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Constructing Some Logical Algebras from *EQ*-Algebras

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Abstract. *EQ*-algebras were introduced by Novák in [16] as an algebraic structure of truth values for fuzzy type theory (FTT). Novák and De Baets in [18] introduced various kinds of *EQ*-algebras such as good, residuated, and lattice ordered *EQ*-algebras. In any logical algebraic structures, by using various kinds of filters, one can construct various kinds of other logical algebraic structures. With this inspirations, by means of fantastic filters of *EQ*-algebras we construct *MV*-algebras. Also, we study prelinear *EQ*-algebras and introduce a new kind of filter and named it prelinear filter. Then, we show that the quotient structure which is introduced by a prelinear filter is a distributive lattice-ordered *EQ*-algebras and under suitable conditions, is a De Morgan algebra, Stone algebra and Boolean algebra.

1. Introduction

Fuzzy type theory was developed as a counterpart of the classical higher-order logic. Since the algebra of truth values is no longer a residuated lattice, a specific algebra called an *EQ*-algebra was proposed by Novák [16–18]. The main primitive operations of *EQ*-algebras are meet, multiplication, and fuzzy equality. Implication is derived from the fuzzy equality and it is not a residuation with respect to multiplication. Consequently, *EQ*-algebras overlap with residuated lattices but are not identical with them. Novák and De Baets in [18] introduced various kinds of *EQ*-algebras and they defined the concept of prefilter on *EQ*-algebras which is the same as filter of other algebraic structures such as residuated lattices, MTL-algebras, and etc. But the binary relation has been introduced by prefilters is not a congruence relation. For solving this problem, they added another condition to the definition of prefilter so filter of *EQ*-algebras is defined. In studying logical algebras, filter theory or ideal theory is very important. In [2–4, 12, 19] different kinds of filters such as implicative, positive implicative and fantastic filters were introduced in various logical algebras. Liu and Zhang in [14], introduced positive implicative and implicative (pre)filters of *EQ*-algebras and showed that these two concepts are the same in *IEQ*-algebras. Xin et al. [20], have studied fantastic (pre)filters of good *EQ*-algebras. In this paper, we investigate properties of fantastic (pre)filters in more general form of *EQ*-algebras and by means of this properties we can construct an *MV*-algebra. El-Zekey in [8] introduced prelinear good *EQ*-algebras and proved that a prelinear good *EQ*-algebra is a distributive lattice. In Section 4, we introduce a new kind of filter, named prelinear filter and we will show that if an

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EQ-algebra is not good or prelinear, then the quotient structure can be distributive lattice. Also, we will see that if a prelinear filter is fantastic, positive implicative, or implicative, then we can construct a Demorgan algebra, Stone algebra or Boolean algebra, respectively.

2. Preliminaries

In this section, we recollect some definitions and results which will be used in this paper [8, 9, 14].

An *EQ-algebra* is an algebraic structure $\mathcal{E}_{II} = (E, \wedge, \otimes, \sim, 1)$ of type $(2, 2, 2, 0)$, where for any $\alpha, \beta, \gamma, \delta \in E$, the following statements hold:
$(E1)$ $(E, \wedge, 1)$ is a $\wedge$-semilattice with top element 1.
$(E2)$ $(E, \otimes, 1)$ is a (commutative) monoid and $\otimes$ is isotone with respect to $\leqslant$.
$(E3)$ $\alpha \sim \alpha = 1$.
$(E4)$ $((\alpha \wedge \beta) \sim \gamma) \otimes (\delta \sim \alpha) \leqslant (\gamma \sim (\delta \wedge \beta))$.
$(E5)$ $(\alpha \sim \beta) \otimes (\gamma \sim \delta) \leqslant (\alpha \sim \gamma) \sim (\beta \sim \delta)$.
$(E6)$ $(\alpha \wedge \beta \wedge \gamma) \sim \alpha \leqslant (\alpha \wedge \beta) \sim \alpha$.
$(E7)$ $\alpha \otimes \beta \leqslant \alpha \sim \beta$.

The operations " $\wedge$ ", " $\otimes$ ", and " $\sim$ " are called *meet*, *multiplication*, and *fuzzy equality*, respectively. For any $\alpha, \beta \in E$, we set $\alpha \leqslant \beta$ if and only if $\alpha \wedge \beta = \alpha$ and we defined the binary operation *implication* on E by, $\alpha \to \beta = (\alpha \wedge \beta) \sim \alpha$. Also, in particular $1 \to \alpha = 1 \sim \alpha = \tilde{\alpha}$. If E has a bottom element 0, we denote it by BEQ-algebra and then an unary operation $\neg$ is defined on E by $\neg\alpha = \alpha \sim 0$.

Let $\mathcal{E}_{II} = (E, \wedge, \otimes, \sim, 1)$ be an EQ-algebra and $\alpha, \beta, \gamma \in E$ are arbitrary elements. Then $\mathcal{E}_{II}$ is called
(i) *separated* if $\alpha \sim \beta = 1$, implies $\alpha = \beta$,
(ii) *good* if $\alpha \sim 1 = \alpha$,
(iii) an *involutive (IEQ-algebra)* if $\mathcal{E}_{II}$ is a BEQ-algebra and for any $\alpha \in E$, $\neg\neg\alpha = \alpha$,
(iv) *residuated*, where $(\alpha \otimes \beta) \wedge \gamma = \alpha \otimes \beta$ if and only if $\alpha \wedge ((\beta \wedge \gamma) \sim \beta) = \alpha$,
(v) *lattice-ordered EQ-algebra* if it has a lattice reduct[1)],
(vi) *prelinear EQ-algebra* if the set $\{(\alpha \to \beta), (\beta \to \alpha)\}$ has the unique upper bound 1,
(vii) *lattice EQ-algebra (or ℓEQ-algebra)* if it is a lattice-ordered EQ-algebra and

$$((\alpha \vee \beta) \sim \gamma) \otimes (\delta \sim \alpha) \leqslant ((\delta \vee \beta) \sim \gamma).$$

Proposition 2.1. [9] *Let $\mathcal{E}_{II}$ be an EQ-algebra. Then, for all $\alpha, \beta, \gamma \in E$, the following properties hold:*
(i) $\alpha \sim \beta = \beta \sim \alpha$.
(ii) $\beta \leqslant \alpha \to \beta$.
(iii) $\alpha \to \beta = \alpha \to (\alpha \wedge \beta)$.
(iv) $\alpha \to \beta \leqslant (\beta \to \gamma) \to (\alpha \to \gamma)$.
(v) $\alpha \to \beta \leqslant (\gamma \to \alpha) \to (\gamma \to \beta)$.
(vi) *If $\alpha \leqslant \beta$, then $\gamma \to \alpha \leqslant \gamma \to \beta$ and $\beta \to \gamma \leqslant \alpha \to \gamma$.*
(vii) *If $\mathcal{E}_{II}$ is separated, then $\alpha \to \beta = 1$ if and only if $\alpha \leqslant \beta$.*
$(viii)$ *If $\mathcal{E}_{II}$ is a BEQ-algebra, then $\neg 0 = 1$ and $\neg\alpha = \alpha \to 0$.*
(ix) *If $\mathcal{E}_{II}$ is a BEQ-algebra, then $\alpha \to \beta \leqslant \neg\beta \to \neg\alpha$ and if $\mathcal{E}_{II}$ is involutive, then $\alpha \to \beta = \neg\beta \to \neg\alpha$.*

An EQ-algebra $\mathcal{E}_{II}$ has *exchange principle condition* if for any $\alpha, \beta, \gamma \in E$, $\alpha \to (\beta \to \gamma) = \beta \to (\alpha \to \gamma)$.

Proposition 2.2. [9, 17] *Let $\mathcal{E}_{II}$ be an EQ-algebra with exchange principle condition. Then, for all indexed families $\{\alpha_i\}_{i \in I} \subseteq E$ and $\gamma \in E$, we have, $(\bigvee_{i \in I} \alpha_i) \to \gamma = \bigwedge_{i \in I} (\alpha_i \to \gamma)$.*

Proposition 2.3. [14] *Let $\mathcal{E}_{II}$ be an EQ-algebra. Then, for all $\alpha, \beta, \gamma \in E$, the following statements are equivalent:*
(i) *$\mathcal{E}_{II}$ is good,*

[1)] Given an algebra $< E, F >$, where F is a set of operations on E and $F' \subseteq F$, then the algebra $< E, F' >$ is called the F'-reduct of $< E, F >$.

(*ii*) $\mathcal{E}_{\amalg}$ *is separated and satisfies exchange principle condition,*
(*iii*) $\mathcal{E}_{\amalg}$ *is separated and has* $\alpha \leqslant (\alpha \to \beta) \to \beta$.

Proposition 2.4. [8] *Let* $\mathcal{E}_{\amalg}$ *be a prelinear and separated EQ-algebra. Then, for any* $\alpha, \beta \in E$, $\alpha \vee \beta = 1$ *if and only if* $\alpha \to \beta = \beta$ *and* $\beta \to \alpha = \alpha$.

Let $\mathcal{E}_{\amalg}$ be an *EQ*-algebra, $\alpha, \beta, \gamma \in E$ and $\emptyset \neq F \subseteq E$. Then;
(*i*) F is called a *prefilter* of $\mathcal{E}_{\amalg}$ if $1 \in F$ and if $\alpha \in F$ and $\alpha \to \beta \in F$, then $\beta \in F$.
(*ii*) F is called an *implicative prefilter* of $\mathcal{E}_{\amalg}$ if $1 \in F$ and if $\gamma \to ((\alpha \to \beta) \to \alpha) \in F$ and $\gamma \in F$, then $\alpha \in F$.
(*iii*) a prefilter F of $\mathcal{E}_{\amalg}$ is called a *filter* of $\mathcal{E}_{\amalg}$ if $\alpha \to \beta \in F$, implies $(\alpha \otimes \gamma) \to (\beta \otimes \gamma) \in F$.
(*iv*) a (pre)filter F of $\mathcal{E}_{\amalg}$ is called a *positive implicative (pre)filter* of $\mathcal{E}_{\amalg}$ if $\alpha \to (\beta \to \gamma) \in F$ and $\alpha \to \beta \in F$, imply $\alpha \to \gamma \in F$.

Remark 2.5. *[18] Let F be a prefilter of EQ-algebra* $\mathcal{E}_{\amalg}$. *If* $\alpha \in F$ *and* $\alpha \leqslant \beta$, *then* $\beta \in F$.

Remark 2.6. *[9] Let* $\mathcal{E}_{\amalg}$ *be a separated EQ-algebra. The singleton subset* $\{1\} \subseteq E$ *is a filter of* $\mathcal{E}_{\amalg}$.

Theorem 2.7. [9] *Let F be a filter of EQ-algebra* $\mathcal{E}_{\amalg}$. *A binary relation* $\approx_F$ *on E which is defined by* $\alpha \approx_F \beta$ *if and only if* $\alpha \sim \beta \in F$, *is a congruence relation on* $\mathcal{E}_{\amalg}$ *and* $\mathcal{E}_{\amalg}/F = (E/F, \wedge_F, \otimes_F, \sim_F, F)$ *is a separated EQ-algebra, where, for any* $\alpha, \beta \in E$, *we have,*

$$[\alpha] \wedge_F [\beta] = [\alpha \wedge \beta] \ , \ [\alpha] \otimes_F [\beta] = [\alpha \otimes \beta] \ , \ [\alpha] \sim_F [\beta] = [\alpha \sim \beta] \ , \ [\alpha] \to_F [\beta] = [\alpha \to \beta]$$

A binary relation $\leqslant_F$ on E/F which is defined by $[\alpha] \leqslant_F [\beta]$ if and only if $[\alpha] \wedge_F [\beta] = [\alpha]$ is a partial order on E/F and for any $[\alpha], [\beta] \in E/F$, $[\alpha] \leqslant_F [\beta]$ if and only if $\alpha \to \beta \in F$ if and only if $[\alpha] \to_F [\beta] = [1]$.

Corollary 2.8. *If an EQ-algebra* $\mathcal{E}_{\amalg}$ *has exchange principle condition, then* $\mathcal{E}_{\amalg}/F$ *is a good EQ-algebra.*

Theorem 2.9. [14] *Let* $\mathcal{E}_{\amalg}$ *be an EQ-algebra and F be a prefilter of* $\mathcal{E}_{\amalg}$. *Then, for any* $\alpha, \beta \in E$, *the following statements are equivalent:*
(*i*) *F is a positive implicative prefilter of* $\mathcal{E}_{\amalg}$,
(*ii*) $(\alpha \wedge (\alpha \to \beta)) \to \beta \in F$.

Theorem 2.10. [14] *Let* $\mathcal{E}_{\amalg}$ *be an EQ-algebra. Then the following statements hold:*
(*i*) *Every implicative (pre)filter of* $\mathcal{E}_{\amalg}$ *is a (pre)filter of* $\mathcal{E}_{\amalg}$.
(*ii*) *Every implicative (pre)filter of* $\mathcal{E}_{\amalg}$ *is a positive implicative (pre)filter of* $\mathcal{E}_{\amalg}$.

Corollary 2.11. [14] *Let* $\mathcal{E}_{\amalg}$ *be a BEQ-algebra and F be a prefilter of* $\mathcal{E}_{\amalg}$. *If* $\mathcal{E}_{\amalg}$ *has exchange principle condition, then for any* $\alpha, \beta \in E$, *the following statments are equivalent:*
(*i*) *F is an implicative prefilter of* $\mathcal{E}_{\amalg}$,
(*ii*) *F is a positive implicative prefilter of* $\mathcal{E}_{\amalg}$, *and* $(\alpha \to \beta) \to \beta \in F$ *implies* $(\beta \to \alpha) \to \alpha \in F$,
(*iii*) $(\alpha \to \beta) \to \alpha \in F$ *implies* $\alpha \in F$.

Notation 2.12. *From now on, in this paper,* $\mathcal{E}_{\amalg} = (E, \wedge, \otimes, \sim, 1)$ *or simply* $\mathcal{E}_{\amalg}$ *is an EQ-algebra, unless otherwise state.*

3. Fantastic (pre)filter of *EQ*-algebras

In [21], Zebardast et al. showed that every good *EQ*-algebra is an equality algebra. On the other hand, in [1], it is proved that one can define another binary operation on any equality algebra which the equality algebra with this new operation become a good *EQ*-algebra. Thus the properties of (pre)filters in good *EQ*-algebras are the same as properties of filters in equality algebras. In [20] Xin, Ma, and Fu introduced the notions of fantastic (pre)filter of *EQ*-algebras and studied it in good *EQ*-algebras. They proved that the quotient structure of good *BEQ*-algebra is an *MV*-algebra. In this section, we investigate some properties of fantasitc (pre)filters of *EQ*-algebras such as every implicative (pre)filter of *EQ*-algebra is a fantastic (pre)filter of *EQ*-algebra and the quotient structure which is introduced by a fantastic filter is a lattice-ordered *EQ*-algebra. Also, we prove that the quotient structure of *BEQ*-algebra with exchange principle condition is an *MV*-algebra.

Definition 3.1. *[20] Let F be a (pre)filter of $\mathcal{E}_{\amalg}$. Then F is called a fantasic (pre)filter of $\mathcal{E}_{\amalg}$, if for any $\alpha, \beta \in E$, $\beta \to \alpha \in F$ implies $((\alpha \to \beta) \to \beta) \to \alpha \in F$.*

Proposition 3.2. *Let F be a (pre)filter of $\mathcal{E}_{\amalg}$. Then, for any $\alpha, \beta, \gamma \in E$, the following conditions are equivalent,*
(i) F is a fantastic (pre)filter of $\mathcal{E}_{\amalg}$,
(ii) if $\alpha \to \gamma \in F$ and $\beta \to \gamma \in F$, then $((\alpha \to \beta) \to \beta) \to \gamma \in F$,
(iii) if $\mathcal{E}_{\amalg}$ has exchange principle condition, then

$$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha) = (\beta \to \alpha) \to (((\alpha \to \beta) \to \beta) \to \alpha) \in F.$$

Proof. ($i \Rightarrow ii$) Suppose that, for $\alpha, \beta, \gamma \in E$, $\alpha \to \gamma \in F$ and $\beta \to \gamma \in F$. Since F is a fantastic (pre)filter of $\mathcal{E}_{\amalg}$, $((\gamma \to \beta) \to \beta) \to \gamma \in F$. On the other hand, by Proposition 2.1(iii), we have,

$$\begin{aligned} \alpha \to \gamma &\leqslant (\gamma \to \beta) \to (\alpha \to \beta) \\ &\leqslant ((\alpha \to \beta) \to \beta) \to ((\gamma \to \beta) \to \beta) \\ &\leqslant (((\gamma \to \beta) \to \beta) \to \gamma) \to (((\alpha \to \beta) \to \beta) \to \gamma). \end{aligned}$$

Since F is a (pre)filter of $\mathcal{E}_{\amalg}$ and $\alpha \to \gamma \in F$, by Remark 2.5, we get

$$(((\gamma \to \beta) \to \beta) \to \gamma) \to (((\alpha \to \beta) \to \beta) \to \gamma) \in F.$$

Moreover, since F is a fantastic (pre)filter of $\mathcal{E}_{\amalg}$ and $((\gamma \to \beta) \to \beta) \to \gamma \in F$, we get $((\alpha \to \beta) \to \beta) \to \gamma \in F$.
($ii \Rightarrow i$) Let $\gamma = \alpha$ in (ii). Then the proof is clear.
($i \Rightarrow iii$) Since $\mathcal{E}_{\amalg}$ has exchange principle condition, for any $\alpha, \beta \in E$,

$$\beta \to ((\beta \to \alpha) \to \alpha) = (\beta \to \alpha) \to (\beta \to \alpha) = 1 \in F.$$

Moreover, since F is a fantastic (pre)filter of $\mathcal{E}_{\amalg}$ and $\beta \to ((\beta \to \alpha) \to \alpha) \in F$, we get

$$(\beta \to \alpha) \to (((((\beta \to \alpha) \to \alpha) \to \beta) \to \beta) \to \alpha) = ((((\beta \to \alpha) \to \alpha) \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha) \in F.$$

Also, by Proposition 2.1(ii) and (vi), $\alpha \leqslant (\beta \to \alpha) \to \alpha$ and so $((\beta \to \alpha) \to \alpha) \to \beta \leqslant \alpha \to \beta$. Hence $(\alpha \to \beta) \to \beta \leqslant (((\beta \to \alpha) \to \alpha) \to \beta) \to \beta$. Then

$$((((\beta \to \alpha) \to \alpha) \to \beta) \to \beta) \to \alpha \leqslant ((\alpha \to \beta) \to \beta) \to \alpha,$$

which implies that,

$$(\beta \to \alpha) \to (((((\beta \to \alpha) \to \alpha) \to \beta) \to \beta) \to \alpha) \leqslant (\beta \to \alpha) \to (((\alpha \to \beta) \to \beta) \to \alpha).$$

Since $\mathcal{E}_{\amalg}$ has exchange principle condition, we get,

$$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha) = (\beta \to \alpha) \to (((\alpha \to \beta) \to \beta) \to \alpha).$$

Since $(\beta \to \alpha) \to (((((\beta \to \alpha) \to \alpha) \to \beta) \to \beta) \to \alpha) \in F$, and F is a (pre)filter of $\mathcal{E}_{\amalg}$, by Remark 2.5, we have $(\beta \to \alpha) \to (((\alpha \to \beta) \to \beta) \to \alpha) \in F$.
($iii \Rightarrow i$) Let $\alpha, \beta \in E$ such that $\beta \to \alpha \in F$. By (ii), $(\beta \to \alpha) \to (((\alpha \to \beta) \to \beta) \to \alpha) \in F$. Then by definition of (pre)filter, $((\alpha \to \beta) \to \beta) \to \alpha \in F$. Hence, F is a fantastic (pre)filter of $\mathcal{E}_{\amalg}$. □

Note. By Proposition 2.3, every good EQ-algebra has exchange principle condition. So there exist a lot of examples of EQ-algebras where have exchange principle condition.

Corollary 3.3. *Let $\mathcal{E}_{\amalg}$ be a BEQ-algebra. If F is a fantastic (pre)filter of $\mathcal{E}_{\amalg}$, then for any $\alpha \in E$, $\neg\neg\alpha \to \alpha \in F$.*

Proof. By Proposition 2.1($viii$), the proof is clear. □

In the next example we can see that the converse of Corollary 3.3, may not be true, generally.

Example 3.4. *Let $E = \{0, \alpha, \beta, \gamma, \delta, \theta, \kappa, 1\}$ be a lattice with a Hesse diagram as Figure 1. For any $x, y \in E$, we define the operations $\otimes$ and $\sim$ as Table 1 and Table 2.*

$\otimes$	0	α	β	γ	δ	θ	κ	1
0	0	0	0	0	0	0	0	0
α	0	0	0	0	0	0	0	α
β	0	0	0	0	0	0	0	β
γ	0	0	0	0	0	0	0	γ
δ	0	0	0	0	δ	δ	δ	δ
θ	0	0	0	0	δ	θ	δ	θ
κ	0	0	0	0	δ	δ	δ	κ
1	0	α	β	γ	δ	θ	κ	1

Table 1

$\sim$	0	α	β	γ	δ	θ	κ	1
0	1	θ	κ	δ	γ	α	β	0
α	θ	1	δ	κ	γ	α	γ	α
β	κ	δ	1	θ	γ	γ	β	β
γ	δ	κ	θ	1	γ	γ	γ	γ
δ	γ	γ	γ	γ	1	κ	θ	δ
θ	α	α	γ	γ	κ	1	δ	θ
κ	β	γ	β	γ	θ	δ	1	κ
1	0	α	β	γ	δ	θ	κ	1

Table 2

$\to$	0	α	β	γ	δ	θ	κ	1
0	1	1	1	1	1	1	1	1
α	θ	1	θ	1	1	1	1	1
β	κ	κ	1	1	1	1	1	1
γ	δ	κ	θ	1	1	1	1	1
δ	γ	γ	γ	γ	1	1	1	1
θ	α	α	γ	γ	κ	1	κ	1
κ	β	γ	β	γ	θ	θ	1	1
1	0	α	β	γ	δ	θ	κ	1

Table 3

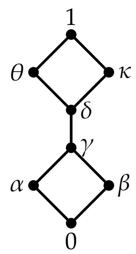

Figure 1

Then $\mathcal{E}_{II} = (E, \wedge, \otimes, \sim, 1)$ is an IEQ-algebra [18] and operation $\to$ is as Table 3. Hence for any $\alpha \in E$, $\neg\neg\alpha = \alpha$ then $\neg\neg\alpha \to \alpha = 1$. But $G = \{1\}$ is not a fantastic (pre)filter of $\mathcal{E}_{II}$. Because $\gamma \to \delta = 1 \in G$ but $((\delta \to \gamma) \to \gamma) \to \delta = 1 \to \delta = \delta \notin G$.

Corollary 3.5. *Let $\mathcal{E}_{II}$ be a BEQ-algebra with exchange principle condition. If F is a fantastic filter of $\mathcal{E}_{II}$, then $\mathcal{E}_{II}/F$ is an IEQ-algebra.*

Proof. By Theorem 2.7 and Corollary 3.3, for any $\alpha \in E$, $[\neg\neg\alpha] \leqslant [\alpha]$. On the other hand, since $\mathcal{E}_{II}$ has exchange principle condition, for any $\alpha \in E$ we have, $\alpha \to \neg\neg\alpha = (\alpha \to 0) \to (\alpha \to 0) = 1 \in F$. Hence, $[\alpha] \leqslant [\neg\neg\alpha]$ and so $[\alpha] = [\neg\neg\alpha]$. Therefore, $\mathcal{E}_{II}/F$ is an *IEQ*-algebra. □

In the following theorem, we show that extended of every fantastic (pre)filter of an *EQ*-algebra is also a fantastic (pre)filter.

Theorem 3.6. *Suppose $\mathcal{E}_{II}$ has exchange principle condition and F and G are two (pre)filters of $\mathcal{E}_{II}$ such that $F \subseteq G$. If F is a fantastic (pre)filter of $\mathcal{E}_{II}$, then G is a fantastic (pre)filter of $\mathcal{E}_{II}$.*

Proof. Let $\alpha, \beta \in E$ such that $\beta \to \alpha \in G$. Since $\beta \to ((\beta \to \alpha) \to \alpha) = (\beta \to \alpha) \to (\beta \to \alpha) = 1 \in F$ and F is a fantastic (pre)filter of $\mathcal{E}_{II}$, we have

$$((((\beta \to \alpha) \to \alpha) \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha) \in F \subseteq G.$$

Since $\mathcal{E}_{II}$ has exchange principle condition, we have

$$(\beta \to \alpha) \to (((((\beta \to \alpha) \to \alpha) \to \beta) \to \beta) \to \alpha) \in G.$$

Moreover, since G is a (pre)filter of $\mathcal{E}_{II}$ and $\beta \to \alpha \in G$, then $(((((\beta \to \alpha) \to \alpha) \to \beta) \to \beta) \to \alpha) \in G$. By Proposition 2.1($ii$), $\alpha \leqslant (\beta \to \alpha) \to \alpha$. Then $\alpha \to ((\beta \to \alpha) \to \alpha) = 1$. Hence, by Proposition 2.1(iv),

$$\begin{aligned} \alpha \to ((\beta \to \alpha) \to \alpha) &\leqslant (((\beta \to \alpha) \to \alpha) \to \beta) \to (\alpha \to \beta) \\ &\leqslant ((\alpha \to \beta) \to \beta) \to ((((\beta \to \alpha) \to \alpha) \to \beta) \to \beta) \\ &\leqslant (((((\beta \to \alpha) \to \alpha) \to \beta) \to \beta) \to \alpha) \to (((\alpha \to \beta) \to \beta) \to \alpha). \end{aligned}$$

Since $\alpha \to ((\beta \to \alpha) \to \alpha) = 1$, we get

$$(((((\beta \to \alpha) \to \alpha) \to \beta) \to \beta) \to \alpha) \to (((\alpha \to \beta) \to \beta) \to \alpha) = 1.$$

Also, since $((((\beta \to \alpha) \to \alpha) \to \beta) \to \beta) \to \alpha \in G$ and G is a (pre)filter of $\mathcal{E}_{II}$, by definition of (pre)filter, $((\alpha \to \beta) \to \beta) \to \alpha \in G$. Hence, G is a fantastic (pre)filter of $\mathcal{E}_{II}$. □

Corollary 3.7. *Consider $\mathcal{E}_{II}$ has exchange principle condition. If $\{1\}$ is a fantastic prefilter of $\mathcal{E}_{II}$, then any prefilter of $\mathcal{E}_{II}$ is a fantastic prefilter of $\mathcal{E}_{II}$.*

Theorem 3.8. *Consider $\mathcal{E}_{II}$ has exchange principle condition. Then,*
(i) any implicative (pre)filter of $\mathcal{E}_{II}$ is a fantastic (pre)filter of $\mathcal{E}_{II}$.
(ii) F is a fantastic and positive implicative prefilter of $\mathcal{E}_{II}$ if and only if F is an implicative prefilter of $\mathcal{E}_{II}$.

Proof. (i) Let F be an implicative (pre)filter of $\mathcal{E}_{II}$ and for $\alpha, \beta \in E$, $\beta \to \alpha \in F$. By Proposition 2.1(ii), $\alpha \leqslant ((\alpha \to \beta) \to \beta) \to \alpha$. Then by Proposition 2.1(vi), $(((\alpha \to \beta) \to \beta) \to \alpha) \to \beta \leqslant \alpha \to \beta$. Let $x = ((\alpha \to \beta) \to \beta) \to \alpha$. Then $x \to \beta \leqslant \alpha \to \beta$ and so $(\alpha \to \beta) \to x \leqslant (x \to \beta) \to x$. On the other hand, by Proposition 2.1(v), $\beta \to \alpha \leqslant ((\alpha \to \beta) \to \beta) \to ((\alpha \to \beta) \to \alpha)$. Then by exchange principle condition,

$$(\alpha \to \beta) \to x = (\alpha \to \beta) \to (((\alpha \to \beta) \to \beta) \to \alpha) = ((\alpha \to \beta) \to \beta) \to ((\alpha \to \beta) \to \alpha) \in F.$$

Since F is a prefilter of $\mathcal{E}_{II}$ and $(\alpha \to \beta) \to x \in F$, by Remark 2.5, $(x \to \beta) \to x \in F$. Moreover, since F is an implicative prefilter of $\mathcal{E}_{II}$, by Corollary 2.11(iii), $x \in F$, and so $((\alpha \to \beta) \to \beta) \to \alpha \in F$. Therefore, F is a fantastic filter of $\mathcal{E}_{II}$.

(ii) If F is an implicative prefilter of $\mathcal{E}_{II}$, then by Theorem 3.8, F is a fantistic prefilter of $\mathcal{E}_{II}$, and by Theorem 2.10(ii), F is a positive implicative prefilter of $\mathcal{E}_{II}$.
Conversely, suppose F is a fantastic and positive implicative prefilter of $\mathcal{E}_{II}$ such that, for $\alpha, \beta \in E$, $(\alpha \to \beta) \to \beta \in F$. Since F is a fantastic prefilter of $\mathcal{E}_{II}$, by Proposition 3.2(iii), $(\beta \to \alpha) \to \alpha \in F$. Moreover, since F is a positive implicative prefilter of $\mathcal{E}_{II}$, by Corollary 2.11(ii), F is an implicative prefilter of $\mathcal{E}_{II}$. □

In the next example, we can see that the converse of Theorem 3.8(i), is generally not correct.

Example 3.9. *Let $E = \{0, \alpha, \beta, \gamma, \delta, 1\}$ be a lattice with a Hesse diagram as Figure 2. For any $x, y \in E$, we define the operations $\otimes$ and $\sim$ on E as Table 4 and Table 5:*

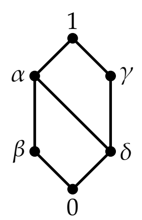

Figure 2

$\otimes$	0	α	β	γ	δ	1
0	0	0	0	0	0	0
α	0	β	β	δ	0	α
β	0	β	β	0	0	β
γ	0	δ	0	γ	δ	γ
δ	0	0	0	δ	0	δ
1	0	α	β	γ	δ	1

Table 4

$\sim$	0	α	β	γ	δ	1
0	1	δ	γ	β	α	0
α	δ	1	α	δ	γ	α
β	γ	α	1	0	δ	β
γ	β	δ	0	1	α	γ
δ	α	γ	δ	α	1	δ
1	0	α	β	γ	δ	1

Table 5

$\to$	0	α	β	γ	δ	1
0	1	1	1	1	1	1
α	δ	1	α	γ	γ	1
β	γ	1	1	γ	γ	1
γ	β	α	β	1	α	1
δ	α	1	α	1	1	1
1	0	α	β	γ	δ	1

Table 6

Then $\mathcal{E}_{II} = (E, \wedge, \otimes, \sim, 1)$ is a good EQ-algebra and operation $\to$ is as Table 6. It is easy to see that $H = \{1\}$ is a fantastic filter of $\mathcal{E}_{II}$, but H is not an implicative filter of $\mathcal{E}_{II}$. Because $(\alpha \to 0) \to \alpha = \delta \to \alpha = 1 \in H$ but $\alpha \notin H$. Also, H is not a positive implicative filter of $\mathcal{E}_{II}$. Because $(\alpha \wedge (\alpha \to 0)) \to 0 = \alpha \notin H$.

Theorem 3.10. *Let $\mathcal{E}_{II}$ has exchange principle condition. If F is a fantastic filter of $\mathcal{E}_{II}$, then $\mathcal{E}_{II}/F = (E/F, \otimes_F, \wedge_F, \sim_F, [1])$ is a lattice-ordered EQ-algebra.*

Proof. By Theorem 2.7, $\mathcal{E}_{II}/F$ is an EQ-algebra. Now, for any $\alpha, \beta \in E$, we define $[\alpha] \vee_f [\beta] = [(\alpha \to \beta) \to \beta]$. We claim that "$\vee_f$" is a join operation on $\mathcal{E}_{II}$. By Proposition 2.1(*ii*), $[\beta] \leqslant [(\alpha \to \beta) \to \beta]$. Since $\mathcal{E}_{II}$ has exchange principle condition, by Proposition 2.3 and Corollary 2.8, $\mathcal{E}_{II}/F$ is a good EQ-algebra and so by Proposition 2.1(*vii*), we have $[\alpha] \leqslant [(\alpha \to \beta) \to \beta]$. Thus, $[\alpha] \vee [\beta] \leqslant [(\alpha \to \beta) \to \beta]$. Suppose that there exists $\delta \in E$ such that $[\alpha] \leqslant [\delta]$ and $[\beta] \leqslant [\delta]$. By Theorem 2.7, we obtain $\alpha \to \delta \in F$ and $\beta \to \delta \in F$. Since F is a fantastic filter of $\mathcal{E}_{II}$, by Proposition 3.2(*iii*), we have $((\alpha \to \beta) \to \beta) \to \delta \in F$, which means $[(\alpha \to \beta) \to \beta] \leqslant [\delta]$. Therefore, "$\vee_f$" is the join operation. □

The next example shows that the quotient structure induced by fantastic filter is not an ℓEQ-algebra, in general.

Example 3.11. *Let $\mathcal{E}_{II}$ be an EQ-algebra as in Example 3.9. By some calculations, we can see that $\{1\}$ is a fantastic prefilter of $\mathcal{E}_{II}$, but $\mathcal{E}_{II}$ is not an ℓEQ-algebra. Because $((\beta \vee \gamma) \sim 1) \otimes (\gamma \sim \delta) = 1 \otimes \alpha = \alpha$ and $(\gamma \vee \gamma) \sim 1 = \gamma$, but α and γ are not comparable.*

An *MV-algebra* [6] is an algebraic structure $(M, \oplus, ^*, 0)$ of type $(2, 1, 0)$ which for any $\alpha, \beta \in M$, satisfies the following conditions:
(MV1) $(M, \oplus, 0)$ is a commutative monoid.
(MV2) $(\alpha^*)^* = \alpha$.
(MV3) $0^* \oplus \alpha = 0^*$.
(MV4) $(\alpha^* \oplus \beta)^* \oplus \beta = (\beta^* \oplus \alpha)^* \oplus \alpha$.

Theorem 3.12. *Let $\mathcal{E}_{II}$ be an BEQ-algebra with exchange principle condition. Let F be a filter of $\mathcal{E}_{II}$ and for any $\alpha, \beta \in E$, binary operation $\oplus$ on $\mathcal{E}_{II}/F$ is defined by $[\alpha] \oplus [\beta] = \neg[\alpha] \to [\beta]$, where $\neg\alpha = \alpha \sim 0$. Then $\mathcal{E}_{II}/F = (E/F, \oplus, \neg, [0])$ is an MV-algebra if and only if F is a fantastic filter of $\mathcal{E}_{II}$.*

Proof. Let F be a fantastic filter of $\mathcal{E}_{II}$. Then by Corollary 3.5, $\mathcal{E}_{II}/F$ is an IEQ-algebra. Hence for any $[\alpha] \in \mathcal{E}_{II}/F$, $\neg(\neg[\alpha]) = [\alpha]$ and so (MV2) holds. Now, we show that the binary operation $\oplus$ is associative. From Proposition 2.1(*ix*) and exchange principle condition, we have

$$\begin{aligned}[\alpha] \oplus ([\beta] \oplus [\gamma]) &= \neg[\alpha] \to (\neg[\beta] \to [\gamma]) = \neg[\alpha] \to (\neg[\gamma] \to [\beta]) \\ &= \neg[\gamma] \to (\neg[\alpha] \to [\beta]) = \neg(\neg[\alpha] \to [\beta]) \to \neg\neg[\gamma] \\ &= \neg(\neg[\alpha] \to [\beta]) \to [\gamma] \\ &= ([\alpha] \oplus [\beta]) \oplus [\gamma].\end{aligned}$$

By Proposition 2.1(*ix*), for any $[\alpha], [\beta] \in \mathcal{E}_{II}/F$, we have $[\alpha] \oplus [\beta] = \neg[\alpha] \to [\beta] = \neg[\beta] \to [\alpha] = [\beta] \oplus [\alpha]$ and $[\alpha] \oplus [0] = \neg[\alpha] \to [0] = ([\alpha] \to [0]) \to [0] = [\alpha]$. Hence, $(E/F, \oplus, 0)$ is a commutative monoid and so (MV1) holds. Also, (MV3) is satisfied, because for any $\alpha \in E$, we have,

$$\neg[0] \oplus [\alpha] = ([0] \to [0]) \oplus [\alpha] = [1] \oplus [\alpha] = [\neg 1 \to \alpha] = [0 \to \alpha] = [1].$$

Now, we show that (MV4) holds. Since $\mathcal{E}_{\amalg}/F$ is an IEQ-algebra, for any $\alpha, \beta \in E$, we get

$$\neg(\neg[\alpha] \oplus [\beta]) \oplus [\beta] = (\neg[\alpha] \oplus [\beta]) \to [\beta] = ([\alpha] \to [\beta]) \to [\beta] = [(\alpha \to \beta) \to \beta].$$

and

$$\neg(\neg[\beta] \oplus [\alpha]) \oplus [\alpha] = (\neg[\beta] \oplus [\alpha]) \to [\alpha] = ([\beta] \to [\alpha]) \to [\alpha] = [(\beta \to \alpha) \to \alpha].$$

Since $\mathcal{E}_{\amalg}$ has exchange principle condition and F is a fantastic filter of $\mathcal{E}_{\amalg}$, by Proposition 3.2(*iii*), $[(\alpha \to \beta) \to \beta] = [(\beta \to \alpha) \to \alpha]$. Hence, $\neg(\neg[\alpha] \oplus [\beta]) \oplus [\beta] = \neg(\neg[\beta] \oplus [\alpha]) \oplus [\alpha]$. Therefore, $\mathcal{E}_{\amalg}/F = (E/F, \oplus, \neg, 0)$ is an MV-algebra.

Conversely, let $\mathcal{E}_{\amalg}/F = (E/F, \oplus, \neg, 0)$ be an MV-algebra. Then by (MV4), for any $\alpha, \beta \in E$, we have,

$$[(\alpha \to \beta) \to \beta] = \neg(\neg[\alpha] \oplus [\beta]) \oplus [\beta] = \neg(\neg[\beta] \oplus [\alpha]) \oplus [\alpha] = [(\beta \to \alpha) \to \alpha].$$

Then, for any $\alpha, \beta \in E$, $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha) \in F$. Thus by Proposition 3.2(*iii*), F is a fantastic filter of $\mathcal{E}_{\amalg}$. □

Example 3.13. *Let $E = \{0, \alpha, \beta, \gamma, \delta, \theta, \kappa, \mu, \nu, 1\}$ be a lattice with the following Hasse digram (Figure 3) and the operations $\otimes$ and $\sim$ are defined on E as Table 7 and Table 8.*

$\otimes$	0	ν	α	β	γ	δ	θ	κ	μ	1
0	0	0	0	0	0	0	0	0	0	0
ν	0	0	0	0	0	0	0	0	0	ν
α	0	0	α	0	α	0	α	0	α	α
β	0	0	0	0	0	0	0	β	β	β
γ	0	0	α	0	α	0	α	β	γ	γ
δ	0	0	0	0	0	β	β	δ	δ	δ
θ	0	0	α	0	α	β	γ	δ	θ	θ
κ	0	0	0	β	β	δ	δ	κ	κ	κ
μ	0	0	α	β	γ	δ	θ	κ	μ	μ
1	0	ν	α	β	γ	δ	θ	κ	μ	1

Table 7

$\sim$	0	ν	α	β	γ	δ	θ	κ	μ	1
0	1	μ	κ	θ	δ	γ	β	α	ν	0
ν	μ	1	κ	θ	δ	γ	β	α	ν	ν
α	κ	κ	1	δ	θ	β	γ	ν	α	α
β	θ	θ	δ	1	κ	θ	δ	γ	β	β
γ	δ	δ	θ	κ	1	δ	θ	β	γ	γ
δ	γ	γ	β	θ	δ	1	κ	θ	δ	δ
θ	β	β	γ	δ	θ	κ	1	δ	θ	θ
κ	α	α	ν	γ	β	θ	δ	1	κ	κ
μ	ν	ν	α	β	γ	δ	θ	κ	1	μ
1	0	ν	α	β	γ	δ	θ	κ	μ	1

Table 8

$\to$	0	ν	α	β	γ	δ	θ	κ	μ	1
0	1	1	1	1	1	1	1	1	1	1
ν	μ	1	1	1	1	1	1	1	1	1
α	κ	κ	1	κ	1	κ	1	κ	1	1
β	θ	θ	θ	1	1	1	1	1	1	1
γ	δ	δ	θ	κ	1	κ	1	κ	1	1
δ	γ	γ	γ	θ	θ	1	1	1	1	1
θ	β	β	γ	δ	θ	κ	1	κ	1	1
κ	α	α	α	γ	γ	θ	θ	1	1	1
μ	ν	ν	α	β	γ	δ	θ	κ	1	1
1	0	ν	α	β	γ	δ	θ	κ	μ	1

Table 9

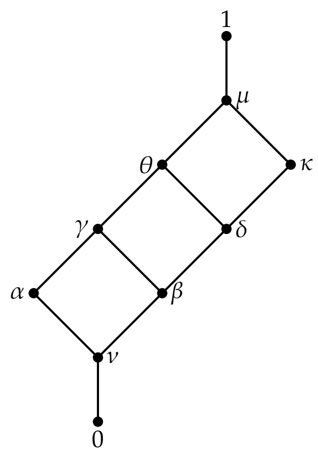

Figure 3

Then $\mathcal{E}_{II} = (E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra and the operation $\rightarrow$ is as Table 9. By some routine calculations, we can see that $F = \{\mu, 1\}$ is a fantastic filter of $\mathcal{E}_{II}$ and $\mathcal{E}_{II}/F = \{[0], [\alpha], [\beta], [\gamma], [\delta], [\theta], [\kappa], [1]\}$ is an MV-algebra. But F is not a positive implicative filter of $\mathcal{E}$. Because, $(\beta \wedge (\beta \rightarrow \nu)) \rightarrow \nu = \theta \notin F$. Thus, by Theorem 2.10(ii), F is not an implicative filter of $\mathcal{E}_{II}$.

4. Prelinear filters of *EQ*-algebras

Every finite *EQ*-algebra is a lattice-ordered *EQ*-algebra [8]. But in which condition an *EQ*-algebra is a $(\wedge, \vee)$-distributive lattice-ordered *EQ*-algebra? In [8], Elzekey proved that one can define a join operation on a prelinear *EQ*-algebra and then the *EQ*-algebra will be $(\wedge, \vee)$-distributive lattice-ordered *EQ*-algebra. In this section, we introduce a new kind of (pre)filter, named *prelinear (pre)filter*. In the rest of this section, we show that the quotient structure induced by a prelinear filter, is a $(\wedge, \vee)$-distributive lattice-ordered *EQ*-algebra. Also, we will show that if this prelinear filter is fantastic, positive implicative, or implicative, then we can construct a De Morgan algebra, Stone algebra or Boolean algebra, respectively.

Definition 4.1. *Let F be a (pre)filter of $\mathcal{E}_{II}$. Then F is called a prelinear (pre)filter of $\mathcal{E}_{II}$ if for any $\alpha, \beta, \gamma \in E$, $((\alpha \rightarrow \beta) \rightarrow \gamma) \rightarrow (((\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \gamma) \in F$.*

Example 4.2. *Let $\mathcal{E}_{II}$ be an EQ-algebra as in Example 3.9. Then $F = \{\alpha, \beta, 1\}$ is a prelinear filter of $\mathcal{E}_{II}$.*

Remark 4.3. *If $\mathcal{E}_{II}$ is a prelinear EQ-algebra with exchange principle condition, then every (pre)filter of $\mathcal{E}_{II}$ is a prelinear (pre)filter.*

In the following examples, we show that the concept of prelinear (pre)filter is not the same as fantastic or (positive)implicative (pre)filter.

Example 4.4. *(i) Let $E = \{0, \alpha, \gamma, \delta, \mu, 1\}$ be a lattice with a Hesse diagram as Figure 3. For any $x, y \in E$, we define the operations $\otimes$ and $\sim$ on E as Table 10 and Table 11:*

$\otimes$	0	α	γ	δ	μ	1
0	0	0	0	0	0	0
α	0	α	0	0	α	α
γ	0	0	γ	γ	γ	γ
δ	0	0	γ	γ	γ	δ
μ	0	α	γ	γ	μ	μ
1	0	α	γ	δ	μ	1

Table 10

$\sim$	0	α	γ	δ	μ	1
0	1	δ	α	α	0	0
α	δ	1	0	0	α	α
γ	α	0	1	μ	δ	γ
δ	α	0	μ	1	δ	δ
μ	0	α	δ	δ	1	μ
1	0	α	γ	δ	μ	1

Table 11

$\rightarrow$	0	α	γ	δ	μ	1
0	1	1	1	1	1	1
α	δ	1	δ	δ	1	1
γ	α	α	1	1	1	1
δ	α	α	μ	1	1	1
μ	0	α	δ	δ	1	1
1	0	α	γ	δ	μ	1

Table 12

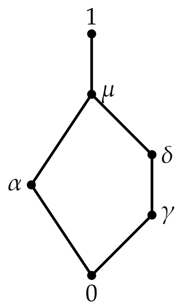

Figure 4

Then $\mathcal{E}_{II} = (E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra and operation $\rightarrow$ is as Table 12. We can see that $\mathcal{E}_{II}$ is not prelinear because, $\alpha \rightarrow \delta = \delta$ and $\delta \rightarrow \alpha = \alpha$ but $\alpha \vee \delta = \mu \neq 1$. Since $\mathcal{E}_{II}$ is good, $G = \{1\}$ is a filter of $\mathcal{E}_{II}$. But G is not a prelinear filter of $\mathcal{E}_{II}$. Because, $((\alpha \rightarrow \delta) \rightarrow \mu) \rightarrow (((\delta \rightarrow \alpha) \rightarrow \mu) \rightarrow \mu = \mu \notin G$.

(ii) Let $\mathcal{E}_{\amalg}$ be an EQ-algebra as in Example 3.4. It is obvious that $\mathcal{E}_{\amalg}$ is a prelinear good EQ-algebra. By Remark 2.6, we obtain $G = \{1\}$ is a prelinear filter of $\mathcal{E}_{\amalg}$. But G is not a fantastic filter of $\mathcal{E}_{\amalg}$. Because, $\alpha \to \delta = 1 \in G$ and $((\delta \to \alpha) \to \alpha) \to \delta = \theta \notin G$.

(iii) Let $E = \{0, \alpha, \beta, 1\}$ be a chain where $0 \leqslant \alpha \leqslant \beta \leqslant 1$. For any $x, y \in E$, we define the operations $\otimes$ and $\sim$ on E as Table 13 and Table 14:

$\otimes$	0	α	β	1
0	0	0	0	0
α	0	0	0	α
β	0	0	0	β
1	0	α	β	1

Table 13

$\sim$	0	α	β	1
0	1	a	0	0
α	α	1	α	α
β	0	α	1	β
1	0	α	β	1

Table 14

$\to$	0	α	β	1
0	1	1	1	1
α	α	1	1	1
β	0	α	1	1
1	0	α	β	1

Table 15

Then $\mathcal{E}_{\amalg} = (E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra and operation $\to$ is as Table 15. Since $\mathcal{E}_{\amalg}$ is a linearly ordered EQ-algebra, we can see that $F = \{1, \beta\}$ is a prelinear filter of $\mathcal{E}_{\amalg}$ but by Theorem 2.9(ii), it is not a positive implicative filter of $\mathcal{E}_{\amalg}$. Because, $(\alpha \wedge (\alpha \to 0)) \to 0 = \alpha \notin F$ and then by Proposition 2.10, F is not an implicative filter of $\mathcal{E}_{\amalg}$, either.

Theorem 4.5. [8] *Let $\mathcal{E}_{\amalg}$ be prelinear and good. If, for any $\alpha, \beta \in E$,*

$$\alpha \vee \beta = ((\alpha \to \beta) \to \beta) \wedge ((\beta \to \alpha) \to \alpha),$$

then $\mathcal{E}_{\amalg}$ is a $(\wedge, \vee)$-distributive ℓEQ-algebra.

Theorem 4.6. [8] *A lattice-ordered separated EQ-algebra $\mathcal{E}_{\amalg}$ is prelinear if and only if, for any $\alpha, \beta, \gamma \in E$:*

$$(\alpha \wedge \beta) \to \gamma = (\alpha \to \gamma) \vee (\beta \to \gamma).$$

Lemma 4.7. *Let $\mathcal{E}_{\amalg}$ be good. Then $\mathcal{E}_{\amalg}$ is prelinear if and only if, for any $\alpha, \beta, \gamma \in E$,*

$$(\alpha \to \beta) \to \gamma \leqslant ((\beta \to \alpha) \to \gamma) \to \gamma.$$

Proof. Suppose $\mathcal{E}_{\amalg}$ is prelinear and good. Then for any $\alpha, \beta \in E$, 1 is the unique upper bound of $\{\alpha \to \beta, \beta \to \alpha\}$ in E. By Proposition 2.1(*ii*) and (*iv*), we have

$$\alpha \to \beta \leqslant ((\beta \to \alpha) \to \gamma) \to (\alpha \to \beta) \leqslant ((\alpha \to \beta) \to \gamma) \to (((\beta \to \alpha) \to \gamma) \to \gamma).$$

On the other hand, since $\mathcal{E}_{\amalg}$ is good, by Proposition 2.3(*ii*), $\mathcal{E}_{\amalg}$ satisfies the exchange principle condition. Then by Proposition 2.1(*ii*) and (*iv*),

$$\begin{aligned}\beta \to \alpha &\leqslant ((\alpha \to \beta) \to \gamma) \to (\beta \to \alpha)\\ &\leqslant ((\beta \to \alpha) \to \gamma) \to (((\alpha \to \beta) \to \gamma) \to \gamma)\\ &= ((\alpha \to \beta) \to \gamma) \to (((\beta \to \alpha) \to \gamma) \to \gamma).\end{aligned}$$

Hence $((\alpha \to \beta) \to \gamma) \to (((\beta \to \alpha) \to \gamma) \to \gamma)$ is an upper bound of $\{\alpha \to \beta, \beta \to \alpha\}$. Since $\mathcal{E}_{\amalg}$ is prelinear and separated, we have $((\alpha \to \beta) \to \gamma) \to (((\beta \to \alpha) \to \gamma) \to \gamma) = 1$ and so,

$$(\alpha \to \beta) \to \gamma \leqslant (((\beta \to \alpha) \to \gamma) \to \gamma.$$

Conversely, suppose that for any $\alpha, \beta, \gamma \in E$, $(\alpha \to \beta) \to \gamma \leqslant ((\beta \to \alpha) \to \gamma) \to \gamma$. Since 1 is the greatest element of $\mathcal{E}_{\amalg}$, it is clear that $((\alpha \to \beta) \to \gamma) \to (((\beta \to \alpha) \to \gamma) \to \gamma) = 1$ is an upper bound of $\{\alpha \to \beta, \beta \to \alpha\}$. We show $\{\alpha \to \beta, \beta \to \alpha\}$ dose not have another upper bound. For this, suppose that there exists $\delta \in E$ such that $\alpha \to \beta \leqslant \delta$ and $\beta \to \alpha \leqslant \delta$. Thus, by Proposition 2.1(*vi*), we have $\delta \to \gamma \leqslant (\alpha \to \beta) \to \gamma$. By the similar way, $\delta \to \gamma \leqslant (\beta \to \alpha) \to \gamma$. Then $((\beta \to \alpha) \to \gamma) \to \gamma \leqslant (\delta \to \gamma) \to \gamma$. Now, by Proposition 2.1(*vi*), we have

$$\begin{aligned} 1 =& ((\alpha \to \beta) \to \gamma) \to (((\beta \to \alpha) \to \gamma) \to \gamma) \\ \leqslant& ((\alpha \to \beta) \to \gamma) \to ((\delta \to \gamma) \to \gamma) \\ \leqslant& (\delta \to \gamma) \to ((\delta \to \gamma) \to \gamma). \end{aligned}$$

Since $\mathcal{E}_{\amalg}$ is separated, by Proposition 2.1(*vii*), for any $\gamma \in E$, we have $\delta \to \gamma \leqslant (\delta \to \gamma) \to \gamma$. Let $\gamma = \delta$. Then $1 \leqslant \delta$ and so $\delta = 1$. Hence, the upper bound of $\{\alpha \to \beta, \beta \to \alpha\}$ is equal to 1. Therefore, $\mathcal{E}_{\amalg}$ is a prelinear EQ-algebra. □

Corollary 4.8. *Let $\mathcal{E}_{\amalg}$ be prelinear with exchange principle condition. Then for any $\alpha, \beta, \gamma \in E$,*

$$((\alpha \to \beta) \to \gamma) \to (((\beta \to \alpha) \to \gamma) \to \gamma) = 1.$$

Proof. By considering the proof of Lemma 4.7 and Proposition 2.3, the separated condition only use to obtain the nonequality from $((\alpha \to \beta) \to \gamma) \to (((\beta \to \alpha) \to \gamma) \to \gamma) = 1$. □

An algebra $(D, \vee, \wedge, \neg, 0, 1)$ of type $(2, 2, 1, 0, 0)$ is called a *De Morgan algebra* [15], if for any $\gamma, \delta \in D$, the following conditions hold:
(D1) $(D, \vee, \wedge, 0, 1)$ is a bounded distributive lattice.
(D2) $\neg\neg\gamma = \gamma$.
(D3) $\neg(\gamma \vee \delta) = \neg\gamma \wedge \neg\delta$, and $\neg(\gamma \wedge \delta) = \neg\gamma \vee \neg\delta$.

Proposition 4.9. *Let $\mathcal{E}_{\amalg}$ has exchange principle condition. If F is a prelinear filter of $\mathcal{E}_{\amalg}$, then:*
(i) $\mathcal{E}_{\amalg}/F$ is good and prelinear.
(ii) If for any $\alpha, \beta \in E$, we define

$$[\alpha] \vee_F [\beta] = [((\alpha \to \beta) \to \beta) \wedge ((\beta \to \alpha) \to \alpha)],$$

then $\mathcal{E}_{\amalg}/F = (E/F, \vee_F, \wedge_F, \neg_F, [0], [1])$ is a distributive lattice which satisfies the De Morgan Laws.
(iii) If F is a fantastic filter of $\mathcal{E}_{\amalg}$, then $\mathcal{E}_{\amalg}/F$ is a De Morgan algebra.

Proof. (*i*) By Theorem 2.7, for any filter F of $\mathcal{E}_{\amalg}$, $\mathcal{E}_{\amalg}/F$ is separated. Since $\mathcal{E}_{\amalg}$ has exchange principle condition, for any $\alpha, \beta, \gamma \in E$, we have

$$[\alpha] \to ([\beta] \to [\gamma]) = [\alpha \to (\beta \to \gamma)] = [\beta \to (\alpha \to \gamma)] = [\beta] \to ([\alpha] \to [\gamma]).$$

Then, $\mathcal{E}_{\amalg}/F$ has exchange principle condition and so by Proposition 2.3(*ii*), $\mathcal{E}_{\amalg}/F$ is a good EQ-algebra. Since F is a prelinear filter of $\mathcal{E}_{\amalg}$, for any $\alpha, \beta, \gamma \in E$,

$$((\alpha \to \beta) \to \gamma) \to (((\beta \to \alpha) \to \gamma) \to \gamma) \in F.$$

Then $[(\alpha \to \beta) \to \gamma] \leqslant [((\beta \to \alpha) \to \gamma) \to \gamma]$. Hence, by Lemma 4.7, $\mathcal{E}_{\amalg}/F$ is a prelinear EQ-algebra.

(*ii*) By Theorems 2.7, 4.5 and (*i*), $\mathcal{E}_{\amalg}/F$ is a $(\wedge_F, \vee_F)$-distributive lattice-ordered EQ-algebra. Since every good EQ-algebra is separated, by Theorem 4.6, for any $\alpha, \beta \in E$, we have $\neg([\alpha] \wedge_F [\beta]) = \neg[\alpha] \vee_F \neg[\beta]$. Since $\mathcal{E}_{\amalg}/F$ has exchange principle condition, from Proposition 2.2, for any $\alpha, \beta \in E$, $\neg([\alpha] \vee_F [\beta]) = \neg[\alpha] \wedge_F \neg[\beta]$. Therefore, $\mathcal{E}_{\amalg}/F$ satisfies the De Morgan Laws.

(*iii*) Since F is a prelinear filter of $\mathcal{E}_{\amalg}$, by Proposition 4.9, $\mathcal{E}_{\amalg}/F$ is a $(\vee_F, \wedge_F)$-distributive lattice which satisfies the De Morgan Laws. Also, F is a fantastic filter of $\mathcal{E}_{\amalg}$, then by Corollary 3.5, $\mathcal{E}_{\amalg}/F$ is an involutive EQ-algebra and $(D2)$ is satisfied. □

Example 4.10. (*i*) *According to Example 3.4, we can see that $\mathcal{E}_{\amalg}$ is a prelinear and involutive EQ-algebra and so it is a De Morgan algebra.*

(*ii*) *Let $E = \{0, \alpha, \beta, 1\}$ be a chain where $0 \leqslant \alpha \leqslant \beta \leqslant 1$. For any $x, y \in E$, we define the operations $\otimes$ and $\sim$ on E as Table 16 and Table 17:*

$\otimes$	0	α	β	1
0	0	0	0	0
α	0	α	α	α
β	0	α	β	β
1	0	α	β	1

Table 16

$\sim$	0	α	β	1
0	1	0	0	0
α	0	1	α	α
β	0	α	1	1
1	0	α	1	1

Table 17

$\rightarrow$	0	α	β	1
0	1	1	1	1
α	0	1	1	1
β	0	α	1	1
1	0	α	1	1

Table 18

By routine calculations, we can see that $\mathcal{E}_{II} = (E, \wedge, \otimes, \sim, 1)$ is a prelinear EQ-algebra and operation $\rightarrow$ is as Table 18. By Proposition 2.3 and Remark 2.6, we know that $\{1\}$ is a filter of $\mathcal{E}_{II}$. Since $\mathcal{E}_{II}$ is not involutive, it is not a De Morgan algebra, either.

(iii) Let $E = \{0, \alpha, \beta, \gamma, \delta, 1\}$ be a chain where $0 \leqslant \alpha \leqslant \beta \leqslant \gamma \leqslant \delta \leqslant 1$. For any $x, y \in E$, we define the operations $\otimes$ and $\sim$ on E as Table 19 and Table 20:

$\otimes$	0	α	β	γ	δ	1
0	0	0	0	0	0	0
α	0	0	0	0	0	α
β	0	0	0	0	α	β
γ	0	0	0	α	α	γ
δ	0	0	α	α	α	δ
1	0	α	β	γ	δ	1

Table 19

$\sim$	0	α	β	γ	δ	1
0	1	γ	β	α	0	0
α	γ	1	β	α	α	α
β	β	β	1	β	β	β
γ	α	α	β	1	γ	γ
δ	0	α	β	γ	1	δ
1	0	α	β	γ	δ	1

Table 20

$\rightarrow$	0	α	β	γ	δ	1
0	1	1	1	1	1	1
α	γ	1	1	1	1	1
β	β	β	1	1	1	1
γ	α	α	β	1	1	1
δ	0	α	β	γ	1	1
1	0	α	β	γ	δ	1

Table 21

By routine calculations, we can see that $\mathcal{E}_{II} = (E, \wedge, \otimes, \sim, 1)$ is a good prelinear and non involutive EQ-algebra and operation $\rightarrow$ is as Table 21. We can see that, $F = \{\gamma, \delta, 1\}$ is a fantastic filter of $\mathcal{E}_{II}$ and $\mathcal{E}_{II}/F = ([0], [\beta], [1])$ is a De Morgan algebra.

Let $(X, \vee, \wedge, 0, 1)$ be a bounded lattice. An element $x^* \in X$ is called a *pseudocomplement* of $x \in X$, if $x \wedge x^* = 0$ and if there exists $y \in X$ such that $x \wedge y = 0$, then $y \leqslant x^*$. If every element of X has a pseudocomplement element, then X is called a *pseudocomplemented lattice*(See [15]).

Theorem 4.11. *Let $\mathcal{E}_{II}$ be a good BEQ-algebra. If F is a prelinear positive implicative filter of $\mathcal{E}_{II}$, then $\mathcal{E}_{II}/F$ is a pseudocomplemented lattice.*

Proof. Since $\mathcal{E}_{II}$ has a bottom element and F is a prelinear filter of $\mathcal{E}_{II}$, by Propositions 2.3 and 4.9, $\mathcal{E}_{II}/F$ is a bounded lattice. Now, for any $[\alpha] \in \mathcal{E}_{II}/F$, we define $[\alpha]^* = \neg[\alpha] = [\neg\alpha]$. Since F is a positive implicative filter of $\mathcal{E}_{II}$, by Theorem 2.9(*ii*), for any $\alpha \in E$, we have $(\alpha \wedge (\alpha \rightarrow 0)) \rightarrow 0 \in F$ and so $[\alpha] \wedge_F [\neg\alpha] = [0]$. Now, suppose that there exists $[\delta] \in E/F$ such that $[\alpha] \wedge_F [\delta] = [0]$. By Propositions 2.3 and 4.9, $\mathcal{E}_{II}/F$ satisfies the De Morgan Laws and so we obtain $[\neg\alpha] \vee_F [\neg\delta] = [1]$. By Proposition 2.4, we have $[\neg\alpha] \rightarrow [\neg\delta] = [\neg\delta]$ and so $(\neg\alpha \rightarrow \neg\delta) \rightarrow \neg\delta \in F$. Since $\mathcal{E}_{II}$ is good, by exchange principle condition, we get

$$(\neg\alpha \rightarrow \neg\delta) \rightarrow \neg\delta = \delta \rightarrow ((\neg\alpha \rightarrow \neg\delta) \rightarrow 0) = \delta \rightarrow ((\delta \rightarrow \neg\neg\alpha) \rightarrow 0).$$

By Proposition 2.1(*iv*), we obtain

$$\delta \rightarrow ((\delta \rightarrow \neg\neg\alpha) \rightarrow 0) \leqslant (((\delta \rightarrow \neg\neg\alpha) \rightarrow 0) \rightarrow \neg\alpha) \rightarrow (\delta \rightarrow \neg\alpha). \tag{1}$$

By exchange principle condition, we have

$$((\delta \rightarrow \neg\neg\alpha) \rightarrow 0) \rightarrow \neg\alpha = \alpha \rightarrow (((\delta \rightarrow \neg\neg\alpha) \rightarrow 0) \rightarrow 0) = \alpha \rightarrow \neg\neg(\delta \rightarrow \neg\neg\alpha).$$

Since $\mathcal{E}_{II}$ is good, by Proposition 2.3(*iii*), we have $\alpha \leqslant \neg\neg\alpha$ and by Proposition 2.1(*ii*), we have $\neg\neg\alpha \leqslant \delta \rightarrow \neg\neg\alpha$. Again by Proposition 2.3(*iii*), we obtain $\delta \rightarrow \neg\neg\alpha \leqslant \neg\neg(\delta \rightarrow \neg\neg\alpha)$. Then, $(\delta \rightarrow \neg\neg\alpha) \rightarrow 0) \rightarrow \neg\alpha = \alpha \rightarrow \neg\neg(\delta \rightarrow \neg\neg\alpha) = 1 \in F$. Hence, by (4.1), we have $\delta \rightarrow \neg\alpha \in F$. Then, $[\delta] \leqslant [\neg\alpha]$ and so, $[\neg\alpha]$ is the greatest element in $\mathcal{E}_{II}/F$ such that $[\alpha] \wedge_F [\neg\alpha] = [0]$. Therefore $\mathcal{E}_{II}/F$ is a pseudocomplemented lattice. □

An algebra $(S, \vee, \wedge, ^*, 0, 1)$ of type $(2, 2, 1, 0, 0)$ is called a *Stone algebra* [15], if:
(S1) $(S, \vee, \wedge, 0, 1)$ is a pseudocomplemented distributive lattice.
(S2) $s^* \vee s^{**} = 1$, for any $s \in S$.

Theorem 4.12. *Let $\mathcal{E}_{II}$ be a good BEQ-algebra. If F is a prelinear positive implicative filter of $\mathcal{E}_{II}$, then $\mathcal{E}_{II}/F = (E/F, \vee_F, \wedge_F, \neg_F, [0], [1])$ is a Stone algebra.*

Proof. Let F be a prelinear positive implicative filter of $\mathcal{E}_{II}$. Then by Proposition 4.9, $\mathcal{E}_{II}/F$ is an $(\vee_F, \wedge_F)$-distributive lattice. By Theorem 4.11, $\mathcal{E}_{II}/F$ is a pseudocomplemented lattice and so, (S1) is satisfied. By Proposition 4.9(*ii*) we have $\neg([\alpha] \wedge [\neg\alpha]) = \neg[0]$ and so $[\neg\alpha] \vee_F [\neg\neg\alpha] = [\neg 0] = [1]$. Hence, (S2) is satisfied. Therefore, $\mathcal{E}_{II}/F$ is a Stone algebra. □

In the following example, we show that the converse of Theorem 4.12 may not be true, in general.

Example 4.13. *Let $E = \{0, \alpha, \beta, 1\}$ be a chain where $0 \leqslant \alpha \leqslant \beta \leqslant 1$. Define the operations $\otimes$ and $\sim$ on E as Table 22 and Table 23:*

$\otimes$	0	α	β	1
0	0	0	0	0
α	0	0	0	α
β	0	0	0	β
1	0	α	β	1

Table 22

$\sim$	0	α	β	1
0	1	0	0	0
α	0	1	β	α
β	0	β	1	β
1	0	α	β	1

Table 23

$\rightarrow$	0	α	β	1
0	1	1	1	1
α	0	1	1	1
β	0	β	1	1
1	0	α	β	1

Table 24

Then $\mathcal{E}_{II} = (E, \wedge, \otimes, \sim, 1)$ is a prelinear good EQ-algebra and it is a Stone algebra. Moreover, operation $\rightarrow$ is as Table 24. Since $\mathcal{E}_{II}$ is a good EQ-algebra, by Remark 2.6, $\{1\}$ is a filter of $\mathcal{E}_{II}$ but, $\{1\}$ is not a positive implicative filter of $\mathcal{E}_{II}$, because $(\beta \wedge (\beta \rightarrow \alpha)) \rightarrow \alpha = \beta \notin \{1\}$.

A *Boolean algebra* [5] is an algebra $(B, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that for any $a, b \in E$;
(B1) $(B, \vee, \wedge)$ is a distributive lattice.
(B2) $a \wedge 0 = 0$, and $a \vee 1 = 1$. (bounded)
(B3) $a \wedge a' = 0$, and $a \vee a' = 1$. (complemented)

Remark 4.14. *Let $\mathcal{E}_{II}$ be a BEQ-algebra with exchange principle condition and F be a prelinear filter of $\mathcal{E}_{II}$. By Proposition 4.9 we know $\mathcal{E}_{II}/F = (E/F, \vee_F, \wedge_F, \neg, [0], [1])$ is a distributive lattice, where for any $\alpha, \beta \in E$, $[\alpha] \vee_F [\beta] = [((\alpha \rightarrow \beta) \rightarrow \beta) \wedge ((\beta \rightarrow \alpha) \rightarrow \alpha)]$. Also, if F is a fantastic filter of $\mathcal{E}_{II}$, then by Proposition 3.2(iii), $[\alpha] \vee_F [\beta] = [((\alpha \rightarrow \beta) \rightarrow \beta]$ and $\mathcal{E}_{II}/F$ is a De Morgan algebra.*

Lemma 4.15. *Let $\mathcal{E}_{II}$ be a BEQ-algebra with exchange principle condition. A (pre)filter F of $\mathcal{E}_{II}$ is an implicative (pre)filter of $\mathcal{E}_{II}$ if and only if for any $\alpha \in E$, $(\neg\alpha \rightarrow \alpha) \rightarrow \alpha \in F$.*

Proof. The proof is similar to the proof of [4, Proposition 3.17]. □

Theorem 4.16. *Let $\mathcal{E}_{II}$ be a BEQ-algebra with exchange principle condition such that F be a prelinear filter of $\mathcal{E}_{II}$. Then F is an implicative filter of $\mathcal{E}_{II}$ if and only if $(\mathcal{E}_{II}/F, \vee_F, \wedge_F, \neg_F, [0], [1])$ is a Boolean algebra.*

Proof. Let F be an implicative filter of $\mathcal{E}_{II}$. By Proposition 3.8, F is a fantastic filter of $\mathcal{E}_{II}$ and so by Proposition 4.9, $(\mathcal{E}_{II}/F, \vee_F, \wedge_F, \neg_F, [0], [1])$ is a De Morgan algebra. By Remark 4.14, for any $\alpha \in F$, we have $[\neg\alpha] \vee_F [\alpha] = [(\neg\alpha \rightarrow \alpha) \rightarrow \alpha]$. Since F is an implicative filter of $\mathcal{E}_{II}$, by Lemma 4.15, $(\neg\alpha \rightarrow \alpha) \rightarrow \alpha \in F$. Hence, $[\neg\alpha] \vee_F [\alpha] = [1]$. Therefore, $(\mathcal{E}_{II}/F, \vee_F, \wedge_F, \neg_F, [0], [1])$ is a Boolean algebra.

Conversely, let $(\mathcal{E}_{II}/F, \vee_F, \wedge_F, \neg_F, [0], [1])$ be a Boolean algebra. Then, for any $\alpha \in E$, $[\alpha \vee_F \neg\alpha] = [1]$. By definition of $\vee_F$, we have $((\alpha \rightarrow \neg\alpha) \rightarrow \neg\alpha) \wedge ((\neg\alpha \rightarrow \alpha) \rightarrow \alpha) \in F$. Since

$$((\alpha \rightarrow \neg\alpha) \rightarrow \neg\alpha) \wedge ((\neg\alpha \rightarrow \alpha) \rightarrow \alpha) \leqslant (\neg\alpha \rightarrow \alpha) \rightarrow \alpha,$$

by Remark 2.5, for any $\alpha \in E$, $(\neg\alpha \rightarrow \alpha) \rightarrow \alpha \in F$. Hence, by Lemma 4.15, F is an implicative filter of $\mathcal{E}_{II}$. □

Example 4.17. (i) *According to Example 4.4*(iii), $\mathcal{E}_{\amalg}$ *is a good prelinear EQ-algebra and* $F = \{1, \beta\}$ *is a filter but, it is not an implicative filter of* $\mathcal{E}_{\amalg}$. *By routine calculations, we can see that* $\mathcal{E}_{\amalg}/F = \{[0], [\alpha], [1]\}$ *is not a Boolean algebra. So the implicative condition is necessary.*

(ii) *According to Example 3.4,* $\mathcal{E}_{\amalg}$ *is a prelinear good IEQ-algebra. By Remark 2.6,* $G = \{1\}$ *is a filter of* $\mathcal{E}_{\amalg}$ *but it is not an implicative filter of* $\mathcal{E}_{\amalg}$. *Because,* $\neg\kappa \to \kappa = \beta \to \kappa = 1 \in G$ *but* $\kappa \notin G$. *Also,* $\mathcal{E}_{\amalg}$ *is not a Boolean algebra, because* $\alpha \vee \neg\alpha = \alpha \vee \theta = \theta \neq 1$.

(iii) *According to Example 3.13,* $\mathcal{E}_{\amalg}$ *is a prelinear good IEQ-algebra and* $F = \{\mu, 1\}$ *is a prelinear and fantastic filter of* $\mathcal{E}_{\amalg}$. *But* $\mathcal{E}_{\amalg}/F$ *is not a Boolean algebra, because* $\neg[\beta] = [\theta]$ *and* $[\beta] \wedge [\theta] = [\beta] \neq [0]$.

5. Conclusions and future works

In this paper, a new kind of filter of *EQ*-algebras was introduced and the quotient structures induced by it were studied.

It was proved that the quotient structure was induced by a fantastic filter is an *MV*-algebra. By using a prelinear filter of an *EQ*-algebra, a distributive lattice was constructed. If the prelinear filter also, was positive implicative or implicative filter, then the quotient structure would be a Stone algebra or a Boolean algebra, respectively.

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