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Product Generalized Local Morrey Spaces and Commutators of Multi-Sublinear Operators Generated by Multilinear Calderon-Zygmund Operators and Local Campanato Functions

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Abstract. The aim of this paper is to get the boundedness of the commutators of multi-sublinear operators generated by local campanato functions and multilinear Calderón-Zygmund operators on the product generalized local Morrey spaces.

1. Introduction and main results

Because of the need for study of the local behavior of solutions of second order elliptic partial differential equations (PDEs) and together with the now well-studied Sobolev spaces, constitute a formidable three parameter family of spaces useful for proving regularity results for solutions to various PDEs, especially for non-linear elliptic systems, in 1938, Morrey [15] introduced the classical Morrey spaces which are natural generalizations of the classical Lebesgue spaces.

Its definition defined by

Definition 1.1. *Let $0 < q \le p < \infty$. For an $L_q^{loc}(\mathbb{R}^n)$-function f and any ball $B = B(x,r)$, the Morrey space $M_q^p(\mathbb{R}^n)$ is the collection of all measurable functions f whose Morrey space norm is*

$$\left\|f\right\|_{M_q^p(\mathbb{R}^n)} = \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} |B|^{\frac{1}{p}-\frac{1}{q}} \left\|f\chi_B\right\|_{L_q(\mathbb{R}^n)} < \infty.$$

Here, we would like to mention that in many research papers, such as in [2, 9] et al., the Morrey space is defined in another way.

Definition 1.2. *Let $0 \le \lambda \le n$ and $0 < q < \infty$. Then for $f \in L_q^{loc}(\mathbb{R}^n)$ and any ball $B = B(x,r)$, the Morrey space $L_{q,\lambda}(\mathbb{R}^n)$ is defined by*

$$\left\|f\right\|_{L_{q,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n}\sup_{r>0} r^{-\frac{\lambda}{q}} \|f\|_{L_q(B)} \equiv \sup_{B} r^{-\frac{\lambda}{q}} \|f\|_{L_q(B)} < \infty.$$

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Conclusion 1.3. *Recall that* $0 < q \le p < \infty$ *and* $0 \le \lambda \le n$. *By checking the definitions of* $M_q^p(\mathbb{R}^n)$ *and* $L_{q,\lambda}(\mathbb{R}^n)$, *it is easy to see that if we take* $\lambda = \left(1 - \frac{q}{p}\right) n \in [0, n]$, *then* $L_{q,\left(1-\frac{q}{p}\right)n}(\mathbb{R}^n) = M_q^p(\mathbb{R}^n)$. *Moreover, if we choose* $p = \frac{qn}{n-\lambda} \le q$, $M_q^{\frac{qn}{n-\lambda}}(\mathbb{R}^n) = L_{q,\lambda}(\mathbb{R}^n)$. *Thus, we conclude that* $M_q^p(\mathbb{R}^n)$ *is equivalent to* $L_{q,\lambda}(\mathbb{R}^n)$.

Remark 1.4. *Obviously, the Morrey space is the generalization of the Lebesgue space that can be seen from the special case* $M_q^q(\mathbb{R}^n) = L_q(\mathbb{R}^n)$ *with* $1 \le q < \infty$.

We also refer to [1] for the latest research on the theory of Morrey spaces associated with Harmonic Analysis. In recent years, more and more researches focus on function spaces based on Morrey spaces to fill in some gaps in the theory of Morrey type spaces (see, for example, [2, 9]). Moreover, various Morrey spaces are defined in the process of study. Also, these spaces are useful in harmonic analysis and PDEs. But, this topic exceeds the scope of this paper. Thus, we omit the details here.

First of all, we recall some explanations and notations used in the paper.

Recall that the concept of the generalized local (central) Morrey space $LM_{p,\varphi}^{\{x_0\}}$ has been studied in [2, 9].

Definition 1.5. ***(Generalized local (central) Morrey space)*** *Let* $\varphi(x, r)$ *be a positive measurable function on* $\mathbb{R}^n \times (0, \infty)$ *and* $1 \le p < \infty$. *For any fixed* $x_0 \in \mathbb{R}^n$, *the generalized local Morrey space* $LM_{p,\varphi}^{\{x_0\}}$ *is defined by*

$$\begin{aligned} LM_{p,\varphi}^{\{x_0\}} &\equiv LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n) \\ &= \left\{ f \in L_p^{loc}(\mathbb{R}^n) : \|f\|_{LM_{p,\varphi}^{\{x_0\}}} = \sup_{r>0} \varphi(x_0, r)^{-1} |B(x_0, r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x_0,r))} < \infty \right\}. \end{aligned}$$

According to this definition, we recover the local Morrey space $LL_{p,\lambda}^{\{x_0\}}$ under the choice $\varphi(x_0, r) = r^{\frac{\lambda-n}{p}}$:

$$LL_{p,\lambda}^{\{x_0\}} = LM_{p,\varphi}^{\{x_0\}} \Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}} .$$

For the properties and applications of generalized local (central) Morrey spaces $LM_{p,\varphi}^{\{x_0\}}$, see also [2, 9].

On the other hand, in 1976, Coifman et al. [4] introduced the commutator $\overline{T}_b$ generated by the Calderón-Zygmund operator $\overline{T}$ and a locally integrable function b as follows:

$$\overline{T}_b f(x) = [b, \overline{T}] f(x) \equiv b(x)\overline{T} f(x) - \overline{T}(bf)(x) = \int_{\mathbb{R}^n} K(x, y) [b(x) - b(y)] f(y) dy, \tag{1}$$

with the kernel K satisfying the following size condition:

$$K(x, y) \le C |x - y|^{-n}, \qquad x \ne y,$$

and some smoothness assumption. A celebrated result is that $\overline{T}$ is bounded operator on L_p space, where $1 < p < \infty$. Sometimes, the commutator defined by (1) is also called the commutator in Coifman et al.'s sense, which has its root in the complex analysis and harmonic analysis (see [4]). The main result from [4] states that, if and only if $b \in BMO$ (bounded mean oscillation space), T_b is a bounded operator on $L_p(\mathbb{R}^n)$, $1 < p < \infty$. It is worth noting that for a constant C, if $\overline{T}$ is linear we have,

$$\begin{aligned} [b + C, \overline{T}] f &= (b + C)\overline{T} f - \overline{T}((b + C) f) \\ &= b\overline{T} f + C\overline{T} f - \overline{T}(bf) - C\overline{T} f \\ &= [b, \overline{T}] f. \end{aligned}$$

This leads one to intuitively look to spaces for which we identify functions which differ by constants, and so it is no surprise that $b \in BMO$ or $LC_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ (local Campanato space) has had the most historical significance.

Also, the definition and some properties of spaces of bounded mean oscillation *BMO* and local Campanato space $LC_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ that we need in the proof of commutators are as follows.

BMO spaces have been, and continue to be, of great interest and a subject of intense research in harmonic analysis. One of the most fascinating aspects of *BMO* spaces is their self-improvement properties, which go back to the work of John and Nirenberg in [10]. Functions of *BMO* were also introduced by John and Nirenberg [10], in connection with differential equations. The definition on $\mathbb{R}^n$ reads as follows:

Definition 1.6. *The space $BMO(\mathbb{R}^n)$ of functions of bounded mean oscillation consists of locally summable functions with finite semi-norm*

$$\|b\|_* \equiv \|b\|_{BMO} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy < \infty, \tag{2}$$

where $b_{B(x,r)}$ is the mean value of the function b on the ball $B(x,r)$ and $\|b\|_*$ is called the BMO-norm of b, and it becomes a norm on after dividing out the constant functions. Bounded functions are in BMO and a BMO-function is locally in $L_p(\mathbb{R})$ for every $p < \infty$. Typical examples of BMO-functions are of the form $\log|P|$ with a polynomial on $\mathbb{R}^n$. Furthermore, BMO is a bit like the space L_∞, but L_∞ is a subspace of BMO. Indeed,

$$\begin{aligned}\frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy &\leq \frac{1}{|B(x,r)|} \int_{B(x,r)} \left|b(y)\right| dy + \frac{1}{|B(x,r)|} \int_{B(x,r)} \left|b_{B(x,r)}\right| dy \\ = \frac{1}{|B(x,r)|} \int_{B(x,r)} \left|b(y)\right| dy + \left|b_{B(x,r)}\right| &\leq 2\frac{1}{|B(x,r)|} \int_{B(x,r)} \left|b(y)\right| dy \leq 2\,\|b\|_{L_\infty}.\end{aligned}$$

As a result, since $\|b\|_* \leq 2\,\|b\|_{L_\infty}$, $L_\infty(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$ is valid.

Remark 1.7. *The fact that precisely the mean value $b_{B(x,r)}$ figures in (2) is inessential and one gets an equivalent seminorm if $b_{B(x,r)}$ is replaced by an arbitrary constant c :*

$$\|b\|_* \approx \sup_{r>0} \inf_{c\in\mathbb{C}} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left|b(y) - c\right| dy. \tag{3}$$

Indeed, it is obvious that (2) implies (3). If (3) holds, then

$$\left|b_{B(x,r)} - c\right| = \left|\frac{1}{|B(x,r)|} \int_{B(x,r)} (b(y) - c)\, dy\right| \leq C,$$

so

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} \left(\left|b(y) - c\right| + \left|c - b_{B(x,r)}\right|\right) dy \leq 2C.$$

Definition 1.8. *[2, 9] Let $1 \leq q < \infty$ and $0 \leq \lambda < \frac{1}{n}$. A local Campanato function $b \in L_q^{loc}(\mathbb{R}^n)$ is said to belong to the $LC_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, if*

$$\|b\|_{LC_{q,\lambda}^{\{x_0\}}} = \sup_{r>0} \frac{1}{|B(x_0,r)|^\lambda} \left(\frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} \left|b(y) - b_{B(x_0,r)}\right|^q dy\right)^{\frac{1}{q}} < \infty, \tag{4}$$

where

$$b_{B(x_0,r)} = \frac{1}{|B(x_0,r)|} \int\limits_{B(x_0,r)} b(y)\,dy.$$

Define

$$LC^{\{x_0\}}_{q,\lambda}(\mathbb{R}^n) = \left\{ b \in L^{loc}_q(\mathbb{R}^n) : \|b\|_{LC^{\{x_0\}}_{q,\lambda}} < \infty \right\}.$$

Remark 1.9. *If two functions which differ by a constant are regarded as a function in the space $LC^{\{x_0\}}_{q,\lambda}(\mathbb{R}^n)$, then $LC^{\{x_0\}}_{q,\lambda}(\mathbb{R}^n)$ becomes a Banach space. The space $LC^{\{x_0\}}_{q,\lambda}(\mathbb{R}^n)$ when $\lambda = 0$ is just the $LC^{\{x_0\}}_{q}(\mathbb{R}^n)$. Apparently, (4) is equivalent to the following condition:*

$$\sup_{r>0} \inf_{c\in\mathbb{C}} \left(\frac{1}{|B(x_0,r)|^{1+\lambda q}} \int\limits_{B(x_0,r)} \left| b(y) - c \right|^q dy \right)^{\frac{1}{q}} < \infty.$$

Also, in [14], Lu and Yang introduced the central BMO space $CBMO_q(\mathbb{R}^n) = LC^{\{0\}}_{q,0}(\mathbb{R}^n)$. Note that $BMO(\mathbb{R}^n) \subset \bigcap\limits_{q>1} LC^{\{x_0\}}_{q}(\mathbb{R}^n)$, $1 \le q < \infty$. Moreover, one can imagine that the behavior of $LC^{\{x_0\}}_{q}(\mathbb{R}^n)$ may be quite different from that of $BMO(\mathbb{R}^n)$, since there is no analogy of the famous John-Nirenberg inequality of $BMO(\mathbb{R}^n)$ for the space $LC^{\{x_0\}}_{q}(\mathbb{R}^n)$.

Lemma 1.10. *[2, 9] Let b be a local Campanato function in $LC^{\{x_0\}}_{q,\lambda}(\mathbb{R}^n)$, $1 \le q < \infty$, $0 \le \lambda < \frac{1}{n}$ and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{|B(x_0,r_1)|^{1+\lambda q}} \int\limits_{B(x_0,r_1)} \left| b(y) - b_{B(x_0,r_2)} \right|^q dy \right)^{\frac{1}{q}} \le C \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right) \|b\|_{LC^{\{x_0\}}_{q,\lambda}}, \tag{5}$$

where $C > 0$ is independent of b, r_1 and r_2.

From this inequality (5), we have

$$\left| b_{B(x_0,r_1)} - b_{B(x_0,r_2)} \right| \le C \left(1 + \ln \frac{r_1}{r_2} \right) |B(x_0,r_1)|^{\lambda} \|b\|_{LC^{\{x_0\}}_{q,\lambda}}, \tag{6}$$

and it is easy to see that

$$\left\| b - (b)_B \right\|_{L_q(B)} \le C \left(1 + \ln \frac{r_1}{r_2} \right) r^{\frac{n}{q}+n\lambda} \|b\|_{LC^{\{x_0\}}_{q,\lambda}}. \tag{7}$$

Remark 1.11. *Let $x_0 \in \mathbb{R}^n$, $1 < p_i, q_i < \infty$, for $i = 1,\ldots,m$ such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m}$ and $\overrightarrow{b} \in LC^{\{x_0\}}_{q_i,\lambda_i}(\mathbb{R}^n)$ for $0 \le \lambda_i < \frac{1}{n}$, $i = 1,\ldots,m$. Then, from Lemma 1.10, it is easy to see that*

$$\left\| b_i - (b_i)_B \right\|_{L_{q_i}(B)} \le C r^{\frac{n}{q_i}+n\lambda_i} \|b_i\|_{LC^{\{x_0\}}_{q_i,\lambda_i}},$$

and

$$\left\| b_i - (b_i)_B \right\|_{L_{q_i}(2B)} \le \left\| b_i - (b_i)_{2B} \right\|_{L_{q_i}(2B)} + \left\| (b_i)_B - (b_i)_{2B} \right\|_{L_{q_i}(2B)} \lesssim r^{\frac{n}{q_i}+n\lambda_i} \|b_i\|_{LC^{\{x_0\}}_{q_i,\lambda_i}}, \tag{8}$$

for $i = 1, 2$.

The singular integral operator theory, which plays a significant part in many respects of harmonic analysis has been extensively studied in recent years, and the results are plentiful and substantial. In the last century, the theory of the Calderón-Zygmund operator is a crucial part of accomplishment in the classical analysis, and it has been fully applied in Fourier analysis, complex analysis, operator theory and so on. The generalized Calderón-Zygmund operator originated in the classical Calderón-Zygmund operator has attracted many researchers to explore it. Lin [13] established the sharp maximal function pointwise estimates for generalized Calderón-Zygmund operators and their commutators with *BMO* functions. On the other hand, multilinear Calderón-Zygmund theory is a natural generalization of the linear case. The initial work on the class of multilinear Calderón-Zygmund operators has been done by Coifman and Meyer in [3]. Moreover, the study of multilinear singular integrals has motivated not only as the generalization of the theory of linear ones but also their natural appearance in analysis. It has received increasing attention and much development in recent years, such as the study of the bilinear Hilbert transform by Lacey and Thiele [11, 12] and the systematic treatment of multilinear Calderón-Zygmund operators by Grafakos-Torres [6–8] and Grafakos-Kalton [5]. Meanwhile, the commutators generated by the multilinear singular integral and *BMO* functions of Lipschitz functions also attract much attention, since the commutator is more singular than the singular integral operator itself.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of points $x=(x_1,...,x_n)$ with norm $|x|=\left(\sum\limits_{i=1}^{n}x_i^2\right)^{\frac{1}{2}}$ and $(\mathbb{R}^n)^m=\mathbb{R}^n\times\ldots\times\mathbb{R}^n$ be the m-fold product spaces ($m\in\mathbb{N}$). For $x\in\mathbb{R}^n$ and $r>0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by $B^C(x,r)$ denote its complement and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)|=v_nr^n$, where $v_n=|B(0,1)|$. Throughout this paper, we denote by $\overrightarrow{y}=(y_1,\ldots,y_m)$, $d\overrightarrow{y}=dy_1\ldots dy_m$, and by $\overrightarrow{f}$ the m-tuple $(f_1,...,f_m)$, m, n the nonnegative integers with $n\geq 2$, $m\geq 1$.

Suppose that $T^{(m)}$ represents a multilinear or a multi-sublinear operator, which satisfies that for any $m\in\mathbb{N}$ and $\overrightarrow{f}=(f_1,\ldots,f_m)$, suppose each f_i ($i=1,\ldots,m$) is integrable on $\mathbb{R}^n$ with compact support and $x\notin\bigcap\limits_{i=1}^{m}supp f_i$,

$$\left|T^{(m)}\left(\overrightarrow{f}\right)(x)\right|\leq c_0\int\limits_{(\mathbb{R}^n)^m}\frac{1}{\left|(x-y_1,\ldots,x-y_m)\right|^{mn}}\left\{\prod_{i=1}^{m}\left|f_i\left(y_i\right)\right|\right\}d\overrightarrow{y},\tag{9}$$

where c_0 is independent of $\overrightarrow{f}$ and x.

We point out that the condition (9) in the case of $m=1$ was first introduced by Soria and Weiss in [16] . The condition (9) is satisfied by many interesting operators in harmonic analysis, such as the m-linear Calderón–Zygmund operators, m-sublinear Carleson's maximal operator, m-sublinear Hardy–Littlewood maximal operator, C. Fefferman's singular multipliers, R. Fefferman's m-linear singular integrals, Ricci–Stein's m-linear oscillatory singular integrals, the m-linear Bochner–Riesz means and so on (see [2, 9, 16] for details).

We are going to be working on $\mathbb{R}^n$. Let's begin with the recalling of the multilinear Calderón-Zygmund operator $\overline{T}^{(m)}$ ($m\in\mathbb{N}$). Let $\overrightarrow{f}\in L^{loc}_{p_1}(\mathbb{R}^n)\times\ldots\times L^{loc}_{p_m}(\mathbb{R}^n)$ and $K(y_0,y_1,\cdots,y_m)$ be a function away from the diagonal in $(\mathbb{R}^n)^{m+1}$. $\overline{T}^{(m)}$ stands for an m-linear singular integral operator defined by

$$\begin{aligned}\overline{T}^{(m)}\left(\overrightarrow{f}\right)(x)&=\overline{T}^{(m)}\left(f_1,\ldots,f_m\right)(x)=\int\limits_{\mathbb{R}^n}\cdots\int\limits_{\mathbb{R}^n}K\left(x,y_1,\ldots,y_m\right)\prod_{i=1}^{m}f_i\left(y_i\right)dy_1\ldots dy_m\\&=\int\limits_{(\mathbb{R}^n)^m}K\left(x,y_1,\ldots,y_m\right)\left\{\prod_{i=1}^{m}f_i\left(y_i\right)\right\}d\overrightarrow{y},\end{aligned}$$

where f_i $(i = 1, \cdots, m)$ are smooth functions with compact support and $x \notin \bigcap_{i=1}^{m} supp f_i$. If the kernel K satisfies the following size and smoothness conditions:

(i) For some $C > 0$ and all $(y_0, y_1, \cdots, y_m) \in (\mathbb{R}^n)^{m+1}$ defined away from the diagonal,

$$\left|K(y_0, y_1, \ldots, y_m)\right| \leq \frac{C}{\left(\sum_{k,l=0}^{m} \left|y_k - y_l\right|\right)^{mn}}; \tag{10}$$

(ii) For some $\epsilon > 0$, whenever $0 \leq i \leq m$ and $\left|y_i - y_i'\right| \leq \frac{1}{2} \max_{0 \leq k \leq m} \left|y_i - y_k\right|$,

$$\left|K(y_0, \ldots, y_i, \cdots, y_m) - K\left(y_0, \ldots, y_i', \cdots, y_m\right)\right| \leq \frac{C\left|y_i - y_i'\right|^{\epsilon}}{\left(\sum_{k,l=0}^{m} \left|y_k - y_l\right|\right)^{mn+\epsilon}},$$

then we call $\overline{T}^{(m)}$ an standard m-linear Calderón-Zygmund operator, where K is an m-Calderón-Zygmund kernel which is a locally integrable function defined away from the diagonal $y_0 = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, see [5, 6] for details.

At the same time, Grafakos and Torres [6, 8] have proved that the multilinear Calderón-Zygmund operator is bounded on the product of Lebesgue spaces.

Theorem 1.12. *[6, 8] Let $\overline{T}^{(m)}$ be an m-linear Calderón-Zygmund operator. Then, for any numbers $1 \leq p_1, \ldots, p_m < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\overline{T}^{(m)}$ can be extended to a bounded operator from $L_{p_1} \times \cdots \times L_{p_m}$ into L_p, and bounded from $L_1 \times \cdots \times L_1$ into $L_{\frac{1}{m},\infty}$.*

Let $\overline{T}^{(m)}$ be an m-linear Calderón-Zygmund operator, $\overrightarrow{b} = (b_1, \ldots, b_m)$ is a group of locally integrable functions and $\overrightarrow{f} = (f_1, \ldots, f_m)$. Then the m-linear iterated commutator generated by $\overline{T}^{(m)}$ and $\overrightarrow{b}$ is defined to be

$$\overline{T}^{(m)}_{\Pi \overrightarrow{b}}\left(\overrightarrow{f}\right) = \left[b_1, [b_2, \cdots [b_{m-1}, [b_m, T]_m]_{m-1} \cdots]_2\right]_1 \left(\overrightarrow{f}\right).$$

Inspired by [6, 8], we will introduce the commutators $\overline{T}^{(m)}_{\Pi \overrightarrow{b}}$ generated by m-linear Calderón-Zygmund operators $\overline{T}^{(m)}$ and local Campanato functions $\overrightarrow{b} = (b_1, \ldots, b_m)$

$$\overline{T}^{(m)}_{\Pi \overrightarrow{b}}\left(\overrightarrow{f}\right)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) \left[\prod_{i=1}^{m} [b_i(x) - b_i(y_i)] f_i(y_i)\right] d\overrightarrow{y},$$

where $K(x, y_1, \ldots, y_m)$ is a m-linear Calderón-Zygmund kernel, $b_i \in LC^{\{x_0\}}_{q_i,\lambda_i}(\mathbb{R}^n)$ (local Campanato spaces) for $0 \leq \lambda_i < \frac{1}{n}$, $i = 1, \ldots, m$. We would like to point out that $\overline{T}_b$ is the special case of $\overline{T}^{(m)}_{\Pi \overrightarrow{b}}$ with taking $m = 1$.

Closely related to the above results, in this paper in the case of $b_i \in LC^{\{x_0\}}_{q_i,\lambda_i}(\mathbb{R}^n)$ for $0 \leq \lambda_i < \frac{1}{n}$, $i = 1, \ldots, m$, we find the sufficient conditions on $(\varphi_1, \ldots \varphi_m, \varphi)$ which ensures the boundedness of the commutator operators $T^{(m)}_{\Pi \overrightarrow{b}}$ from $LM^{\{x_0\}}_{p_1,\varphi_1} \times \cdots \times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $LM^{\{x_0\}}_{p,\varphi}$, where $1 < p_i, q_i < \infty$, for $i = 1, \ldots, m$ such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m}$. In fact, in this paper the results of [2] and [9] (by taking $\Omega \equiv 1$ there) will be generalized to the multilinear case; we omit the details here.

Remark 1.13. *Our results in this paper remain true for the nonhomogeneous versions of local Campanato spaces* $LC_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ *for* $0 \leq \lambda < \frac{1}{n}$ *and generalized local Morrey spaces* $LM_{p,\varphi}^{\{x_0\}}$.

We now make some conventions. Throughout this paper, we use the symbol $A \lesssim B$ to denote that there exists a positive consant C such that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, we then write $A \approx B$ and say that A and B are equivalent. For a fixed $p \in [1,\infty)$, p' denotes the dual or conjugate exponent of p, namely, $p' = \frac{p}{p-1}$ and we use the convention $1' = \infty$ and $\infty' = 1$.

Our main results can be formulated as follows.

Theorem 1.14. *Let* $x_0 \in \mathbb{R}^n$, $1 < p_i, q_i < \infty$, *for* $i = 1, \ldots, m$ *such that* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m}$ *and* $\overrightarrow{b} \in LC_{q_i,\lambda_i}^{\{x_0\}}(\mathbb{R}^n)$ *for* $0 \leq \lambda_i < \frac{1}{n}$, $i = 1, \ldots, m$. *Let also,* $T^{(m)}$ $(m \in \mathbb{N})$ *be a multilinear operator satisfying condition (9), bounded from* $L_{p_1} \times \cdots \times L_{p_m}$ *into* L_p *for* $p_i > 1$ $(i = 1, \ldots, m)$. *Then the inequality*

$$\|T_{\Pi\overrightarrow{b}}^{(m)}\left(\overrightarrow{f}\right)\|_{L_p(B(x_0,r))} \lesssim \prod_{i=1}^{m} \|\overrightarrow{b}\|_{LC_{q_i,\lambda_i}^{\{x_0\}}} r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right)^m \prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\sum_{i=1}^{n}\frac{1}{p_i} - \sum_{i=1}^{n}\lambda_i\right)+1}} \tag{11}$$

holds for any ball $B(x_0,r)$ *and for all* $\overrightarrow{f} \in L_{p_1}^{loc}(\mathbb{R}^n) \times \cdots L_{p_m}^{loc}(\mathbb{R}^n)$.

Theorem 1.15. *Let* $x_0 \in \mathbb{R}^n$, $1 < p_i, q_i < \infty$, *for* $i = 1, \ldots, m$ *such that* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m}$ *and* $\overrightarrow{b} \in LC_{q_i,\lambda_i}^{\{x_0\}}(\mathbb{R}^n)$ *for* $0 \leq \lambda_i < \frac{1}{n}$, $i = 1, \ldots, m$. *Let also,* $T^{(m)}$ $(m \in \mathbb{N})$ *be a multilinear operator satisfying condition (9), bounded from* $L_{p_1} \times \cdots \times L_{p_m}$ *into* L_p *for* $p_i > 1$ $(i = 1, \ldots, m)$. *If functions* $\varphi, \varphi_i : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, $(i = 1, \ldots, m)$ *and* $(\varphi_1, \ldots \varphi_m, \varphi)$ *satisfies the condition*

$$\int_{r}^{\infty} \left(1 + \ln\frac{t}{r}\right)^m \frac{\operatorname*{essinf}_{t<\tau<\infty} \prod_{i=1}^{m} \varphi_i(x_0,\tau)\tau^{\frac{n}{p_i}}}{t^{n\left(\sum_{i=1}^{n}\frac{1}{p_i} - \sum_{i=1}^{n}\lambda_i\right)+1}} dt \leq C\varphi(x_0,r), \tag{12}$$

where C *does not depend on* r.

Then the operator $T_{\Pi\overrightarrow{b}}^{(m)}$ $(m \in \mathbb{N})$ *is bounded from product space* $LM_{p_1,\varphi_1}^{\{x_0\}} \times \cdots \times LM_{p_m,\varphi_m}^{\{x_0\}}$ *to* $LM_{p,\varphi}^{\{x_0\}}$ *for* $p_i > 1$ $(i = 1, \ldots, m)$. *Moreover, we have for* $p_i > 1$ $(i = 1, \ldots, m)$

$$\left\|T_{\Pi\overrightarrow{b}}^{(m)}\left(\overrightarrow{f}\right)\right\|_{LM_{p,\varphi}^{\{x_0\}}} \lesssim \prod_{i=1}^{m} \left\|\overrightarrow{b}\right\|_{LC_{q_i,\lambda_i}^{\{x_0\}}} \prod_{i=1}^{m} \left\|f_i\right\|_{LM_{p_i,\varphi_i}^{\{x_0\}}}. \tag{13}$$

For the m-sublinear commutator of the m-sublinear maximal operator

$$M_{\Pi\overrightarrow{b}}^{(m)}\left(\overrightarrow{f}\right)(x) = \sup_{t>0} \frac{1}{|B(x,t)|} \int_{B(x,t)} \prod_{i=1}^{m} \left[b_i(x) - b_i(y_i)\right] \left|f_i(y_i)\right| d\overrightarrow{y}$$

from Theorem 1.15 we get the following new results.

Corollary 1.16. *Let* $x_0 \in \mathbb{R}^n$, $1 < p_i, q_i < \infty$, *for* $i = 1, \ldots, m$ *such that* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m}$ *and* $\overrightarrow{b} \in LC_{q_i,\lambda_i}^{\{x_0\}}(\mathbb{R}^n)$ *for* $0 \leq \lambda_i < \frac{1}{n}$, $i = 1, \ldots, m$ *and* $(\varphi_1, \ldots \varphi_m, \varphi)$ *satisfies condition (12). Then, the operators* $M_{\Pi\overrightarrow{b}}^{(m)}$ *and* $\overline{T}_{\Pi\overrightarrow{b}}^{(m)}$ $(m \in \mathbb{N})$ *are bounded from product space* $LM_{p_1,\varphi_1}^{\{x_0\}} \times \cdots \times LM_{p_m,\varphi_m}^{\{x_0\}}$ *to* $LM_{p,\varphi}^{\{x_0\}}$ *for* $p_i > 1$ $(i = 1, \ldots, m)$.

Remark 1.17. *Note that, in the case of $m = 1$ Theorem 1.15 and Corollary 1.16 have been proved in [2, 9].*

Corollary 1.18. *Let $m \geq 2$, $T^{(m)}$ $(m \in \mathbb{N})$ be a multilinear operator satisfying condition (9), bounded from $L_{r_1} \times \cdots \times L_{r_m}$ into $L_{r,\infty}$ for fixed $1 \leq r_1, \cdots, r_m < \infty$ with $\frac{1}{r} = \frac{1}{r_1} + \cdots + \frac{1}{r_m}$. Let $x_0 \in \mathbb{R}^n$ and $\overrightarrow{b} \in BMO^m(\mathbb{R}^n)$. If functions $\varphi, \varphi_i : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, $(i = 1, \ldots, m)$ and $(\varphi_1, \ldots \varphi_m, \varphi)$ satisfies the condition*

$$\prod_{i=1}^{m} \int_{r}^{\infty} \left(1 + \ln \frac{t_i}{r}\right) \frac{\operatorname*{essinf}_{t_i<\tau_i<\infty} \varphi_i(x_0,\tau_i)\tau_i^{\frac{n}{p_i}}}{t_i^{\frac{n}{p_i}+1}} dt_i \leq C\varphi(x_0, r),$$

where C does not depend on r. Then the operator $T^{(m)}_{\Pi \overrightarrow{b}}$ $(m \in \mathbb{N})$ is bounded from product space $LM^{\{x_0\}}_{p_1,\varphi_1} \times \cdots \times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $LM^{\{x_0\}}_{p,\varphi}$ for any $1 < p_i < \infty$ $(i = 1, \ldots, m)$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $1 < p < \infty$. Moreover, we have

$$\left\| T^{(m)}_{\Pi \overrightarrow{b}} \left(\overrightarrow{f}\right) \right\|_{LM^{\{x_0\}}_{p,\varphi}} \lesssim \prod_{j=1}^{m} \|b_j\|_{BMO} \prod_{i=1}^{m} \|f_i\|_{LM^{\{x_0\}}_{p_i,\varphi_i}}.$$

Corollary 1.19. *Let $1 < p_i, q_i < \infty$, for $i = 1, \ldots, m$ such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\overrightarrow{b} \in BMO^m(\mathbb{R}^n)$ for $i = 1, \ldots, m$. Let also, $T^{(m)}$ $(m \in \mathbb{N})$ be a multilinear operator satisfying condition (9), bounded from $L_{p_1} \times \cdots \times L_{p_m}$ into L_p for $p_i > 1$ $(i = 1, \ldots, m)$. If functions $\varphi, \varphi_i : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, $(i = 1, \ldots, m)$ and $(\varphi_1, \ldots \varphi_m, \varphi)$ satisfies the condition*

$$\int_{r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^{m} \frac{\operatorname*{essinf}_{t<\tau<\infty} \prod_{i=1}^{m} \varphi_i(x,\tau)\tau^{\frac{n}{p_i}}}{t^{n \sum_{i=1}^{n} \frac{1}{p_i}+1}} dt \leq C\varphi(x, r), \tag{14}$$

where C does not depend on r.

Then the operator $T^{(m)}_{\Pi \overrightarrow{b}}$ $(m \in \mathbb{N})$ is bounded from product space $M_{p_1,\varphi_1} \times \cdots \times M_{p_m,\varphi_m}$ to $M_{p,\varphi}$ for $p_i > 1$ $(i = 1, \ldots, m)$. Moreover, we have for $p_i > 1$ $(i = 1, \ldots, m)$

$$\left\| T^{(m)}_{\Pi \overrightarrow{b}} \left(\overrightarrow{f}\right) \right\|_{M_{p,\varphi}} \lesssim \prod_{j=1}^{m} \|b_j\|_{BMO} \prod_{i=1}^{m} \|f_i\|_{M_{p_i,\varphi_i}}.$$

Corollary 1.20. *Let $1 < p_i, q_i < \infty$, for $i = 1, \ldots, m$ such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\overrightarrow{b} \in BMO^m(\mathbb{R}^n)$ for $i = 1, \ldots, m$ and also $(\varphi_1, \ldots \varphi_m, \varphi)$ satisfies condition (14). Then, the operators $M^{(m)}_{\Pi \overrightarrow{b}}$ and $\overline{T}^{(m)}_{\Pi \overrightarrow{b}}$ $(m \in \mathbb{N})$ are bounded from product space $M_{p_1,\varphi_1} \times \cdots \times M_{p_m,\varphi_m}$ to $M_{p,\varphi}$ for $p_i > 1$ $(i = 1, \ldots, m)$.*

Corollary 1.21. *Let $m \geq 2$, $T^{(m)}$ $(m \in \mathbb{N})$ be a multilinear operator satisfying condition (9), bounded from $L_{r_1} \times \cdots \times L_{r_m}$ into $L_{r,\infty}$ for fixed $1 \leq r_1, \cdots, r_m < \infty$ with $\frac{1}{r} = \frac{1}{r_1} + \cdots + \frac{1}{r_m}$. If $\overrightarrow{b} \in BMO^m(\mathbb{R}^n)$, then the operator $T^{(m)}_{\Pi \overrightarrow{b}}$ $(m \in \mathbb{N})$ is bounded from product space $M^{q_1}_{p_1} \times \cdots \times M^{q_m}_{p_m}$ to M^{q}_{p} for any $p_j \leq q_j < \infty$ $(j = 1, \ldots, m)$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$ and $1 < p \leq q < \infty$.*

2. Proofs of the main results

2.1. *Proof of Theorem 1.14.*

Proof. In order to simplify the proof, we consider only the situation when $m = 2$. Actually, a similar procedure works for all $m \in \mathbb{N}$. Thus, without loss of generality, it is sufficient to show that the conclusion holds for $T^{(2)}_{\Pi \overrightarrow{b}} \left(\overrightarrow{f}\right) = T^{(2)}_{(b_1,b_2)}(f_1, f_2)$.

We just consider the case $p_i, q_i > 1$ for $i = 1, 2$. For any $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r and $2B = B(x_0, 2r)$. Thus, we have the following decomposition,

$$\begin{aligned} T^{(2)}_{(b_1,b_2)}(f_1, f_2)(x) &= [(b_1(x) - \{b_1\}_B)][(b_2(x) - \{b_2\}_B)]\, T^{(2)}(f_1, f_2)(x) \\ &- [(b_1(x) - \{b_1\}_B)]\, T^{(2)}[f_1, (b_2(\cdot) - \{b_2\}_B) f_2](x) \\ &- [(b_2(x) - \{b_2\}_B)]\, T^{(2)}[(b_1(\cdot) - \{b_1\}_B) f_1, f_2](x) \\ &+ T^{(2)}[(b_1(\cdot) - \{b_1\}_B) f_1, (b_2(\cdot) - \{b_2\}_B) f_2](x) \\ &\equiv H_1(x) + H_2(x) + H_3(x) + H_4(x). \end{aligned}$$

Thus,

$$\left\| T^{(2)}_{(b_1,b_2)}(f_1, f_2) \right\|_{L_p(B(x_0,r))} = \left(\int_B \left| T^{(2)}_{(b_1,b_2)}(f_1, f_2)(x) \right|^p dx \right)^{\frac{1}{p}} \le \sum_{i=1}^{4} \left(\int_B |H_i(x)|^p dx \right)^{\frac{1}{p}} = \sum_{i=1}^{4} G_i. \tag{15}$$

One observes that the estimate of G_2 is analogous to that of G_3. Thus, we will only estimate G_1, G_2 and G_4.

Indeed, we also decompose f_i as $f_i(y_i) = f_i(y_i)\chi_{2B} + f_i(y_i)\chi_{(2B)^C}$ for $i = 1, 2$. And, we write $f_1 = f_1^0 + f_1^\infty$ and $f_2 = f_2^0 + f_2^\infty$, where $f_i^0 = f_i\chi_{2B}$, $f_i^\infty = f_i\chi_{(2B)^C}$, for $i = 1, 2$.

(i) For $G_1 = \left\| T^{(2)}_{(b_1,b_2)}\left(f_1^0, f_2^0\right) \right\|_{L_p(B(x_0,r))}$, we decompose it into four parts as follows:

$$\begin{aligned} G_1 &\lesssim \left\| [(b_1 - \{b_1\}_B)][(b_2 - \{b_2\}_B)]\, T^{(2)}\left(f_1^0, f_2^0\right) \right\|_{L_p(B(x_0,r))} \\ &+ \left\| [(b_1 - \{b_1\}_B)]\, T^{(2)}\left[f_1^0, (b_2 - \{b_2\}_B) f_2^0\right] \right\|_{L_p(B(x_0,r))} \\ &+ \left\| [(b_2 - \{b_2\}_B)]\, T^{(2)}\left[(b_1 - \{b_1\}_B) f_1^0, f_2^0\right] \right\|_{L_p(B(x_0,r))} \\ &+ \left\| T^{(2)}\left[(b_1 - \{b_1\}_B) f_1^0, (b_2 - \{b_2\}_B) f_2^0\right] \right\|_{L_p(B(x_0,r))} \\ &\equiv G_{11} + G_{12} + G_{13} + G_{14}. \end{aligned}$$

Firstly, $1 < \overline{p}, \overline{q} < \infty$, such that $\frac{1}{\overline{p}} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{\overline{q}} = \frac{1}{q_1} + \frac{1}{q_2}$. Then, using Hölder's inequality and from the boundedness of $T^{(2)}$ from $L_{p_1} \times L_{p_2}$ into $L_{\overline{p}}$ (see Theorem 1.12) it follows that:

$$\begin{aligned} G_{11} &\lesssim \|[(b_1 - \{b_1\}_B)][(b_2 - \{b_2\}_B)]\|_{L_{\overline{q}}(B)} \left\| T^{(2)}\left(f_1^0, f_2^0\right) \right\|_{L_{\overline{p}}(B)} \\ &\lesssim \|(b_1 - \{b_1\}_B)\|_{L_{q_1}(B)} \|(b_2 - \{b_2\}_B)\|_{L_{q_2}(B)} \left\| f_1 \right\|_{L_{p_1}(2B)} \left\| f_2 \right\|_{L_{p_2}(2B)} \\ &\lesssim \|(b_1 - \{b_1\}_B)\|_{L_{q_1}(B)} \|(b_2 - \{b_2\}_B)\|_{L_{q_2}(B)}\, r^{n\left(\frac{1}{p_1} + \frac{1}{p_2}\right)} \int_{2r}^{\infty} \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\frac{1}{p_1} + \frac{1}{p_2}\right)+1}} \\ &\lesssim \prod_{i=1}^{2} \|\overrightarrow{b}\|_{LC^{\{x_0\}}_{q_i,\lambda_i}} r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\left(\frac{1}{p_1} + \frac{1}{p_2}\right) - (\lambda_1 + \lambda_2)\right)+1}}. \end{aligned}$$

Secondly, for G_{12}, let $1 < \tau < \infty$, such that $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{\tau}$. Then similar to the estimates for G_{11}, we have

$$\begin{aligned} G_{12} &\lesssim \|(b_1 - \{b_1\}_B)\|_{L_{q_1}(B)} \left\| T^{(2)}\left[f_1^0, (b_2 - \{b_2\}_B) f_2^0\right] \right\|_{L_\tau(B)} \\ &\lesssim \|(b_1 - \{b_1\}_B)\|_{L_{q_1}(B)} \left\| f_1^0 \right\|_{L_{p_1}(\mathbb{R}^n)} \left\| (b_2 - \{b_2\}_{2B}) f_2^0 \right\|_{L_k(\mathbb{R}^n)} \\ &\lesssim \|(b_1 - \{b_1\}_B)\|_{L_{q_1}(B)} \|(b_2 - \{b_2\}_B)\|_{L_{q_2}(2B)} \left\| f_1 \right\|_{L_{p_1}(2B)} \left\| f_2 \right\|_{L_{p_2}(2B)}, \end{aligned}$$

where $1 < k < \infty$, such that $\frac{1}{k} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{\tau} - \frac{1}{p_1}$.

Hence, we get

$$G_{12} \lesssim \prod_{i=1}^{2} \|\overrightarrow{b}\|_{LC^{\{x_0\}}_{q_i,\lambda_i}} r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-(\lambda_1+\lambda_2)\right)+1}}.$$

Similarly, G_{13} has the same estimate above, here we omit the details, thus following inequality

$$G_{13} \lesssim \prod_{i=1}^{2} \|\overrightarrow{b}\|_{LC^{\{x_0\}}_{q_i,\lambda_i}} r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-(\lambda_1+\lambda_2)\right)+1}}$$

is valid.

At last, we consider the term G_{14}. Let $1 < \tau_1, \tau_2 < \infty$, such that $\frac{1}{\tau_1} = \frac{1}{p_1} + \frac{1}{q_1}$ and $\frac{1}{\tau_2} = \frac{1}{p_2} + \frac{1}{q_2}$. It is easy to see that $\frac{1}{p} = \frac{1}{\tau_1} + \frac{1}{\tau_2}$. Then by the boundedness of $T^{(2)}$ from $L_{\tau_1} \times L_{\tau_2}$ into L_p (see Theorem 1.12), Hölder's inequality and (8), we obtain

$$\begin{aligned} G_{14} &\lesssim \left\|(b_1 - \{b_1\}_B) f_1^0\right\|_{L_{\tau_1}(\mathbb{R}^n)} \left\|(b_2 - \{b_2\}_B) f_2^0\right\|_{L_{\tau_2}(\mathbb{R}^n)} \\ &\lesssim \|(b_1 - \{b_1\}_B)\|_{L_{q_1}(2B)} \|(b_2 - \{b_2\}_B)\|_{L_{q_2}(2B)} \left\|f_1\right\|_{L_{p_1}(2B)} \left\|f_2\right\|_{L_{p_2}(2B)} \\ &\lesssim \prod_{i=1}^{2} \|\overrightarrow{b}\|_{LC^{\{x_0\}}_{q_i,\lambda_i}} r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-(\lambda_1+\lambda_2)\right)+1}}. \end{aligned}$$

Combining all the estimates of G_{11}, G_{12}, G_{13}, G_{14}; there is

$$\begin{aligned} G_1 = \left\|T^{(2)}_{(b_1,b_2)}\left(f_1^0, f_2^0\right)\right\|_{L_p(B(x_0,r))} &\lesssim \prod_{i=1}^{2} \|\overrightarrow{b}\|_{LC^{\{x_0\}}_{q_i,\lambda_i}} r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \\ &\times \frac{dt}{t^{n\left(\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-(\lambda_1+\lambda_2)\right)+1}}. \end{aligned}$$

(ii) For $G_2 = \left\|T^{(2)}_{(b_1,b_2)}\left(f_1^0, f_2^\infty\right)\right\|_{L_p(B(x_0,r))}$, we also write

$$\begin{aligned} G_2 &\lesssim \left\|[(b_1 - \{b_1\}_B)]\,[(b_2 - \{b_2\}_B)]\, T^{(2)}\left(f_1^0, f_2^\infty\right)\right\|_{L_p(B(x_0,r))} \\ &+ \left\|[(b_1 - \{b_1\}_B)]\, T^{(2)}\left[f_1^0, (b_2 - \{b_2\}_B) f_2^\infty\right]\right\|_{L_p(B(x_0,r))} \\ &+ \left\|[(b_2 - \{b_2\}_B)]\, T^{(2)}\left[(b_1 - \{b_1\}_B) f_1^0, f_2^\infty\right]\right\|_{L_p(B(x_0,r))} \\ &+ \left\|T^{(2)}\left[(b_1 - \{b_1\}_B) f_1^0, (b_2 - \{b_2\}_B) f_2^\infty\right]\right\|_{L_p(B(x_0,r))} \\ &\equiv G_{21} + G_{22} + G_{23} + G_{24}. \end{aligned}$$

Let $1<\overline{p},\overline{q}<\infty$, such that $\frac{1}{\overline{p}}=\frac{1}{p_1}+\frac{1}{p_2}$ and $\frac{1}{\overline{q}}=\frac{1}{q_1}+\frac{1}{q_2}$. Then, using Hölder's inequality we have

$$\begin{aligned}
G_{21} &= \left\|\left[(b_1-\{b_1\}_B)\right]\left[(b_2-\{b_2\}_B)\right]T^{(2)}\left(f_1^0,f_2^\infty\right)\right\|_{L_p(B(x_0,r))} \\
&\lesssim \|\left[(b_1-\{b_1\}_B)\right]\left[(b_2-\{b_2\}_B)\right]\|_{L_{\overline{q}}(B)}\left\|T^{(2)}\left(f_1^0,f_2^\infty\right)\right\|_{L_{\overline{p}}(B)} \\
&\lesssim \|(b_1-\{b_1\}_B)\|_{L_{q_1}(B)}\|(b_2-\{b_2\}_B)\|_{L_{q_2}(B)}\, r^{\frac{n}{\overline{p}}}\int_{2r}^{\infty}\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{\frac{n}{\overline{p}}+1}} \\
&\lesssim \prod_{i=1}^{2}\|\overrightarrow{b}\|_{LC^{\{x_0\}}_{q_i,\lambda_i}} r^{n\left(\frac{1}{q_1}+\frac{1}{q_2}\right)+n(\lambda_1+\lambda_2)} r^{n\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\frac{1}{p_1}+\frac{1}{p_2}\right)+1}} \\
&\lesssim \prod_{i=1}^{2}\|\overrightarrow{b}\|_{LC^{\{x_0\}}_{q_i,\lambda_i}} r^{\frac{n}{p}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-(\lambda_1+\lambda_2)\right)+1}},
\end{aligned}$$

where in the second inequality we have used the following fact:

It is clear that $\left|(x_0-y_1,\ x_0-y_2)\right|^{2n}\geq\left|x_0-y_2\right|^{2n}$. By the condition (9) with $m=2$ and Hölder's inequality, we have

$$\begin{aligned}
\left|T^{(2)}\left(f_1^0,f_2^\infty\right)(x)\right| &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\left|f_1^0(y_1)\right|\left|f_2^\infty(y_2)\right|}{\left|(x-y_1,x-y_2)\right|^{2n}}dy_1dy_2 \\
&\lesssim \int_{2B}\left|f_1(y_1)\right|dy_1\int_{(2B)^C}\frac{\left|f_2(y_2)\right|}{\left|x_0-y_2\right|^{2n}}dy_2 \\
&\approx \int_{2B}\left|f_1(y_1)\right|dy_1\int_{(2B)^C}\left|f_2(y_2)\right|\int_{\left|x_0-y_2\right|}^{\infty}\frac{dt}{t^{2n+1}}dy_2 \\
&\lesssim \left\|f_1\right\|_{L_{p_1}(2B)}|2B|^{1-\frac{1}{p_1}}\int_{2r}^{\infty}\left\|f_2\right\|_{L_{p_2}(B(x_0,t))}|B(x_0,t)|^{1-\frac{1}{p_2}}\frac{dt}{t^{2n+1}} \\
&\lesssim \int_{2r}^{\infty}\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{\frac{n}{p}+1}},
\end{aligned}$$

where $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. Thus, the inequality

$$\left\|T^{(2)}\left(f_1^0,f_2^\infty\right)\right\|_{L_{\overline{p}}(B(x_0,r))}\lesssim r^{\frac{n}{p}}\int_{2r}^{\infty}\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{\frac{n}{p}+1}}$$

is valid.

On the other hand, for the estimates used in G_{22}, G_{23}, we have to prove the below inequality:

$$\left|T^{(2)}\left[f_1^0,(b_2(\cdot)-\{b_2\}_B)f_2^\infty\right](x)\right|\lesssim\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-\lambda_2\right)+1}}. \tag{16}$$

Indeed, it is clear that $\left|(x_0-y_1,\ x_0-y_2)\right|^{2n} \geq \left|x_0-y_2\right|^{2n}$. Moreover, using the conditions (10) and (9) with $m=2$, we have

$$\left|T^{(2)}\left[f_1^0,(b_2(\cdot)-\{b_2\}_B)f_2^\infty\right](x)\right|$$
$$\lesssim \int\limits_{2B}\left|f_1(y_1)\right|dy_1 \int\limits_{(2B)^C}\frac{\left|b_2(y_2)-\{b_2\}_B\right|\left|f_2(y_2)\right|}{\left|x_0-y_2\right|^{2n}}dy_2.$$

It's obvious that

$$\int\limits_{2B}\left|f_1(y_1)\right|dy_1 \lesssim \left\|f_1\right\|_{L_{p_1}(2B)}|2B|^{1-\frac{1}{p_1}}, \tag{17}$$

and using Hölder's inequality and by (8)

$$\int\limits_{(2B)^C}\frac{\left|b_2(y_2)-\{b_2\}_B\right|\left|f_2(y_2)\right|}{\left|x_0-y_2\right|^{2n}}dy_2$$
$$\lesssim \int\limits_{(2B)^C}\left|b_2(y_2)-\{b_2\}_B\right|\left|f_2(y_2)\right|\left[\int\limits_{|x_0-y_2|}^{\infty}\frac{dt}{t^{2n+1}}\right]dy_2$$
$$\lesssim \int\limits_{2r}^{\infty}\left\|b_2(y_2)-\{b_2\}_{B(x_0,t)}\right\|_{L_{q_2}(B(x_0,t))}\left\|f_2\right\|_{L_{p_2}(B(x_0,t))}|B(x_0,t)|^{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}\frac{dt}{t^{2n+1}}$$
$$+\left|\{b_2\}_{B(x_0,t)}-\{b_2\}_{B(x_0,r)}\right|\left\|f_2\right\|_{L_{p_2}(B(x_0,t))}|B(x_0,t)|^{1-\frac{1}{p_2}}\frac{dt}{t^{2n+1}}$$
$$\lesssim \|b_2\|_{LC_{q_2,\lambda_2}^{\{x_0\}}}\int\limits_{2r}^{\infty}|B(x_0,t)|^{\frac{1}{q_2}+\lambda_2}\left\|f_2\right\|_{L_{p_2}(B(x_0,t))}|B(x_0,t)|^{1-\left(\frac{1}{p_2}+\frac{1}{q_2}\right)}\frac{dt}{t^{2n+1}}$$
$$\lesssim \|b_2\|_{LC_{q_2,\lambda_2}^{\{x_0\}}}\int\limits_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)|B(x_0,t)|^{\lambda_2}\left\|f_2\right\|_{L_{p_2}(B(x_0,t))}|B(x_0,t)|^{1-\frac{1}{p_2}}\frac{dt}{t^{2n+1}}$$

$$\lesssim \|b_2\|_{LC_{q_2,\lambda_2}^{\{x_0\}}}\int\limits_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\left\|f_2\right\|_{L_{p_2}(B(x_0,t))}\frac{dt}{t^{n\left(1+\frac{1}{p_2}-\lambda_2\right)+1}}. \tag{18}$$

Hence, by (17) and (18), it follows that:

$$\left|T^{(2)}\left[f_1^0,(b_2(\cdot)-\{b_2\}_B)f_2^\infty\right](x)\right|$$
$$\lesssim \|b_2\|_{LC_{q_2,\lambda_2}^{\{x_0\}}}\left\|f_1\right\|_{L_{p_1}(2B)}|2B|^{1-\frac{1}{p_1}}\int\limits_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\left\|f_2\right\|_{L_{p_2}(B(x_0,t))}\frac{dt}{t^{n\left(1+\frac{1}{p_2}-\lambda_2\right)+1}}$$
$$\lesssim \|b_2\|_{LC_{q_2,\lambda_2}^{\{x_0\}}}\int\limits_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-\lambda_2\right)+1}}.$$

This completes the proof of inequality (16).

Thus, let $1 < \tau < \infty$, such that $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{\tau}$. Then, to estimate G_{22}, similar to the estimates for G_{11}, using Hölder's inequality and from (18), we get

$$\begin{aligned} G_{22} &= \left\| \left[(b_1 - \{b_1\}_B) \right] T^{(2)} \left[f_1^0, (b_2 - \{b_2\}_B) f_2^\infty \right] \right\|_{L_p(B(x_0,r))} \\ &\lesssim \| (b_1 - \{b_1\}_B) \|_{L_{q_1}(B)} \left\| T^{(2)} \left[f_1^0, (b_2 - \{b_2\}_B) f_2^\infty \right] \right\|_{L_\tau(B)} \\ &\lesssim \prod_{i=1}^{2} \| \vec{b} \|_{LC^{\{x_0\}}_{q_i,\lambda_i}} |B|^{\lambda_1 + \frac{1}{q_1} + \frac{1}{\tau}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\left(\frac{1}{p_1} + \frac{1}{p_2}\right) - \lambda_2\right) + 1}} \\ &\lesssim \prod_{i=1}^{2} \| \vec{b} \|_{LC^{\{x_0\}}_{q_i,\lambda_i}} r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\left(\frac{1}{p_1} + \frac{1}{p_2}\right) - (\lambda_1 + \lambda_2)\right) + 1}}. \end{aligned}$$

Similarly, G_{23} has the same estimate above, here we omit the details, thus the inequality

$$\begin{aligned} G_{23} &= \left\| \left[(b_2 - \{b_2\}_B) \right] T^{(2)} \left[(b_1 - \{b_1\}_B) f_1^0, f_2^\infty \right] \right\|_{L_p(B(x_0,r))} \\ &\lesssim \prod_{i=1}^{2} \| \vec{b} \|_{LC^{\{x_0\}}_{q_i,\lambda_i}} r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\left(\frac{1}{p_1} + \frac{1}{p_2}\right) - (\lambda_1 + \lambda_2)\right) + 1}} \end{aligned}$$

is valid.

Now we turn to estimate G_{24}. Similar to (18), we have to prove the following estimate for G_{24}:

$$\begin{aligned} &\left| T^{(2)} \left[(b_1 - \{b_1\}_B) f_1^0, (b_2 - \{b_2\}_B) f_2^\infty \right] (x) \right| \\ &\lesssim \prod_{i=1}^{2} \| \vec{b} \|_{LC^{\{x_0\}}_{q_i,\lambda_i}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\left(\frac{1}{p_1} + \frac{1}{p_2}\right) - (\lambda_1 + \lambda_2)\right) + 1}}. \end{aligned} \tag{19}$$

Firstly, using the condition (9) with $m = 2$, we have

$$\begin{aligned} &\left| T^{(2)} \left[(b_1 - \{b_1\}_B) f_1^0, (b_2 - \{b_2\}_B) f_2^\infty \right] (x) \right| \\ &\lesssim \int_{2B} \left| b_1(y_1) - \{b_1\}_B \right| \left| f_1(y_1) \right| dy_1 \int_{(2B)^C} \frac{\left| b_2(y_2) - \{b_2\}_B \right| \left| f_2(y_2) \right|}{\left| x_0 - y_2 \right|^{2n}} dy_2. \end{aligned}$$

It's obvious that

$$\int_{2B} \left| b_1(y_1) - \{b_1\}_B \right| \left| f_1(y_1) \right| dy_1 \lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} |B|^{\lambda_1 + 1 - \frac{1}{p_1}} \left\| f_1 \right\|_{L_{p_1}(2B)}. \tag{20}$$

Then, by (18) and (20) we get (19). This completes the proof of inequality (19). Therefore, by (19) we deduce that

$$\begin{aligned} G_{24} &= \left\| T^{(2)} \left[(b_1 - \{b_1\}_B) f_1^0, (b_2 - \{b_2\}_B) f_2^\infty \right] \right\|_{L_p(B(x_0,r))} \\ &\lesssim \prod_{i=1}^{2} \| \vec{b} \|_{LC^{\{x_0\}}_{q_i,\lambda_i}} r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\left(\frac{1}{p_1} + \frac{1}{p_2}\right) - (\lambda_1 + \lambda_2)\right) + 1}}. \end{aligned}$$

Considering estimates G_{21}, G_{22}, G_{23}, G_{24} together, we get the desired conclusion

$$G_2 = \left\| T^{(2)}_{(b_1,b_2)}\left(f_1^0, f_2^\infty\right)\right\|_{L_p(B(x_0,r))} \lesssim \prod_{i=1}^{2} \|\overrightarrow{b}\|_{LC^{\{x_0\}}_{q_i,\lambda_i}} r^{\frac{n}{p}} \int\limits_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \times \frac{dt}{t^{n\left(\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-(\lambda_1+\lambda_2)\right)+1}}.$$

Similar to G_2, we can also get the estimates for F_3,

$$G_3 = \left\| T^{(2)}_{(b_1,b_2)}\left(f_1^\infty, f_2^0\right)\right\|_{L_p(B(x_0,r))} \lesssim \prod_{i=1}^{2} \|\overrightarrow{b}\|_{LC^{\{x_0\}}_{q_i,\lambda_i}} r^{\frac{n}{p}} \int\limits_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \times \frac{dt}{t^{n\left(\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-(\lambda_1+\lambda_2)\right)+1}}.$$

Finally, for $G_4 = \left\| T^{(2)}_{(b_1,b_2)}\left(f_1^\infty, f_2^\infty\right)\right\|_{L_p(B(x_0,r))}$, we write

$$\begin{aligned} G_4 &\lesssim \left\| \left[(b_1-\{b_1\}_B)\right]\left[(b_2-\{b_2\}_B)\right] T^{(2)}\left(f_1^\infty, f_2^\infty\right)\right\|_{L_p(B(x_0,r))} \\ &+ \left\| \left[(b_1-\{b_1\}_B)\right] T^{(2)}\left[f_1^\infty, (b_2-\{b_2\}_B) f_2^\infty\right]\right\|_{L_p(B(x_0,r))} \\ &+ \left\| \left[(b_2-\{b_2\}_B)\right] T^{(2)}\left[(b_1-\{b_1\}_B) f_1^\infty, f_2^\infty\right]\right\|_{L_p(B(x_0,r))} \\ &+ \left\| T^{(2)}\left[(b_1-\{b_1\}_B) f_1^\infty, (b_2-\{b_2\}_B) f_2^\infty\right]\right\|_{L_p(B(x_0,r))} \\ &\equiv G_{41}+G_{42}+G_{43}+G_{44}. \end{aligned}$$

Now, let us estimate G_{41}, G_{42}, G_{43}, G_{44}, respectively.

For the term G_{41}, let $1<\tau<\infty$, such that $\frac{1}{p} = \left(\frac{1}{q_1}+\frac{1}{q_2}\right)+\frac{1}{\tau}$, $\frac{1}{\tau} = \frac{1}{p_1}+\frac{1}{p_2}$. Then, by Hölder's inequality we get

$$\begin{aligned} G_{41} &= \left\| \left[(b_1-\{b_1\}_B)\right]\left[(b_2-\{b_2\}_B)\right] T^{(2)}\left(f_1^\infty, f_2^\infty\right)\right\|_{L_p(B(x_0,r))} \\ &\lesssim \|(b_1-\{b_1\}_B)\|_{L_{q_1}(B)} \|(b_2-\{b_2\}_B)\|_{L_{q_2}(B)} \left\| T^{(2)}\left(f_1^\infty, f_2^\infty\right)\right\|_{L_\tau(B)} \\ &\lesssim \prod_{i=1}^{2} \|\overrightarrow{b}\|_{LC^{\{x_0\}}_{q_i,\lambda_i}} |B|^{(\lambda_1+\lambda_2)+\left(\frac{1}{q_1}+\frac{1}{q_2}\right)} r^{\frac{n}{\tau}} \int\limits_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{\frac{n}{\tau}+1}} \\ &\lesssim \prod_{i=1}^{2} \|\overrightarrow{b}\|_{LC^{\{x_0\}}_{q_i,\lambda_i}} r^{\frac{n}{p}} \int\limits_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t^{n\left(\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-(\lambda_1+\lambda_2)\right)+1}}, \end{aligned}$$

where in the second inequality we have used the following fact:

Noting that $\left|(x_0-y_1,\ x_0-y_2)\right|^{2n} \geq \left|x_0-y_1\right|^n \left|x_0-y_2\right|^n$. Using the condition (9) with $m=2$ and by

Hölder's inequality, we get

$$
\begin{aligned}
&\left|T_{\alpha}^{(2)}\left(f_1^{\infty}, f_2^{\infty}\right)(x)\right| \\
&\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left|f_1(y_1)\chi_{(2B)^c}\right|\left|f_2(y_2)\chi_{(2B)^c}\right|}{\left|(x_0-y_1, x_0-y_2)\right|^{2n}} dy_1 dy_2 \\
&\lesssim \int_{(2B)^c}\int_{(2B)^c} \frac{\left|f_1(y_1)\right|\left|f_2(y_2)\right|}{\left|x_0-y_1\right|^{n}\left|x_0-y_2\right|^{n}} dy_1 dy_2 \\
&\lesssim \sum_{j=1}^{\infty}\prod_{i=1}^{2}\int_{B\left(x_0,2^{j+1}r\right)\setminus B\left(x_0,2^{j}r\right)} \frac{\left|f_i(y_i)\right|}{\left|x_0-y_i\right|^{n}} dy_i \\
&\lesssim \sum_{j=1}^{\infty}\prod_{i=1}^{2}\left(2^{j}r\right)^{-n}\int_{B\left(x_0,2^{j+1}r\right)} \left|f_i(y_i)\right| dy_i \\
&\lesssim \sum_{j=1}^{\infty}\left(2^{j}r\right)^{-2n}\prod_{i=1}^{2}\left\|f_i\right\|_{L_{p_i}\left(B\left(x_0,2^{j+1}r\right)\right)}\left|B\left(x_0,2^{j+1}r\right)\right|^{1-\frac{1}{p_i}} \\
&\lesssim \sum_{j=1}^{\infty}\int_{2^{j+1}r}^{2^{j+2}r}\left(2^{j+1}r\right)^{-2n-1}\prod_{i=1}^{2}\left\|f_i\right\|_{L_{p_i}\left(B\left(x_0,2^{j+1}r\right)\right)}\left|B\left(x_0,2^{j+1}r\right)\right|^{1-\frac{1}{p_i}} dt \\
&\lesssim \sum_{j=1}^{\infty}\int_{2^{j+1}r}^{2^{j+2}r}\prod_{i=1}^{2}\left\|f_i\right\|_{L_{p_i}(B(x_0,t))}\left|B(x_0,t)\right|^{1-\frac{1}{p_i}}\frac{dt}{t^{2n+1}} \\
&\lesssim \int_{2r}^{\infty}\prod_{i=1}^{2}\left\|f_i\right\|_{L_{p_i}(B(x_0,t))}\left|B(x_0,t)\right|^{2-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}\frac{dt}{t^{2n+1}} \\
&\lesssim \int_{2r}^{\infty}\left\|f_1\right\|_{L_{p_1}(B(x_0,t))}\left\|f_2\right\|_{L_{p_2}(B(x_0,t))}\frac{dt}{t^{\frac{n}{\tau}+1}},
\end{aligned}
$$

where $\frac{1}{\tau}=\frac{1}{p_1}+\frac{1}{p_2}$. Thus, for $p_1, p_2 \in [1,\infty)$ the inequality

$$
\left\|T^{(2)}\left(f_1^{\infty}, f_2^{\infty}\right)\right\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{\tau}}\int_{2r}^{\infty}\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{\frac{n}{\tau}+1}}
$$

is valid.

For the terms G_{42}, G_{43}, similar to the estimates used for (16), we have to prove the following inequality:

$$
\left|T^{(2)}\left[f_1^{\infty}, (b_2-\{b_2\}_B) f_2^{\infty}\right](x)\right| \lesssim \|b_2\|_{LC_{q_2,\lambda_2}^{\{x_0\}}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-\lambda_2\right)+1}}. \tag{21}
$$

Indeed, noting that $\left|(x_0-y_1,\ x_0-y_2)\right|^{2n} \geq \left|x_0-y_1\right|^{n}\left|x_0-y_2\right|^{n}$. Recalling the estimates used for G_{22}, G_{23},

G_{24} and also using the condition (9) with $m = 2$, we have

$$\begin{aligned}
&\left|T^{(2)}\left[f_1^{\infty},(b_2-\{b_2\}_B)\,f_2^{\infty}\right](x)\right| \\
&\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\left|b_2(y_2)-\{b_2\}_B\right|\left|f_1(y_1)\,\chi_{(2B)^C}\right|\left|f_2(y_2)\,\chi_{(2B)^C}\right|}{\left|(x_0-y_1,\ x_0-y_2)\right|^{2n}}dy_1dy_2 \\
&\lesssim \int_{(2B)^C}\int_{(2B)^C}\frac{\left|b_2(y_2)-\{b_2\}_B\right|\left|f_1(y_1)\right|\left|f_2(y_2)\right|}{\left|x_0-y_1\right|^n\left|x_0-y_2\right|^n}dy_1dy_2 \\
&\lesssim \sum_{j=1}^{\infty}\int_{B(x_0,2^{j+1}r)\setminus B(x_0,2^jr)}\frac{\left|f_1(y_1)\right|}{\left|x_0-y_1\right|^n}dy_1\int_{B(x_0,2^{j+1}r)\setminus B(x_0,2^jr)}\frac{\left|b_2(y_2)-\{b_2\}_B\right|\left|f_2(y_2)\right|}{\left|x_0-y_2\right|^n}dy_2 \\
&\lesssim \sum_{j=1}^{\infty}\left(2^jr\right)^{-2n}\int_{B(x_0,2^{j+1}r)}\left|f_1(y_1)\right|dy_1\int_{B(x_0,2^{j+1}r)}\left|b_2(y_2)-\{b_2\}_B\right|\left|f_2(y_2)\right|dy_2.
\end{aligned}$$

It's obvious that

$$\int_{B(x_0,2^{j+1}r)}\left|f_1(y_1)\right|dy_1 \lesssim \left\|f_1\right\|_{L_{p_1}(B(x_0,2^{j+1}r))}\left|B\left(x_0,2^{j+1}r\right)\right|^{1-\frac{1}{p_1}}, \tag{22}$$

and using Hölder's inequality and by (8)

$$\begin{aligned}
&\int_{B(x_0,2^{j+1}r)}\left|b_2(y_2)-\{b_2\}_B\right|\left|f_2(y_2)\right|dy_2 \\
&\lesssim \left\|b_2(y_2)-\{b_2\}_{B(x_0,2^{j+1}r)}\right\|_{L_{q_2}(B(x_0,2^{j+1}r))}\left\|f_2\right\|_{L_{p_2}(B(x_0,2^{j+1}r))}\left|B\left(x_0,2^{j+1}r\right)\right|^{1-\left(\frac{1}{p_2}+\frac{1}{q_2}\right)} \\
&+\left|\{b_2\}_{B(x_0,2^{j+1}r)}-\{b_2\}_{B(x_0,r)}\right|\left\|f_2\right\|_{L_{p_2}(B(x_0,2^{j+1}r))}\left|B\left(x_0,2^{j+1}r\right)\right|^{1-\frac{1}{p_2}} \\
&\lesssim \|b_2\|_{LC_{q_2,\lambda_2}^{\{x_0\}}}\left|B\left(x_0,2^{j+1}r\right)\right|^{\frac{1}{q_2}+\lambda_2}\left\|f_2\right\|_{L_{p_2}(B(x_0,2^{j+1}r))}\left|B\left(x_0,2^{j+1}r\right)\right|^{1-\left(\frac{1}{p_2}+\frac{1}{q_2}\right)} \\
&+\|b_2\|_{LC_{q_2,\lambda_2}^{\{x_0\}}}\left(1+\ln\frac{2^{j+1}r}{r}\right)\left|B\left(x_0,2^{j+1}r\right)\right|^{\lambda_2}\left\|f_2\right\|_{L_{p_2}(B(x_0,2^{j+1}r))}\left|B\left(x_0,2^{j+1}r\right)\right|^{1-\frac{1}{p_2}}
\end{aligned}$$

$$\lesssim \|b_2\|_{LC_{q_2,\lambda_2}^{\{x_0\}}}\int_{2r}^{\infty}\left(1+\ln\frac{2^{j+1}r}{r}\right)^2\left|B\left(x_0,2^{j+1}r\right)\right|^{\lambda_2-\frac{1}{p_2}+1}\left\|f_2\right\|_{L_{p_2}(B(x_0,2^{j+1}r))}. \tag{23}$$

Hence, by (22) and (23), it follows that:

$$
\begin{aligned}
&\left|T^{(2)}\left[f_1^{\infty},(b_2-\{b_2\}_B)f_2^{\infty}\right](x)\right| \\
&\lesssim \sum_{j=1}^{\infty}\left(2^j r\right)^{-2n}\int\limits_{B(x_0,2^{j+1}r)}\left|f_1(y_1)\right|dy_1\int\limits_{B(x_0,2^{j+1}r)}\left|b_2(y_2)-\{b_2\}_B\right|\left|f_2(y_2)\right|dy_2 \\
&\lesssim \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\sum_{j=1}^{\infty}\left(2^j r\right)^{-2n}\left(1+\ln\frac{2^{j+1}r}{r}\right)^2\left|B\left(x_0,2^{j+1}r\right)\right|^{\lambda_2-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)+2}\prod_{i=1}^{2}\left\|f_i\right\|_{L_{p_i}(B(x_0,2^{j+1}r))} \\
&\lesssim \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\sum_{j=1}^{\infty}\int\limits_{2^{j+1}r}^{2^{j+2}r}\left(2^{j+1}r\right)^{-2n-1}\left(1+\ln\frac{2^{j+1}r}{r}\right)^2\left|B\left(x_0,2^{j+1}r\right)\right|^{\lambda_2-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)+2} \\
&\times\prod_{i=1}^{2}\left\|f_i\right\|_{L_{p_i}(B(x_0,2^{j+1}r))}dt \\
&\lesssim \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\sum_{j=1}^{\infty}\int\limits_{2^{j+1}r}^{2^{j+2}r}\left(1+\ln\frac{2^{j+1}r}{r}\right)^2\left|B\left(x_0,2^{j+1}r\right)\right|^{\lambda_2-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)+2}\prod_{i=1}^{2}\left\|f_i\right\|_{L_{p_i}(B(x_0,2^{j+1}r))}\frac{dt}{t^{2n+1}} \\
&\lesssim \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\int\limits_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\left|B(x_0,t)\right|^{\lambda_2-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)+2}\prod_{i=1}^{2}\left\|f_i\right\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{2n+1}} \\
&\lesssim \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\int\limits_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-\lambda_2\right)+1}}.
\end{aligned}
$$

This completes the proof of inequality (21).

Now we turn to estimate G_{42}. Let $1<\tau<\infty$, such that $\frac{1}{p}=\frac{1}{q_1}+\frac{1}{\tau}$. Then, by Hölder's inequality and (21), we obtain

$$
\begin{aligned}
G_{42} &= \left\|\left[(b_1-\{b_1\}_B)\right]T^{(2)}\left[f_1^{\infty},(b_2-\{b_2\}_B)f_2^{\infty}\right]\right\|_{L_p(B(x_0,r))} \\
&\lesssim \|(b_1-\{b_1\}_B)\|_{L_{q_1}(B)}\left\|T^{(2)}\left[f_1^{\infty},(b_2-\{b_2\}_B)f_2^{\infty}\right]\right\|_{L_\tau(B)} \\
&\lesssim \prod_{i=1}^{2}\|\vec{b}\|_{LC^{\{x_0\}}_{q_i,\lambda_i}}r^{\frac{n}{p}}\int\limits_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-(\lambda_1+\lambda_2)\right)+1}}.
\end{aligned}
$$

Similarly, G_{43} has the same estimate above, here we omit the details, thus the inequality

$$
\begin{aligned}
G_{43} &= \left\|\left[(b_2-\{b_2\}_B)\right]T^{(2)}\left[(b_1-\{b_1\}_B)f_1^{\infty},f_2^{\infty}\right]\right\|_{L_p(B(x_0,r))} \\
&\lesssim \prod_{i=1}^{2}\|\vec{b}\|_{LC^{\{x_0\}}_{q_i,\lambda_i}}r^{\frac{n}{p}}\int\limits_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-(\lambda_1+\lambda_2)\right)+1}}.
\end{aligned}
$$

is valid.

Finally, to estimate G_{44}, similar to the estimate of (21), we have

$$\begin{aligned}&\left|T^{(2)}\left[(b_1-\{b_1\}_B)\,f_1^{\infty},(b_2-\{b_2\}_B)\,f_2^{\infty}\right](x)\right|\\&\lesssim\sum_{j=1}^{\infty}\left(2^jr\right)^{-2n}\left[\int\limits_{B(x_0,2^{j+1}r)}\left|b_1(y_1)-\{b_1\}_B\right|\left|f_1(y_1)\right|dy_1\right]\left[\int\limits_{B(x_0,2^{j+1}r)}\left|b_2(y_2)-\{b_2\}_B\right|\left|f_2(y_2)\right|dy_2\right]\\&\lesssim\prod_{i=1}^{2}\|\overrightarrow{b}\|_{LC_{q_i,\lambda_i}^{\{x_0\}}}\int\limits_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-(\lambda_1+\lambda_2)\right)+1}}.\end{aligned}$$

Thus, we have

$$\begin{aligned}G_{44}&=\left\|T^{(2)}\left[(b_1-\{b_1\}_B)\,f_1^{\infty},(b_2-\{b_2\}_B)\,f_2^{\infty}\right]\right\|_{L_p(B(x_0,r))}\\&\lesssim\prod_{i=1}^{2}\|\overrightarrow{b}\|_{LC_{q_i,\lambda_i}^{\{x_0\}}}r^{\frac{n}{p}}\int\limits_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-(\lambda_1+\lambda_2)\right)+1}}.\end{aligned}$$

By the estimates of G_{4j} above, where $j=1,2,3,4$. We know that

$$\begin{aligned}G_4&=\left\|T_{(b_1,b_2)}^{(2)}\left(f_1^{\infty},f_2^{\infty}\right)\right\|_{L_p(B(x_0,r))}\lesssim\prod_{i=1}^{2}\|\overrightarrow{b}\|_{LC_{q_i,\lambda_i}^{\{x_0\}}}r^{\frac{n}{p}}\int\limits_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\\&\quad\times\frac{dt}{t^{n\left(\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-(\lambda_1+\lambda_2)\right)+1}}.\end{aligned}$$

Recalling (15), and combining all the estimates for G_1, G_2, G_3, G_4, we get

$$\left\|T_{(b_1,b_2)}^{(2)}(f_1,f_2)\right\|_{L_p(B(x_0,r))}\lesssim\prod_{i=1}^{2}\|\overrightarrow{b}\|_{LC_{q_i,\lambda_i}^{\{x_0\}}}r^{\frac{n}{p}}\int\limits_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-(\lambda_1+\lambda_2)\right)+1}}.$$

Therefore, Theorem 1.14 is completely proved. □

2.2. ***Proof of Theorem 1.15.***

Proof. To prove Theorem 1.15, we will use the following relationship between essential supremum and essential infimum

$$\left(\operatorname*{essinf}_{x\in E}f(x)\right)^{-1}=\operatorname*{esssup}_{x\in E}\frac{1}{f(x)},\tag{24}$$

where f is any real-valued nonnegative function and measurable on E (see [17], page 143). Indeed, since $\overrightarrow{f}\in LM_{p_1,\varphi_1}^{\{x_0\}}\times\cdots\times LM_{p_m,\varphi_m}^{\{x_0\}}$, by (24) and the non-decreasing, with respect to t, of the norm $\prod_{i=1}^{m}\|f_i\|_{L_{p_i}(B(x_0,t))}$, we get

$$\frac{\prod_{i=1}^{m}\|f_i\|_{L_{p_i}(B(x_0,t))}}{\operatorname*{essinf}_{0<t<\tau<\infty}\prod_{i=1}^{m}\varphi_i(x_0,\tau)\tau^{\frac{n}{p_i}}} \leq \operatorname*{esssup}_{0<t<\tau<\infty}\frac{\prod_{i=1}^{m}\|f_i\|_{L_{p_i}(B(x_0,t))}}{\prod_{i=1}^{m}\varphi_i(x_0,\tau)\tau^{\frac{n}{p_i}}}$$

$$\leq \operatorname*{esssup}_{0<\tau<\infty}\frac{\prod_{i=1}^{m}\|f_i\|_{L_{p_i}(B(x_0,t))}}{\prod_{i=1}^{m}\varphi_i(x_0,\tau)\tau^{\frac{n}{p_i}}} \leq \prod_{i=1}^{m}\left\|f_i\right\|_{LM^{\{x_0\}}_{p_i,\varphi_i}}. \tag{25}$$

For $1 < p_1, \dots, p_m < \infty$, since $(\varphi_1, \dots, \varphi_m, \varphi)$ satisfies (12) and by (25), we have

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right)^m \prod_{i=1}^{m}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\sum_{i=1}^{n}\frac{1}{p_i}-\sum_{i=1}^{n}\lambda_i\right)+1}}$$

$$\leq \int_r^\infty \left(1+\ln\frac{t}{r}\right)^m \frac{\prod_{i=1}^{m}\|f_i\|_{L_{p_i}(B(x_0,t))}}{\operatorname*{essinf}_{t<\tau<\infty}\prod_{i=1}^{m}\varphi_i(x_0,\tau)\tau^{\frac{n}{p_i}}}\,\frac{\operatorname*{essinf}_{t<\tau<\infty}\prod_{i=1}^{m}\varphi_i(x_0,\tau)\tau^{\frac{n}{p_i}}}{t^{n\left(\sum_{i=1}^{n}\frac{1}{p_i}-\sum_{i=1}^{n}\lambda_i\right)}}\frac{dt}{t}$$

$$\leq C\prod_{i=1}^{m}\left\|f_i\right\|_{LM^{\{x_0\}}_{p_i,\varphi_i}}\int_r^\infty \left(1+\ln\frac{t}{r}\right)^m \frac{\operatorname*{essinf}_{t<\tau<\infty}\prod_{i=1}^{m}\varphi_i(x_0,\tau)\tau^{\frac{n}{p}}}{t^{n\left(\sum_{i=1}^{n}\frac{1}{p_i}-\sum_{i=1}^{n}\lambda_i\right)+1}}dt$$

$$\leq C\prod_{i=1}^{m}\left\|f_i\right\|_{LM^{\{x_0\}}_{p_i,\varphi_i}}\varphi(x_0,r). \tag{26}$$

Then by (11) and (26), we get

$$\left\|T^{(m)}_{\Pi\vec{b}}\left(\vec{f}\right)\right\|_{LM^{\{x_0\}}_{p,\varphi}} = \sup_{r>0}\varphi(x_0,r)^{-1}\left|B(x_0,r)\right|^{-\frac{1}{p}}\left\|T^{(m)}_{\vec{b}}\left(\vec{f}\right)\right\|_{L_p(B(x_0,r))}$$

$$\lesssim \prod_{i=1}^{m}\left\|\vec{b}\right\|_{LC^{\{x_0\}}_{q_i,\lambda_i}}\sup_{r>0}\varphi(x_0,r)^{-1}\int_r^\infty \left(1+\ln\frac{t}{r}\right)^m \prod_{i=1}^{m}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t^{n\left(\sum_{i=1}^{n}\frac{1}{p_i}-\sum_{i=1}^{n}\lambda_i\right)+1}}$$

$$\lesssim \prod_{i=1}^{m}\left\|\vec{b}\right\|_{LC^{\{x_0\}}_{q_i,\lambda_i}}\prod_{i=1}^{m}\left\|f_i\right\|_{LM^{\{x_0\}}_{p_i,\varphi_i}}.$$

Thus we obtain (13). Hence the proof is completed. □

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