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Fatou and Julia Like Sets II

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Abstract. This paper is a continuation of authors work: *Fatou and Julia like sets, Ukranian Math. J., to appear/arXiv:2006.08308[math.CV](see [5])*. Here, we introduce escaping like set and generalized escaping like set for a family of holomorphic functions on an arbitrary domain, and establish some distinctive properties of these sets. The connectedness of the Julia like set is also proved.

1. Introduction

We shall have the following notations throughout the paper:

- $\mathcal{M}(D)$: the class of all meromorphic functions on a domain $D \subseteq \overline{\mathbb{C}}$;
- $\mathcal{H}(D)$, the class of all holomorphic functions on a domain $D \subseteq \mathbb{C}$; and
- $\mathbb{D}$: the open unit disk in $\mathbb{C}$.

A family $\mathcal{F}$ of meromorphic functions defined on a domain $D \subseteq \overline{\mathbb{C}}$ is said to be normal in D if every sequence in $\mathcal{F}$ contains a subsequence that converges locally uniformly on D with respect to the spherical metric. $\mathcal{F}$ is said to be normal at a point $z \in D$ if it is normal in some neighborhood of z in D (see [25, 27]). Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a meromorphic function and denote by f^n the *nth* iterate of f for $n \in \mathbb{N}$. Then $f^n(z)$ is defined for all $z \in \mathbb{C}$ except for a countable set which consists of the poles of $f, f^2, \ldots, f^{n-1}$. The basic objects studied in iteration theory are the Fatou set $F = F(f)$ and the Julia set $J = J(f)$ of a meromorphic function f defined as follows:

$$F := \left\{z \in \overline{\mathbb{C}} : \{f^n : n \in \mathbb{N}\} \text{ is defined and normal in some neighborhood of } z\right\}$$

and

$$J := \overline{\mathbb{C}} \setminus F.$$

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The requirement that f^n be defined is always satisfied if f is rational and so does for transcendental entire functions, where f^n is defined for all $z \in \mathbb{C}$, and in this case, we always have $\infty \in J$. The study of Fatou and Julia sets of f is a subject matter of iteration theory of meromorphic functions which was initiated by P. Fatou and G. Julia (see [11, 17] and was kept alive mainly by I. N. Baker (for his contributions one may refer to [19]. For recent developments in Fatou Julia theory the reader is urged to refer to [1, 2, 4, 19, 26].

A natural generalization of the Fatou and Julia theory is the dynamics of semigroups of meromorphic functions initiated by A. Hinkkanen and G. Martin (see [13, 14]) for rational functions and for transcendental entire functions this study is initiated by K.K. Poon (see [23, 24], also see [9, 13, 14, 21]). A semigroup $\mathcal{F}$ of entire functions is a semigroup with binary operation defined by the function composition. If the semigroup $\mathcal{F}$ is generated by the functions $f_1, f_2, \ldots$, then we denote it by $\langle f_1, f_2, \ldots \rangle$. The *Fatou set* of the semigroup $\mathcal{F}$, denoted by $F(\mathcal{F})$ is defined as

$$F(\mathcal{F}) := \{z \in \mathbb{C} : \mathcal{F} \text{ is a normal family at } z\}$$

and the complement $\mathbb{C} \setminus F(\mathcal{F})$ of $F(f)$ is called the *Julia set* of $\mathcal{F}$ and is denoted by $J(\mathcal{F})$. $F(\mathcal{F})$ is an open subset of $\mathbb{C}$ and $J(\mathcal{F})$ is a closed subset of $\mathbb{C}$. The study of Fatou and Julia sets of semigroups of entire and rational functions is a subject matter of Dynamics of Semigroups of meromorphic functions. It has been seen that there are significant differences between the dynamics of a rational function (as well as transcendental entire function) and that of its semigroup and hence the study of the dynamics of semigroups is not merely a generalization.

A natural question arises:

Question 1.1. *Can one have a Fatou and Julia like theory on $(\mathcal{F}, D)$, where $\mathcal{F}$ is a subfamily of $\mathcal{M}(D)$ and D is an arbitrary domain in $\mathbb{C}$?*

In [5], the authors have initiated their work on Question 1.1. For a subfamily $\mathcal{F}$ of $\mathcal{H}(D)$, the authors in [5] introduced Fatou like set $F(\mathcal{F})$ and Julia like set $J(\mathcal{F})$ of the family $\mathcal{F}$ as follows:
Fatou like set $F(\mathcal{F})$ of $\mathcal{F}$ is defined to be a subset of D on which $\mathcal{F}$ is a normal family and *Julia like set* $J(\mathcal{F})$ of $\mathcal{F}$ is the complement $D \setminus F(\mathcal{F})$ of $F(\mathcal{F})$. If $\mathcal{F}$ happens to be a family of iterates of an entire function f, then $F(\mathcal{F})$ and $J(\mathcal{F})$ reduce to the Fatou set of f and the Julia set of f respectively. Various interesting properties of the sets $F(\mathcal{F})$ and $J(\mathcal{F})$ are studied there, and it is found that this generalization does not work as smoothly as of semigroups.

The present paper is a continuation of our work in [5] and introduces the *escaping like set* and *generalized escaping like set* for a family of holomorphic functions on an arbitrary domain. We have divided our findings into the following four sections:

- Properties of Julia like set $J(\mathcal{F})$ of $\mathcal{F}$;
- Escaping like set, generalized escaping like set and their properties;
- Discussion on limit functions and fixed points of $\mathcal{F}$; and
- Concluding Remarks.

2. Properties of Julia like set $J(\mathcal{F})$ of $\mathcal{F}$

Let $\mathcal{F}$ be a subfamily of $\mathcal{H}(D)$ and $z \in \mathbb{C}$. We define the *backward orbit* of z with respect to $\mathcal{F}$ as

$$O^-_{\mathcal{F}}(z) := \{w \in D : f(w) = z, \text{ for some } f \in \mathcal{F}\}$$

and the *exceptional set* of $\mathcal{F}$ is defined as

$$E(\mathcal{F}) := \left\{z \in \mathbb{C} : O^-_{\mathcal{F}}(z) \text{ is finite}\right\}.$$

If $\mathcal{F}$ is a semigroup of entire functions and $z \in J(\mathcal{F}) \setminus E(\mathcal{F})$, then the backward invariance of $J(\mathcal{F})$ (see [13], Theorem 2.1) implies that $O^-_{\mathcal{F}}(z) \subseteq J(\mathcal{F})$. The other way inclusion is true for any $\mathcal{F} \subseteq \mathcal{H}(D)$ with $E(\mathcal{F}) \neq \emptyset$ (see, [5], Theorem 1.9). Thus we have:

Theorem 2.1. *Suppose that $\mathcal{F}$ is a semigroup of entire functions with $E(\mathcal{F}) \neq \emptyset$, and $z \in J(\mathcal{F}) \setminus E(\mathcal{F})$. Then $J(\mathcal{F}) = \overline{O^{-}_{\mathcal{F}}(z)}$.*

We know (see [5], Theorem 1.1) that if N is a neighborhood of a point $z_0 \in J(\mathcal{F})$, then $\mathbb{C} \setminus \cup_{f\in\mathcal{F}} f(N)$ contains at most one point. If $J(\mathcal{F})$ has an isolated point, the following counterpart holds:

Theorem 2.2. *(a) Suppose that $J(\mathcal{F})$ has an isolated point. Then $\mathbb{C} \setminus \bigcup_{f\in\mathcal{F}} f(U)$ has at most one point, for some open set $U \subseteq F(\mathcal{F})$.*

(b) Suppose that N is a neighborhood of a point in $J(\mathcal{F})$. If $E(\mathcal{F}) \neq \phi$, then

$$\mathbb{C} \setminus \bigcup_{f\in\mathcal{F}} f(N) \subset E(\mathcal{F}).$$

Proof. Let $z_0 \in J(\mathcal{F})$ be an isolated point. Then we can choose a neighborhood N of z_0 such that $U := N \setminus \{z_0\} \subseteq F(\mathcal{F})$. Since $\mathcal{F}$ is not normal at z_0, by an extension of Montel's theorem (see [3], p. 203), $\mathcal{F}$ omits at most one point in $N \setminus \{z_0\}$. This proves (a).
Let N be a neighborhood of $z_0 \in J(\mathcal{F})$ and $w_0 \in \mathbb{C} \setminus \bigcup_{f\in\mathcal{F}} f(N)$. Suppose that $w_0 \notin E(\mathcal{F})$ and let $w_1 \in E(\mathcal{F})$. Then we can choose a deleted neighborhood $N_1 \subset N$ of z_0 such that $O^{-}_{\mathcal{F}}(w_1) \cap N_1 = \phi$ showing that $\mathcal{F}$ omits two points w_0, w_1 in the deleted neighborhood N_1 of z_0. Now by extension of Montel's Theorem, $z_0 \in F(\mathcal{F})$, a contradiction. This proves (b). □

Example 2.3. *Consider the family $\mathcal{F} := \{nz : n \in \mathbb{N}\}$ of entire functions . Then $F(\mathcal{F}) = \mathbb{C} \setminus \{0\}$. For any deleted neighborhood N of 0, $\bigcup_{f\in\mathcal{F}} f(N) = \mathbb{C} \setminus \{0\}$ and the set $\mathbb{C} \setminus \cup_{f\in\mathcal{F}} f(N)$ contains exactly one point.*

Theorem 2.4. *Let $\mathcal{F}$ be a family of transcendental entire functions with nonempty backward invariant Julia like set $J(\mathcal{F})$. Then $J(\mathcal{F})$ is a singleton or an infinite set. If $J(\mathcal{F})$ is a singleton $\{z_0\}$, say, then for any $f \in \mathcal{F}$, z_0 is a fixed point of f or a Picard exceptional value of f, and if $J(\mathcal{F})$ is infinite, then $J(\mathcal{F})$ has no isolated points.*

Proof. Suppose that $J(\mathcal{F})$ is finite and has at least two points. Then there is some $z \in J(\mathcal{F})$ and $f \in \mathcal{F}$ such that $f^{-1}(\{z\})$ is infinite. Backward invariance of $J(\mathcal{F})$ implies that $f^{-1}(\{z\}) \subseteq J(\mathcal{F})$, which is a contradiction. Hence $J(\mathcal{F})$ reduces to a singleton, $\{z_0\}$, say. Since for any $f \in \mathcal{F}$, $f^{-1}(\{z_0\}) \subseteq J(\mathcal{F})$, $f^{-1}(\{z_0\}) = \{z_0\}$ or $f^{-1}(\{z_0\}) = \emptyset$.

Next, if $J(\mathcal{F})$ is infinite and has an isolated point w_0, say, then by Theorem 2.2, there exists an open subset U in $F(\mathcal{F})$ such that $\mathbb{C} \setminus \left(\cup_{f\in\mathcal{F}} f(U)\right)$ has at most one point. We claim that $f(U) \cap J(\mathcal{F}) = \emptyset$ for any $f \in \mathcal{F}$. For, suppose $f(U) \cap J(\mathcal{F}) \neq \emptyset$ for some $f \in \mathcal{F}$. Then there is $w \in f(U) \cap J(\mathcal{F})$ such that $w = f(z)$ for some $z \in U$. Since $w \in J(\mathcal{F})$, $z \in f^{-1}(\{w\}) \subseteq J(\mathcal{F})$, a contradiction. Thus it follows that $J \subseteq \mathbb{C} \setminus \left(\cup_{f\in\mathcal{F}} f(U)\right)$, showing that $J(\mathcal{F})$ is finite which is not the case. Hence $J(\mathcal{F})$ has no isolated points. □

2.1. Connectedness of Julia like set

Kisaka [18] characterized the connectedness of the Julia set of a transcendental entire function, as a subset of $\mathbb{C}$. Here, we also characterize the connectedness of Julia like set $J(\mathcal{F})$ of a family $\mathcal{F}$ of holomorphic functions on a simply connected domain in $\mathbb{C}$.

Let D be a domain in $\mathbb{C}$. Let D_0 be a subset of D. We shall denote by ∂D_0, the set of boundary points of D_0 in D and denote by $\overline{D}_0$, the set of adherent points of D_0 in D.

Lemma 2.5. *Let D be a simply connected domain in $\mathbb{C}$. Let D_1 and D_2 be two disjoint open connected subsets of D such that $\partial D_1 \subset \partial D_2$. Then ∂D_1 is connected.*

Proof. Suppose on the contrary that $\partial D_1 = A \cup B$, where A and B be two nonempty disjoint closed subsets of ∂D_1. Let $\zeta_1 \in B$. Since A is closed and $\{\zeta_1\}$ is compact, $d(A, \zeta_1) = \epsilon > 0$ (where d is the Euclidean metric) and so we can choose $z_1 \in D_1$ and $z_2 \in D_2$ with $d(\zeta_1, z_i) < \frac{\epsilon}{2}$ $(i = 1, 2)$ and a line segment L_1 joining z_1 and z_2. Clearly $L_1 \cap A = \phi$. Similarly, we can choose $\zeta_2 \in A$, $z_1^{'} \in D_1, z_2^{'} \in D_2$ and a line segment L_2 joining $z_1^{'}$ and $z_2^{'}$ with $L_2 \cap B = \phi$. Since z_1 and $z_1^{'}$ are in D_1, there is a curve $\gamma_1 \subset D_1$ joining z_1 and $z_1^{'}$ such that γ_1 does not intersect L_1 and L_2. Similarly, we can choose a curve $\gamma_2 \subset D_2$ joining z_2 and $z_2^{'}$ such that γ_2 does not intersect L_1 and L_2. Let U be the region bounded by the closed curve $\gamma_1 \cup \gamma_2 \cup L_1 \cup L_2$. Then $A \cap \overline{U}$ and $B \cap \overline{U}$ are compact and hence are at a positive distance apart. Now from this it follows that we can choose a curve $\gamma_3 \subset D$ joining a point at γ_1 and a point at γ_2 and which does not intersect $\partial D_1 = A \cup B$ which implies that $D_1 \cap D_2 \neq \phi$, a contradiction. □

Simple connectedness of D in Theorem 2.5 is essential:

Example 2.6. *Consider the annulus $D = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$, where $0 < r_1 < r_2$ and consider $D_1 = D \cap \{z : Im(z) > 0\}$ and $D_2 = D \cap \{z : Im(z) < 0\}$ as two disjoint open connected subsets of D. Then $\partial D_1 \subset \partial D_2$, and ∂D_1 is not connected.*

As an immediate consequence of Lemma 2.5, we have

Lemma 2.7. *Let D be a simply connected domain in $\mathbb{C}$ and D_1 be an open connected subset of D. If U is a component of $D \setminus \overline{D}_1$, then ∂U is connected.*

Theorem 2.8. *Let K be a closed subset of a simply connected domain D in $\mathbb{C}$. Then K is connected if, and only if the boundary of each component of the complement $D \setminus K$ of K is connected.*

Proof. The connectedness of K is achieved-without any significant modification-by following the proof of Proposition 1 in [18]. The converse is proved by using the ideas of Newman([22], Theorem 14.4) as follows: Suppose on the contrary that there is a component G of $D \setminus K$ with disconnected boundary. Let A be the component of ∂G and put $B := \partial G \setminus A$. Since $\partial(D \setminus \overline{G}) = \partial \overline{G} \subset \partial G$, Lemma 2.7 implies that the boundary of any component of $D \setminus \overline{G}$ does not meet A and B simultaneously which leads to a natural division of the class C of components of $D \setminus \overline{G}$ into two subclasses:

$$C_1 := \{U \in C : \partial U \subset A\},$$

and

$$C_2 := \{V \in C : \partial V \subset B\}.$$

Put

$$U_1 := \bigcup_{U \in C_1} U,$$

and

$$U_2 := \bigcup_{V \in C_2} V.$$

Claim: $U_1 \cup A$ and $U_2 \cup B$ are closed sets.
First, we show that $\partial U_1 \subset A$. For, let $z_0 \in \partial U_1$. We consider the following two cases:
Case-I: There exists a component $D_0 \in C_1$ such that $z_0 \in \partial D_0$ and hence $\partial U_1 \subset A$.
Cases-II: There does not exist a component $D_0 \in C_1$ such that $z_0 \in \partial D_0$. Then for each neighborhood $N_0 := \{z : |z - z_0| < \epsilon\}$ of z_0, we see that $N_0 \cap U_1 \neq \phi$. Thus, there exists a component $D_1 \in C_1$ such that $N_0 \cap D_1 \neq \phi$. This implies that for each $n \in \mathbb{N}$ there is a component $D_n \in C_1$ with $z_0 \notin \partial D_n$ such that $N_n \cap D_n \neq \phi$, where $N_n = \{|z - z_1| < \epsilon/n\}$. Hence $N_n \cap \partial D_n \neq \phi$. Since $\partial D_n \subset A$, $N_n \cap A \neq \phi$. That is, $N_n \cap A \neq \phi, \forall\, n \in \mathbb{N}$. This implies that $z_0 \in \overline{A} = A$. Thus $\partial U_1 \subset A$, as desired.

Thus, $U_1 \cup \partial U_1 \cup A = U_1 \cup A$ is closed. Similarly, $U_2 \cup B$ is closed, and hence the claim. Further, we have

$$(U_1 \cup A) \cup (U_2 \cup B) = \left(D \setminus \overline{D}_1\right) \cup \partial D_1 = (D \setminus D_1)^o \cup \partial (D \setminus D_1) = D \setminus D_1.$$

Since $D \setminus D_1$ contains K, the union $(U_1 \cup A) \cup (U_2 \cup B)$ contains K. Since A and B are non empty disjoint subsets of K, $(U_1 \cup A) \cap K$ and $(U_2 \cup B) \cap K$ are non empty disjoint closed subsets of K whose union is equal to K showing that K is disconnected, a contradiction. □

From Theorem 2.8, we immediately obtain the connectedness of $J(\mathcal{F})$ as follows:

Theorem 2.9. *Let $\mathcal{F}$ be a family of holomorphic functions in a simply connected domain D. Then $J(\mathcal{F})$ is connected if, and only if the boundary of each component of $F(\mathcal{F})$ is connected.*

3. Escaping like set, generalized escaping like set and their properties

In the following discussion, by an infinite sequence in a subfamily $\mathcal{F} \subseteq \mathcal{H}(D)$ we mean a sequence $\{f_n\} \subset \mathcal{F}$ with $f_m \neq f_n$ for $m \neq n$.

Definition 3.1. *For a subfamily $\mathcal{F} \subseteq \mathcal{H}(D)$, we define escaping like set and generalized escaping like set of $\mathcal{F}$ as:*

$$I(\mathcal{F}) := \{z \in D : f_n(z) \to \infty \text{ for every infinite sequence } \{f_n\} \text{ in } \mathcal{F}\},$$

and

$$U(\mathcal{F}) := \{z \in D : f_n(z) \to \infty \text{ for some sequence } \{f_n\} \text{ in } \mathcal{F}\},$$

respectively.

Remark 3.2. (*i*) $I(\mathcal{F}) \subset U(\mathcal{F})$.
(*ii*) *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two subfamilies of $\mathcal{H}(D)$, then the following hold:*

1. *If $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then $I(\mathcal{F}_2) \subset I(\mathcal{F}_1)$ and $U(\mathcal{F}_1) \subset U(\mathcal{F}_2)$.*
2. *$I(\mathcal{F}_1 \cup \mathcal{F}_2) = I(\mathcal{F}_1) \cap I(\mathcal{F}_2)$ and $U(\mathcal{F}_1 \cup \mathcal{F}_2) = U(\mathcal{F}_1) \cup U(\mathcal{F}_2)$.*
3. *If $\mathcal{F}_1 \cap \mathcal{F}_2$ is infinite, then $I(\mathcal{F}_1 \cap \mathcal{F}_2) \supseteq I(\mathcal{F}_1) \cup I(\mathcal{F}_2)$.*
4. *$U(\mathcal{F}_1 \cap \mathcal{F}_2) \subset U(\mathcal{F}_1) \cap U(\mathcal{F}_2)$.*

Following examples show that the equality need not hold in (3) and (4) in Remark 3.2:

Example 3.3. *Consider*

$$\mathcal{F}_1 := \{e^{nz} : n \in \mathbb{N}\} \cup \{z^n : n \in \mathbb{N}\}$$

and

$$\mathcal{F}_2 := \{e^{nz} : n \in \mathbb{N}\} \cup \{(z - x)^n : n \in \mathbb{N}\},$$

where $x > 1$ is chosen such that the disk $\{z : |z - x| < 1\}$ intersects the unit disk $\{z : |z| < 1\}$. Then

$$I(\mathcal{F}_1) = \{z : Re(z) > 0 \text{ and } |z| > 1\},$$

$$I(\mathcal{F}_2) = \{z : Re(z) > 0 \text{ and } |z - x| > 1\}$$

and

$$I(\mathcal{F}_1 \cap \mathcal{F}_2) = \{z : Re(z) > 0\}.$$

Clearly, $I(\mathcal{F}_1 \cap \mathcal{F}_2) \neq I(\mathcal{F}_1) \cup I(\mathcal{F}_2)$.

Example 3.4. *Consider the subfamilies*

$$\mathcal{F}_1 := \{nz : n \in \mathbb{N}\}$$

and

$$\mathcal{F}_2 := \left\{n(z - \frac{1}{2}) : n \in \mathbb{N}\right\}$$

of $\mathcal{H}(\mathbb{D})$. *Then* $U(\mathcal{F}_1 \cap \mathcal{F}_2) = \phi$ *and* $U(\mathcal{F}_1) \cap U(\mathcal{F}_2) = \{z : |z| < 1\} \setminus \left\{0, \frac{1}{2}\right\}$. *Therefore,* $U(\mathcal{F}_1 \cap \mathcal{F}_2) \neq U(\mathcal{F}_1) \cap U(\mathcal{F}_2)$.

Further, $I(\mathcal{F})$ and $F(\mathcal{F})$ possess the following-easy to verify-properties:

(a) $F(\mathcal{F}_1 + \mathcal{F}_2) = F(\mathcal{F}_1) \cap F(\mathcal{F}_2)$.

(b) $I(\mathcal{F}_1 + \mathcal{F}_2) \subseteq I(\mathcal{F}_1) \cap I(\mathcal{F}_2)$.

(c) $F(\mathcal{F}_1\mathcal{F}_2) = F(\mathcal{F}_1) \cup F(\mathcal{F}_2)$.

(d) $I(\mathcal{F}_1\mathcal{F}_2) \subseteq I(\mathcal{F}_1) \cup I(\mathcal{F}_2)$.

If $z \in I(\mathcal{F}) \cap F(\mathcal{F})$, then by the definition of $I(\mathcal{F})$ and the normality of $\mathcal{F}$ at z imply that the component of $F(\mathcal{F})$ which contains z is contained in $I(\mathcal{F})$. This conclusion also holds for $U(\mathcal{F})$. That is,

1. If $I(\mathcal{F}) \cap F(\mathcal{F}) \neq \emptyset$, then $I(\mathcal{F})$ has non-empty interior. Moreover, if $U \cap I(\mathcal{F}) \neq \emptyset$ for some component U of $F(\mathcal{F})$, then $U \subseteq I(\mathcal{F})$.

2. If $U(\mathcal{F}) \cap F(\mathcal{F}) \neq \emptyset$, then $U(\mathcal{F})$ has non-empty interior. Moreover, if $V \cap U(\mathcal{F}) \neq \emptyset$ for some component V of $F(\mathcal{F})$, then $V \subseteq U(\mathcal{F})$.

As a consequence of the above conclusions, one can see that if $J(\mathcal{F}) = \emptyset$, then $I(\mathcal{F})$ and $U(\mathcal{F})$ are open subsets of D.

In general, $I(\mathcal{F})$ is neither forward invariant nor backward invariant, for example, consider the family $\mathcal{F} := \{e^{nz} : n \in \mathbb{N}\}$. Then

$$I(\mathcal{F}) = \{z \in \mathbb{C} : Re(z) > 0\}.$$

Since exponential function maps vertical lines onto circles, $I(\mathcal{F})$ is not forward invariant. Again, since exponential function maps horizontal lines onto rays emanating from the origin, $I(\mathcal{F})$ is not backward invariant. However, we have

Theorem 3.5. *If* $\mathcal{F}$ *is a family of entire functions such that* $f \circ g = g \circ f$, *for each* $f, g \in \mathcal{F}$, *then* $I(\mathcal{F})$ *and* $U(\mathcal{F})$ *are backward invariant.*

Proof. Let $w \in I(\mathcal{F})$ and $g \in \mathcal{F}$. Let $z \in g^{-1}(\{w\})$ be such that $z \notin I(\mathcal{F})$. Then there exists a sequence $\{f_n\}$ in $\mathcal{F}$ which is bounded at z. Since g is continuous, $\{g \circ f_n\}$ is bounded at z. But $g \circ f_n = f_n \circ g$, so the sequence f_n is bounded at $g(z) = w$, a contradiction. This proves that $I(\mathcal{F})$ is backward invariant.

Let $w \in U(\mathcal{F})$ and $g \in \mathcal{F}$. Let $z \in g^{-1}(\{w\})$ be such that $z \notin U(\mathcal{F})$. Then each sequence $\{f_n\}$ in $\mathcal{F}$ is bounded at z. By the same argument as above, we find that $U(\mathcal{F})$ is backward invariant. □

For semigroups of transcendental entire functions, we have

Theorem 3.6. *Let* $\mathcal{F}$ *be a semigroup of transcendental entire functions. Then* $U(\mathcal{F})$ *is non-empty and backward invariant. Further, if* $\mathcal{F} = \langle f_1, \cdots, f_m \rangle$, *where* f_i *are transcendental entire functions, then for each* $z \in U(\mathcal{F})$, *there exists* $f_i \in \{f_1, \cdots, f_m\}$ *such that* $f_i(z) \in U(\mathcal{F})$.

Proof. Let $f \in \mathcal{F}$. Then $I(f) \neq \phi$ by Theorem 1 of Eremenko[10] and hence $U(\mathcal{F}) \neq \phi$. Let $z_1 \in U(\mathcal{F})$ and $f \in \mathcal{F}$. Put $w_1 \in f^{-1}(\{z_1\})$. Then there is a sequence $\{f_n\}$ in $\mathcal{F}$ such that $f_n(z_1) \to \infty$ as $n \to \infty$. Put $g_n = f_n \circ f$. Then $g_n \in \mathcal{F}$, $\forall\, n \in \mathbb{N}$. Further, $g_n(w_1) = f_n(z_1) \to \infty$, as $n \to \infty$ showing that $w_1 \in U(\mathcal{F})$ and hence $U(\mathcal{F})$ is backward invariant.

Further, let $z_0 \in U(\mathcal{F})$. Then there is a sequence $\{g_n\} \subset \mathcal{F}$ such that $g_n(z_0) \to \infty$, as $n \to \infty$ and hence there exists an $n_0 \in \mathbb{N}$ such that $g_n \neq f_i, \forall\, i = 1, \ldots, m, \forall\, n \geq n_0$. This implies that for each $n \geq n_0$,

$$g_n = h_n \circ f_i, \text{ for some } i \in \{1, \ldots, m\}, \text{ and for some } h_n \in \mathcal{F}.$$

Then we can choose a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that $g_{n_k} = h_{n_k} \circ f_{i_0}$, for some fixed $i_0 \in \{1, \ldots, m\}$. Let $w_0 = f_{i_0}(z_0)$. Then $h_{n_k}(w_0) = h_{n_k} \circ f_{i_0}(z_0) = g_{n_k}(z_0) \to \infty$, as $k \to \infty$ showing that $w_0 \in U(\mathcal{F})$. Hence $f_i(z_0) \in U(\mathcal{F})$, for some $f_i \in \{f_1, \ldots, f_m\}$. □

If $I(\mathcal{F})$ and $U(\mathcal{F})$ are not open subsets of D, then one can easily see that $I(\mathcal{F})$ as well as $U(\mathcal{F})$ intersect $J(\mathcal{F})$. Converse of this statement does not hold as seen through the following examples:

Example 3.7. (*i*) *Let*

$$U = \{z : |z - 2| < 1/2\},$$

$$f_n(z) := \left[\left(2 - \frac{1}{n}\right) - z\right]^2 z^n, \; z \in U,$$

and consider the family

$$\mathcal{F} := \{f_n : n \in \mathbb{N}\}.$$

Then $f_n(z) \to \infty$, as $n \to \infty$, $z \in U$. But $\{f_n(z)\}$ does not tend to infinity uniformly in any neighborhood of 2*. Thus* $2 \in J(\mathcal{F})$ *and* $I(\mathcal{F}) = U$*. Also, note that* $I(\mathcal{F})$ *is open.*

(*ii*) *Consider*

$$\mathcal{F} := \{nz : n \in \mathbb{N}\} \cup \{n(z-1) : n \in \mathbb{N}\}.$$

Then $J(\mathcal{F}) = \{0, 1\}$ *and* $U(\mathcal{F}) = \mathbb{C}$*. Thus* $U(\mathcal{F}) \cap J(\mathcal{F}) \neq \phi$ *and* $U(\mathcal{F})$ *is open.*

Following example shows that $I(\mathcal{F})$ may be empty or non-empty independent of whether $J(\mathcal{F})$ is empty or non-empty:

Example 3.8. (*i*) *If* $\mathcal{F}$ *is locally uniformly bounded family of holomorphic functions on a domain D, then* $J(\mathcal{F}) = \emptyset$ *and* $I(\mathcal{F}) = \emptyset$*.*

(*ii*) *Consider the subfamily*

$$\mathcal{F} := \{n(z-2) : n \in \mathbb{N}\}$$

of $\mathcal{H}(\mathbb{D})$*. Then* $J(\mathcal{F}) = \emptyset$ *and* $I(\mathcal{F}) = \mathbb{D}$*, which is non-empty.*

(*iii*) *Consider the subfamily*

$$\mathcal{F} := \{nz : n \in \mathbb{N}\} \cup \{n(z-1) : n \in \mathbb{N}\}$$

of $\mathcal{H}(\mathbb{D})$*. Then* $J(\mathcal{F}) = \{0, 1\}$ *and* $I(\mathcal{F}) = \mathbb{D} \setminus \{0, 1\}$*. Thus both* $J(\mathcal{F})$ *and* $I(\mathcal{F})$ *are non-empty.*

(*iv*) *For the family* $\mathcal{F} := \cup_{|a|<1} \{n(z-a) : n \in \mathbb{N}\}$ *in* $\mathcal{H}(\mathbb{D})$*, we see that* $J(\mathcal{F}) = \mathbb{D}$ *and* $I(\mathcal{F}) = \emptyset$*.*

If $z \in \partial I(\mathcal{F})$, then clearly $\mathcal{F}$ is not normal at z and hence $\partial I(\mathcal{F}) \subseteq J(\mathcal{F})$. The other way inclusion may not hold, see (*i*) of Example 3.7.

Theorem 3.9. *Suppose that* $\mathcal{F}$ *is a semigroup of entire functions and* $I(\mathcal{F})$ *has at least two points and is invariant. Then* $J(\mathcal{F}) = \partial I(\mathcal{F})$*.*

Proof. Let $z \in \mathbb{C} \setminus I(\mathcal{F})$ and let $f \in \mathcal{F}$ be such that $f(z) \in I(\mathcal{F})$. Then backward invariance of $I(\mathcal{F})$ implies that $z \in I(\mathcal{F})$, a contradiction. This implies that $\mathcal{F}$ omits $I(\mathcal{F})$ on $\mathbb{C} \setminus I(\mathcal{F})$. By Montel's theorem, open subsets of $\mathbb{C} \setminus I(\mathcal{F})$ are contained in $F(\mathcal{F})$.

Since transcendental entire function has infinitely many periodic points, $\mathbb{C} \setminus I(\mathcal{F})$ has at least two points. Forward invariance of $I(\mathcal{F})$ implies that $\mathcal{F}$ omits $\mathbb{C} \setminus I(\mathcal{F})$ on $I(\mathcal{F})$. and hence by Motel's theorem, open subsets of $I(\mathcal{F})$ are contained in $F(\mathcal{F})$. This implies that $J(\mathcal{F}) \subseteq \partial I(\mathcal{F})$. □

Question 3.10. *Under the hypothesis of Theorem 3.9, can $J(\mathcal{F})$ be empty? In the dynamics of entire functions, it is always non-empty.*

When $J(\mathcal{F}) \neq \emptyset$, the following result holds:

Theorem 3.11. *If $\mathcal{F}$ is a subfamily of $\mathcal{H}(D)$ such that $J(\mathcal{F})$ has an isolated point, then $U(\mathcal{F}) \neq \emptyset$.*

Proof. Suppose that z_0 is an isolated point of $J(\mathcal{F})$ and let N be a neighborhood of z_0 such that $N \cap J(\mathcal{F}) \setminus \{z_0\} = \phi$. Let $\{f_n\}$ be a sequence in $\mathcal{F}$ such that it has no uniformly convergent subsequence in N.

We shall show that $f_n(z) \to \infty$ in $N \setminus \{z_0\}$. Suppose on the contrary that there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a point $z_1 \in N \setminus \{z_0\}$ such that $|f_{n_k}(z_1)| \leq M$ for all $k \in \mathbb{N}$ and for some $M > 0$. By [[8], Lemma 2.9], we see that $\{f_{n_k}\}$ is locally uniformly bounded in $N \setminus \{z_0\}$. Take a circle C with center z_0 and radius ϵ in $N \setminus \{z_0\}$, there exists a constant $M_1 > 0$ such that $|f_{n_k}(z)| \leq M_1$ for all $z \in C$ and $n \in \mathbb{N}$. Then by Maximum Modulus Principle, $|f_{n_k}(z)| \leq M_1$ for all $z \in \{z : |z - z_0| < \epsilon\}$ and for all $n \in \mathbb{N}$. Thus $\{f_{n_k}\}$ is normal at z_0, a contradiction. □

Example 3.12. *Let $\mathcal{F} := \{f_n(z) = nz : n \in \mathbb{N}\}$. Then $J(\mathcal{F}) = \{0\}$ and $f_n(z) \to \infty$, $n \to \infty$, in any deleted neighborhood of* 0*.*

If $J(\mathcal{F})$ has an isolated point, it is implicit in the proof of Theorem 3.11 that $U(\mathcal{F})$ has non-empty interior. Consequently, we have:

Corollary 3.13. *If $U(\mathcal{F})$ has empty interior, then $J(\mathcal{F})$ is either empty or a perfect set.*

4. Discussion on limit functions and fixed points of $\mathcal{F}$

Let $\mathcal{F}$ be a subfamily of $\mathcal{H}(D)$ and let U be a component of $F(\mathcal{F})$. A holomorphic function f on D is said to be a limit function of $\mathcal{F}$ on U if there is a sequence $\{f_n\}$ in $\mathcal{F}$ which converges locally uniformly on U to f. If there is a sequence in $\mathcal{F}$ which converges locally uniformly to ∞, then ∞ also qualifies to be a limit function of $\mathcal{F}$. By $\mathcal{L}_{\mathcal{F}}(U)$, we denote the set of finite limit functions of $\mathcal{F}$ on U.

Suppose that $f \circ g = g \circ f$ for every $f, g \in \mathcal{F}$ and U is a forward invariant component of $F(\mathcal{F})$. If a constant c is a limit function of $\mathcal{F}$ on U, then one can see that either $c = \infty$ or c is a fixed point of every $f \in \mathcal{F}$. Further, if $\mathcal{L}_{\mathcal{F}}(U)$ contains only constant functions, then $\mathcal{L}_{\mathcal{F}}(U)$ is a singleton.

A point $z_0 \in D$ is said to be a fixed point of a subfamily $\mathcal{F}$ of $\mathcal{H}(D)$ if z_0 is a fixed point of each $f \in \mathcal{F}$. Classification of fixed points of an entire function can be extended to the fixed points of a family of holomorphic functions. In classical dynamics, if z_0 is an attracting or repelling fixed point of f, then z_0 is in Fatou set $F(f)$ or Julia set $J(f)$ of f respectively. This is not true in this situation, even a super attracting fixed point may not be in the Fatou like set $F(\mathcal{F})$. For example,

(i) 0 is an attracting (not super attracting) fixed point of

$$\mathcal{F} := \left\{ f_n(z) = \left(\frac{1}{2} + \frac{1}{3n}\right) z e^{nz} : n \in \mathbb{N} \right\}$$

and $0 \in J(\mathcal{F})$;

(ii) 0 is a super attracting fixed point of

$$\mathcal{F} := \left\{ f_n(z) = nz^2 : n \in \mathbb{N} \right\}$$

and $0 \in J(\mathcal{F})$;

(iii) 0 is a repelling fixed point of

$$\mathcal{F} := \{nz : n \geq 2\}$$

and $0 \in J(\mathcal{F})$.

If z_0 is a super attracting fixed point of a family $\mathcal{F}$ of holomorphic functions on a domain D and g is a non constant limit function of $\mathcal{F}$, then clearly z_0 is super attracting fixed point of g. But the same is not true if z_0 is an attracting fixed point of $\mathcal{F}$, for example 0 is the attracting fixed point of

$$\mathcal{F} := \left\{ f_n(z) = ze^z \left(1 - \frac{1}{2n}\right) : n \in \mathbb{N} \right\}$$

but 0 is not an attracting fixed point of the limit function $g(z) = ze^z$ of $\mathcal{F}$. With regard to repelling fixed points, the Fatou like set may contain the repelling fixed points of $\mathcal{F}$, for example 0 is a repelling fixed point of

$$\mathcal{F} := \left\{ f_n(z) = a\left(1 + \frac{1}{n}\right) ze^z : n \in \mathbb{N} \right\},\ |a| > 1.$$

It is well known that a Fatou component contains at most one fixed point. But this is not true in Fatou like sets. That is, a component U of $F(\mathcal{F})$ can contain two fixed points, for example 0 is an indifferent fixed point and 1/2 is a repelling fixed point of

$$\mathcal{F} := \left\{ f_n(z) = z^n(z - \frac{1}{2}) + z : n \in \mathbb{N} \right\}$$

and both lie in $F(\mathcal{F})$ since $J(\mathcal{F}) = \{z : |z| = 1\}$. The Fatou like set may contain two attracting fixed points of $\mathcal{F}$, for example consider $g(z) = a + (z - a)h(z)$, where

$$h(z) = \frac{(a - b)(z - b) + (z - a)}{(b - a)},\ a, b \in \mathbb{R} : 0 < b - a < \frac{1}{2}.$$

Then a and b are attracting fixed points of g. Let $a = 0.1$ and $b = -0.1$, and let $f_n(z) = g(z) + ((z - a)(z - b))^n$, $\forall\, n \in \mathbb{N}$. Then $a,\ b$ are attracting fixed points of $\{f_n\}$. Moreover, $\{f_n(z)\}$ converges uniformly to $g(z)$ in $\{z : |z| < 0.3\}$. Let U be the component of $F(\{f_n\})$ containing $\{z : |z| < 0.3\}$. Then U contains two attracting fixed points a and b of $\{f_n\}$.

5. Concluding Remarks

The investigations initiated in [5] and the present paper are just the initial stages and there are many aspects to be looked into. For example, one can look into the domains like the wandering domains and Baker's domains of transcendental semigroups (see [9, 15]), and quasi-nested wandering domains(see [12] and [20]). The interesting point is that this study differs from dynamics of a meromorphic function as well as the dynamics of semigroups in the sense that many properties of the dynamics of meromorphic function and that of dynamics of transcendental semigroups fail to hold in the present Fatou and Julia like theory. One may also try to look at the Fatou and Julia like theory in the context of families of bicomplex holomorphic functions (see [6, 7]).

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