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# Numerical Ranges of Conjugations and Antilinear Operators on a Banach Space

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**Abstract.** In this paper, we prove that the numerical range of a conjugation on Banach spaces, using the connected property, is either the unit circle or the unit disc depending the dimension of the given Banach space. When a Banach space is reflexive, we have the same result for the numerical range of a conjugation by applying path-connectedness which is applicable to the Hilbert space setting. In addition, we show that the numerical ranges of antilinear operators on Banach spaces are contained in annuli.

## 1. Introduction and preliminaries

Let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. In 1918, O. Toeplitz [17] introduced the notion of the numerical range of a bounded linear operator $T$ on $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$, the *numerical range* $W(T)$ of $T$ is defined by

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}. \tag{1}$$

Toeplitz-Hausdorff theorem [11, 12, 17] establishes the convexity of the numerical range for any bounded linear operator on a Hilbert space. The authors in [7, 16] present some properties and further developments of the numerical ranges of bounded linear operators on a Hilbert space. In particular, diverse convex sets are discussed in [16] such that they become the numerical ranges of some linear operators.

In 1961 and 1962, Bauer [1] and Lumer [14] extended the concept of the numerical range on a Banach space $\mathcal{X}$. Let $\mathcal{X}^*$ be the dual space of $\mathcal{X}$ and let $T^*$ be the adjoint operator of $T \in \mathcal{L}(\mathcal{X})$, where $\mathcal{L}(\mathcal{X})$ is the algebra of all bounded linear operators on $\mathcal{X}$. The set $\Pi$ is defined by

$$\Pi = \{ (x, f) \in \mathcal{X} \times \mathcal{X}^* \, : \, \|f\| = f(x) = \|x\| = 1 \}. \tag{2}$$

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For $T \in \mathcal{L}(\mathcal{X})$, the *numerical range* $V(T)$ of $T$ is defined by

$$V(T) = \{\, f(Tx) \,:\, (x, f) \in \Pi \,\}.$$

It is a natural extension of the numerical range $W(T)$ of a bounded linear operator $T$ on a Hilbert space $\mathcal{H}$ to a Banach space $\mathcal{X}$. It turns out that $V(T)$ is not convex in general (see [3, Example 1, page 98]), but it is connected due to the connectedness of $\Pi$ (see [2] and [3, Corollary 5, page 102] or Proposition 2.3 below). Note that it is an open problem if $V(T)$ is path-connected or not (see [3, (7), page 129]).

One of the reasons to study numerical ranges is due to their relation to spectra. Let $\sigma(T)$ denote the spectrum of $T \in \mathcal{L}(\mathcal{X})$. For a subset $M$ of $\mathbb{C}$, we denote the closure of $M$ by $\overline{M}$. Note that for any $T \in \mathcal{L}(\mathcal{X})$, $\sigma(T) \subset \overline{V(T)}$ holds (see [18] or [3, Theorem 1, page 88]). If $H$ is a Hermitian operator on $\mathcal{X}$ (i.e, $V(H) \subset \mathbb{R}$), then $V(H) = \mathrm{co}\,\sigma(H)$ ([3, Corollary 11, page 53]), where $\mathrm{co}\,\sigma(H)$ is the convex hull of $\sigma(H)$. Moreover, even though $H$ is Hermitian, $H^2$ may not be Hermitian (see [3, Example 1, Page 58]).

In contrast to numerical ranges of linear operators, the ones of antilinear operators seem to have easier structure. An *antilinear* operator $A$ on $\mathcal{H}$ is defined by

$$A(\alpha x + \beta y) = \overline{\alpha}Ax + \overline{\beta}Ay$$

for $x, y \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$. A typical example of antilinear operators is a conjugation. A *conjugation* on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ with $C^2 = I$ which satisfies $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$ (see [9] and [8–10] for more details). For a conjugation $C$ on $\mathcal{H}$, the numerical range $W(C)$ is defined by

$$W(C) = \{\, \langle Cx, x\rangle \,:\, x \in \mathcal{H},\ \|x\| = 1 \,\}, \tag{3}$$

similar to the definition of those for linear operators. Recently, the authors in [13] provide the numerical range of a conjugation on a Hilbert space as follows.

**Theorem 1.1.** *([13, Theorem 2.1]) Let C be a conjugation on $\mathcal{H}$. Then the numerical range $W(C)$ of C is the following:*

(I) $W(C) = \{z : |z| = 1\}$, *when* $\dim \mathcal{H} = 1$ (*equivalently*, $\mathcal{H} = \mathbb{C}$).
(II) $W(C) = \{z : |z| \le 1\}$ *for* $\dim \mathcal{H} \ge 2$.

The same paper [13] also shows that the numerical ranges of antilinear operators on $\mathcal{H}$ (which are defined similarly to (3)) are contained in annuli (see [13, Theorem 2.5]). Diverse convex sets can be numerical ranges $W(T)$ for $T \in \mathcal{L}(\mathcal{H})$, but the ones for antilinear operators look circular due to the circular property (6).

In this manuscript we would like to extend all these to Banach spaces. Recently, Chō and Tanahashi [6] extend the concept of conjugations to a complex Banach space $\mathcal{X}$ (with its norm $\|\cdot\|$) as antilinear involutions whose operator norms are at most 1. More precisely, any operator $C : \mathcal{X} \to \mathcal{X}$ is called a *conjugation* on $\mathcal{X}$, if $C$ satisfies

$$C^2 = I, \quad \|C\| \le 1, \quad C(x + y) = Cx + Cy, \quad C(\lambda x) = \overline{\lambda}Cx, \tag{4}$$

for $x, y \in \mathcal{X}$ and $\lambda \in \mathbb{C}$. Note that (4) implies that $\|Cx\| = \|x\|$ for all $x \in \mathcal{X}$. With this extension of $C$, Theorems 2.4 and 3.1 tell us the numerical ranges $V(C)$ and $V(A)$ of conjugations $C$ and antilinear operators $A$ on $\mathcal{X}$, respectively.

This paper is organized as follows. In Section 2 we would like to present the numerical range for a conjugation on a Banach space $\mathcal{X}$. Section 3 leads us to the numerical ranges for antilinear operators on $\mathcal{X}$.

## 2. Conjugations

Before considering the numerical range of a conjugation on a Banach space $\mathcal{X}$, we would like to give the alternative proof of Theorem 1.1(II) on a numerical range of a conjugation on a Hilbert space $\mathcal{H}$. This is because this proof is shorter than the original proof in [13] and a similar argument will be applied later when a Banach space is reflexive.

*Proof.* [Proof of Theorem 1.1(II)] By [9], there exists a unit vector $x$ (so called an isotropic vector) such that $\langle Cx, x\rangle = 0$ which means that $0 \in W(C)$. Next, for nonzero vector $x$, since $x + Cx \neq 0$ or $x - Cx \neq 0$, it follows that $C(x + Cx) = x + Cx$ and $C(ix + C(ix)) = C(i(x - Cx)) = ix + C(ix)$. Thus $1 \in W(C)$. Therefore, we can choose $x_0, x_1$ which are unit vectors such that $\langle Cx_0, x_0\rangle = 0$ and $\langle Cx_1, x_1\rangle = 1$. Let $\langle Cx_0, x_1\rangle = re^{i\theta}$ and let $x_0' = e^{i\theta}x_0$. Then $x_0'$ is a unit vector and it satisfies $\langle Cx_0', x_1\rangle = r$. Since $x_0', x_1$ are linearly independent, we have $(1-t)x_0' + tx_1 \neq 0$. If not, $(1-t)x_0' + tx_1 = 0$ implies that $|1-t| = |t|$, so $t = \frac{1}{2}$. This means that $x_0' = -x_1$, which contradicts to the fact that $\langle Cx_0, x_0\rangle = 0$ and $\langle Cx_1, x_1\rangle = 1$. Indeed, $1 = \langle Cx_0', x_0'\rangle = e^{-2i\theta}\langle Cx_0, x_0\rangle = 0$ which is impossible.

Let us now connect $x_0'$ and $x_1$ on the unit sphere of $\mathcal{X}$. Put

$$y(t) := \frac{(1-t)x_0' + tx_1}{\|(1-t)x_0' + tx_1\|} \quad \text{for } \ 0 \leq t \leq 1.$$

Then $y(t)$ is a unit vector and it satisfies

$$\begin{aligned} &\langle Cy(t), y(t)\rangle \\ = \ & \frac{1}{\|(1-t)x_0' + tx_1\|^2}\Big((1-t)^2\langle Cx_0', x_0'\rangle + 2t(1-t)\langle Cx_0', x_1\rangle + t^2\langle Cx_1, x_1\rangle\Big) \\ = \ & \frac{1}{\|(1-t)x_0' + tx_1\|^2}\Big(2t(1-t)r + t^2\Big) \subset [0,1]. \end{aligned}$$

Since $t \in [0,1]$ is arbitrary, $\langle Cy(0), y(0)\rangle = 0$, and $\langle Cy(1), y(1)\rangle = 1$, it follows that the continuity of $y(t)$ in $t$ says that

$$[0,1] \subset W(C). \tag{5}$$

Next we discuss a circular structure of $V(C)$. Let $r \in [0,1]$ and let $x$ be a unit vector with $\langle Cx, x\rangle = r$. For any real number $\theta$, since $e^{i\theta}x$ is a unit vector and

$$\langle C(e^{i\theta}x), e^{i\theta}x\rangle = e^{-2i\theta}\langle Cx, x\rangle = e^{-2i\theta}r, \tag{6}$$

it follows that $e^{-2i\theta}r \in W(C)$. Moreover, since $\theta$ is arbitrary, it holds

$$\{\, z \in \mathbb{C} \, : \, |z| = r \,\} \subset W(C). \tag{7}$$

From (5) and (7), we have $W(C) = \{\, z \in \mathbb{C} \, : \, |z| \leq 1 \,\}$. Hence the proof is completed. □

To extend Theorem 1.1 to a Banach space $\mathcal{X}$, recall that the numerical range $V(C)$ of $C$,

$$V(C) = \{\, f(Cx) : \, (x, f) \in \Pi \,\},$$

where $\Pi$ is denoted as in (2). To proceed we need the following lemma.

**Lemma 2.1.** *If* $\dim \mathcal{X} \geq 2$, *then both* 0 *and* 1 *are in* $V(C)$.

*Proof.* By Lemma 2.2 on [5], there exists an isotropic vector (say $x_1$) if $\dim \mathcal{X} \geq 2$, i.e., there exists $(x_1, f) \in \Pi$ such that $f(x_1) = 1$ and $f(Cx_1) = 0$. This means that $V(C)$ contains 0. To see $1 \in V(C)$, let us first show that there exists a nonzero unit vector $x_2 \in \mathcal{X}$ such that $Cx_2 = x_2$. We may assume that, for a nonzero vector $x$, $x + Cx \neq 0$. Put $x_2 := (x + Cx)/\|x + Cx\|$. Then $x_2$ is a unit vector and it satisfies $Cx_2 = x_2$. Hence, by Hahn-Banach theorem, there exists $(x_2, g) \in \Pi$ such that $g(x_2) = 1 = g(Cx_2)$, which means $1 \in V(C)$. □

A topological space $X$ is called *connected* if there is no separation by open subsets, that is, if there are two open subsets $A$ and $B$ in $X$ such that $X = A \cup B$ and $A \cap B = \emptyset$, then either $A = \emptyset$ or $B = \emptyset$. One of the crucial properties of $V(T)$ for $T \in \mathcal{L}(\mathcal{X})$ is that $V(T)$ is connected. This topological property follows from the connectedness of $\Pi$ in (2).

**Theorem 2.2.** *([4, Theorem 11.4]) Let $\mathcal{X}$ be a complex Banach space. Then $\Pi$ is a connected subset of $\mathcal{X} \times \mathcal{X}^*$ with the norm $\times$ weak$^*$ topology.*

Similar to [4, Corollary 11.5], the mapping from $\Pi$ to $V(A)$ is continuous with any antilinear bounded operator $A$.

**Proposition 2.3.** *Let $\mathcal{X}$ be a complex Banach space and let $C$ be a conjugation on $\mathcal{X}$. Then $V(C)$ is connected in the complex plane $\mathbb{C}$.*

*Proof.* For $(x, f), (y, g) \in \Pi$, since

$$|f(Cx) - g(Cy)| \leq \|Cx - Cy\| + |(f - g)(Cy)|,$$

the mapping $(x, f) \rightarrow f(Cx)$ is a continuous mapping of $\Pi$ with the relative norm $\times$ weak$^*$ topology onto $V(C)$. Hence, by Theorem 2.2, $V(C)$ is connected in the complex plane $\mathbb{C}$. □

We are now ready to figure out $V(C)$ on $\mathcal{X}$. Let us denote by $B(0, \lambda)$ the open ball with center 0 and radius $\lambda$.

**Theorem 2.4.** *Let $\mathcal{X}$ be a complex Banach space and let $C$ be a conjugation on $\mathcal{X}$. Then the numerical range $V(C)$ of a conjugation $C$ is the following:*

(i) $V(C) = \{z : |z| = 1\}$, *when* $\dim \mathcal{X} = 1$ (*equivalently*, $\mathcal{X} = \mathbb{C}$).
(ii) $V(C) = \{z : |z| \leq 1\}$ *for* $\dim \mathcal{X} \geq 2$.

Before proving, recall the circular property (6), that is, if $(x, f) \in \Pi$, then $(e^{i\theta}x, e^{-i\theta}f) \in \Pi$ for any real number $\theta$. Therefore if $f(Ax) \in V(A)$, then $e^{-2i\theta}f(Ax) \in V(A)$ for any antilinear operator $A$. In other words,

$$\{z \in \mathbb{C} \, : \, |z| = |f(Ax)|\} \subset V(A).$$

*Proof.* (i) Since one dimensional Banach space is a Hilbert space, we have this result from Theorem 1.1(I).
(ii) The isometric property of $C$ implies that

$$|f(Cx)| \leq \|f\|\|x\| = 1,$$

which says that $V(C) \subset \overline{B(0, 1)}$. Now let us show that $\overline{B(0, 1)} \subset V(C)$. With the help of the circular property, Lemma 2.1 says that $V(C)$ contains 0 and the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. This means that it suffices to show that $B(0, 1) \setminus \{0\}$ is in $V(C)$. Suppose not, i.e., there exists a nonzero complex number $\lambda$ in $B(0, 1)$ which does not belong to $V(C)$. Due to the circular property again, the circle $\{z \in \mathbb{C} : |z| = |\lambda|\}$ has no intersection point with $V(C)$, i.e.,

$$V(C) \cap \{z \in \mathbb{C} : |z| = |\lambda|\} = \emptyset.$$

Put $G := V(C) \cap B(0, \lambda)$ and $F := V(C) \cap (\mathbb{C} \setminus \overline{B(0, \lambda)})$. By construction these two sets $G$ and $F$ are (relatively) open in $V(C)$ and disjoint (i.e., $F \cap G = \emptyset$). Since $0 \in G$ and $1 \in F$, $G$ and $F$ are nonempty. All this means that there are two nonempty open subsets $G$ and $F$ such that $V(C) = G \cup F$ and $F \cap G = \emptyset$, i.e., $V(C)$ has a separation by nonempty open sets. So $V(C)$ is not connected, which contradicts to Proposition 2.3. Then $V(C)$ contains $\overline{B(0, 1)}$ and therefore $V(C) = \overline{B(0, 1)}$. □

For a conjugation $C$ on $\mathcal{X}$, we define the dual conjugation $C^*$ on $\mathcal{X}^*$ of $C$ by

$$(C^* f)(x) = \overline{f(Cx)} \quad (x \in \mathcal{X}),$$

where $\overline{f(Cx)}$ is the complex conjugation of the complex number $f(Cx)$. In general, $V(T) \subset V(T^*)$ for $T \in \mathcal{L}(\mathcal{X})$ and its adjoint operator $T^*$ on $\mathcal{X}^*$. However, due to Theorem 2.4, $V(C) = V(C^*)$.

**Corollary 2.5.** *Let $\mathcal{X}$ be a complex Banach space and let $C$ be a conjugation on $\mathcal{X}$. Then $V(C) = V(C^*)$.*

From now on let us show the same result of Theorem 2.4 when $\mathcal{X}$ is reflexive but with path-connected argument. A space $\mathcal{X}$ is called *path-connected* if for any two points $x$ and $y$ in $\mathcal{X}$ there exists a continuous path $f$ from $[0,1]$ to $\mathcal{X}$ such that $f(0) = x$ and $f(1) = y$. Recall that $\mathcal{X}$ is a reflexive Banach space if $\mathcal{X}^{**} = \{\hat{x} : x \in \mathcal{X}\}$, where $\hat{x}$ is the Gelfand transformation of $x$, i.e., $x^*(f) = f(x)$ for $f \in \mathcal{X}^*$.

**Remark 2.6.** *In general, there is no relation between connectedness and path-connectedness. For example, topologist's sine curve (which is a subset of $\mathbb{C}$), i.e.,*

$$\{\, x + i \sin \frac{1}{x} \;:\; 0 < x \leq 1 \,\} \cup \{\, iy \;:\; -1 \leq y \leq 1 \,\} \subset \mathbb{C}$$

*is connected but not path-connected (even though $\mathbb{C}$ is path-connected).*

Luna [15] and Weigel [19] show the path-connectedness of $\Pi$ and then $V(T)$.

**Theorem 2.7.** *([15, Corollary 7] or [19, Theorem 2]) Let $\mathcal{X}$ be a complex reflexive Banach space with $\dim \mathcal{X} \geq 2$ and let $T \in \mathcal{L}(\mathcal{X})$. Then $V(T)$ is path-connected.*

Note that the theorem above is also true for any antilinear bounded operators, since [15] shows that $\Pi$ is path-connected when $\mathcal{X}$ is reflexive and therefore a similar argument to Proposition 2.3 says that $V(A)$ is path-connected. We now provide the another proof of Theorem 2.4 (ii) by using path-connectedness of $V(C)$, when $\mathcal{X}$ is reflexive. The reason is that this proof is very similar to the one on a Hilbert space.

**Theorem 2.8.** *With the same hypothesis as in Theorem 2.4, if $\mathcal{X}$ is reflexive with $\dim \mathcal{X} \geq 2$. Then the numerical range $V(C)$ of $C$ is $V(C) = \{z : |z| \leq 1\}$.*

*Proof.* From Lemma 2.1, it holds that $0, 1 \in V(C)$. Since, by Theorem 2.7, $V(C)$ is path-connected, there exists a continuous curve $y = y(t)$ such that $y(0) = 0$ and $y(1) = 1$ on the complex plane. Then the circular property implies $V(C) = \overline{B(0,1)}$. More precisely, for any $a$ ($0 \leq a \leq 1$), there exists $t_0$ such that $|y(t_0)| = a$. Put $y(t_0) = z$. Hence there exists $(x, f) \in \Pi$ such that $f(Cx) = z$. Let $\theta$ be any real number. Since $(e^{-i\theta}x, e^{i\theta}f) \in \Pi$, it follows that

$$e^{i\theta} f(Ce^{-i\theta}x) = e^{2i\theta} f(Cx) = e^{2i\theta} z \in V(C).$$

Moreover, since $\theta$ is any real number, we have $\{\, w \in \mathbb{C} \;:\; |w| = a \,\}$. Therefore, it holds $V(C) = \{z : |z| \leq 1\}$. □

**Remark 2.9.** *It does not seem easy to apply the idea of the original proof on Toeplitz-Hausdorff theorem directly to show Theorem 2.4 (ii) (unlike Hilbert space setting or Theorem 1.1).*

## 3. Antilinear operators

In this section, we investigate the numerical ranges of antilinear operators on a Banach space $\mathcal{X}$. As the case of conjugations, the numerical ranges of any antilinear operators on a Banach space $\mathcal{X}$ have a circular structure due to (6).

**Theorem 3.1.** *Let $A$ be a bounded antilinear operator on $\mathcal{X}$. Put $a := \inf\{|f(Ax)| : (x, f) \in \Pi\}$ and $b := \sup\{|f(Ax)| : (x, f) \in \Pi\}$. Then its numerical range $V(A)$ of $A$ is the following:*

- *(i) When $\dim \mathcal{X} = 1$ (equivalently, $\mathcal{X} = \mathbb{C}$), $a = b$ and $V(A) = \{z : |z| = a\}$.*
- *(ii) For $\dim \mathcal{X} \geq 2$, $V(A)$ is contained in the annulus whose boundaries are two circles $\{z : |z| = a\}$ and $\{z : |z| = b\}$. Inner or outer boundary circle is in $V(A)$ if and only if the infimum or supremum becomes the minimum or maximum, respectively.*

*Proof.* (i) Choose any $(x, f), (y, g) \in \Pi$. Since $\dim \mathcal{X} = 1$, there exists $\theta \in \mathbb{R}$ such that $y = e^{i\theta}x$. Since $1 = g(y) = g(e^{i\theta}x) = e^{i\theta}g(x)$, it holds $g(x) = e^{-i\theta} = (e^{-i\theta}f)(x)$ and hence $g = e^{-i\theta}f$. Therefore, $g(Ay) = (e^{-i\theta}f)(A(e^{i\theta}x)) = e^{-2i\theta}f(Ax)$, which indicates that $a = b$. By the circular property, we have a proof of (i).
(ii) Suppose not, i.e., there exists a nonzero complex number $\lambda$ between $a$ and $b$ which does not belong to $V(A)$. By the circular property

$$\{z \in \mathbb{C} : |z| = b\} \cap V(A) = \emptyset.$$

Then set $G := V(A) \cap B(0, \lambda)$ and $F := V(A) \cap (\mathbb{C} \setminus \overline{B(0, \lambda)})$ where $B(0, \lambda)$ is the open ball with center 0 and radius $\lambda$ with $a \le \lambda \le b$. By construction these two sets $G$ and $F$ are (relatively) open in $V(A)$ and disjoint (i.e., $F \cap G = \emptyset$). Since $a \in G$ and $b \in F$, $G$ and $F$ are nonempty. It means that there are two nonempty open subsets $G$ and $F$ such that $V(A) = G \cup F$ and $F \cap G = \emptyset$, i.e., $V(A)$ has a separation by nonempty open sets. So $V(A)$ is not connected, which contradicts to Proposition 2.3. Hence $V(A)$ is contained in the annulus. □

It is notable that the proof above indicates that $V(A)$ is path-connected, which was unknown for numerical ranges $V(T)$ of linear bounded operators $T \in \mathcal{L}(\mathcal{X})$.

For an antilinear operator $A$ on $\mathcal{X}$, we define the adjoint operator $A^*$ of $A$ by

$$(A^*f)(x) = \overline{f(Ax)}, \quad (x \in \mathcal{X},\ f \in \mathcal{X}^*),$$

where $\overline{f(Ax)}$ is the complex conjugation of the complex number $f(Ax)$. Then $A^*$ is an antilinear operator on $\mathcal{X}^*$. Similar to bounded linear operators we have the following corollary.

**Corollary 3.2.** *Let $A$ be an antilinear operator on $\mathcal{X}$. Then $V(A) \subseteq V(A^*)$ and the equality holds when $\mathcal{X}$ is reflexive.*

**Remark 3.3.** *If $\mathcal{X}$ is non-reflexive, then $\Pi(\mathcal{X})$ is strictly smaller than $\Pi(\mathcal{X}^*)$ in the sense that there exists $f \in \mathcal{X}^*$ such that it does not have $x \in \mathcal{X}$ satisfying $(x, f) \in \Pi(\mathcal{X})$. Due to this, for example, it is possible that, even though $V(A)$ does not contain $a$ in (ii) on Theorem 3.1, $V(A^*)$ may contain $a$. Therefore $V(A^*)$ may be strictly greater than $V(T)$.*

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