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On Three-Dimensional (m, ρ)-Quasi-Einstein $N(\kappa)$-Contact Metric Manifold

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Abstract. (m, ρ)-quasi-Einstein $N(\kappa)$-contact metric manifolds have been studied and it is established that if such a manifold is a (m, ρ)-quasi-Einstein manifold, then the manifold is a manifold of constant sectional curvature κ. Further analysis has been done for gradient Einstein soliton, in particular. Obtained results are supported by an illustrative example.

1. Introduction

In an attempt to solve the Poincare conjecture, in 1982, Hamilton[18] developed the idea Ricci flow which is given by

$$\frac{\partial}{\partial t} g = -2Ric,$$

where Ric is the Ricci tensor of the matric g, satisfying a prescribed initial condition. The method of Ricci flow was used by Perelman[26] to solve 'Poincare conjecture'completely. The self-similar solution of Ricci flow is Ricci soliton[10, 20]. A Ricci soliton (g, W, λ) is given by

$$\frac{1}{2}\pounds_W g + Ric = \lambda g, \tag{1}$$

where $\pounds_W$ denotes the Lie derivative in the direction of the vector field W and λ is a real number. The soliton is considered expanding, steady or shrinking according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively. The soliton is called gradient when W is a gradient vector field associated with a smooth function ψ, and it is described by

$$Ric + \nabla^2\psi = \lambda g.$$

Here ∇^2 is the Hessian operator of g. In particular, if ψ is a constant, it is said that the soliton is trivial. In [11], the author gave the idea of m-Bakry-Emery Ricci curvature. When $m > 0$ and $\psi : M \to \mathbb{R}$ is a smooth

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function, the m-Bakery Ricci tensor Ric_{ψ}^{m} is defined by

$$\mathrm{Ric}_{\psi}^{m} = Ric + \nabla^2\psi - \frac{1}{m}d\psi \otimes d\psi.$$

A differentiable manifold (M^n, g), $n \geq 3$ is defined to be a generalized quasi-Einstein manifold if there are three smooth functions ψ, α, β satisfying

$$Ric + \nabla^2\psi - \alpha d\psi \otimes d\psi = \beta g.$$

The idea of generalized quasi-Einstein manifold was developed by Catino[12]. When $\alpha = 0$, $\beta \in \mathbb{R}$, then generalized quasi-Einstein manifold reduces to a gradient Ricci soliton[27] and m-quasi Einstein[2] when $\alpha = \frac{1}{m}, m \in \mathbb{N}$ and $\beta \in \mathbb{R}$. Here we study a particular case of generalized quasi-Einstein soliton which is called (m, ρ)-quasi-Einstein manifold introduced by Huang and Wei[19].

Definition 1.1. A Riemannian manifold (M^n, g) is called a (m, ρ)-quasi-Einstein manifold if there is a smooth function $\psi : M \to \mathbb{R}$ and three constants m, ρ, λ with $0 < m \leq \infty$ satisfying

$$Ric + \nabla^2\psi - \frac{1}{m}d\psi \otimes d\psi = \beta g = (\rho r + \lambda)g, \tag{2}$$

where r is the scalar curvature of the metric g. When $m = \infty, \rho = 0$, then the manifold reduces to exactly gradient Ricci soliton and gradient ρ-Einstein soliton when $m = \infty$. We denote (m, ρ)-quasi-Einstein manifold by (M^n, g, ψ, λ). If the potential function ψ is constant, then (m, ρ)-quasi-Einstein manifold is called trivial. In [19], the authors gave some classification of (m, ρ)-quasi-Einstein manifold whenever it is bach-flat.

In 2016, Catino-Mazzieri[13] introduced the notion of Einstein solitons which is generated by self-similar solutions to Einstein flow

$$\frac{\partial}{\partial t}g = -2\left(Ric - \frac{1}{2}rg\right). \tag{3}$$

Definition 1.2. ([13]) Let (M, g) be a Riemannian manifold of dimension $n \geq 3$. Then M is called a gradient Einstein soliton, denoted by (g, ψ, λ) if there is a smooth function $\psi : M \to \mathbb{R}$ such that

$$Ric - \frac{1}{2}rg + \nabla^2\psi = \lambda g. \tag{4}$$

If the scalar curvature r of the manifold is constant, then the gradient Einstein soliton (g, ψ, λ) reduces to a gradient Ricci soliton $\left(g, \psi, \lambda + \frac{1}{2}r\right)$.

Catino-Mazzieri[13] showed that every compact gradient Einstein, Schouten or traceless Ricci soliton is trivial. They also proved that every gradient ρ-Einstein soliton is rectifiable. Next, they classified three-dimensional gradient shrinking Schouten soliton and proved that it is isometric to a finite quotient of either $\mathbb{S}^3$ or $\mathbb{R}^3$ or $\mathbb{R} \times \mathbb{S}^2$.

In the paper [9], Blaga studied gradient η-Einstein solitons. In the paper [17], Ghosh studied (m, ρ)-quasi-Einstein metrices in the framework of K-contact manifolds and showed that in a complete (m, ρ)-quasi-Einstein manifold with $m \neq 1$, the potential function ψ is constant and the manifold is compact, Einstein and Sasakian. Motivated by these works in this paper we study (m, ρ)-quasi-Einstein matrices on three dimensional $N(\kappa)$-contact metric manifolds. We also are interested to study gradient Einstein solitons on 3-dimensional $N(\kappa)$-contact metric manifolds.

Now we state the main results of the paper :

Theorem 1.1. *If a three-dimensional $N(\kappa)$-contact metric manifold (M, g, ψ, λ) is a (m, ρ)-quasi-Einstein manifold, then M is a manifold of constant sectional curvature κ and either $\lambda = (m + 2 - 6\rho)\kappa$ or, ψ is a constant.*

Theorem 1.2. *If the metric of a three-dimensional $N(\kappa)$-contact metric manifold $M^3(g, \psi, \lambda)$ is a gradient Einstein soliton, then M is a manifold of constant sectional curvature κ. Moreover, either M is flat or, ψ is a constant.*

2. Preliminaries

On a $(2n+1)$-dimensional manifold M^{2n+1}, by an almost contact structure, we mean the triplet (φ, ζ, θ), where φ is a $(1,1)$ tensor field, ζ is a global vector field and θ is a 1-form, and

$$\varphi^2 + I = \theta \otimes \zeta, \quad \theta(\zeta) = 1, \tag{5}$$

which implies that

$$\varphi\zeta = 0, \quad \theta \circ \varphi = 0, \text{ and } \operatorname{rank}(\varphi) = 2n. \tag{6}$$

The manifold M^{2n+1} equipped with the structure (φ, ζ, θ) is called an almost contact manifold [3, 4]. When $[\varphi, \varphi] + 2d\theta \otimes \zeta$ vanishes identically, then almost contact manifold is said to be normal. If, in addition, the manifold is endowed with a Riemannian metric such that

$$g(U, V) = g(\varphi U, \varphi V) + \theta(U)\theta(V) \tag{7}$$

for all vector fields U, V on M, then $(M, \varphi, \zeta, \theta, g)$ is called an almost contact metric manifold. Putting $V = \zeta$ in (7), we find that

$$g(U, \varphi V) + g(\varphi U, V) = 0, \quad \theta(U) = g(U, \zeta). \tag{8}$$

If $g(U, \varphi V) = d\theta(U, V)$ for all U, V on M, then the almost contact metric manifold $(M, \varphi, \zeta, \theta, g)$ is called a contact metric manifold. In this case, the volume form $\theta \wedge (d\theta)^n \neq 0$ everywhere on M. We denote by ∇ the Riemannian connection of g and by K the corresponding curvature tensor given by

$$K(U, V) = [\nabla_U, \nabla_V] - \nabla_{[U,V]} \tag{9}$$

for all vector fields U, V on M. A normal contact metric manifold is known as Sasakian manifold. A necessary and sufficient condition for an almost contact metric manifold $(M, \varphi, \zeta, \theta, g)$ to be Sasakian is that

$$(\nabla_U \varphi)V = g(U, V)\zeta - \theta(V)U.$$

On the other hand for a Sasakian manifold, we have

$$K(U, V)\zeta = \theta(V)U - \theta(U)V.$$

For a contact metric manifold, we can define a (1,1)-tensor field $h = \frac{1}{2}£_\zeta\varphi$ which is symmetric and satisfy

$$\varphi h + h\varphi = 0, \quad \operatorname{tr} h = \operatorname{tr} \varphi h = 0 \tag{10}$$

and

$$\nabla_U \zeta = -\varphi U - \varphi h U \tag{11}$$

for all vector field U on M. $h = 0$ if and only if the characteristic vector field ζ is a Killing vector field, that is $£_\zeta g = 0$. In this case the contact metric manifold is called K-contact. Every Sasakian manifold is K-contact, but the converse is true. However every 3-dimensional K-contact manifold is Sasakian[22].

The (κ, μ)-nullity distribution $N(\kappa, \mu)$[7] of a contact metric manifold $(M, \varphi, \zeta, \theta, g)$ is the distribution

$$\begin{aligned} & N(\kappa, \mu) : p \to N_p(\kappa, \mu) \\ = \; & \{W \in T_pM : K(U, V)W = (\kappa I + \mu h)(g(V, W)U - g(U, W)V)\} \end{aligned}$$

for all vector field U, V on M, where $(\kappa, \mu) \in \mathbb{R}^2$. If the characteristic vector field $\zeta \in N(\kappa, \mu)$ then the manifold M is called a (κ, μ)-contact metric manifold. For a (κ, μ)-contact metric manifold, we have

$$K(U, V)W = (\kappa I + \mu h)(\theta(V)U - \theta(U)V). \tag{12}$$

On a (κ,μ)-contact metric manifold, $\kappa \leq 1$.When $\kappa = 1$, the structure is Sasakian. If $\mu = 0$, the (κ,μ)-nullity distribution $N(\kappa,\mu)$ is reduced to the κ-nullity distribution $N(\kappa)$[28]. The κ-nullity distribution $N(\kappa)$ of a Riemannian manifold is defined by [28]

$$\begin{aligned} N(\kappa): \quad p &\rightarrow N_p(\kappa) \\ &= \{W \in T_pM : K(U,V)W = \kappa(g(V,W)U - g(U,W)V)\}, \end{aligned}$$

where κ is a real number. If $\zeta \in N(\kappa)$, then a contact metric manifold is called an $N(\kappa)$-contact metric manifold. $N(\kappa)$-contact metric manifolds have been studied by several authors such as [14–16, 21, 23–25] and many others.

For $N(\kappa)$-contact metric manifolds M^{2n+1} the following relations hold[7]:

$$h^2 = (\kappa - 1)\varphi^2, \quad \kappa \leq 1, \tag{13}$$

$$(\nabla_U\varphi)V = g(U + hU, V)\zeta - \theta(V)(U + hU), \tag{14}$$

$$K(U,V)\zeta = \kappa(\theta(V)U - \theta(U)V), \tag{15}$$

$$\begin{aligned} Ric(U,V) &= 2(n-1)g(U,V) + 2(n-1)g(hU,V) \\ &+ [2n\kappa - 2(n-1)]\theta(U)\theta(V), \end{aligned} \tag{16}$$

$$Ric(U,\zeta) = 2n\kappa\theta(U), \tag{17}$$

$$(\nabla_U\theta)V = g(U + hU, \varphi V) \tag{18}$$

for any vector fields U, V on M. The curvature tensor K in a 3-dimensional Riemannian manifold is given by

$$\begin{aligned} K(U,V)W &= Ric(V,W)U - Ric(U,W)V + g(V,W)QU \\ &- g(U,W)QV - \frac{r}{2}\{g(V,W)U - g(U,W)V\}. \end{aligned} \tag{19}$$

In [6], the authors proved that in a 3-dimensional $N(\kappa)$-contact metric manifold M, the following relations hold :

$$QU = \left(\frac{r}{2} - \kappa\right)U + \left(3\kappa - \frac{r}{2}\right)\theta(U)\zeta, \tag{20}$$

where Q is the Ricci operator defined by $Ric(U,V) = g(QU,V)$.

$$\begin{aligned} K(U,V)W &= \left(\frac{r}{2} - 2\kappa\right)(g(V,W)U - g(U,W)V) \\ &+ \left(3\kappa - \frac{r}{2}\right)(g(V,W)\theta(U)\zeta - g(U,W)\theta(V)\zeta \\ &+ \theta(V)\theta(W)U - \theta(U)\theta(W)V). \end{aligned} \tag{21}$$

$$\nabla_U\zeta = -(1+\alpha)\varphi U \tag{22}$$

for all vector fields U, V, W on M, where $\alpha = \pm\sqrt{1-\kappa}$. From (21), it follows that, a 3-dimensional $N(\kappa)$-contact metric manifold is of constant curvature if and only if $r = 6\kappa$.

Lemma 2.1. *[8] Let $M^{2n+1}(\varphi,\zeta,\theta,g)$ be a contact metric manifold and suppose that $K(U,V)\zeta = 0$ for all vector fields U and V. Then M^{2n+1} is locally the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of positive constant curvature 4 for $n > 1$ and flat for $n = 1$.*

3. Proof of the main results

Before proving the main result, we need the following lemma.

Lemma 3.1. *If* (M^n, g, ψ, λ), $n \geq 3$ *be a* (m, ρ)*-quasi-Einstein manifold, then the curvature tensor K satisfies*

$$\begin{aligned} K(U,V)D\psi &= (\nabla_V Q)U - (\nabla_U Q)V + (U\beta)V - (V\beta)U \\ &+ \frac{1}{m}\{(U\psi)QV - (V\psi)QU\} \\ &+ \frac{\beta}{m}\{(V\psi)U - (U\psi)V\}. \end{aligned} \tag{23}$$

Proof. From (2), we have

$$\nabla_V D\psi = -QV + \frac{1}{m}g(D\psi, V)D\psi + \beta V. \tag{24}$$

Differentiating covarianly (24) along the vector field U, we get

$$\begin{aligned} \nabla_U\nabla_V D\psi &= -\nabla_U(QV) + \beta\nabla_U V + (U\beta)V \\ &+ \frac{1}{m}\{g(\nabla_U D\psi, V)D\psi + g(D\psi, \nabla_U V)D\psi + (V\psi)\nabla_U D\psi\}. \end{aligned} \tag{25}$$

Interchanging U and V in the previous equation, we obtain

$$\begin{aligned} \nabla_V\nabla_U D\psi &= -\nabla_V(QU) + \beta\nabla_V U + (V\beta)U \\ &+ \frac{1}{m}\{g(\nabla_V D\psi, U)D\psi + g(D\psi, \nabla_V U)D\psi + (U\psi)\nabla_V D\psi\}. \end{aligned} \tag{26}$$

Substituting the values (24)-(26) in (9), we get the result. □

Proof of Theorem 1.1. From (20), we get

$$\begin{aligned} (\nabla_V Q)U &= \frac{1}{2}(Vr)U - \frac{1}{2}(Vr)\theta(U)\zeta \\ &+ \left(\frac{r}{2} - 3\kappa\right)\{g(U, \varphi V + \varphi hV)\zeta + \theta(U)(\varphi V + \varphi hV)\}. \end{aligned} \tag{27}$$

Putting (27) and (20) in (23) and taking inner product with ζ, we obtain

$$\begin{aligned} g(K(U,V)D\psi, \zeta) &= (U\beta)\theta(V) - (V\beta)\theta(U) + (r - 6\kappa)g(U, \varphi V) \\ &+ \frac{1}{m}(\beta - 2\kappa)\{(V\psi)\theta(U) - (U\psi)\theta(V)\}. \end{aligned} \tag{28}$$

Using (15) in (28), it follows that

$$\begin{aligned} \frac{(m+2)\kappa - \beta}{m}\{(V\psi)\theta(U) - (U\psi)\theta(V)\} &= (U\beta)\theta(V) - (V\beta)\theta(U) \\ &+ (r - 6\kappa)g(U, \varphi V). \end{aligned} \tag{29}$$

Replacing U by φU and V by φV in the foregoing equation, we have

$$(r - 6\kappa)d\theta(U, V) = 0.$$

As $d\theta$ is non-vanishing on any contact metric manifold, from the above we get

$$r = 6\kappa.$$

From (21), we see that M is a manifold of constant sectional curvature κ. Since the scalar curvature $r = 6\kappa$ is constant the function $\beta = \rho r + \lambda = 6\rho\kappa + \lambda$ becomes a constant. Substituting $U = \zeta$ in (29)

$$\frac{(m + 2 - 6\rho)\kappa - \lambda}{m}(V\psi - (\zeta\psi)\theta(V)) = 0. \tag{30}$$

From the above, we have either $\lambda = (m + 2 - 6\rho)\kappa$ or, $D\psi = (\zeta\psi)\zeta$.
Suppose that $\lambda \neq (m + 2 - 6\rho)\kappa$. Differentiating $D\psi = (\zeta\psi)\zeta$ along the vector U and using (11), we get

$$\nabla_U D\psi = U(\zeta\psi)\zeta - (\zeta\psi)(\varphi U + \varphi hU). \tag{31}$$

In the previous equation, applying Poincare Lemma ($d^2 = 0$), we infer that

$$U(\zeta\psi)\theta(V) - V(\zeta\psi)\theta(U) + 2(\zeta\psi)g(U, \varphi V) = 0.$$

Replacing U by φU and V by φV in the above equation, we have

$$(\zeta\psi)d\theta(U, V) = 0.$$

Since $d\theta \neq 0$ for any contact manifold, we find that $\zeta\psi = 0$. Consequently, $D\psi = (\zeta\psi)\zeta = 0$. This completes the proof.

Corollary 3.1. *If the metric of a 3-dimensional compact $N(\kappa)$-contact metric manifold (M, g, ψ, λ) with $\kappa > 0$ is a (m, ρ)-quasi-Einstein manifold then ψ is a constant.*

Proof. Since $r = 6\kappa$, from (20), $QV = 2\kappa V$. By proof of Theorem 1.1, either $\lambda = (m+2-6\rho)\kappa$ or ψ is constant. If $\lambda = (m + 2 - 6\rho)\kappa$, from (24) it follows that

$$\nabla_V D\psi = \frac{1}{m}g(D\psi, V)D\psi + m\kappa V.$$

Taking trace over V, we have

$$\Delta\psi + \frac{1}{m}|D\psi|^2 + 3m\kappa = 0.$$

where $\Delta = -\mathrm{Div}\, D$ is the Laplacian operator of g. Integrating over M and using divergence theorem, we obtain

$$\int_M |D\psi|^2\, dM = -3m^2\kappa \int_M dM. \tag{32}$$

Since M is orientable, the volume element dM on M is always positive and hence right hand side of (32) is negative. While the integrand on the left hand side is non-negative, a contradiction. This completes the proof. □

Proof of Theorem 1.2. From (4), it follows that

$$QU - \frac{1}{2}rU + \nabla_U D\psi = \lambda U, \tag{33}$$

where D is the gradient operator of g.
Using (20) in (33), we get

$$\nabla_U D\psi = (\kappa + \lambda)U + \left(\frac{r}{2} - 3\kappa\right)\theta(U)\zeta. \tag{34}$$

Differentiating the equation (34) covariantly with respect to the vector field V and using (11), we obtain

$$\begin{aligned} \nabla_V\nabla_U D\psi &= (\lambda+1)\nabla_V U + \frac{1}{2}(Vr)\theta(U)\zeta \\ &+ \left(\frac{r}{2}-3\kappa\right)\{(V\theta(U))\zeta - \theta(U)(\varphi V + \varphi hV)\}. \end{aligned} \tag{35}$$

Interchanging U and V in the previous equation, we have

$$\begin{aligned} \nabla_U\nabla_V D\psi &= (\lambda+1)\nabla_U V + \frac{1}{2}(Ur)\theta(V)\zeta \\ &+ \left(\frac{r}{2}-3\kappa\right)\{(U\theta(V))\zeta - \theta(V)(\varphi U + \varphi hU)\}. \end{aligned} \tag{36}$$

Putting the values (34)-(36) in (9) and using (18) and (10), we infer that

$$\begin{aligned} K(U,V)D\psi &= \frac{1}{2}(Ur)\theta(V)\zeta - \frac{1}{2}(Vr)\theta(U)\zeta \\ &+ (r-6\kappa)g(U,\varphi V)\zeta - \left(\frac{r}{2}-3\kappa\right)\theta(V)(\varphi U + \varphi hU) \\ &+ \left(\frac{r}{2}-3\kappa\right)\theta(U)(\varphi V + \varphi hV). \end{aligned} \tag{37}$$

Taking inner product of (37) with ζ and using (6) and (15), we get

$$\begin{aligned} \kappa(\theta(U)V\psi - \theta(V)U\psi) &= \frac{1}{2}(Ur)\theta(V) - \frac{1}{2}(Vr)\theta(U) \\ &+ (r-6\kappa)g(U,\varphi V). \end{aligned} \tag{38}$$

Replacing U by φU and V by φV in (38), we obtain

$$(r-6\kappa)d\theta(U,V) = 0.$$

Since $d\theta$ is non-vanishing on any contact manifold, from above equation it follows that

$$r = 6\kappa.$$

Putting the value of r in (21), we see that M is a manifold of constant sectional curvature κ.
Substituting U by ζ in (38), we get

$$\kappa\{V\psi - \theta(V)\zeta\psi\} = 0. \tag{39}$$

This shows that either $\kappa = 0$ or $D\psi = (\zeta\psi)\zeta$. If $\kappa = 0$, then $K(U,V)\zeta = 0$ for all vector fields U and V. Using Lemma 2.1 we conclude that M is flat.
Suppose that $\kappa \neq 0$. By the same proof of Theorem 1.1 we conclude that ψ is constant. This completes the proof.

If $\kappa = 1$, then M is a Sasakian manifold. Thus, we are in a position to state the following :

Corollary 3.2. *If the metric of a Sasakian 3-manifold is a gradient Einstein soliton with potential function ψ, then the manifold is a Sasaki-Einstein manifold and ψ is a constant.*

4. Example

Let $M = \{(x,y,z) \in \mathbb{R} : x \neq 0\}$ be a three-dimensional manifold.
Suppose

$$E_1 = \frac{\partial}{\partial x}, \quad E_2 = 2e^{-z}\frac{\partial}{\partial x} + e^y\frac{\partial}{\partial y} + 2xe^z\frac{\partial}{\partial z}, \quad E_3 = e^z\frac{\partial}{\partial z}.$$

here E_1, E_2, E_3 are linearly independent at each point of M. We have

$$[E_1, E_2] = 2E_3, \quad [E_1, E_3] = 0, \quad [E_2, E_3] = 2E_1.$$

Let g be the Riemannian metric such that

$$g(E_i, E_j) = \delta_{ij}, \quad i, j \in \{1, 2, 3\}.$$

Let φ be the (1,1) tensor field and θ be the 1-form defined by

$$\varphi(E_1) = 0, \quad \varphi(E_2) = E_3, \quad \varphi(E_3) = -E_2, \quad \theta = dx - 2e^{-y-x}dz.$$

By linearity of φ and g, we have

$$\varphi^2 V = -V + \theta(V)E_1, \quad \theta(E_1) = 1,$$
$$g(\varphi U, \varphi V) = g(U, V) - \theta(U)\theta(V),$$
$$d\theta(U, V) = g(U, \varphi V)$$

for all vector fields U, V on M. Thus for $E_1 = \zeta$, $M(\varphi, \zeta, \theta, g)$ is a contact metric manifold. The tensor h is given by

$$hE_1 = 0, \quad hE_2 = E_2, \quad hE_3 = -E_3.$$

By Koszul formula,

$$\nabla_{E_1}E_1 = 0, \quad \nabla_{E_1}E_2 = 0, \quad \nabla_{E_1}E_3 = 0,$$
$$\nabla_{E_2}E_1 = -2E_3, \quad \nabla_{E_2}E_2 = 0, \quad \nabla_{E_2}E_3 = 2E_1,$$
$$\nabla_{E_3}E_1 = 0, \quad \nabla_{E_3}E_2 = 0, \quad \nabla_{E_3}E_3 = 0.$$

From the above we see that

$$\nabla_V \zeta = -\varphi V - \varphi h V$$

for all $V \in \chi(M)$. The Riemannian curvature tensor K vanishes identically. Consequently M is a $N(0)$-contact metric manifold. Also, the Ricci tensor Ric and the scalar curvature r vanish.
Suppose that $\psi = \frac{\lambda}{2}(x^2 + e^{-2y} + e^{-2z})$. By straightforward calculations we have $\nabla^2\psi = \lambda g$. This shows that g is a gradient Einstein soliton.

References

[1] Baikoussis, C., Blair, D. E. and Koufogiorgos, T., *A decomposition of the curvature tensor of a contact manifold satisfying $R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y)$,* Math. Technical Reports, University of Ioannian, 1992.
[2] Barros, A., Gomes, J. N., *Triviality of compact m-quasi-Einstein manifolds,* Res. Math. **71**(2017), 241-250.
[3] Blair, D. E., *Contact manifold in Riemannian geometry,* Lecture note in math., **509,** Springer-verlag, Berlin-New york, 1976.
[4] Blair, D. E., *Riemannian geometry of contact and symplectic manifolds*, Progress in math. **203,** Birkhäuser, 2010.
[5] Blair, D. E., Kim, J.-S. and Tripathi, M. M., *On the concircular curvature tensor of a contact metric manifold,* J. Korean Math. Soc., **42**(2005), 883-992.
[6] Blair, D. E., Koufogiorgos, T. and Sharma, R., *A classification of 3-dimensional contact metric manifolds with $Q\varphi = \varphi Q$,* Kodai Math. J., **13**(1990), 391-401.
[7] Blair, D. E., Koufogiorgos, T. and Papantoniou, B. J., *Contact metric manifolds satisfying a nullity condition,* Israel J. Math., **91**(1995), 189-214.
[8] Blair, D. E., *Two remarks on contact metric structures,* Tohoku Math. J., **29**(1977), 319-324.
[9] Blaga, A. M.,*On gradient η-Einstein solitons,* Kragujevac J. of Math., **42**(2018), 229-237.
[10] Cao, H. D., *Recent progress on Ricci soliton,* Adv. Lect. Math., **11**(2009), 1-38.
[11] Case, J. S., *Singularity theorems and the Lorentzian splitting theorem for the Bakry-Emery Ricci tensor,* J. Geom. and Phys., **60**(2010), 477-490.
[12] Catino, G., *Generalized quasi-Einstein manifolds with harmonic Weyl tensor,* Math. Z., **271**(2012), 751-756.
[13] Catino, G. and Mazzieri, L., *Gradient Einstein Solitons,* Nonlinear Anal., **132**(2016), 66-94.
[14] De, U. C. and Majhi, P., *On a type of contact metric manifolds,* Lobachevskii J. Math., **34**(2013), 89-98.
[15] De, U. C. and Sarkar, A. *On the quasi-conformal curvature tensor of a (κ, μ)-contact metric manifold,* Math. Reports, **14**(2012), 115-129.
[16] De, U. C., Yildiz, A. and Ghosh, S., *On a class of $N(\kappa)$-contact metric manifolds,* Math. Reports, **16**(2014), 207-217.
[17] Ghosh, A., *(m, ρ)-quasi-Einstein metrices in the framework of K-contact manifolds,* Math. Phys. Anal. and Geom., **17**(2014), 369-376.

[18] Hamilton, R. S., *The Ricci flow on surfaces,* Mathematics and general Relativity, Contemp. Math., **71**(1998), 237-262.
[19] Huang G. and Wei, Y., *The classification of* (m, ρ)*-quasi-Einstein manifolds,* Ann. Global Anal. Geom., **44**(2013), 269-282.
[20] Ivey, T., *Ricci solitons on compact 3-manifolfds,* Diff. Geom. and its Appl., **3**(1993), 301-307.
[21] Jun, J. B. and De, A., *On* $N(\kappa)$*-contact metric manifolds satisfying certain curvature conditions,* Kyungpook Math. J., **51**(2011), 457-468.
[22] Jun, J. B., Kim, I.-B. and Kim, U. K. *On 3-dimensional almost contact metric manifolds,* Kyungpook Math. J., **34**(199), 293-301.
[23] Murathan, C., De, U. C. and Arsalan, K., *On the Wely projective curvature tensor of an* $N(\kappa)$*-contact metric manifold,* Mathematica Panonoica, **21**(2010), 129-142.
[24] Özgür, C., *Contact metric manifolds with cyclic-parallel Ricci tensor,* Diff. Geom. Dynamical Systems, **4**(2002), 21-25.
[25] Özgür, C. and Sular, S., *On* $N(\kappa)$*-contact metric manifolds satisfying certain conditions,* SUT J. Math., **44**(2008), 89-99.
[26] Perelman, G., *The entropy formula for the Ricci flow and its geometric application*, arXiv:math/**0211159**.
[27] Sharma, R., *Certain results on K-contact and* (κ, μ)*-contact manifolds,* J. of Geom., **89**(2008), 138-147.
[28] Tanno, S., *Ricci curvature of contact Riemannian manifolds,* Tohoku Math. J., **40**(1998), 441-448.