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The Minus Order for Projections

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Abstract. Let $\mathcal{B}(\mathcal{H})^{Id}$ be the set of all projections on a Hilbert space $\mathcal{H}$. The necessary and sufficient conditions are presented for the existence of the supremum, as well as the infimum, of two arbitrary projections in $\mathcal{B}(\mathcal{H})^{Id}$ with respect to the minus order $\preceq$. For a projection Q in $\mathcal{B}(\mathcal{H})^{Id}$, the properties of the sets $\{P : P$ is an orthogonal projection on $\mathcal{H}$ and $Q \preceq P\}$ and $\{P : P$ is an orthogonal projection on $\mathcal{H}$ and $P \preceq Q\}$ are further explored.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be separable complex Hilbert spaces, $\mathcal{B}(\mathcal{H}, \mathcal{K})$ be the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$, and abbreviate $\mathcal{B}(\mathcal{H}, \mathcal{H})$ to $\mathcal{B}(\mathcal{H})$. For an operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, T^*, $\mathcal{N}(T)$ and $\mathcal{R}(T)$ denote the adjoint, the null space and the range of T, respectively. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be positive, if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$, where $\langle \cdot, \cdot\rangle$ is the inner product of $\mathcal{H}$. The set of all positive operators in $\mathcal{B}(\mathcal{H})$ is denoted by $\mathcal{B}(\mathcal{H})^+$ and we write $A \leq B$ if $B - A \in \mathcal{B}(\mathcal{H})^+$. For $A \in \mathcal{B}(\mathcal{H})$, A is said to be a projection (or idempotent) if $A^2 = A$. Let $\mathcal{B}(\mathcal{H})^{Id}$ be the set of all projections in $\mathcal{B}(\mathcal{H})$ and $\mathcal{P}(\mathcal{H})$ be the set of all orthogonal projections (self-adjoint projections) in $\mathcal{B}(\mathcal{H})$. Also, $P_{\mathcal{M}}$ denotes the orthogonal projection of $\mathcal{H}$ onto the closed subspace $\mathcal{M}$.

The minus partial order for matrices was introduced by Hartwig ([12]) and independently by Nambooripad ([19]), as a generalization of some classical partial orders. It was extended to operators on infinite dimensional Hilbert spaces in [4, 7, 21]. In particular, the minus partial order on $\mathcal{B}(\mathcal{H})$ is defined by $A \preceq B$ if there exist $E, F \in \mathcal{B}(\mathcal{H})^{Id}$ such that $A = EB$ and $A^* = FB^*$ (see [7]). It is clear that $A \preceq B$ if and only if $A^* \preceq B^*$. Djikić et al. also proved in [7] that $A \preceq B$ if and only if $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and there exists a projection E in $\mathcal{B}(\mathcal{H})^{Id}$ such that $A = EB$. So $A \preceq B$ implies the inclusions $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$. Moreover, for $P, Q \in \mathcal{B}(\mathcal{H})^{Id}$,

$$P \preceq Q \Longleftrightarrow PQ = QP = P \Longleftrightarrow \mathcal{R}(P) \subseteq \mathcal{R}(Q) \text{ and } \mathcal{N}(Q) \subseteq \mathcal{N}(P).$$

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Also, if $E, F \in \mathcal{P}(\mathcal{H})$, it is trivial that $E \leq F$ if and only if $E \leq F$.

For the usual operator order, the lattice properties of $\mathcal{B}(\mathcal{H})$ have been studied in different contexts (see [2, 10, 11, 15]). In [15], Kadison showed that for self-adjoint operators A and B, the infimum $A \wedge B$ exists in the set of all self-adjoint operators with respect to operator order $\leq$ if and only if A and B are comparable ($A \leq B$ or $B \leq A$). After many years, the characterization of pairs of positive bounded operators that admit the infimum over the cone of positive operators was given in different contexts by Ando ([2]), Gheondea et al. ([11]) and Du et al. ([10]). In recent years, the lattice properties of $\mathcal{B}(\mathcal{H})$ endowed with the star partial order were studied thoroughly in [3, 8, 9, 22]. Moreover, with respect to the star partial order, it was showed that the lattice properties of $\mathcal{B}(\mathcal{H})^{Id}$ is different from the lattice properties of $\mathcal{B}(\mathcal{H})$, and necessary and sufficient conditions were given for the existence of the supremum within $\mathcal{B}(\mathcal{H})^{Id}$ (see [23]). However, the lattice properties of the minus order seem not have been revealed earlier, though the minus order is a well known order defined and studied for matrices and later on for operators in $\mathcal{B}(\mathcal{H})$ by many authors (see [4, 7, 19, 20]).

In this paper we study the minus order on $\mathcal{B}(\mathcal{H})^{Id}$, and the lattice properties of the poset $(\mathcal{B}(\mathcal{H})^{Id}, \leq)$ is a subject of our interest. For $P, Q \in \mathcal{B}(\mathcal{H})^{Id}$, we denote by $P \underset{\leq}{\vee} Q$ the least upper bound (the supremum) of P and Q within $\mathcal{B}(\mathcal{H})^{Id}$, if it exists. To be more precise, $P \underset{\leq}{\vee} Q \in \mathcal{B}(\mathcal{H})^{Id}$ is an upper bound of P and Q such that $P \underset{\leq}{\vee} Q \leq R$ for every other upper bound $R \in \mathcal{B}(\mathcal{H})^{Id}$ of P and Q. Analogously, $P \underset{\leq}{\wedge} Q$ denotes the greatest lower bound (the infimum) of P and Q within $\mathcal{B}(\mathcal{H})^{Id}$, if it exists.

The contents of this paper are as follows. In Section 2, we study the lattice properties of the poset $(\mathcal{B}(\mathcal{H})^{Id}, \leq)$. For $P, Q \in \mathcal{B}(\mathcal{H})^{Id}$, we present some necessary and sufficient conditions for the existence of $P \underset{\leq}{\vee} Q$, as well as of $P \underset{\leq}{\wedge} Q$. In this case, $P \underset{\leq}{\vee} Q$ and $P \underset{\leq}{\wedge} Q$ are given.

In Section 3, we explore the properties of the sets $\{P : P \in \mathcal{P}(\mathcal{H}) \text{ and } Q \leq P\}$ and $\{P : P \in \mathcal{P}(\mathcal{H}) \text{ and } P \leq Q\}$ for a given projection Q in $\mathcal{B}(\mathcal{H})^{Id}$. We first show that the sets $\{P : P \leq Q \text{ and } P \in \mathcal{P}(\mathcal{H})\}$ and $\{P : Q \leq P \text{ and } P \in \mathcal{P}(\mathcal{H})\}$ have the maximum and the minimum with respect to the minus order, respectively. Then we study in detail the maximum $\underset{\leq}{max}\{P : P \leq Q \text{ and } P \in \mathcal{P}(\mathcal{H})\}$ and the minimum $\underset{\leq}{min}\{P : Q \leq P \text{ and } P \in \mathcal{P}(\mathcal{H})\}$.

2. $P \underset{\leq}{\vee} Q$ and $P \underset{\leq}{\wedge} Q$

If $\mathcal{M}$ and $\mathcal{N}$ are closed subspaces of $\mathcal{H}$ and $\mathcal{M} \cap \mathcal{N} = \{0\}$, the direct sum of $\mathcal{M}$ and $\mathcal{N}$ is usually denoted by $\mathcal{M} \dotplus \mathcal{N}$. If more $\mathcal{M} \dotplus \mathcal{N} = \mathcal{H}$, then there exists a (unique) projection in $\mathcal{B}(\mathcal{H})^{Id}$ with range $\mathcal{M}$ and nullspace $\mathcal{N}$, say $P_{\mathcal{M}//\mathcal{N}}$. In this case, $P_{\mathcal{M}//\mathcal{N}}$ is called the projection onto $\mathcal{M}$ along $\mathcal{N}$. On the other hand, the orthogonal sum of $\mathcal{M}$ and $\mathcal{N}$ is denoted by $\mathcal{M} \oplus \mathcal{N}$, and $\mathcal{M} \ominus \mathcal{N} := \mathcal{M} \cap (\mathcal{M} \cap \mathcal{N})^{\perp}$.

It is well known that $\mathcal{M} + \mathcal{N}$ is a closed subspace of $\mathcal{H}$ if $\mathcal{M}$ is a finite dimensional subspace and $\mathcal{N}$ is a closed subspace of $\mathcal{H}$. Moreover, we have the following result.

Lemma 2.1. *Let $\mathcal{M}$ be a finite dimensional subspace of $\mathcal{H}$ and let $\mathcal{N}$ be a closed subspace of $\mathcal{H}$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$. Then $\mathcal{M} \dotplus [\mathcal{N} \oplus (\mathcal{M}^{\perp} \cap \mathcal{N}^{\perp})] = \mathcal{H}$.*

Proof. Since $(\mathcal{M} + \mathcal{N})^{\perp} = \mathcal{M}^{\perp} \cap \mathcal{N}^{\perp}$, $\mathcal{M} + (\mathcal{N} \oplus (\mathcal{M}^{\perp} \cap \mathcal{N}^{\perp})) = (\mathcal{M} + \mathcal{N}) \oplus (\mathcal{M}^{\perp} \cap \mathcal{N}^{\perp}) = \mathcal{H}$.

It remains to prove that $\mathcal{M} \cap (\mathcal{N} \oplus (\mathcal{M}^{\perp} \cap \mathcal{N}^{\perp})) = \{0\}$. Suppose that $x \in \mathcal{M} \cap (\mathcal{N} \oplus (\mathcal{M}^{\perp} \cap \mathcal{N}^{\perp}))$. Then $x \in \mathcal{N} \oplus (\mathcal{M}^{\perp} \cap \mathcal{N}^{\perp})$ implies $x = y + z$ for some $y \in \mathcal{N}$ and $z \in \mathcal{M}^{\perp} \cap \mathcal{N}^{\perp}$. It follows that $z = x - y \in \mathcal{M} + \mathcal{N}$, and since $z \in \mathcal{M}^{\perp} \cap \mathcal{N}^{\perp}$, $z = 0$. Thus $x(= y) \in \mathcal{M} \cap \mathcal{N}$, and hence $x = 0$. So $\mathcal{M} \cap (\mathcal{N} \oplus (\mathcal{M}^{\perp} \cap \mathcal{N}^{\perp})) = \{0\}$. □

Lemma 2.2. *Let $P, Q \in \mathcal{B}(\mathcal{H})^{Id}$.*

(a) $P \underset{\leq}{\vee} Q = I$ if and only if $\mathcal{N}(P) \cap \mathcal{N}(Q) \subseteq \overline{\mathcal{R}(P) + \mathcal{R}(Q)}$.

(b) $P \underset{\leq}{\vee} Q = P_{\overline{\mathcal{R}(P)+\mathcal{R}(Q)}//\mathcal{N}(P)\cap\mathcal{N}(Q)}$ if and only if $(\mathcal{N}(P) \cap \mathcal{N}(Q)) \dotplus \overline{\mathcal{R}(P) + \mathcal{R}(Q)} = \mathcal{H}$.

Proof. (a) Assume $\mathcal{N}(P) \cap \mathcal{N}(Q) \subseteq \overline{\mathcal{R}(P) + \mathcal{R}(Q)}$. If R is a projection in $\mathcal{B}(\mathcal{H})^{Id}$ such that $P \leq R$ and $Q \leq R$, then

$$\mathcal{N}(R) \subseteq \mathcal{N}(P) \cap \mathcal{N}(Q) \subseteq \overline{\mathcal{R}(P) + \mathcal{R}(Q)} \subseteq \mathcal{R}(R).$$

It follows that $\mathcal{N}(R) = \{0\}$, and hence $R = I$. Thus $P \underset{\leq}{\vee} Q = I$.

For the converse, assume $P \underset{\leq}{\vee} Q = I$. If $\mathcal{N}(P) \cap \mathcal{N}(Q) \nsubseteq \overline{\mathcal{R}(P) + \mathcal{R}(Q)}$, then we can pick a nonzero element $x \in (\mathcal{N}(P) \cap \mathcal{N}(Q)) \setminus \overline{\mathcal{R}(P) + \mathcal{R}(Q)}$. Let $\mathcal{M} := \overline{\{x\}}$ be the one-dimensional subspace spanned by x. It is clear that $\mathcal{M} \cap \overline{\mathcal{R}(P) + \mathcal{R}(Q)} = \{0\}$, and by Lemma 2.1,

$$\mathcal{M} \dotplus [\overline{\mathcal{R}(P) + \mathcal{R}(Q)} \oplus (\mathcal{M}^{\perp} \cap \overline{\mathcal{R}(P) + \mathcal{R}(Q)}^{\perp})] = \mathcal{H}.$$

Then the projection $P_{\overline{\mathcal{R}(P)+\mathcal{R}(Q)} \oplus (\mathcal{M}^{\perp} \cap \overline{\mathcal{R}(P)+\mathcal{R}(Q)}^{\perp})\backslash\backslash \mathcal{M}}$ is a common upper bound of P and Q. Moreover, since $\mathcal{N}(P_{\overline{\mathcal{R}(P)+\mathcal{R}(Q)} \oplus (\mathcal{M}^{\perp} \cap \overline{\mathcal{R}(P)+\mathcal{R}(Q)}^{\perp})\backslash\backslash \mathcal{M}}) = \mathcal{M} \neq \{0\}$,

$$P \underset{\leq}{\vee} Q \leq P_{\overline{\mathcal{R}(P)+\mathcal{R}(Q)} \oplus (\mathcal{M}^{\perp} \cap \overline{\mathcal{R}(P)+\mathcal{R}(Q)}^{\perp})\backslash\backslash \mathcal{M}} \neq I.$$

We have arrived at a contradiction.

(b) If $P \underset{\leq}{\vee} Q = P_{\overline{\mathcal{R}(P)+\mathcal{R}(Q)}//\mathcal{N}(P)\cap\mathcal{N}(Q)}$, then

$$\overline{\mathcal{R}(P) + \mathcal{R}(Q)} \dotplus (\mathcal{N}(P) \cap \mathcal{N}(Q)) = \mathcal{R}(P_{\overline{\mathcal{R}(P)+\mathcal{R}(Q)}//\mathcal{N}(P)\cap\mathcal{N}(Q)}) \dotplus \mathcal{N}(P_{\overline{\mathcal{R}(P)+\mathcal{R}(Q)}//\mathcal{N}(P)\cap\mathcal{N}(Q)}) = \mathcal{H}.$$

Now assume $(\mathcal{N}(P) \cap \mathcal{N}(Q)) \dotplus \overline{\mathcal{R}(P) + \mathcal{R}(Q)} = \mathcal{H}$. It is easy to check that the projection $P_{\overline{\mathcal{R}(P)+\mathcal{R}(Q)}//\mathcal{N}(P)\cap\mathcal{N}(Q)}$ is a common upper bound of P and Q. Moreover, if R is a common upper bound of P and Q, then

$$\mathcal{N}(R) \subseteq \mathcal{N}(P) \cap \mathcal{N}(Q) = \mathcal{N}(P_{\overline{\mathcal{R}(P)+\mathcal{R}(Q)}//\mathcal{N}(P)\cap\mathcal{N}(Q)})$$

and

$$\mathcal{R}(R) \supseteq \overline{\mathcal{R}(P) + \mathcal{R}(Q)} = \mathcal{R}(P_{\overline{\mathcal{R}(P)+\mathcal{R}(Q)}//\mathcal{N}(P)\cap\mathcal{N}(Q)}),$$

and hence $P_{\overline{\mathcal{R}(P)+\mathcal{R}(Q)}//\mathcal{N}(P)\cap\mathcal{N}(Q)} \leq R$. So $P \underset{\leq}{\vee} Q = P_{\overline{\mathcal{R}(P)+\mathcal{R}(Q)}//\mathcal{N}(P)\cap\mathcal{N}(Q)}$. □

Lemma 2.3. *Let $P, Q \in \mathcal{B}(\mathcal{H})^{Id}$. If $P \underset{\leq}{\vee} Q$ exists, then either $\mathcal{N}(P) \cap \mathcal{N}(Q) \subseteq \overline{\mathcal{R}(P) + \mathcal{R}(Q)}$ or $(\mathcal{N}(P) \cap \mathcal{N}(Q)) \dotplus \overline{\mathcal{R}(P) + \mathcal{R}(Q)} = \mathcal{H}$.*

Proof. Let $R = P \underset{\leq}{\vee} Q$. Then $\mathcal{N}(R) \subseteq \mathcal{N}(P) \cap \mathcal{N}(Q)$ and $\overline{\mathcal{R}(P) + \mathcal{R}(Q)} \subseteq \mathcal{R}(R)$. In light of Lemma 2.2, it suffices to prove that $R = I$ or $R = P_{\overline{\mathcal{R}(P)+\mathcal{R}(Q)}//\mathcal{N}(P)\cap\mathcal{N}(Q)}$.

Claim 2.4. *If $R \neq I$, then $\mathcal{N}(R) = \mathcal{N}(P) \cap \mathcal{N}(Q)$.*

We only need to prove $\mathcal{N}(P) \cap \mathcal{N}(Q) \subseteq \mathcal{N}(R)$. Let $x \in \mathcal{N}(P) \cap \mathcal{N}(Q)$. Without lose of generality, we may assume $R(x) \neq x$. Indeed, if $R(x) = x$, then we can find a nonzero element $y \in \mathcal{N}(R)$ because $R \neq I$, and consider $x + y \in \mathcal{N}(P) \cap \mathcal{N}(Q)$. Clearly, $R(x + y) = R(x) = x \neq x + y$.

Since $\overline{\{x\}} \cap \overline{\mathcal{R}(P) + \mathcal{R}(Q)} \subseteq \overline{\{x\}} \cap \mathcal{R}(R) = \{0\}$, we conclude by Lemma 2.1 that

$$\overline{\{x\}} \dotplus [\overline{\mathcal{R}(P) + \mathcal{R}(Q)} \oplus (\overline{\{x\}}^{\perp} \cap \overline{\mathcal{R}(P) + \mathcal{R}(Q)}^{\perp})] = \mathcal{H}.$$

Then the projection $P_{\overline{\mathcal{R}(P)+\mathcal{R}(Q)} \oplus (\overline{\{x\}}^{\perp} \cap \overline{\mathcal{R}(P)+\mathcal{R}(Q)}^{\perp})//\overline{\{x\}}}$ is a common upper bound of P and Q, and hence $R = P \underset{\leq}{\vee} Q \leq P_{\overline{\mathcal{R}(P)+\mathcal{R}(Q)} \oplus (\overline{\{x\}}^{\perp} \cap \overline{\mathcal{R}(P)+\mathcal{R}(Q)}^{\perp})//\overline{\{x\}}}$. It follows that

$$\mathcal{N}(P_{\overline{\mathcal{R}(P)+\mathcal{R}(Q)} \oplus (\overline{\{x\}}^{\perp} \cap \overline{\mathcal{R}(P)+\mathcal{R}(Q)}^{\perp})//\overline{\{x\}}}) = \overline{\{x\}} \subseteq \mathcal{N}(R).$$

This implies $x \in \mathcal{N}(R)$. So $\mathcal{N}(P) \cap \mathcal{N}(Q) \subseteq \mathcal{N}(R)$.

Claim 2.5. *If $R \neq I$, then $\mathcal{R}(R) = \overline{\mathcal{R}(P) + \mathcal{R}(Q)}$.*

Conversely, if $\overline{\mathcal{R}(P) + \mathcal{R}(Q)} \subsetneq \mathcal{R}(R)$, then we can find a nonzero element $x \in \mathcal{R}(R) \setminus \overline{\mathcal{R}(P) + \mathcal{R}(Q)}$. It is clear that $\overline{\{x\}} \cap \overline{\mathcal{R}(P) + \mathcal{R}(Q)} = \{0\}$. Let $\mathcal{M} = \mathcal{R}(R) \ominus (\overline{\{x\}} + \overline{\mathcal{R}(P) + \mathcal{R}(Q)})$. For a nonzero element y in $\mathcal{N}(P) \cap \mathcal{N}(Q)$, we have

$$\begin{aligned} & (\overline{\{x+y\}} + (\overline{\mathcal{R}(P) + \mathcal{R}(Q)} \oplus \mathcal{M})) \dotplus (\mathcal{N}(P) \cap \mathcal{N}(Q)) \\ = \; & ((\overline{\{x\}} + \overline{\mathcal{R}(P) + \mathcal{R}(Q)}) \oplus \mathcal{M}) \dotplus (\mathcal{N}(P) \cap \mathcal{N}(Q)) \end{aligned}$$

and by Claim 2.4, we see that

$$(\overline{\{x+y\}} + (\overline{\mathcal{R}(P) + \mathcal{R}(Q)} \oplus \mathcal{M})) \dotplus (\mathcal{N}(P) \cap \mathcal{N}(Q)) = \mathcal{R}(R) \dotplus \mathcal{N}(R) = \mathcal{H}.$$

Then the projection $P_{(\overline{\{x+y\}} + (\overline{\mathcal{R}(P)+\mathcal{R}(Q)} \oplus \mathcal{M}))//\mathcal{N}(P)\cap\mathcal{N}(Q)}$ is a common upper bound of P and Q, and hence $R = P \underset{\leq}{\vee} Q \leq P_{(\overline{\{x+y\}} + (\overline{\mathcal{R}(P)+\mathcal{R}(Q)} \oplus \mathcal{M}))//\mathcal{N}(P)\cap\mathcal{N}(Q)}$. So

$$\mathcal{R}(R) \subseteq \mathcal{R}(P_{(\overline{\{x+y\}} + (\overline{\mathcal{R}(P)+\mathcal{R}(Q)} \oplus \mathcal{M}))//\mathcal{N}(P)\cap\mathcal{N}(Q)}) = \overline{\{x+y\}} + (\overline{\mathcal{R}(P) + \mathcal{R}(Q)} \oplus \mathcal{M}),$$

which yields $x \in \overline{\{x+y\}} + (\overline{\mathcal{R}(P) + \mathcal{R}(Q)} \oplus \mathcal{M})$. Then $x = 0$ follows from the fact $((\overline{\{x\}} + \overline{\mathcal{R}(P) + \mathcal{R}(Q)}) \oplus \mathcal{M}) \cap (\mathcal{N}(P) \cap \mathcal{N}(Q)) = \{0\}$. This contradiction indicates $\mathcal{R}(R) = \overline{\mathcal{R}(P) + \mathcal{R}(Q)}$.

By Claim 2.4 and Claim 2.5, we have $R = I$ or $R = P_{\overline{\mathcal{R}(P)+\mathcal{R}(Q)}//\mathcal{N}(P)\cap\mathcal{N}(Q)}$. □

In the finite dimensional case, the existence of $P \underset{\leq}{\vee} Q$ has been considered in [21]. Combining Lemma 2.2 and Lemma 2.3, we have the following theorem.

Theorem 2.6. *Let $P, Q \in \mathcal{B}(\mathcal{H})^{Id}$. Then $P \underset{\leq}{\vee} Q$ exists if and only if either $\mathcal{N}(P) \cap \mathcal{N}(Q) \subseteq \overline{\mathcal{R}(P) + \mathcal{R}(Q)}$ or $(\mathcal{N}(P) \cap \mathcal{N}(Q)) \dotplus \overline{\mathcal{R}(P) + \mathcal{R}(Q)} = \mathcal{H}$. Moreover, if $\mathcal{N}(P) \cap \mathcal{N}(Q) \subseteq \overline{\mathcal{R}(P) + \mathcal{R}(Q)}$, then $P \underset{\leq}{\vee} Q = I$; if $(\mathcal{N}(P) \cap \mathcal{N}(Q)) \dotplus \overline{\mathcal{R}(P) + \mathcal{R}(Q)} = \mathcal{H}$, then $P \underset{\leq}{\vee} Q = P_{\overline{\mathcal{R}(P)+\mathcal{R}(Q)}//\mathcal{N}(P)\cap\mathcal{N}(Q)}$.*

Lemma 2.7. *Let $P, Q \in \mathcal{B}(\mathcal{H})^{Id}$. Then $P \underset{\leq}{\vee} Q = R$ if and only if $(I-P) \underset{\leq}{\wedge} (I-Q) = I - R$.*

Proof. Assume that $P \underset{\leq}{\vee} Q = R$. Then we have $P \leq R$ and $P \leq R$, and hence $I - R \leq I - P$ and $I - R \leq I - Q$. So $I - R$ is a lower bound of $I - P$ and $I - Q$. Moreover, if S is projection in $\mathcal{B}(\mathcal{H})^{Id}$ such that $S \leq I - P$ and $S \leq I - Q$, then $P \leq I - S$ and $Q \leq I - S$. It follows that $R = P \underset{\leq}{\vee} Q \leq I - S$, or equivalently, $S \leq I - R$. So $(I-P) \underset{\leq}{\wedge} (I-Q) = I - R$. The converse follows by a similar proof. □

Theorem 2.8. *Let $P, Q \in \mathcal{B}(\mathcal{H})^{Id}$. Then $P \underset{\leq}{\wedge} Q$ exists if and only if either $\mathcal{R}(P) \cap \mathcal{R}(Q) \subseteq \overline{\mathcal{N}(P) + \mathcal{N}(Q)}$ or $(\mathcal{R}(P) \cap \mathcal{R}(Q)) \dotplus \overline{\mathcal{N}(P) + \mathcal{N}(Q)} = \mathcal{H}$. Moreover, if $\mathcal{R}(P) \cap \mathcal{R}(Q) \subseteq \overline{\mathcal{N}(P) + \mathcal{N}(Q)}$, then $P \underset{\leq}{\wedge} Q = 0$; if $(\mathcal{R}(P) \cap \mathcal{R}(Q)) \dotplus \overline{\mathcal{N}(P) + \mathcal{N}(Q)} = \mathcal{H}$, then $P \underset{\leq}{\wedge} Q = P_{\mathcal{R}(P)\cap\mathcal{R}(Q)//\overline{\mathcal{N}(P)+\mathcal{N}(Q)}}$.*

Proof. By Lemma 2.7, $P \underset{\leq}{\wedge} Q$ exists if and only if $(I-P) \underset{\leq}{\vee} (I-Q)$ exists, and by Theorem 2.6, this is the case if and only if $\mathcal{N}(I-P) \cap \mathcal{N}(I-Q) \subseteq \overline{\mathcal{R}(I-P) + \mathcal{R}(I-Q)}$ or $(\mathcal{N}(I-P) \cap \mathcal{N}(I-Q)) \dotplus \overline{\mathcal{R}(I-P) + \mathcal{R}(I-Q)} = \mathcal{H}$, or equivalently, if and only if $\mathcal{R}(P) \cap \mathcal{R}(Q) \subseteq \overline{\mathcal{N}(P) + \mathcal{N}(Q)}$ or $(\mathcal{R}(P) \cap \mathcal{R}(Q)) \dotplus \overline{\mathcal{N}(P) + \mathcal{N}(Q)} = \mathcal{H}$.

Moreover, if $\mathcal{R}(P) \cap \mathcal{R}(Q) \subseteq \overline{\mathcal{N}(P) + \mathcal{N}(Q)}$, that is, $\mathcal{N}(I-P) \cap \mathcal{N}(I-Q) \subseteq \overline{\mathcal{R}(I-P) + \mathcal{R}(I-Q)}$, then by Lemma 2.7 and Theorem 2.6, $P \underset{\leq}{\wedge} Q = I - (I-P) \underset{\leq}{\vee} (I-Q) = 0$.

Similarly, $(\mathcal{R}(P) \cap \mathcal{R}(Q)) \dotplus \overline{\mathcal{N}(P) + \mathcal{N}(Q)} = \mathcal{H}$ implies $(\mathcal{N}(I-P) \cap \mathcal{N}(I-Q)) \dotplus \overline{\mathcal{R}(I-P) + \mathcal{R}(I-Q)} = \mathcal{H}$, which yields

$$\begin{aligned} P \underset{\leq}{\wedge} Q = I - (I-P) \underset{\leq}{\vee} (I-Q) &= I - P_{\overline{\mathcal{R}(I-P)+\mathcal{R}(I-Q)}//\mathcal{N}(I-P)\cap\mathcal{N}(I-Q)} \\ &= I - P_{\overline{\mathcal{N}(P)+\mathcal{N}(Q)}//\mathcal{R}(P)\cap\mathcal{R}(Q)} \\ &= P_{\mathcal{R}(P)\cap\mathcal{R}(Q)//\overline{\mathcal{N}(P)+\mathcal{N}(Q)}}. \end{aligned}$$

□

Corollary 2.9. *Let $P \in \mathcal{B}(\mathcal{H})^{Id}$.*

(a) $P \underset{\leq}{\vee} (I-P) = I$ and $P \underset{\leq}{\wedge} (I-P) = 0$.

(b) $P \underset{\leq}{\vee} P^* = P_{\overline{\mathcal{R}(P+P^*)}}$ and $P \underset{\leq}{\wedge} P^* = P_{\mathcal{R}(P)\cap\mathcal{R}(P^*)}$.

Proof. (*a*) Since $\mathcal{N}(P) \cap \mathcal{N}(I-P) = \mathcal{N}(P) \cap \mathcal{R}(P) = \{0\}$, we have $P \underset{\leq}{\vee} (I-P) = I$ by Theorem 2.6. Moreover, Lemma 2.7 yields $P \underset{\leq}{\wedge} (I-P) = I - (I-P) \underset{\leq}{\vee} P = 0$.

(*b*) Since $(\mathcal{N}(P) \cap \mathcal{N}(P^*)) \oplus \overline{\mathcal{R}(P) + \mathcal{R}(P^*)} = \mathcal{H}$, we conclude by Theorem 2.6 that

$$P \underset{\leq}{\vee} P^* = P_{\overline{\mathcal{R}(P)+\mathcal{R}(P^*)}//\mathcal{N}(P)\cap\mathcal{N}(P^*)} = P_{\overline{\mathcal{R}(P+P^*)}}.$$

On the other hand, since $(\mathcal{R}(P)\cap\mathcal{R}(P^*))\oplus\overline{\mathcal{N}(P) + \mathcal{N}(P^*)} = \mathcal{H}$, Theorem 2.8 yields $P \underset{\leq}{\wedge} P^* = P_{\mathcal{R}(P)\cap\mathcal{R}(P^*)//\overline{\mathcal{N}(P)+\mathcal{N}(P^*)}} = P_{\mathcal{R}(P)\cap\mathcal{R}(P^*)}$. □

By Lemma 2.2, $P \underset{\leq}{\vee} Q = I$ if and only if $\mathcal{N}(P) \cap \mathcal{N}(Q) \subseteq \overline{\mathcal{R}(P) + \mathcal{R}(Q)}$. The following corollary gives characterizations of $P,\ Q \in \mathcal{B}(\mathcal{H})^{Id}$ such that $P \underset{\leq}{\vee} Q \in \mathcal{P}(\mathcal{H}) \setminus \{I\}$.

Corollary 2.10. *Let $P,\ Q \in \mathcal{B}(\mathcal{H})^{Id}$. Then the following statements are equivalent.*

(a) $P \underset{\leq}{\vee} Q \in \mathcal{P}(\mathcal{H}) \setminus \{I\}$.

(b) $\mathcal{N}(P) \cap \mathcal{N}(Q) = \mathcal{N}(P^*) \cap \mathcal{N}(Q^*) \neq \{0\}$.

(c) $\overline{\mathcal{R}(PP^* + QQ^*)} = \overline{\mathcal{R}(P^*P + Q^*Q)} \neq \mathcal{H}$.

(d) $\{0\} \neq (\mathcal{N}(P) \cap \mathcal{N}(Q)) \cup (\mathcal{N}(P^*) \cap \mathcal{N}(Q^*)) \subseteq \mathcal{N}(P^* + P) \cap \mathcal{N}(Q^* + Q)$.

Proof. (*a*) ⇔ (*b*) : By Theorem 2.6, $P \underset{\leq}{\vee} Q \in \mathcal{P}(\mathcal{H}) \setminus \{I\}$ if and only if

$$\mathcal{N}(P) \cap \mathcal{N}(Q) \neq \{0\} \text{ and } (\mathcal{N}(P) \cap \mathcal{N}(Q)) \oplus \overline{\mathcal{R}(P) + \mathcal{R}(Q)} = \mathcal{H},$$

and this is the case if and only if

$$\{0\} \neq \mathcal{N}(P) \cap \mathcal{N}(Q) = \overline{\mathcal{R}(P) + \mathcal{R}(Q)}^{\perp} = \mathcal{N}(P^*) \cap \mathcal{N}(Q^*).$$

(*b*) ⇔ (*c*) : Since

$$\overline{\mathcal{R}(PP^* + QQ^*)} = (\mathcal{N}(PP^* + QQ^*))^{\perp} = (\mathcal{N}(P^*) \cap \mathcal{N}(Q^*))^{\perp}$$

and

$$\overline{\mathcal{R}(P^*P + Q^*Q)} = (\mathcal{N}(P^*P + Q^*Q))^{\perp} = (\mathcal{N}(P) \cap \mathcal{N}(Q))^{\perp},$$

$\overline{\mathcal{R}(PP^* + QQ^*)} = \overline{\mathcal{R}(P^*P + Q^*Q)} \neq \mathcal{H}$ if and only if $\mathcal{N}(P) \cap \mathcal{N}(Q) = \mathcal{N}(P^*) \cap \mathcal{N}(Q^*) \neq \{0\}$.

(*b*) ⇔ (*d*) : We observe that $\mathcal{N}(P^* + P) = \mathcal{N}(P) \cap \mathcal{N}(P^*)$. Indeed, it is clear that $\mathcal{N}(P^* + P) \supseteq \mathcal{N}(P) \cap \mathcal{N}(P^*)$. For the converse, assume $x \in \mathcal{N}(P^* + P)$. Then we have

$$(P^* + P^*P + PP^* + P)x = (P^* + P)^2 x = 0,$$

and hence $(P^*P + PP^*)x = 0$. It follows that $x \in \mathcal{N}(P) \cap \mathcal{N}(P^*)$. So $\mathcal{N}(P^* + P) \subseteq \mathcal{N}(P) \cap \mathcal{N}(P^*)$.

Now, we see that $\{0\} \neq (\mathcal{N}(P)\cap\mathcal{N}(Q))\cup(\mathcal{N}(P^*)\cap\mathcal{N}(Q^*)) \subseteq \mathcal{N}(P^*+P)\cap\mathcal{N}(Q^*+Q)$ if and only if $\{0\} \neq (\mathcal{N}(P)\cap \mathcal{N}(Q)) \subseteq (\mathcal{N}(P) \cap \mathcal{N}(P^*)) \cap (\mathcal{N}(Q) \cap \mathcal{N}(Q^*))$ and $\{0\} \neq (\mathcal{N}(P^*) \cap \mathcal{N}(Q^*)) \subseteq (\mathcal{N}(P) \cap \mathcal{N}(P^*)) \cap (\mathcal{N}(Q) \cap \mathcal{N}(Q^*))$, or equivalently, if and only if $\{0\} \neq \mathcal{N}(P) \cap \mathcal{N}(Q) = \mathcal{N}(P^*) \cap \mathcal{N}(Q^*)$. □

An operator $J \in \mathcal{B}(\mathcal{H})$ is said to be a symmetry (or self-adjoint unitary operator) if $J = J^* = J^{-1}$. In this case, $J^+ = \frac{I+J}{2}$ and $J^- = \frac{I-J}{2}$ are mutually annihilating orthogonal projections. If J is a non-scalar symmetry, then an indefinite inner product is defined by

$$[x, y] := \langle Jx, y\rangle \qquad (x, y \in \mathcal{H})$$

and $(\mathcal{H}, J)$ is called a Krein space (see [1]).

Corollary 2.11. *Let $P, Q \in \mathcal{B}(\mathcal{H})^{Id}$ and J be a symmetry in $\mathcal{B}(\mathcal{H})$. If P and Q commute with J and $P \underset{\leq}{\vee} Q$ exists, then $P \underset{\leq}{\vee} Q$ commutes with J and*

$$P \underset{\leq}{\vee} Q = \underset{\leq}{min}\{Q' \in \mathcal{B}(\mathcal{H})^{Id} : P, Q \leq Q' \text{ and } Q' \text{ commutes with } J\}.$$

Proof. If the symmetry J is represented as a 2×2 operator matrix relative to $\mathcal{H} = \mathcal{N}(I - J) \oplus \mathcal{N}(I + J)$, then

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Since P and Q commute with J, P and Q can be written as 2×2 operator matrices

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \text{ and } Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix},$$

where $P_1, Q_1 \in \mathcal{B}(\mathcal{N}(I-J))^{Id}$ and $P_2, Q_2 \in \mathcal{B}(\mathcal{N}(I+J))^{Id}$. Moreover, since $P \underset{\leq}{\vee} Q$ exists, we conclude by Theorem 2.6 that $P_i \underset{\leq}{\vee} Q_i$ exists for $i = 1, 2$, and

$$P \underset{\leq}{\vee} Q = (P_1 \underset{\leq}{\vee} Q_1) \oplus (P_2 \underset{\leq}{\vee} Q_2).$$

Thus $P \underset{\leq}{\vee} Q$ commutes with J and

$$P \underset{\leq}{\vee} Q = \underset{\leq}{min}\{Q' \in \mathcal{B}(\mathcal{H})^{Id} : P, Q \leq Q' \text{ and } Q' \text{ commutes with } J\}.$$

□

3. Q_{or} and Q^{or}

Let $Q \in \mathcal{B}(\mathcal{H})^{Id}$. In this section, we study the sets $\{P : P \leq Q \text{ and } P \in \mathcal{P}(\mathcal{H})\}$ and $\{P : Q \leq P \text{ and } P \in \mathcal{P}(\mathcal{H})\}$. Firstly, we show that the sets $\{P : P \leq Q \text{ and } P \in \mathcal{P}(\mathcal{H})\}$ and $\{P : Q \leq P \text{ and } P \in \mathcal{P}(\mathcal{H})\}$ have the maximum and the minimum with respect to the minus order, respectively.

Theorem 3.1. *Let $Q \in \mathcal{B}(\mathcal{H})^{Id}$.*
(a) $\underset{\leq}{max}\{P : P \leq Q \text{ and } P \in \mathcal{P}(\mathcal{H})\} = P_{\mathcal{R}(Q)\cap\mathcal{R}(Q^*)}$.
(b) $\underset{\leq}{min}\{P : Q \leq P \text{ and } P \in \mathcal{P}(\mathcal{H})\} = P_{\mathcal{N}(Q+Q^*)^\perp}$.

Proof. (a) Since $QP_{\mathcal{R}(Q)\cap\mathcal{R}(Q^*)} = P_{\mathcal{R}(Q)\cap\mathcal{R}(Q^*)} = P_{\mathcal{R}(Q)\cap\mathcal{R}(Q^*)}Q$, $P_{\mathcal{R}(Q)\cap\mathcal{R}(Q^*)} \leq Q$. Moreover, if $P \in \mathcal{P}(\mathcal{H})$ and $P \leq Q$, then $PQ = QP = P$. This implies $\mathcal{R}(P) \subseteq \mathcal{R}(Q) \cap \mathcal{R}(Q^*)$, and hence $P \leq P_{\mathcal{R}(Q)\cap\mathcal{R}(Q^*)}$. So $\underset{\leq}{max}\{P : P \leq Q \text{ and } P \in \mathcal{P}(\mathcal{H})\} = P_{\mathcal{R}(Q)\cap\mathcal{R}(Q^*)}$.
(b) Using (a) and the equality $\mathcal{N}(Q) \cap \mathcal{N}(Q^*) = \mathcal{N}(Q + Q^*)$, we have

$$\begin{aligned} \underset{\leq}{min}\{P : Q \leq P \text{ and } P \in \mathcal{P}(\mathcal{H})\} &= I - \underset{\leq}{max}\{P : Q \leq I - P \text{ and } P \in \mathcal{P}(\mathcal{H})\} \\ &= I - \underset{\leq}{max}\{P : P \leq I - Q \text{ and } P \in \mathcal{P}(\mathcal{H})\} \\ &= I - P_{\mathcal{R}(I-Q)\cap\mathcal{R}(I-Q^*)} \\ &= I - P_{\mathcal{N}(Q)\cap\mathcal{N}(Q^*)} = P_{\mathcal{N}(Q+Q^*)^\perp}. \end{aligned}$$

□

Let $Q \in \mathcal{B}(\mathcal{H})^{Id}$. We write $Q_{or} := \max_{\leq}\{P : P \leq Q \text{ and } P \in \mathcal{P}(\mathcal{H})\}$ and $Q^{or} := \min_{\leq}\{P : Q \leq P \text{ and } P \in \mathcal{P}(\mathcal{H})\}$. By the proof of Theorem 3.1 (b), we have $Q^{or} = I - (I - Q)_{or}$. Moreover, if $P \in \mathcal{P}(\mathcal{H})$, then

$$P \leq Q \Longleftrightarrow P \leq Q_{or} \Longleftrightarrow P \leq Q_{or}$$

and

$$Q \leq P \Longleftrightarrow Q^{or} \leq P \Longleftrightarrow Q^{or} \leq P.$$

Remark 3.2. *Let $E, F \in \mathcal{P}(\mathcal{H})$. According to [13], E and F have the least upper bound $E \vee F$ within the set $\mathcal{P}(\mathcal{H})$ (with respect to the operator order $\leq$), and $E \vee F = P_{\overline{\mathcal{R}(E)+\mathcal{R}(F)}}$. Moreover, we have*

$$E \underset{\leq}{\vee} F = E \vee F.$$

Indeed, it is clear that $E, F \leq E \vee F$. If Q is a projection in $\mathcal{B}(\mathcal{H})^{Id}$ such that $E, F \leq Q$, then $E, F \leq Q_{or}$. It follows that $E \vee F \leq Q_{or}$, and since $E \vee F$ and Q_{or} are orthogonal projections, $E \vee F \leq Q_{or}$. So $E \vee F \leq Q$. Thus $E \underset{\leq}{\vee} F = E \vee F$.

Analogously, E and F have the greatest lower bound $E \wedge F$ within the set $\mathcal{P}(\mathcal{H})$ (with respect to the operator order $\leq$) and

$$E \underset{\leq}{\wedge} F = E \wedge F = P_{\mathcal{R}(E)\cap\mathcal{R}(F)}.$$

So we obtain

$$(I - E) \underset{\leq}{\wedge} (I - F) = P_{\mathcal{R}(I-E)\cap\mathcal{R}(I-F)} = P_{\mathcal{N}(E)\cap\mathcal{N}(F)} = I - E \underset{\leq}{\vee} F.$$

Then it follows from Kaplansky formula ([14, Theorem 6.1.7]) that $E \underset{\leq}{\vee} F - F \sim E - E \underset{\leq}{\wedge} F$, where $\sim$ represents Murray-von Neumann equivalent of two orthogonal projections (see [6]).

The following result shows the specificity of $Q - P \in \mathcal{B}(\mathcal{H})^+$, when $P \leq Q$ for $P,\ Q \in \mathcal{B}(\mathcal{H})^{Id}$.

Proposition 3.3. *Let $P,\ Q \in \mathcal{B}(\mathcal{H})^{Id}$. If $P \leq Q$, then the following statements are equivalent.*

(a) $Q - P \geq 0$.

(b) $Q - P$ *is self-adjoint.*

(c) $Q - P$ *is an orthogonal projection.*

(d) $Q + Q^* \geq P + P^*$.

Proof. It is clear that $(a) \Rightarrow (b)$.

$(b) \Rightarrow (c)$: As $PQ = QP = P$, we know that

$$(Q - P)^2 = (Q - P)(Q - P) = Q^2 - QP - PQ + P^2 = Q - P.$$

Thus (b) implies that $Q - P$ is an orthogonal projection as desired.

$(c) \Rightarrow (d)$: It is clear that

$$Q + Q^* - (P + P^*) = (Q - P) + (Q - P)^* = 2(Q - P) \geq 0,$$

so $Q + Q^* \geq P + P^*$.

$(d) \Rightarrow (a)$: Let $A = Q - P$. Since $A^2 = A$, A has the operator matrix form

$$A = \begin{pmatrix} I & A_1 \\ 0 & 0 \end{pmatrix}$$

with respect to $\mathcal{H} = \mathcal{R}(A) \oplus \mathcal{R}(A)^{\perp}$. It follows that

$$A + A^* = \begin{pmatrix} 2I & A_1 \\ A_1^* & 0 \end{pmatrix},$$

and since $A + A^* = (Q + Q^*) - (P + P^*) \geq 0$, we have $A_1 = 0$. Thus $Q - P = A \geq 0$. □

Let J be a symmetry in $\mathcal{B}(\mathcal{H})$. A projection $P \in \mathcal{B}(\mathcal{H})^{Id}$ is said to be a J-projection, if $P = JP^*J$. The existence of J-projections and its properties are studied in [16–18].

If $A \in \mathcal{B}(\mathcal{H})$ and $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, say that $\mathcal{M}$ is a reducing subspace for A if $A\mathcal{M} \subseteq \mathcal{M}$ and $A\mathcal{M}^\perp \subseteq \mathcal{M}^\perp$. $\mathcal{M}$ is a reducing subspace for A if and only if $A\mathcal{M} \subseteq \mathcal{M}$ and $A^*\mathcal{M} \subseteq \mathcal{M}$, or equivalently, if and only if $AP_\mathcal{M} = P_\mathcal{M}A$ (see [5, Chapter II, Section 3]).

Theorem 3.4. *Let J be a symmetry in $\mathcal{B}(\mathcal{H})$ and let $Q \in \mathcal{B}(\mathcal{H})^{Id}$ be a J-projection.*

(a) $\max_{\leq}\{P : P \leq Q,\ P \in \mathcal{P}(\mathcal{H}),\ P \text{ is a } J\text{-projection}\} = P_{\mathcal{R}(Q)\cap\mathcal{R}(Q^*)}$.

(b) $\min_{\leq}\{P : Q \leq P,\ P \in \mathcal{P}(\mathcal{H}),\ P \text{ is a } J\text{-projection}\} = P_{\mathcal{N}(Q+Q^*)^\perp}$.

Proof. (*a*) By Theorem 3.1 (*a*), it suffices to show that $P_{\mathcal{R}(Q)\cap\mathcal{R}(Q^*)}$ is a J-projection. Let $x \in \mathcal{R}(Q) \cap \mathcal{R}(Q^*)$. Then we have $Qx = Q^*x = x$, and since Q is a J-projection,

$$QJx = JQ^*x = Jx \text{ and } Q^*Jx = JQx = Jx.$$

It follows that $Jx \in \mathcal{R}(Q) \cap \mathcal{R}(Q^*)$. So $J(\mathcal{R}(Q) \cap \mathcal{R}(Q^*)) \subseteq \mathcal{R}(Q) \cap \mathcal{R}(Q^*)$, and since J is self-adjoint, $\mathcal{R}(Q) \cap \mathcal{R}(Q^*)$ is a reducing subspace for J. Now, we have $JP_{\mathcal{R}(Q)\cap\mathcal{R}(Q^*)} = P_{\mathcal{R}(Q)\cap\mathcal{R}(Q^*)}J$, and hence $P_{\mathcal{R}(Q)\cap\mathcal{R}(Q^*)}$ is a J-projection.

The proof of (*b*) is similar. □

If $Q \in \mathcal{B}(\mathcal{H})^{Id}$ is a J-projection, then Theorem 3.4 yields that Q^{or} is a J-projection. Conversely, the following theorem study the problem of whether there is a J-projection $Q \in \mathcal{B}(\mathcal{H})^{Id}\backslash\{\mathcal{P}(\mathcal{H})\}$ such that $Q^{or} = P$, if $P \in \mathcal{P}(\mathcal{H})$ is a J-projection.

Theorem 3.5. *Let J be a symmetry in $\mathcal{B}(\mathcal{H})$ and let $P \in \mathcal{P}(\mathcal{H})$ be a J-projection.*

(a) There exists a J-projection $Q \in \mathcal{B}(\mathcal{H})^{Id}\backslash\{\mathcal{P}(\mathcal{H})\}$ such that $Q^{or} = P$ if and only if $dim\mathcal{R}(P) \geq 2$ and $(I \pm J)P \neq 0$.

(b) There exists a J-projection $Q' \in \mathcal{B}(\mathcal{H})^{Id}\backslash\{\mathcal{P}(\mathcal{H})\}$ such that $Q'_{or} = P$ if and only if $dim\mathcal{R}(I-P) \geq 2$ and $(I \pm J)(I-P) \neq 0$.

Proof. (*a*) Assume $dim\mathcal{R}(P) \geq 2$ and $(I \pm J)P \neq 0$. Then $PJ = JP \neq \pm P$, and hence J has the operator matrix form

$$J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} : \mathcal{R}(P) \oplus \mathcal{R}(P)^\perp,$$

where $J_1 \in \mathcal{B}(\mathcal{R}(P))$, $J_2 \in \mathcal{B}(\mathcal{R}(P)^\perp)$ are symmetries with $J_1 \neq \pm I_1$. Thus there exist unit vectors $x_1, x_2 \in \mathcal{R}(P)$ such that $x_1 \perp x_2$,

$$Jx_1 = J_1x_1 = x_1 \text{ and } Jx_2 = J_1x_2 = -x_2.$$

With respect to $\mathcal{H} = \overline{\{x_1\}} \oplus \overline{\{x_2\}} \oplus (\mathcal{R}(P) \ominus \overline{\{x_1, x_2\}}) \oplus \mathcal{R}(P)^\perp$, J has the operator matrix form

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & J_{11} & 0 \\ 0 & 0 & 0 & J_2 \end{pmatrix},$$

where $J_{11} \in \mathcal{B}(\mathcal{R}(P) \ominus \overline{\{x_1, x_2\}})$ is a symmetry. Let

$$Q = \begin{pmatrix} \frac{3}{2} & \frac{\sqrt{-3}}{2} & 0 & 0 \\ \frac{\sqrt{-3}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to $\mathcal{H} = \overline{\{x_1\}} \oplus \overline{\{x_2\}} \oplus (\mathcal{R}(P) \ominus \overline{\{x_1, x_2\}}) \oplus \mathcal{R}(P)^\perp$. Then it is easy to check that $JQ = Q^*J$ and $Q \in \mathcal{B}(\mathcal{H})^{Id}\backslash\{\mathcal{P}(\mathcal{H})\}$, and hence Q is a J-projection. Moreover, we see that $\mathcal{N}(Q + Q^*) = \mathcal{R}(P)^\perp$, and by Theorem 3.1 (b), $Q^{or} = P$.

For the converse, assume that $Q \in \mathcal{B}(\mathcal{H})^{Id} \backslash \{\mathcal{P}(\mathcal{H})\}$ is J-projection and $P = Q^{or} = P_{\mathcal{N}(Q+Q^*)^\perp}$. Then $Q \leq P$, and hence $\mathcal{R}(Q) \subseteq \mathcal{R}(P)$.

If $dim\mathcal{R}(P) = 1$, then $dim\mathcal{R}(Q) = 1$. So there exist a unit vector x and non-zero vectors y and z in $\mathcal{H}$ such that

$$P = x \otimes x \text{ and } Q = y \otimes z,$$

where $u \otimes v$ is the rank-one operator in $\mathcal{B}(\mathcal{H})$ defined by $(u \otimes v)w = \langle w, v\rangle u$ for all $w \in \mathcal{H}$. It follows that

$$QP = (y \otimes z)(x \otimes x) = \langle x, z\rangle (y \otimes x) = y \otimes z = Q$$

and

$$PQ = (x \otimes x)(y \otimes z) = \langle y, x\rangle (x \otimes z) = y \otimes z = Q.$$

Thus $z = \langle z, x\rangle x$ and $y = \langle y, x\rangle x$, and hence $Q = y \otimes z = \lambda(x \otimes x)$ for some $0 \neq \lambda \in \mathbb{C}$. So we have $Q^2 = \lambda^2(x \otimes x) = \lambda(x \otimes x) = Q$. This implies $\lambda = 1$, and hence $Q = x \otimes x \in \mathcal{P}(\mathcal{H})$. This is a contradiction with the fact $Q \in \mathcal{B}(\mathcal{H})^{Id} \backslash \{\mathcal{P}(\mathcal{H})\}$. So $dim\mathcal{R}(P) \geq 2$.

It remains to show that $JP \neq \pm P$. If $JP = P$, then $PJ = JP = P$. So J has the operator matrix form

$$J = \begin{pmatrix} I & 0 \\ 0 & J' \end{pmatrix} : \mathcal{R}(P) \oplus \mathcal{R}(P)^\perp,$$

where $J' \in \mathcal{B}(\mathcal{R}(P)^\perp)$ is a symmetry. Let

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} : \mathcal{R}(P) \oplus \mathcal{R}(P)^\perp.$$

Since $P = P_{\mathcal{N}(Q+Q^*)^\perp}$, we get that $\mathcal{N}(Q + Q^*) = \mathcal{R}(P)^\perp$. So for $x \in \mathcal{R}(P)^\perp$, we have

$$(Q + Q^*)\begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} Q_{11} + Q_{11}^* & Q_{12} + Q_{21}^* \\ Q_{21} + Q_{12}^* & Q_{22} + Q_{22}^* \end{pmatrix}\begin{pmatrix} 0 \\ x \end{pmatrix} = 0.$$

It follows that

$$Q_{12} + Q_{21}^* = 0 \quad \text{and} \quad Q_{22} + Q_{22}^* = 0. \tag{1}$$

On the other hand, since $JQ = Q^*J$,

$$\begin{pmatrix} Q_{11} & Q_{12} \\ J'Q_{21} & J'Q_{22} \end{pmatrix} = \begin{pmatrix} Q_{11}^* & Q_{21}^*J' \\ Q_{12}^* & Q_{22}^*J' \end{pmatrix}.$$

So we have

$$Q_{11} = Q_{11}^* \qquad \text{and} \qquad Q_{12} = Q_{21}^*J'. \tag{2}$$

Combining (1) and (2), we see that

$$Q = \begin{pmatrix} Q_{11} & Q_{21}^*J' \\ -J'Q_{21} & Q_{22} \end{pmatrix} = \begin{pmatrix} Q_{11}^* & Q_{21}^*J' \\ -J'Q_{21} & -Q_{22}^* \end{pmatrix} \in \mathcal{B}(\mathcal{H})^{Id}.$$

By a direct calculation, we obtain

$$Q^2 = \begin{pmatrix} Q_{11}^2 - Q_{21}^*Q_{21} & Q_{11}Q_{21}^*J' + Q_{21}^*J'Q_{22} \\ -J'Q_{21}Q_{11} - Q_{22}J'Q_{21} & Q_{22}^2 - J'Q_{21}Q_{21}^*J' \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = Q,$$

which implies

$$Q_{11}^2 - Q_{21}^*Q_{21} = Q_{11} \qquad \text{and} \qquad Q_{22}^2 - J'Q_{21}Q_{21}^*J' = Q_{22}. \tag{3}$$

So $Q_{22} = Q_{22}^2 - J'Q_{21}Q_{21}^*J' = (Q_{22}^*)^2 - J'Q_{21}Q_{21}^*J' = Q_{22}^*$. Then by (1), $Q_{22} = 0$, and hence $J'Q_{21}Q_{21}^*J' = 0$, that is, $Q_{21} = 0$. Thus we get $Q_{11}^2 = Q_{11}$ by (3). Using (2) again, $Q_{12} = 0$ and $Q_{11} \in \mathcal{P}(\mathcal{R}(P))$, which means

$$Q = \begin{pmatrix} Q_{11} & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{P}(\mathcal{H}).$$

This is a contradiction with the assumption $Q \in \mathcal{B}(\mathcal{H})^{Id} \backslash \{\mathcal{P}(\mathcal{H})\}$. Therefore, $JP \neq P$. In a similar way, we can prove that $JP \neq -P$.

Part (*b*) follows by (*a*) and the equality $Q^{or} = I - (I - Q)_{or}$. □

Lemma 3.6. *Let $Q \in \mathcal{B}(\mathcal{H})^{Id}$ and let $\mathcal{M} = \mathcal{R}(Q) \cap \mathcal{R}(Q^*)$. Then Q has the operator matrix*

$$Q = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & Q_1 \\ 0 & 0 & 0 \end{pmatrix} \tag{4}$$

with respect to the space decomposition $\mathcal{H} = \mathcal{M} \oplus (\mathcal{R}(Q) \ominus \mathcal{M}) \oplus \mathcal{R}(Q)^\perp$, where Q_1 is an operator in $\mathcal{B}(\mathcal{R}(Q)^\perp, \mathcal{R}(Q) \ominus M)$ with dense range.

Proof. It is easy to check that $\mathcal{M}$ reduces Q, $P_{R(Q)^\perp}Q = 0$ and $Q\mid_{\mathcal{R}(Q)} = I$. So Q has the operator matrix form (4). We are left to prove $\mathcal{N}(Q_1^*) = 0$.

If $y \in \mathcal{R}(Q) \ominus M$ and $Q_1^* y = 0$, then

$$Q\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} = Q^*\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}.$$

It follows that $y \in \mathcal{M}$, and hence $y = 0$. So $\mathcal{N}(Q_1^*) = 0$. □

For $A \in \mathcal{B}(\mathcal{H})$, let $|A| := (A^*A)^{\frac{1}{2}}$ be the absolute value of A. If Q is a projection as in (4), then

$$|Q| = \begin{pmatrix} I_1 & 0 \\ 0 & \left|\begin{pmatrix} I_2 & Q_1 \\ 0 & 0 \end{pmatrix}\right| \end{pmatrix}.$$

The following result is an extension of [18, Proposition 1].

Proposition 3.7. *Let $Q \in \mathcal{B}(\mathcal{H})^{Id}$. Then $Q_{or} = P_{\mathcal{N}(I-|Q|)} = P_{\mathcal{N}(2I-Q-Q^*)}$.*

Proof. Write Q in (4). If

$$\begin{pmatrix} I_2 & 0 \\ Q_1^* & 0 \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \left|\begin{pmatrix} I_2 & 0 \\ Q_1^* & 0 \end{pmatrix}\right| = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} (I_2 + Q_1Q_1^*)^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix}$$

is the polar decomposition of $\begin{pmatrix} I_2 & 0 \\ Q_1^* & 0 \end{pmatrix}$, we have

$$U_{11} = (I_2 + Q_1Q_1^*)^{-\frac{1}{2}} \text{ and } U_{21} = Q_1^*(I_2 + Q_1Q_1^*)^{-\frac{1}{2}}.$$

Since $Q_1^*(I_2 + Q_1Q_1^*) = (I_3 + Q_1^*Q_1)Q_1^*$ implies $Q_1^*(I_2 + Q_1Q_1^*)^{\frac{1}{2}} = (I_3 + Q_1^*Q_1)^{\frac{1}{2}}Q_1^*$, we also have

$$U_{21} = (I_3 + Q_1^*Q_1)^{-\frac{1}{2}}Q_1^*.$$

Then polar decomposition theorem yields that

$$\left|\begin{pmatrix} I_2 & Q_1 \\ 0 & 0 \end{pmatrix}\right| = \begin{pmatrix} I_2 & 0 \\ Q_1^* & 0 \end{pmatrix}\begin{pmatrix} U_{11}^* & U_{21}^* \\ U_{12}^* & U_{22}^* \end{pmatrix} = \begin{pmatrix} (I_2+Q_1Q_1^*)^{-\frac{1}{2}} & (I_2+Q_1Q_1^*)^{-\frac{1}{2}}Q_1 \\ Q_1^*(I_2+Q_1Q_1^*)^{-\frac{1}{2}} & (I_3+Q_1^*Q_1)^{-\frac{1}{2}}Q_1^*Q_1 \end{pmatrix},$$

and hence

$$|Q| = \begin{pmatrix} I_1 & 0 \\ 0 & \left|\begin{pmatrix} I_2 & Q_1 \\ 0 & 0 \end{pmatrix}\right| \end{pmatrix} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & (I_2+Q_1Q_1^*)^{-\frac{1}{2}} & (I_2+Q_1Q_1^*)^{-\frac{1}{2}}Q_1 \\ 0 & Q_1^*(I_2+Q_1Q_1^*)^{-\frac{1}{2}} & (I_3+Q_1^*Q_1)^{-\frac{1}{2}}Q_1^*Q_1 \end{pmatrix}.$$

Let

$$\widetilde{Q} = \begin{pmatrix} (I_2+Q_1Q_1^*)^{-\frac{1}{2}} & (I_2+Q_1Q_1^*)^{-\frac{1}{2}}Q_1 \\ Q_1^*(I_2+Q_1Q_1^*)^{-\frac{1}{2}} & (I_3+Q_1^*Q_1)^{-\frac{1}{2}}Q_1^*Q_1 \end{pmatrix}.$$

It is clear that $\mathcal{N}(I-|Q|) = (\mathcal{R}(Q)\cap\mathcal{R}(Q^*))\oplus\mathcal{N}(I-\widetilde{Q})$.

Claim 3.8. $\mathcal{N}(\widetilde{Q}-I) = \{0\}$.

If $\widetilde{Q}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, then we have

$$\begin{cases} (I_2+Q_1Q_1^*)^{-\frac{1}{2}}x + (I_2+Q_1Q_1^*)^{-\frac{1}{2}}Q_1y = x, \\ Q_1^*(I_2+Q_1Q_1^*)^{-\frac{1}{2}}x + (I_3+Q_1^*Q_1)^{-\frac{1}{2}}Q_1^*Q_1y = y. \end{cases} \tag{5}$$

Since $Q_1^*(I_2+Q_1Q_1^*)^{-\frac{1}{2}}x = (I_3+Q_1^*Q_1)^{-\frac{1}{2}}Q_1^*x$, we get that

$$x + Q_1y = (I_2+Q_1Q_1^*)^{\frac{1}{2}}x \quad \text{and} \quad Q_1^*x + Q_1^*Q_1y = (I_3+Q_1^*Q_1)^{\frac{1}{2}}y,$$

and hence

$$(I_3+Q_1^*Q_1)^{\frac{1}{2}}y = Q_1^*(I_2+Q_1Q_1^*)^{\frac{1}{2}}x = (I_3+Q_1^*Q_1)^{\frac{1}{2}}Q_1^*x,$$

which means $y = Q_1^*x$. Using the first equation of (5), we see that

$$(I_2+Q_1Q_1^*)^{\frac{1}{2}}x = (I_2+Q_1Q_1^*)^{-\frac{1}{2}}x + (I_2+Q_1Q_1^*)^{-\frac{1}{2}}Q_1Q_1^*x = x.$$

This implies $Q_1Q_1^*x = 0$, and since Q_1^* is injective, $x = 0$. It follows also that $y = Q_1^*x = 0$. So $\mathcal{N}(\widetilde{Q}-I) = \{0\}$.
So $\mathcal{N}(I-|Q|) = (\mathcal{R}(Q)\cap\mathcal{R}(Q^*))$, and by Theorem 3.1, we have

$$Q_{or} = P_{\mathcal{N}(I-|Q|)} \text{ and } Q_{or} = I-(I-Q)^{or} = P_{\mathcal{N}(2I-Q-Q^*)}.$$

□

Lemma 3.9. *Let $P, Q \in \mathcal{B}(\mathcal{H})^{Id}$. Then $P^{or} \preceq Q_{or}$ if and only if there is a projection Q_1 in $\mathcal{B}(\mathcal{H})^{Id}$ such that $P^{or}Q_1 = Q_1P^{or} = 0$ and $Q = P^{or} + Q_1$.*

Proof. If $P^{or}Q_1 = Q_1P^{or} = 0$ and $Q = P^{or} + Q_1$, then $P^{or} \preceq Q$; hence $P^{or} \preceq Q_{or}$.
For the converse, assume $P^{or} \preceq Q_{or}$. Let $Q_1 = Q - P^{or}$. Then we have

$$Q_1^2 = (Q-P^{or})^2 = Q^2 - QP^{or} - P^{or}Q + P^{or} = Q - P^{or} = Q_1,$$

and hence $Q_1 \in \mathcal{B}(\mathcal{H})^{Id}$. Moreover, since $P^{or} \preceq Q_{or}$, we get $P^{or} \preceq Q$. It follows that

$$P^{or}Q_1 = P^{or}Q - P^{or} = 0 = QP^{or} - P^{or} = Q_1P^{or}.$$

□

Let $P \in \mathcal{B}(\mathcal{H})^{Id}$. The following theorem gives necessary and sufficient conditions under which $P^{or} \leq Q_{or}$ for all $Q \in \mathcal{B}(\mathcal{H})^{Id}$ with $P \prec Q$, where $P \prec Q$ signifies that $P \leq Q$ and $P \neq Q$.

Theorem 3.10. *Let $P \in \mathcal{B}(\mathcal{H})^{Id}$. Then $P^{or} \leq Q_{or}$ for all $Q \in \mathcal{B}(\mathcal{H})^{Id}$ with $P \prec Q$ if and only if $P \in \mathcal{P}(\mathcal{H})$ or $dim\mathcal{R}(P)^{\perp} \leq 1$.*

Proof. If $P \in \mathcal{P}(\mathcal{H})$ and $P \prec Q$, then $P^{or} = P \leq Q_{or}$. If $dim\mathcal{R}(P)^{\perp} = 0$, then $P = I$; hence there is nothing to prove. If $dim\mathcal{R}(P)^{\perp} = 1$ and $P \prec Q$, then $Q = I$ and $P^{or} \leq I = Q_{or}$.

Now assume $P^{or} \leq Q_{or}$ for all $Q \in \mathcal{B}(\mathcal{H})^{Id}$ with $P \prec Q$. By Lemma 3.6, we can represent P as a 3×3 operator matrix

$$P = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & P_1 \\ 0 & 0 & 0 \end{pmatrix} : \mathcal{M} \oplus (\mathcal{R}(P) \ominus \mathcal{M}) \oplus \mathcal{R}(P)^{\perp},$$

where $\mathcal{M} = \mathcal{R}(P) \cap \mathcal{R}(P^*)$ and $P_1 \in \mathcal{B}(\mathcal{R}(P)^{\perp}, (\mathcal{R}(Q) \ominus M))$ has dense range.

Case 1. $\mathcal{N}(P_1) = 0$. If $dim\mathcal{R}(P)^{\perp} \geq 2$, then there exists $Q_2 \in \mathcal{B}(\mathcal{R}(P)^{\perp})^{Id}$ such that $Q_2 \neq 0, I$. Let

$$Q = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & P_1 - P_1 Q_2 \\ 0 & 0 & Q_2 \end{pmatrix}$$

with respect to $\mathcal{H} = \mathcal{M} \oplus (\mathcal{R}(P) \ominus \mathcal{M}) \oplus \mathcal{R}(P)^{\perp}$. By a direct calculation, we have

$$Q^2 = Q \text{ and } PQ = QP = P,$$

and hence $P \prec Q$. So $P^{or} \leq Q_{or}$, and by Lemma 3.9, $P^{or}(Q - P^{or}) = (Q - P^{or})P^{or} = 0$.

On the other hand, since $\mathcal{N}(P_1) = \{0\}$ and $\mathcal{N}(P_1^*) = \{0\}$, $\mathcal{N}(P + P^*) = \{0\}$. We conclude by Theorem 3.1 (b) that $P^{or} = I$, and hence $(Q - P^{or})P^{or} = Q - I \neq 0$. This is a contradiction. So $dim\mathcal{R}(P)^{\perp} \leq 1$.

Case 2. $\mathcal{N}(P_1) \neq 0$. We have

$$\mathcal{H} = \mathcal{M} \oplus (\mathcal{R}(P) \ominus \mathcal{M}) \oplus \mathcal{N}(P_1)^{\perp} \oplus \mathcal{N}(P_1),$$

and with respect to this space decomposition, P has the operator matrix form

$$P = \begin{pmatrix} I_1 & 0 & 0 & 0 \\ 0 & I_2 & P_{11} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $P_{11} \in \mathcal{B}(\mathcal{N}(P_1)^{\perp}, \mathcal{R}(P) \ominus \mathcal{M})$ is injective and has dense range. Since $\mathcal{N}(P_{11}) = \{0\}$ and $\mathcal{N}(P_{11}^*) = \{0\}$, we see that $\mathcal{N}(P + P^*) = \mathcal{N}(P_1)$. Then Theorem 3.1 (b) yields $P^{or} = P_{\mathcal{N}(P_1)^{\perp}} = diag(I_1, I_2, I_3, 0)$.

If $P \notin \mathcal{P}(\mathcal{H})$ and $dim\mathcal{R}(P)^{\perp} \geq 2$, then there exists $0 \neq Q_{11} \in \mathcal{B}(\mathcal{N}(P_1), \mathcal{N}(P_1)^{\perp})$. Let

$$Q' = \begin{pmatrix} I_1 & 0 & 0 & 0 \\ 0 & I_2 & 0 & -P_{11}Q_{11} \\ 0 & 0 & I_3 & Q_{11} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to $\mathcal{H} = \mathcal{M} \oplus (\mathcal{R}(P) \ominus \mathcal{M}) \oplus \mathcal{N}(P_1)^{\perp} \oplus \mathcal{N}(P_1)$, where $0 \neq Q_{11} \in \mathcal{B}(\mathcal{N}(P_1), \mathcal{N}(P_1)^{\perp})$. A direct calculation shows

$$Q'^2 = Q' \text{ and } PQ' = Q'P = P,$$

and hence $P \prec Q'$. So $P^{or} \leq Q'_{or}$, and by Lemma 3.9, $P^{or}(Q' - P^{or}) = (Q' - P^{or})P^{or} = 0$. However, since $P^{or} = diag(I_1, I_2, I_3, 0)$, it is clear that $P^{or}(Q' - P^{or}) \neq 0$. This contradiction implies $P \in \mathcal{P}(\mathcal{H})$ or $dim\mathcal{R}(P)^{\perp} \leq 1$. □

In closing this section, we present a result about the continuity of the map: $Q \to Q^{or}$. Let $\{Q_n\}$ be a sequence in $\mathcal{B}(\mathcal{H})$. Then $\{Q_n\}$ converges to Q in weak operator topology (in symbols, $Q_n \overset{WOT}{\longrightarrow} Q$) if $\langle Q_n x, y\rangle \to \langle Qx, y\rangle$ for all $x, y \in \mathcal{H}$. If more $Q_n \leq Q_{n+1}$ (resp. $Q_{n+1} \leq Q_n$) for $n = 1, 2, \cdots$, we write $Q_n \overset{WOT}{\nearrow} Q$ (resp. $Q_n \overset{WOT}{\searrow} Q$).

Proposition 3.11. *Let J be a symmetry in $\mathcal{B}(\mathcal{H})$, and let Q_n be a sequence of J-projections in $\mathcal{B}(\mathcal{H})^{Id}$ and $Q \in \mathcal{B}(\mathcal{H})^{Id}$.*
(a) If $Q_n \overset{WOT}{\nearrow} Q$, then Q is a J-projection and $Q_n^{or} \overset{WOT}{\nearrow} Q^{or}$.
(b) If $Q_n \overset{WOT}{\searrow} Q$, then Q is a J-projection and $(Q_n)_{or} \overset{WOT}{\searrow} Q_{or}$.

Proof. (a) For vectors $x, y \in \mathcal{H}$, we have

$$\langle JQ_n x, y\rangle = \langle Q_n x, Jy\rangle \to \langle Qx, Jy\rangle$$

and

$$\langle Q_n^* Jx, y\rangle = \langle Jx, Q_n y\rangle \to \langle Jx, Qy\rangle.$$

Since $JQ_n = Q_n^* J$ for all $n = 1, 2, \cdots$, it follows that $\langle Qx, Jy\rangle = \langle Jx, Qy\rangle$. So $JQ = Q^* J$, and hence Q is a J-projection.

For any $n_0 \in \mathbb{Z}^+$, if $n \geq n_0$, then $Q_{n_0} \leq Q_n$ implies $Q_{n_0} Q_n = Q_n Q_{n_0} = Q_{n_0}$. So

$$\langle Q_{n_0} x, y\rangle = \langle Q_{n_0} Q_n x, y\rangle \to \langle Q_{n_0} Qx, y\rangle,$$

and we see that $\langle Q_{n_0} x, y\rangle = \langle Q_{n_0} Qx, y\rangle$. Analogously, we get $\langle Q_{n_0} x, y\rangle = \langle QQ_{n_0} x, y\rangle$. Thus $QQ_{n_0} = Q_{n_0} Q = Q_{n_0}$. It follows that $Q_{n_0} \leq Q$, and hence $Q_{n_0}^{or} \leq Q^{or}$. Since $\{Q_n^{or}\}$ is an increasing sequence, there exists an orthogonal projection P in $\mathcal{B}(\mathcal{H})$ such that $Q_n^{or} \overset{WOT}{\nearrow} P$ (see [5, Chapter IX, Section 1]), and hence $P \leq Q^{or}$. On the other hand, it is clear that

$$\begin{aligned} \langle (PQ - Q)x, y\rangle &= \langle (PQ - PQ_n)x, y\rangle + \langle (PQ_n - Q)x, y\rangle \\ &= \langle P(Q - Q_n)x, y\rangle + \langle (PQ_n^{or} Q_n - Q)x, y\rangle \\ &= \langle P(Q - Q_n)x, y\rangle + \langle (Q_n - Q)x, y\rangle \to 0, \end{aligned}$$

so $PQ = Q$. Similarly, we get $QP = Q$. Therefore, $Q \leq P$, and it follows that $Q^{or} \leq P$. Now we have $P = Q^{or}$ and $Q_n^{or} \overset{WOT}{\nearrow} Q^{or}$. The proof of (b) is similar. □

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