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Nonlinear Maps Preserving the Mixed Product $[A \bullet B, C]_*$ on Von Neumann Algebras

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be two von Neumann algebras. For $A, B \in \mathcal{A}$, define by $[A, B]_* = AB - BA^*$ and $A \bullet B = AB + BA^*$ the new products of A and B. Suppose that a bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi([A \bullet B, C]_*) = [\Phi(A) \bullet \Phi(B), \Phi(C)]_*$ for all $A, B, C \in \mathcal{A}$. In this paper, it is proved that if $\mathcal{A}$ and $\mathcal{B}$ be two von Neumann algebras with no central abelian projections, then the map $\Phi(I)\Phi$ is a sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism, where $\Phi(I)$ is a self-adjoint central element in $\mathcal{B}$ with $\Phi(I)^2 = I$. If $\mathcal{A}$ and $\mathcal{B}$ are two factor von Neumann algebras, then Φ is a linear $*$-isomorphism, or a conjugate linear $*$-isomorphism, or the negative of a linear $*$-isomorphism, or the negative of a conjugate linear $*$-isomorphism.

1. Introduction

Let $\mathcal{A}$ be a $*$-algebra over the complex field $\mathbb{C}$. For $A, B \in \mathcal{A}$, define the skew Lie product of A and B by $[A, B]_* = AB - BA^*$ and the Jordan $*$-product of A and B by $A \bullet B = AB + BA^*$. The skew Lie product and the Jordan $*$-product are fairly meaningful and important in some research topics (see [10–14, 25]). They were extensively studied because they naturally arise in the problem of representing quadratic functionals with sesquilinear functionals (see [17–19]) and in the problem of characterizing ideals (see [2, 16]). Particular attention has been paid to understanding maps which preserve the skew Lie product or the Jordan $*$-product on $*$-algebras (see [1, 3, 4, 7]). For example, J. Cui and C. K. Li [3] showed that every bijective map preserving the skew Lie product on factor von Neumann algebras is a $*$-ring isomorphism. Bai and Du [1] proved that any bijective map preserving the skew Lie product between von Neumann algebras with no central abelian projections is a sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism. C. Li et al. [7] considered maps which preserve the Jordan $*$-product and proved that such a bijective map between factor von Neumann algebras is a $*$-ring isomorphism. These results show that the skew Lie product or the Jordan $*$-product structure is enough to determine the algebraic structure.

Recently, nonlinear maps preserving the products of the mixture of (skew) Lie product and Jordan $*$-product have received a fair amount of attention (see [5, 6, 8, 9, 21–24]). For example, C. Li et al. studied the nonlinear maps preserving the skew Lie triple product $[[A, B]_*, C]_*$ (see [6, 9]) and the Jordan triple $*$-product $A \bullet B \bullet C$ (see [8, 24]) on von Neumann algebras. Z. Yang and J. Zhang in [21, 22] studied the nonlinear maps

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preserving the mixed skew Lie triple product $[[A, B]_*, C]$ and $[[A, B], C]_*$ on factor von Neumann algebras. In the present paper, we will establish the structure of the nonlinear maps preserving the mixed product $[A \bullet B, C]_*$ on von Neumann algebras.

2. Main results

Before stating the main results, we need some notations and preliminaries. A von Neumann algebra $\mathcal{A}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space H containing the identity operator I. The set $\mathcal{Z}(\mathcal{A}) = \{S \in \mathcal{A} : ST = TS \text{ for all } T \in \mathcal{A}\}$ is called the center of $\mathcal{A}$. A projection P is called a central abelian projection if $P \in \mathcal{Z}(\mathcal{A})$ and $P\mathcal{A}P$ is abelian. Recall that the central carrier of A, denoted by $\overline{A}$, is the smallest central projection P satisfying $PA = A$. It is not difficult to see that the central carrier of A is the projection onto the closed subspace spanned by $\{BA(x) : B \in \mathcal{A}, x \in H\}$. If A is self-adjoint, then the core of A, denoted by $\underline{A}$, is $\sup\{S \in \mathcal{Z}(\mathcal{A}) : S = S^*, S \le A\}$. If P is a projection, it is clear that $\underline{P}$ is the largest central projection Q satisfying $Q \le P$. A projection P is said to be core-free if $\underline{P} = 0$. It is easy to see that $\underline{P} = 0$ if and only if $\overline{I - P} = I$.

Lemma 2.1. *[15] Let $\mathcal{A}$ be a von Neumann algebra with no central abelian projections. Then there exists a projection $P \in \mathcal{A}$ such that $\underline{P} = 0$ and $\overline{P} = I$.*

Lemma 2.2. *[4] Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space H. Let A be an operator in $\mathcal{A}$ and $P \in \mathcal{A}$ is a projection with $\overline{P} = I$. If $ABP = 0$ for all $B \in \mathcal{A}$, then $A = 0$.*

Lemma 2.3. *Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space and $A \in \mathcal{A}$. If $AB + BA^* = 0$ for all self-adjoint or conjugate self-adjoint elements $B \in \mathcal{A}$, then $A = -A^* \in \mathcal{Z}(\mathcal{A})$.*

Proof. Since for every element B in $\mathcal{A}$, B can be written as the linear sum of two self-adjoint or conjugate self-adjoint elements in $\mathcal{A}$. Hence $AB + BA^* = 0$ for all $B \in \mathcal{A}$. Now we take $B = I$, then $A = -A^*$. So $AB = BA$ for all $B \in \mathcal{A}$, and then $A \in \mathcal{Z}(\mathcal{A})$. □

Our main result in this paper reads as follows.

Theorem 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two von Neumann algebras with no central abelian projections. Suppose that a bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi([A \bullet B, C]_*) = [\Phi(A) \bullet \Phi(B), \Phi(C)]_*$ for all $A, B, C \in \mathcal{A}$. Then the map $\Phi(I)\Phi$ is a sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism, where $\Phi(I)$ is a self-adjoint central element in $\mathcal{B}$ with $\Phi(I)^2 = I$.*

Proof. The proof will be organized in some claims.
Claim 1. $\Phi(0) = 0$.
Since Φ is surjective, there exists $A \in \mathcal{A}$ such that $\Phi(A) = 0$. So

$$\Phi(0) = \Phi([0 \bullet A, A]_*) = [\Phi(0) \bullet 0, 0]_* = 0.$$

Claim 2. For each $A \in \mathcal{A}$, $A = A^*$ if and only if $\Phi(A) = \Phi(A)^*$.
Since Φ is surjective, there exists $B \in \mathcal{A}$ such that $\Phi(B) = I$. For any A in $\mathcal{A}$, we have that

$$\begin{aligned} 0 &= \Phi([iI \bullet A, B]_*) \\ &= [\Phi(iI) \bullet \Phi(A), I]_* \\ &= \Phi(iI)(\Phi(A) - \Phi(A)^*) + (\Phi(A) - \Phi(A)^*)\Phi(iI)^* \end{aligned}$$

holds true for all $A \in \mathcal{A}$. So $\Phi(iI)B + B\Phi(iI)^* = 0$ holds true for all $B = -B^* \in \mathcal{B}$. It follows from Lemma 2.3 that $\Phi(iI) = -\Phi(iI)^* \in \mathcal{Z}(\mathcal{B})$. Similarly, we have $\Phi^{-1}(iI) \in \mathcal{Z}(\mathcal{A})$.
Let $A = A^* \in \mathcal{A}$ and $\Phi(B) = I$. Since $0 = [B \bullet A, \Phi^{-1}(iI)]_*$, it follows that

$$0 = \Phi([B \bullet A, \Phi^{-1}(iI)]_*) = [I \bullet \Phi(A), iI]_* = 2i(\Phi(A) - \Phi(A)^*).$$

This implies that $\Phi(A)=\Phi(A)^*$. Similarly, if $\Phi(A)=\Phi(A)^*$, then

$$0=\Phi^{-1}([\Phi(I)\bullet\Phi(A),\Phi(iI)]_*)=[I\bullet A,(iI)]_*=2i(A-A^*),$$

and so $A=A^*$.

Claim 3. $\Phi(\mathcal{Z}(\mathcal{A}))=\mathcal{Z}(\mathcal{B})$.

Let $Z\in\mathcal{Z}(\mathcal{A})$ be arbitrary and $\Phi(B)=I$. For every $A=A^*\in\mathcal{A}$, we obtain that

$$0=\Phi([B\bullet A,Z]_*)=[I\bullet\Phi(A),\Phi(Z)]_*=2(\Phi(A)\Phi(Z)-\Phi(Z)\Phi(A)^*).$$

So $\Phi(A)\Phi(Z)=\Phi(Z)\Phi(A)^*$ holds true for all $A=A^*\in\mathcal{A}$. It follows from Claim 2 that $C\Phi(Z)=\Phi(Z)C$ holds true for all $C=C^*\in\mathcal{A}$. Since for every $C\in\mathcal{B}$, we have $C=C_1+iC_2$, where $C_1=\frac{C+C^*}{2}$ and $C_2=\frac{C-C^*}{2i}$ are self-adjoint elements. Hence $C\Phi(Z)=\Phi(Z)C$ holds true for all $C\in\mathcal{A}$. Then $\Phi(Z)\in\mathcal{Z}(\mathcal{B})$. Applying the similar process to Φ^{-1}, we get $\Phi(\mathcal{Z}(\mathcal{A}))=\mathcal{Z}(\mathcal{B})$.

In the following, we will show the additivity of Φ. First we give a key technique. Suppose that $A_1,A_2,...,A_n$ and T are in $\mathcal{A}$ such that $\Phi(T)=\sum_{i=1}^n\Phi(A_i)$. Then for all $S_1,S_2\in\mathcal{A}$, we have

$$\Phi([S_1\bullet S_2,T]_*)=[\Phi(S_1)\bullet\Phi(S_2),\Phi(T)]_*=\sum_{i=1}^n\Phi([S_1\bullet S_2,A_i]_*),\tag{1}$$

$$\Phi([S_1\bullet T,S_2]_*)=[\Phi(S_1)\bullet\Phi(T),\Phi(S_2)]_*=\sum_{i=1}^n\Phi([S_1\bullet A_i,S_2]_*),\tag{2}$$

and

$$\Phi([T\bullet S_1,S_2]_*)=[\Phi(T)\bullet\Phi(S_1),\Phi(S_2)]_*=\sum_{i=1}^n\Phi([A_i\bullet S_1,S_2]_*).\tag{3}$$

By Lemma 2.1, there exists a projection P such that $\underline{P}=0$ and $\overline{P}=I$. Let $P_1=P$ and $P_2=I-P$. Denote $\mathcal{A}_{ij}=P_i\mathcal{A}P_j$. Then $\mathcal{A}=\sum_{i,j=1}^2\mathcal{A}_{ij}$. In all that follows, when we write A_{ij}, it indicates that $A_{ij}\in\mathcal{A}_{ij}$.

Claim 4. For every $A_{12}\in\mathcal{A}_{12},B_{21}\in\mathcal{A}_{21}$, we have

$$\Phi(A_{12}+B_{21})=\Phi(A_{12})+\Phi(B_{21}).$$

Choose $T=\sum_{i,j=1}^2T_{ij}\in\mathcal{A}$ such that

$$\Phi(T)=\Phi(A_{12})+\Phi(B_{21}).$$

Since

$$[I\bullet(i(P_2-P_1)),A_{12}]_*=[I\bullet(i(P_2-P_1)),B_{21}]_*=0,$$

it follows from Eq. (1) that

$$\Phi([I\bullet(i(P_2-P_1)),T]_*)=0.$$

From this, we get $[I\bullet(i(P_2-P_1)),T]_*=0$. So $T_{11}=T_{22}=0$.

Since $[I\bullet A_{12},P_1]_*=0$, it follows from Eq. (2) that

$$\Phi([I\bullet T,P_1]_*)=\Phi([I\bullet B_{21},P_1]_*).$$

By the injectivity of Φ, we obtain that

$$2(TP_1-P_1T^*)=[I\bullet T,P_1]_*=[I\bullet B_{21},P_1]_*=2(B_{21}-B_{21}^*).$$

Hence $T_{21} = B_{21}$. Similarly, $T_{12} = A_{12}$, proving the claim.

Claim 5. For every $A_{11} \in \mathcal{A}_{11}, B_{12} \in \mathcal{A}_{12}, C_{21} \in \mathcal{A}_{21}, D_{22} \in \mathcal{A}_{22}$, we have

$$\Phi(A_{11} + B_{12} + C_{21}) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21})$$

and

$$\Phi(B_{12} + C_{21} + D_{22}) = \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22}).$$

Let $T = \sum_{i,j=1}^{2} T_{ij} \in \mathcal{A}$ be such that

$$\Phi(T) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}).$$

It follows from Eq. (1) and Claim 4 that

$$\begin{aligned}
&\Phi(2i(P_2T + TP_2)) \\
&= \Phi([I \bullet (iP_2), T]_*) \\
&= \Phi([I \bullet (iP_2), A_{11}]_*) + \Phi([I \bullet (iP_2), B_{12}]_*) + \Phi([I \bullet (iP_2), C_{21}]_*) \\
&= \Phi(2iB_{12}) + \Phi(2iC_{21}) \\
&= \Phi(2i(B_{12} + C_{21})).
\end{aligned}$$

Thus $P_2T + TP_2 = B_{12} + C_{21}$, which implies $T_{22} = 0, T_{12} = B_{12}, T_{21} = C_{21}$. Now we get $T = T_{11} + B_{12} + C_{21}$.

Since

$$[I \bullet (i(P_2 - P_1)), B_{12}]_* = [I \bullet (i(P_2 - P_1)), C_{21}]_* = 0,$$

it follows from Eq. (1) that

$$\Phi([I \bullet (i(P_2 - P_1)), T]_*) = \Phi([I \bullet (i(P_2 - P_1)), A_{11}]_*),$$

from which we get $T_{11} = A_{11}$. Consequently, $\Phi(A_{11} + B_{12} + C_{21}) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21})$.

Similarly, we can get that $\Phi(B_{12} + C_{21} + D_{22}) = \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22})$.

Claim 6. For every $A_{11} \in \mathcal{A}_{11}, B_{12} \in \mathcal{A}_{12}, C_{21} \in \mathcal{A}_{21}, D_{22} \in \mathcal{A}_{22}$, we have

$$\Phi(A_{11} + B_{12} + C_{21} + D_{22}) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22}).$$

Let $T = \sum_{i,j=1}^{2} T_{ij} \in \mathcal{A}$ be such that

$$\Phi(T) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22}).$$

It follows from Eq. (1) and Claim 5 that

$$\begin{aligned}
\Phi(2iP_1T + 2iTP_1) &= \Phi([I \bullet (iP_1), T]_*) \\
&= \Phi([I \bullet (iP_1), A_{11}]_*) + \Phi([I \bullet (iP_1), B_{12}]_*) \\
&+ \Phi([I \bullet (iP_1), C_{21}]_*) + \Phi([I \bullet (iP_1), D_{22}]_*) \\
&= \Phi(4iA_{11}) + \Phi(2iB_{12}) + \Phi(2iC_{21}) \\
&= \Phi(4iA_{11} + 2iB_{12} + 2iC_{21}).
\end{aligned}$$

Thus

$$P_1T + TP_1 = 2A_{11} + B_{12} + C_{21},$$

and then $T_{11} = A_{11}, T_{12} = B_{12}, T_{21} = C_{21}$.

Similarly, we can get

$$\Phi(2iP_2T + 2iTP_2) = \Phi(4iD_{22} + 2iB_{12} + 2iC_{21}).$$

From this, we get $T_{22} = D_{22}$, proving the claim.

Claim 7. For every $A_{jk}, B_{jk} \in \mathcal{A}_{jk}, 1 \leq j \neq k \leq 2$, we have

$$\Phi(A_{jk} + B_{jk}) = \Phi(A_{jk}) + \Phi(B_{jk}).$$

For every $A_{jk}, B_{jk} \in \mathcal{A}_{jk}$, since

$$[\frac{I}{2} \bullet (P_j + A_{jk}), P_k + B_{jk}]_* = (A_{jk} + B_{jk}) + A_{jk}^* + B_{jk}A_{jk}^*,$$

we get from Claim 6 that

$$\begin{aligned}
&\Phi(A_{jk} + B_{jk}) + \Phi(A_{jk}^*) + \Phi(B_{jk}A_{jk}^*) \\
&= \Phi([\frac{I}{2} \bullet (P_j + A_{jk}), P_k + B_{jk}]_*) \\
&= [\Phi(\frac{I}{2}) \bullet \Phi(P_j + A_{jk}), \Phi(P_k + B_{jk})]_* \\
&= [\Phi(\frac{I}{2}) \bullet (\Phi(P_j) + \Phi(A_{jk})), \Phi(P_k) + \Phi(B_{jk})]_* \\
&= [\Phi(\frac{I}{2}) \bullet \Phi(P_j), \Phi(P_k)]_* + [\Phi(\frac{I}{2}) \bullet \Phi(P_j), \Phi(B_{jk})]_* \\
&+ [\Phi(\frac{I}{2}) \bullet \Phi(A_{jk}), \Phi(P_k)]_* + [\Phi(\frac{I}{2} \bullet \Phi(A_{jk}), \Phi(B_{jk})]_* \\
&= \Phi(B_{jk}) + \Phi(A_{jk} + A_{jk}^*) + \Phi(B_{jk}A_{jk}^*) \\
&= \Phi(B_{jk}) + \Phi(A_{jk}) + \Phi(A_{jk}^*) + \Phi(B_{jk}A_{jk}^*).
\end{aligned}$$

Then

$$\Phi(A_{jk} + B_{jk}) = \Phi(A_{jk}) + \Phi(B_{jk}).$$

Claim 8. For every $A_{jj}, B_{jj} \in \mathcal{A}_{jj}, 1 \leq j \leq 2$, we have

$$\Phi(A_{jj} + B_{jj}) = \Phi(A_{jj}) + \Phi(B_{jj}).$$

Let $T = \sum_{i,j=1}^{2} T_{ij} \in \mathcal{A}$ be such that

$$\Phi(T) = \Phi(A_{jj}) + \Phi(B_{jj}).$$

For $1 \leq j \neq k \leq 2$, it follows from Eq. (1) that

$$\Phi([I \bullet (iP_k), T]_*) = \Phi([I \bullet (iP_k), A_{jj}]_*) + \Phi([I \bullet (iP_k), B_{jj}]_*) = 0.$$

Hence $P_kT + TP_k = 0$, which implies $T_{jk} = T_{kj} = T_{kk} = 0$. Now we get $T = T_{jj}$.

For every $C_{jk} \in \mathcal{A}_{jk}$, $j \neq k$, it follows from Eq. (2) and Claim 7 that

$$\begin{aligned}
&\Phi(2T_{jj}C_{jk}) = \Phi([P_j \bullet T_{jj}, C_{jk}]_*) \\
&= \Phi([P_j \bullet A_{jj}, C_{jk}]_*) + \Phi([P_j \bullet B_{jj}, C_{jk}]_*) \\
&= \Phi(2A_{jj}C_{jk}) + \Phi(2B_{jj}C_{jk}) \\
&= \Phi(2(A_{jj}C_{jk} + B_{ii}C_{jk})).
\end{aligned}$$

Hence

$$(T_{jj} - A_{jj} - B_{jj})C_{jk} = 0$$

for all $C_{jk} \in \mathcal{A}_{jk}$, that is, $(T_{jj} - A_{jj} - B_{jj})CP_j = 0$ for all $C \in \mathcal{A}$. It follows from Lemma 2.2 that $T_{jj} = A_{jj} + B_{jj}$, proving the claim.

Claim 9. Φ is additive.

Let $A = \sum_{i,j=1}^{2} A_{ij}, B = \sum_{i,j=1}^{2} B_{ij} \in \mathcal{A}$. By Claim 6, Claim 7 and Claim 8, we have

$$\begin{aligned}\Phi(A+B) &= \Phi(\sum_{i,j=1}^{2} A_{ij} + \sum_{i,j=1}^{2} B_{ij}) = \Phi(\sum_{i,j=1}^{2}(A_{ij}+B_{ij})) \\ &= \sum_{i,j=1}^{2}\Phi(A_{ij}+B_{ij}) = \sum_{i,j=1}^{2}\Phi(A_{ij}) + \sum_{i,j=1}^{2}\Phi(B_{ij}) \\ &= \Phi(\sum_{i,j=1}^{2} A_{ij}) + \Phi(\sum_{i,j=1}^{2} B_{ij}) = \Phi(A)+\Phi(B).\end{aligned}$$

Claim 10. $\Phi(I)^2 = I$.

By Claim 2 and Claim 3, $\Phi(I)$ is a self-adjoint central element. For all $A \in \mathcal{A}$, it follows from Claim 9 that

$$2\Phi(A - A^*) = \Phi([I \bullet A, I]_*) = [\Phi(I) \bullet \Phi(A), \Phi(I)]_* = 2\Phi(I)^2(\Phi(A) - \Phi(A)^*). \tag{4}$$

Consequently, for every $A = -A^* \in \mathcal{A}$,

$$\Phi(A) = \Phi(I)^2(\Phi(\frac{A}{2}) - \Phi(\frac{A}{2})^*), \tag{5}$$

which ensures that $\Phi(A) = -\Phi(A)^*$. Note that Φ^{-1} has the same properties as Φ, we have that Φ preserves the conjugate self-adjoint elements in both directions, i.e., $A = -A^*$ if and only if $\Phi(A) = -\Phi(A)^*$. It follows from the additivity of Φ and Eq. (5) that $\Phi(A) = \Phi(I)^2\Phi(A)$. By choosing $\Phi(A) = iI$, we have $\Phi(I)^2 = I$.

Now, defining a map $\phi : \mathcal{A} \to \mathcal{B}$ by $\phi(A) = \Phi(I)\Phi(A)$ for all $A \in \mathcal{A}$. It is easy to see that ϕ is an additive bijection with $\phi(I) = I$, and satisfies

$$\phi([A \bullet B, C]_*) = [\phi(A) \bullet \phi(B), \phi(C)]_*$$

for all $A, B, C \in \mathcal{A}$.

Claim 11 For all $A, B \in \mathcal{A}$, we have $\phi([A, B]_*) = [\phi(A), \phi(B)]_*$.

Indeed, for all $A, B \in \mathcal{A}$, we get that

$$2\phi([A, B]_*) = \phi(2[A, B]_*) = \phi([I \bullet A, B]_*) = [I \bullet \phi(A), \phi(B)]_* = 2[\phi(A), \phi(B)]_*.$$

Then $\phi([A, B]_*) = [\phi(A), \phi(B)]_*$.

Now, by the main result of [1], we have that the map $\phi = \Phi(I)\Phi$ is a sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism. □

$\mathcal{A}$ is a factor von Neumann algebra means that its center only contains the scalar operators. It is well known that the factor von Neumann algebra $\mathcal{A}$ is prime, in the sense that $A\mathcal{A}B = 0$ for $A, B \in \mathcal{A}$ implies either $A = 0$ or $B = 0$.

Theorem 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two factor von Neumann algebras with $\dim\mathcal{A} \geq 2$. Suppose that a bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi([A \bullet B, C]_*) = [\Phi(A) \bullet \Phi(B), \Phi(C)]_*$ for all $A, B, C \in \mathcal{A}$. Then Φ is a linear $*$-isomorphism, or a conjugate linear $*$-isomorphism, or the negative of a linear $*$-isomorphism, or the negative of a conjugate linear $*$-isomorphism.*

Proof. Let P be a nontrivial projection in $\mathcal{A}$. Since A is prime, then $ABP = 0$ for all $B \in \mathcal{A}$ implies $A = 0$. So Lemma 2.2 holds true for factor von Neumann algebras. It is easy to check that all claims of Theorem 2.4 hold true for factor von Neumann algebras. Since $\Phi(I)$ is a self-adjoint central element and $\Phi(I)^2 = I$, we get $\Phi(I) = I$ or $\Phi(I) = -I$. It follows from Claim 11 that Φ or $-\Phi$ is a map preserving the skew Lie product on factor von Neumann algebras. Now, by the main result of [3], we have that Φ or $-\Phi$ is a $*$-ring isomorphism. It is easy to show that Φ or $-\Phi$ is a map preserving the absolute value. Now, by Theorem 2.5 of [20], Φ or $-\Phi$ is a linear $*$-isomorphism or a conjugate linear $*$-isomorphism. Now, we have proved the theorem. □

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