)+\Im(\ell_2)}{2}\left[J^{\alpha}_{\frac{1}{\ell_2}^{+}}\beth(\ell_1)+J^{\alpha}_{\frac{1}{\ell_1}^{-}}\beth(\ell_2)\right].$$

Proof. For $\Re\left(\frac{1}{\varepsilon}\right)=1$, for all $\varepsilon\in\left[\frac{1}{\ell_2},\frac{1}{\ell_1}\right]$, from the inequality (28), we attain

$$\begin{aligned}
&J^{\alpha}_{\frac{1}{\ell_2}^{+}}(\Im\beth)(\ell_1)+J^{\alpha}_{\frac{1}{\ell_1}^{-}}(\Im\beth)(\ell_2)\\
&\le\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^2\frac{\Im(\ell_1)+\Im(\ell_2)}{\Gamma(\alpha)}\left\{\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha-1}\left[\left(\frac{1}{\ell_1}-\varepsilon\right)^2+\left(\varepsilon-\frac{1}{\ell_2}\right)^2\right]\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon\right.\\
&\left.+2\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha}\left(\varepsilon-\frac{1}{\ell_2}\right)\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon\right\}\\
&=\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^2\frac{\Im(\ell_1)+\Im(\ell_2)}{\Gamma(\alpha)}\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha-1}\left[\left(\frac{1}{\ell_1}-\varepsilon\right)^2+\left(\varepsilon-\frac{1}{\ell_2}\right)^2+2\left(\frac{1}{\ell_1}-\varepsilon\right)\left(\varepsilon-\frac{1}{\ell_2}\right)\right]\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon\\
&=\frac{\Im(\ell_1)+\Im(\ell_2)}{\Gamma(\alpha)}\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha-1}\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon=\frac{\Im(\ell_1)+\Im(\ell_2)}{2}\left[J^{\alpha}_{\frac{1}{\ell_2}^{+}}\beth(\ell_1)+J^{\alpha}_{\frac{1}{\ell_1}^{-}}\beth(\ell_2)\right.
\end{aligned}\tag{35}$$

□

Theorem 3.3. *Suppose that conditions of Theorem 3.1 holds, then*

$$\begin{aligned}
&2\Im\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right)\Re\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right)\left[J^{\alpha}_{\frac{1}{\ell_2}^{+}}(\beth)(\ell_1)+J^{\alpha}_{\frac{1}{\ell_1}^{-}}(\beth)(\ell_2)\right]\le\left[J^{\alpha}_{\frac{1}{\ell_2}^{+}}(\Im\Re\beth)(\ell_1)+J^{\alpha}_{\frac{1}{\ell_1}^{-}}(\Im\Re\beth)(\ell_2)\right]\\
&+\frac{1}{\Gamma(\alpha)}\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^2\left\{M(\ell_1,\ell_2)\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha-1}\left[\left(\frac{1}{\ell_1}-\varepsilon\right)^2+\left(\varepsilon-\frac{1}{\ell_2}\right)^2\right]\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon\right.\\
&\left.+2N(\ell_1,\ell_2)\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha}\left(\varepsilon-\frac{1}{\ell_2}\right)\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon\right\},
\end{aligned}\tag{36}$$

or equivalently

$$\begin{aligned}
&2\Im\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right)\Re\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right)\left[J^{\alpha}_{\frac{1}{\ell_2}^{+}}(\beth)(\ell_1)+J^{\alpha}_{\frac{1}{\ell_1}^{-}}(\beth)(\ell_2)\right]\le J^{\alpha}_{\frac{1}{\ell_2}^{+}}(\Im\Re\beth)(\ell_1)+J^{\alpha}_{\frac{1}{\ell_1}^{-}}(\Im\Re\beth)(\ell_2)\\
&+\frac{1}{\Gamma(\alpha)}\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^2\left\{M(\ell_1,\ell_2)\int_{\ell_1}^{\ell_2}\left(\frac{1}{\ell_1}-\frac{1}{\varepsilon}\right)^{\alpha-1}\left[\left(\frac{1}{\ell_1}-\frac{1}{\varepsilon}\right)^2+\left(\frac{1}{\varepsilon}-\frac{1}{\ell_2}\right)^2\right]\frac{\beth(\varepsilon)}{\varepsilon^2}d\varepsilon\right.\\
&\left.+2N(\ell_1,\ell_2)\int_{\ell_1}^{\ell_2}\left(\frac{1}{\ell_1}-\frac{1}{\varepsilon}\right)^{\alpha}\left(\frac{1}{\varepsilon}-\frac{1}{\ell_2}\right)\frac{\beth(\varepsilon)}{\varepsilon^2}d\varepsilon\right\}.
\end{aligned}\tag{37}$$

Proof. Based on the assumption that $\beth$ is integrable, non-negative and symmetric $\frac{2\ell_1\ell_2}{\ell_1+\ell_2}$, it is obvious that $h\left(\frac{1}{\varepsilon}\right)=\frac{1}{\Gamma(\alpha)}\left[\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha-1}+\left(\varepsilon-\frac{1}{\ell_2}\right)^{\alpha-1}\right]\beth\left(\frac{1}{\varepsilon}\right)$ is integrable, non-negative and symmetric about $\frac{2\ell_1\ell_2}{\ell_1+\ell_2}$. Thus by using Theorem 3.1, we attain

$$\Im\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right)\Re\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right)\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}} h\left(\frac{1}{\varepsilon}\right)d\varepsilon \leq \int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}} \Im\left(\frac{1}{\varepsilon}\right)\Re\left(\frac{1}{\varepsilon}\right)h\left(\frac{1}{\varepsilon}\right)d\varepsilon$$
$$+\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^2\left[M(\ell_1,\ell_2)\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^2 h\left(\frac{1}{\varepsilon}\right)d\varepsilon \;+N(\ell_1,\ell_2)\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)\left(\varepsilon-\frac{1}{\ell_2}\right)h\left(\frac{1}{\varepsilon}\right)d\varepsilon\right]. \quad (38)$$

That is, we attain

$$\frac{2}{\Gamma(\alpha)}\Im\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right)\Re\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right)\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left[\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha-1}+\left(\varepsilon-\frac{1}{\ell_2}\right)^{\alpha-1}\right]\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left[\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha-1}+\left(\varepsilon-\frac{1}{\ell_2}\right)^{\alpha-1}\right]\Im\left(\frac{1}{\varepsilon}\right)\Re\left(\frac{1}{\varepsilon}\right)\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon$$
$$+\left(\frac{\ell_1\ell_2}{\ell_2-\ell_1}\right)^2\frac{1}{\Gamma(\alpha)}\left\{M(\ell_1,\ell_2)\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)^2\left[\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha-1}+\left(\varepsilon-\frac{1}{\ell_2}\right)^{\alpha-1}\right]\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon\right.$$
$$\left.+N(\ell_1,\ell_2)\int_{\frac{1}{\ell_2}}^{\frac{1}{\ell_1}}\left(\frac{1}{\ell_1}-\varepsilon\right)\left(\varepsilon-\frac{1}{\ell_2}\right)\left[\left(\frac{1}{\ell_1}-\varepsilon\right)^{\alpha-1}+\left(\varepsilon-\frac{1}{\ell_2}\right)^{\alpha-1}\right]\beth\left(\frac{1}{\varepsilon}\right)d\varepsilon\right\}. \quad (39)$$

From the Definition 1.6, and using the identities (32)-(34), we attain the desired result (36). □

Remark 3.4. *If we elect $\alpha = 1$ in Theorem 3.1, then (28) cut down to (8).*

Corollary 3.5. *If we elect $\Re\left(\frac{1}{\varepsilon}\right) = 1$ for all $\varepsilon \in \left[\frac{1}{\ell_2}, \frac{1}{\ell_1}\right]$, in Theorem 3.3, then*

$$2\Im\left(\frac{2\ell_1\ell_2}{\ell_1+\ell_2}\right)\left[J^{\alpha}_{\frac{1}{\ell_2}^+}\beth(\ell_1)+J^{\alpha}_{\frac{1}{\ell_1}^-}\beth(\ell_2)\right] \leq J^{\alpha}_{\frac{1}{\ell_2}^+}(\Im\beth)(\ell_1)+J^{\alpha}_{\frac{1}{\ell_1}^-}(\Im\beth)(\ell_2)$$
$$+\frac{\Im(\ell_1)+\Im(\ell_2)}{2}\left[J^{\alpha}_{\frac{1}{\ell_2}^+}\beth(\ell_1)+J^{\alpha}_{\frac{1}{\ell_1}^-}\beth(\ell_2)\right]. \quad (40)$$

Proof. The proof is obvious from the inequality (35). □

Remark 3.6. *If we elect $\alpha = 1$ in Theorem 3.3, then (36) cut down to (21).*

References

[1] S. Abramovich, G. Farid, and J. Pečarié, More about Hermite- Hadamard inequalities, Cauchy's means, and superquadracity, Journal of Inequalities and Applications, 2010(2010), Article ID 102467, 14 pages.

[2] A. Akkurt and H. Yildirim, On Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals, Mathematica Moravica, 21 (1) (2017), 105-123.

[3] S. M. Aslani, M. R. Delavar and S. M. Vaezpour, Inequalities of Fejér type related to generalized convex functions with applications, International Journal of Analysis and Applications, 16 (1) (2018), 38-49.

[4] H. Budak, On Fejér type inequalities for products two convex functions, Available at: https://www.researchgate.net/publication/331113291.

[5] F. Chen and S. Wu, Fejér and Hermite-Hadamard type inequalities for harmonically convex functions, J. Appl. Math., 2014 (2014), Article ID 386806.

[6] M. R. Delavar , S. M. Aslani and M. D. L. Sen, Hermite-Hadamard-Fejér Inequality related to generalized convex functions via fractional integrals, Journal of Mathematics, 2018 (2018), Article ID 5864091, 10 pages.

[7] M. R. Delavar and M. D. L. Sen, Some generalizations of Hermite-Hadamard type inequalities, Springer Plus, 5 (1) (2016), article no. 1661.

[8] L. Fejér, Über die Fourierreihen, II, Math. Naturwiss. Anz. Ungar. Akad. Wiss., 24 (1906), 369-390.

[9] İ. İşcan and S. Wu, Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals, Applied Mathematics and Computation, 238 (1) (2014), 237-244.
[10] İ. İşcan and M. Kunt, Hermite-Hadamard-Fejér type inequalities for harmonically convex functions via fractional integrals, RGMIA Research Report Collection 18 (2015), Article no. 107, 1-16.
[11] İ. İşcan, Hermite-Hadamard type inequalities for harmonically convex functions, Hacettepe Journal of Mathematics and Statistics, 43 (6) (2014), 935-942.
[12] İ. İşcan, Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals, Studia Universitatis Babeş-Bolyai Mathematica, 60 (3) (2015), 355-366.
[13] İ. İşcan, Hermite-Hadamards inequalities for preinvex functions via fractional integrals and related fractional inequalities, American Journal of Mathematical Analysis, 1 (3) (3013), 33-38.
[14] A. A. Kilbas, H. M. Srivastava and J. J. truillo, Theory and applications of fractional differential equations, Elsevier, Amsterdam 2006.
[15] M. Kunt and İ. İşcan, On new Hermite-Hadamard-Fejér type inequalities for p-convex functions via fractional integrals, Communication in Mathematical Modeling and Applications, 2 (1) (2017), 1-15.
[16] M. A. Latif, S. S. Dragomir and E. Momoniat, Fejér type inequalities for harmonically-convex functions with applications, Journal of Applied Analysis and Computation, 7 (3) (2017), 795-813.
[17] M. A. Latif and S. S. Dragomir, New inequalities of Hermite-Hadamard and Fejér type preinvexity, J. Comput. Anal. Appl., 19 (1) (2015), 725-739.
[18] J. Park, Inequalities of type for convex functions via fractional integrals, International Journal of Mathematical Analysis, 8 (59) (2014), 2927-2937.
[19] E. Set, İ. İşcan, M. Z. Sarikaya and M. Emin Ozdemir, On new inequalities of Hermite–Hadamard-Fejér type for convex functions via fractional integrals, Applied Mathematics and Computation, 259 (2015), 875-881.
[20] E. Set, İ. İşcan, M. Z. Sarikaya and M. E. Özdemir, On new inequalities of Hermite-Hadamard-Fejér type for convex functions via fractional integrals, Appl. Math. Compt., 259 (2015), 875-881.
[21] M. Z. Sarikaya and S. Erden, On the Hermite-Hadamard-Fejér type integral inequality for convex function, Turk. J. Anal. and Number Theory, 2 (3) (2014), 85-89.
[22] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, Mathematical and Computer Modelling, 57 (9-10) (2013), 2403-2407.
[23] M. Z. Sarikaya, On new Hermite-Hadamard-Fejér type integral inequalities, Studia Universitatis Babeş-Bolyai Series Mathematica, 57 (3) (2012), 377-386.
[24] J. Wang, X. Li, M. Fečkan, and Y. Zhou, Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity, Journal of Applied Analysis, 92 (11) (2012), 2241-2253.