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Norm Bounds for the Inverse for Generalized Nekrasov Matrices in Point-Wise and Block Case

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Abstract. Lower-semi-Nekrasov matrices represent a generalization of Nekrasov matrices. For the inverse of lower-semi-Nekrasov matrices, a max-norm bound is proposed. Numerical examples are given to illustrate that new norm bound can give tighter results compared to already known bounds when applied to Nekrasov matrices. Also, we presented new max-norm bounds for the inverse of lower-semi-Nekrasov matrices in the block case. We considered two types of block generalizations and illustrated the results with numerical examples.

1. Introduction

Defining new upper bounds for the maximum norm of the inverse matrix for matrices belonging to some specific matrix classes is related to bounding the condition number - the quantity determined as the product of a matrix norm and a norm of the inverse matrix and it has been a very active field of research, see [4–6, 8, 11, 15, 16, 18, 20].

The Varah bound was the starting point for many results of this type in the literature. Result of Varah defines an upper bound for the norm of the inverse for matrices that are strictly diagonally dominant.

It is well-known that a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is strictly diagonally dominant (SDD) matrix if, for each $i \in N$, it holds that

$$|a_{ii}| > r_i(A) = \sum_{k \in N, k \neq i} |a_{ik}|,$$

or, in other words, $d(A) > r(A)$, where $d(A) = [|a_{11}|, ..., |a_{nn}|]^T$ and $r(A) = [r_1(A), ..., r_n(A)]^T$ is the vector of deleted row sums. Classes of matrices that we deal with in this paper represent different generalizations of SDD class and they are all subclasses in the class of (nonsingular) H-matrices. A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is an H-matrix if its comparison matrix $\langle A \rangle = [m_{ij}]$ defined by

$$\langle A \rangle = [m_{ij}] \in \mathbb{C}^{n,n}, \qquad m_{ij} = \begin{cases} |a_{ii}|, & i = j \\ -|a_{ij}|, & i \neq j, \end{cases}$$

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is an M-matrix, i.e., $\langle A\rangle^{-1} \geq 0$. According to [12], a matrix A is an H-matrix if and only if there exists a diagonal nonsingular matrix W with the property that AW is an SDD matrix. The diagonal matrix W, that transforms the given matrix A into SDD matrix, is called a scaling matrix for the given matrix A. For some special subclasses of $H-$matrices we know how to construct a corresponding scaling matrix, which can be used further in obtaining eigenvalue localizations, investigating Schur complement properties, see [7, 9, 10, 22, 24], or in determining error bounds for linear complementarity problems, see [13]. As stated in [3], for any nonsingular H-matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $|A^{-1}| \leq \langle A\rangle^{-1}$.

One of the well-known generalizations of SDD class is the class of Nekrasov matrices. Nekrasov matrices, see [14, 17], are defined by condition

$$|a_{ii}| > h_i(A), \quad \text{for all } \ i \in N, \tag{1}$$

or, in vector form, $d(A) > h(A)$, where the sums $h_i(A)$, $i \in N$ are defined recursively by

$$h_1(A) = r_1(A), \quad h_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n} |a_{ij}|, \ \ i = 2, 3, \ldots n, \tag{2}$$

and $h(A) = [h_1(A), ..., h_n(A)]^T$. For a given matrix A, $A = D - L - U$ represents the standard splitting of A into its diagonal (D), strictly lower ($-L$) and strictly upper ($-U$) triangular parts. Given a permutation matrix, P, a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is called P-Nekrasov, if P^TAP is a Nekrasov matrix, i.e., if $|(P^TAP)_{ii}| > h_i(P^TAP)$, for all $i \in N$, or, in other words, $d(P^TAP) > h(P^TAP)$. The union of all P-Nekrasov classes by permutation matrices P is known as Gudkov class, see [14].

The following two theorems recall well-known upper bounds for the maximum norm of the inverse matrix, for SDD matrices and Nekrasov matrices, respectively.

Theorem 1.1 ([23]). *Given an SDD matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ the following bound applies,*

$$\|A^{-1}\|_\infty \leq \frac{1}{\min_{i\in N} (|a_{ii}| - r_i(A))}.$$

This is the result of Varah that was the starting point for obtaining upper bounds for maximum norm of the inverse matrix for matrices belonging to different matrix classes, see [5, 15, 18].

Theorem 1.2 ([16]). *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be a Nekrasov matrix. Then*

$$\|A^{-1}\|_\infty \leq \max_{i\in N} \frac{z_i(A)}{|a_{ii}| - h_i(A)},$$

where $z_1(A) = 1$ *and* $z_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{z_j(A)}{|a_{jj}|} + 1, \ \ i = 2, 3, \ldots n.$

In order to relax SDD condition, one idea is to replace strict inequalities with nonstrict inequalities. As we know that H-matrices have at least one SDD row, it is natural to consider the following class of matrices, called diagonally dominant (DD) matrices.

Definition 1.3. *A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a diagonally dominant (DD) matrix if*

$$|a_{ii}| \geq r_i(A), \quad \textit{for all } \ i \in N,$$

and for at least one index $k \in N$,

$$|a_{kk}| > r_k(A).$$

In 1948, Olga Taussky-Todd introduced the notion of irreducibility, a graph-theoretic property of matrices, which, together with DD, forms a sufficient condition for nonsingularity.

Definition 1.4. *A matrix* $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, *is reducible if there exists a permutation matrix* $P \in \mathbb{R}^{n,n}$, *such that*

$$P^T AP = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathbb{C}^{l,l}$, $A_{22} \in \mathbb{C}^{n-l,n-l}$, *for some* $1 \leq l < n$. *If such a permutation matrix does not exist, we say that A is irreducible. For* $A = [a_{ij}] \in \mathbb{C}^{1,1}$, *A is irreducible if its (only) entry is nonzero.*

It is a well-known result that a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is irreducible if and only if its graph is strongly connected.

With irreducibility added, irreducibly diagonally dominant (IDD) matrices are defined as follows.

Definition 1.5. *An irreducible matrix* $A = [a_{ij}] \in \mathbb{C}^{n,n}$ *is called irreducibly diagonally dominant (IDD) matrix if*

$$|a_{ii}| \geq r_i(A), \quad \text{for all } \; i \in N,$$

and for at least one index $k \in N$,

$$|a_{kk}| > r_k(A).$$

Theorem 1.6 ([24]). *Given any* $A = [a_{ij}] \in \mathbb{C}^{n,n}$, *if* A *is irreducibly diagonally dominant (IDD) matrix then* A *is nonsingular, moreover, it is an* H*-matrix.*

When considering Nekrasov condition, one would expect that by replacing ordinary deleted row sums, $r_i(A)$, in the definition of IDD matrices with recursively defined sums, $h_i(A)$, this new condition would be sufficient for nonsingularity. However, this is not true, as it was shown and further discussed in [1], [21] and [17].

When considering nonstrict conditions sufficient for nonsingularity of matrices, one interesting concept would be semi-strict diagonal dominance (semi-SDD) introduced by Beauwens in 1976 in the paper [2]. It turned out that this condition of Beauwens guarantees nonsingularity even if ordinary deleted row sums are replaced by Nekrasov row sums.

Let us introduce a notation for the part of deleted row sum before the diagonal and for the part of deleted row sum after the diagonal, as follows.

$$l_1(A) = 0, \quad l_i(A) = \sum_{j=1}^{i-1} |a_{ij}|, \quad i = 2, 3, ..., n,$$

$$u_i(A) = \sum_{j=i+1}^{n} |a_{ij}|, \quad i = 1, 2, ..., n-1, \quad u_n(A) = 0.$$

Then, obviously,

$$r_i(A) = l_i(A) + u_i(A), \quad i = 1, 2, ..., n,$$

and, also,

$$l_n(A) = r_n(A),$$

$$u_1(A) = r_1(A).$$

Definition 1.7. *Given* $A = [a_{ij}] \in \mathbb{C}^{n,n}$, A *is a lower semi-strictly diagonally dominant matrix, (lower semi-SDD), if the following conditions hold:*

$$|a_{ii}| \geq r_i(A), \; i = 1, 2, ..., n,$$

$$|a_{ii}| > l_i(A), \; i = 1, 2, ..., n.$$

Definition 1.8. *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$, and let P be a given n-by-n permutation matrix. If $P^T AP$ is a lower semi-SDD matrix, we say that A is a P-semi-SDD matrix.*

Definition 1.9. *Given $A = [a_{ij}] \in \mathbb{C}^{n,n}$, we say that A is a semi-strictly diagonally dominant matrix, (semi-SDD), if there exists a permutation matrix P such that A is a P-semi-SDD matrix.*

It is a well-known fact that lower semi-SDD matrices form a subclass of nonsingular H-matrices. Clearly, the same statement holds for semi-SDD matrices.

Let us introduce notation for the lower part of a Nekrasov row sum.

$$q_1(A) = 0,$$

$$q_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|}, \quad i = 2, 3, ..., n.$$

Then, obviously,

$$h_i(A) = q_i(A) + u_i(A), \quad i = 1, 2, ..., n,$$

and, also

$$h_1(A) = r_1(A) = u_1(A),$$

$$h_n(A) = q_n(A).$$

Definition 1.10. *Given $A = [a_{ij}] \in \mathbb{C}^{n,n}$, we say that A is a lower semi-Nekrasov matrix if the following conditions hold:*

$$|a_{ii}| \geq h_i(A), \; i = 1, 2, ..., n,$$

$$|a_{ii}| > q_i(A), \; i = 1, 2, ..., n.$$

It is proved that the following relation between classes of lower semi-SDD and lower semi-Nekrasov matrices holds, based on a comparison of deleted row sums and Nekrasov row sums.

Theorem 1.11 ([21]). *If $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a lower semi-SDD matrix, then A is also a lower semi-Nekrasov matrix.*

Definition 1.12. *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$, and let P be a given n-by-n permutation matrix. If $P^T AP$ is a lower semi-Nekrasov matrix, we say that A is a P-semi-Nekrasov matrix.*

Definition 1.13. *Given $A = [a_{ij}] \in \mathbb{C}^{n,n}$, we say that A is a semi-Nekrasov matrix if there exists a permutation matrix P such that A is a P-semi-Nekrasov matrix.*

In the paper [21], the following interesting result is proved.

Theorem 1.14 ([21]). *If $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a lower semi-Nekrasov matrix, then A is a Gudkov matrix and therefore nonsingular.*

The proof of this statement, given in [21], is based on a step-by-step construction of the permutation matrix that transforms the given lower semi-Nekrasov matrix, $A = [a_{ij}] \in \mathbb{C}^{n,n}$, to a Nekrasov matrix.

Also, a direct corollary of this is the next statement.

Theorem 1.15 ([21]). *If $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a semi-Nekrasov matrix, then A is a Gudkov matrix and hence a nonsingular H-matrix.*

Notice that, if a matrix A is a semi-SDD matrix, it cannot always be transformed to SDD only by means of simultaneous permutations of rows and columns, as the set of values of deleted row sums, $r_i(A)$, $i = 1, 2, ..., n$, is invariant under such permutations. The set of entries in each row of a given matrix does not change under simultaneous permutations (only their order is changed) and diagonal entries remain on the diagonal. However, as lower semi-SDD matrices are also H-matrices, one can transform a lower semi-SDD matrix to an SDD matrix by diagonal scaling.

On the other hand, from Theorem 1.14, we see that if A is a semi-Nekrasov matrix it can be transformed to a Nekrasov matrix only by simultaneous permutations of rows and columns, because the set of values of Nekrasov row sums, $h_i(A)$, $i = 1, 2, ..., n$, does change when the order of rows is changed.

2. A new bound for the norm of the inverse for lower semi-Nekrasov matrices

The next result gives another relation between lower semi-Nekrasov and lower semi-SDD matrices and we will apply it in order to obtain a new upper bound for the maximum norm of the inverse for a lower semi-Nekrasov matrix.

Lemma 2.1. *If $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a lower semi-Nekrasov matrix, $A = D - L - U$ represents the standard splitting of A into its diagonal (D), strictly lower ($-L$) and strictly upper ($-U$) triangular parts and I is the identity matrix of order n, then $I - (|D| - |L|)^{-1}|U|$ is a lower semi-SDD matrix.*

Proof: Assume that A is a lower semi-Nekrasov matrix. Denote by

$$T := (|D| - |L|)^{-1}|U|.$$

Then,

$$(Te)_i = \frac{h_i(A)}{|a_{ii}|} \leq 1.$$

Denote $T = [t_{ij}]$. Then, for all $i = 1, 2, ..., n$ it holds that

$$1 - t_{ii} \geq \sum_{j \neq i} t_{ij}.$$

Since $T \geq 0$, it means that

$$|(I - T)_{ii}| \geq r_i(I - T), \quad i = 1, 2, ..., n.$$

For all indices i for which $|a_{ii}| > h_i(A)$ holds, obviously $|(I - T)_{ii}| > r_i(I - T)$ holds as well, so for all such indices we have

$$|(I - T)_{ii}| > l_i(I - T).$$

So, it remains to analyze the indices i for which $|a_{ii}| = h_i(A)$ holds. Since $|a_{ii}| > q_i(A)$, this implies $u_i(A) > 0$, and the existence of an index $k \in \{i + 1, i + 2, ..., n\}$, such that $a_{ik} \neq 0$, follows. By the construction of the matrix T, we see that in this case

$$(I - T)_{ik} \neq 0,$$

because, from

$$(|D| - |L|)T = |U|,$$

it follows that

$$((|D| - |L|)T)_{ik} = |U|_{ik}$$

and

$$\sum_{j \leq i} (|D| - |L|)_{ij} T_{jk} = |a_{ik}|.$$

This implies

$$|a_{ii}| T_{ik} = \sum_{j < i} |a_{ij}| T_{jk} + |a_{ik}|.$$

We conclude that, if $|a_{ik}| > 0$, then $T_{ik} > 0$ and $(I - T)_{ik} \neq 0$. Therefore,

$$|I - T|_{ii} > l_i(I - T).$$

This completes the proof.□

Notice that the matrix $I - T$ has all positive diagonal entries and nonpositive off–diagonal entries, and it is, therefore, a nonsingular M-matrix.

A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a weakly chained diagonally dominant (wcdd) matrix if

$$|a_{ii}| \geq r_i(A), \;\; i = 1, 2, ..., n,$$

and for all $i \in N$ such that i does not belong to the set $\beta(A) = \{j||a_{jj}| > r_j(A)\}$ there exist indices $i_1, i_2, ..., i_k$ in N with $a_{i_r,i_{r+1}} \neq 0, 0 \leq r \leq k-1$, where $i_0 = i$ and $i_k \in \beta(A)$.

Also, $A = [a_{ij}] \in \mathbb{C}^{n,n}$ with

$$|a_{ii}| \geq r_i(A), \;\; i = 1, 2, ..., n,$$

is an $H-$matrix if and only if A is wcdd, see [17].

The authors in [20] provided an upper bound for the infinity norm of the inverse of wcdd M-matrix with $u_j(A) < a_{jj}, j = 1, 2, ..., n$.

$$||A^{-1}||_\infty \leq \sum_{i=1}^{n} \left[a_{ii} \prod_{j=1}^{i} \left(1 - \frac{u_j(A)}{a_{jj}} \right) \right]^{-1}.$$

We apply this bound to define a new upper bound for the norm of the inverse for lower-semi Nekrasov matrix.

Theorem 2.2. *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be a lower semi-Nekrasov matrix and let $A = D - L - U$ be the standard splitting of A. Let $T = (|D| - |L|)^{-1}|U|$ and $B = I - T$. Then*

$$||A^{-1}||_\infty \leq \max_{i \in N} \frac{z_i(A)}{|a_{ii}|} \sum_{i=1}^{n} \left[b_{ii} \prod_{j=i}^{n} \left(1 - \frac{l_j(B)}{b_{jj}} \right) \right]^{-1},$$

where

$$z_1(A) = 1, \;\; z_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{z_j(A)}{|a_{jj}|} + 1, \;\; i = 2, 3, \ldots n.$$

Proof: Consider the matrix B as follows:

$$B = I - T = I - (|D| - |L|)^{-1}|U| = (|D| - |L|)^{-1}(|D| - |L| - |U|) = (|D| - |L|)^{-1}\langle A \rangle.$$

This implies

$$\langle A \rangle = (|D| - |L|)B.$$

Therefore,

$$\langle A \rangle^{-1} = B^{-1}(|D| - |L|)^{-1}.$$

As A is a lower semi-Nekrasov matrix, then, A is also an $H-$matrix and therefore,

$$||A^{-1}||_\infty \leq ||\langle A \rangle^{-1}||_\infty = ||B^{-1}(|D| - |L|)^{-1}||_\infty \leq ||B^{-1}||_\infty ||(|D| - |L|)^{-1}||_\infty.$$

From Lemma 2.1 we know that B is lower-semi-SDD and a nonsingular $M-$matrix. Therefore, according to [17] B is wcdd. Consider the matrix

$$S = P^T B P$$

where P is the counteridentical permutation matrix of order n. Then, as

$$b_{ii} > l_i(B), i = 1, 2, ..., n,$$

it holds

$$s_{ii} > u_i(S), i = 1, 2, ..., n.$$

As the matrix S is also wcdd M−matrix, we can apply the bound from [20] to the matrix S and obtain

$$\|B^{-1}\|_\infty = \|S^{-1}\|_\infty \leq \sum_{i=1}^{n}\left[s_{ii}\prod_{j=1}^{i}\left(1-\frac{u_j(S)}{s_{jj}}\right)\right]^{-1} = \sum_{k=1}^{n}\left[b_{kk}\prod_{m=k}^{n}\left(1-\frac{l_m(B)}{b_{mm}}\right)\right]^{-1}.$$

As

$$((|D|-|L|)^{-1}e)_i = \frac{z_i(A)}{|a_{ii}|},$$

then

$$\|(|D|-|L|)^{-1}\|_\infty = \max_{i\in N}\frac{z_i(A)}{|a_{ii}|}.$$

This completes the proof.□

3. New bounds for the norm of the inverse in the block case

In this section we consider two types of block generalizations of lower semi-Nekrasov matrices and we present new upper bounds for the maximum norm of the corresponding inverse matrices.

For a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ and a partition $\pi = \{p_j\}_{j=0}^{l}$, of the index set N, where

$$p_0 = 0 < p_1 < p_2 < \ldots < p_l = n,$$

one can present A in the block form as $[A_{ij}]_{l\times l}$. The given partition π defines the partition of $\mathbb{C}^n$ into a direct sum of subspaces W_i, as follows.

$$\mathbb{C}^n = W_1 \oplus W_2 \oplus \ldots \oplus W_l,$$

where

$$W_j = span\{e^k \mid p_{j-1}+1 \leq k \leq p_j\}, \quad j \in L = \{1,2,\ldots,l\}.$$

Here, $\{e^k\}_{k=1}^{n}$ is the standard basis of $\mathbb{C}^n$.

For rectangular blocks, $\|A_{ij}\|_\infty$ is defined as follows:

$$\|A_{ij}\|_\infty = \sup_{x\in W_j,\, x\neq 0}\frac{\|A_{ij}x\|_\infty}{\|x\|_\infty} = \sup_{\|x\|_\infty=1}\|A_{ij}x\|_\infty.$$

Also, denote

$$(\|A_{ii}^{-1}\|_\infty)^{-1} = \inf_{x\in W_i,\, x\neq 0}\frac{\|A_{ii}x\|_\infty}{\|x\|_\infty},\ i \in L,$$

where the last quantity is zero if A_{ii} is singular.

Now, let us recall two different ways of introducing the $l\times l$ comparison matrix for a given $A = [a_{ij}] \in \mathbb{C}^{n,n}$ and a partition $\pi = \{p_j\}_{j=0}^{l}$ of the index set N.

The comparison matrix of type I is denoted by $\rangle A\langle^{\pi} = [p_{ij}]$, where

$$p_{ii} = (\|A_{ii}^{-1}\|_\infty)^{-1},\ p_{ij} = -\|A_{ij}\|_\infty,\ i,j \in L,\ i \neq j.$$

The comparison matrix of type II is denoted by $\langle A\rangle^{\pi} = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1, & i = j \text{ and } \det A_{ii} \neq 0, \\ -\|A_{ii}^{-1}A_{ij}\|_\infty, & i \neq j \text{ and } \det A_{ii} \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Block H-matrices were researched in [19].

For a given $A = [a_{ij}] \in \mathbb{C}^{n,n}$, and a given partition $\pi = \{p_j\}_{j=0}^{l}$ of the index set N, we say that A is a block π H-matrix of type I if $\rangle A\langle^{\pi}$ is an H-matrix.

For a given $A = [a_{ij}] \in \mathbb{C}^{n,n}$, and a given partition $\pi = \{p_j\}_{j=0}^{l}$ of the index set N, we say that A is a block π H-matrix of type II if $\langle A\rangle^{\pi}$ is an H-matrix.

For a given $A = [a_{ij}] \in \mathbb{C}^{n,n}$, given partition $\pi = \{p_j\}_{j=0}^{l}$ of the index set N, we say that A is a block $\pi-$lower-semi-Nekrasov matrix of type I if $\rangle A\langle^{\pi}$ is lower-semi-Nekrasov matrix.

For a given $A = [a_{ij}] \in \mathbb{C}^{n,n}$, given partition $\pi = \{p_j\}_{j=0}^{l}$ of the index set N, we say that A is a block $\pi-$lower-semi-Nekrasov matrix of type II if $\langle A\rangle^{\pi}$ is lower-semi-Nekrasov matrix.

In [4] the following results can be found.

Theorem 3.1 ([4]). *If $A = [A_{ij}]_{n\times n}$ is a block π H-matrix of type I and $\rangle A\langle^{\pi}$ is its comparison matrix of type I, then*

$$\|A^{-1}\|_\infty \le \|(\rangle A\langle^{\pi})^{-1}\|_\infty.$$

Theorem 3.2 ([4]). *If $A = [A_{ij}]_{n\times n}$ is a block π H-matrix of type II and $\langle A\rangle^{\pi}$ is its comparison matrix of type II, then*

$$\|A^{-1}\|_\infty \le \max_{i\in L} \|A_{ii}^{-1}\|_\infty \|(\langle A\rangle^{\pi})^{-1}\|_\infty.$$

Applying these two results together with our new bound for point-wise case given in Theorem 2.2, we obtain upper bounds for the norm of the inverse for block lower-semi-Nekrasov matrices of type I and type II.

Theorem 3.3. *If $A = [A_{ij}]_{n\times n}$ is a block π lower-semi-Nekrasov matrix of type I for a given partition $\pi = \{p_j\}_{j=0}^{l}$ of the index set N and $B = I - T = I - (|D| - |L|)^{-1}|U|$, where $\rangle A\langle^{\pi} = D - L - U$ is the standard splitting of $\rangle A\langle^{\pi}$, then*

$$\|A^{-1}\|_\infty \le \max_{i\in L} \frac{z_i(\rangle A\langle^{\pi})}{|\rangle A\langle^{\pi}_{ii}|} \sum_{i=1}^{l} \left[b_{ii} \prod_{j=i}^{l} \left(1 - \frac{l_j(B)}{b_{jj}}\right)\right]^{-1}.$$

Theorem 3.4. *If $A = [A_{ij}]_{n\times n}$ is a block π lower-semi-Nekrasov matrix of type II for the given partition $\pi = \{p_j\}_{j=0}^{l}$ of the index set N and $B = I - T = I - (|D| - |L|)^{-1}|U|$, where $\langle A\rangle^{\pi} = D - L - U$ is the standard splitting of $\langle A\rangle^{\pi}$, then*

$$\|A^{-1}\|_\infty \le \max_{i\in L} \|A_{ii}^{-1}\|_\infty \max_{i\in L} \frac{z_i(\langle A\rangle^{\pi})}{|\langle A\rangle^{\pi}_{ii}|} \sum_{i=1}^{l} \left[b_{ii} \prod_{j=i}^{l} \left(1 - \frac{l_j(B)}{b_{jj}}\right)\right]^{-1}.$$

3.1. Numerical examples

Example 1. Consider the following matrix:

$$A_1 = \begin{pmatrix} 7 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & 7 & 1 & 6 \\ 7 & 1 & 7 & \frac{10}{7} \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & 7 \end{pmatrix}$$

Matrix A_1 is a lower-semi Nekrasov matrix. Also, $P^T A_1 P$ does satisfy the Nekrasov condition, when P is the following permutation matrix

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We give max-norm bounds of the inverse matrix obtained using Theorem 1.2 from [16] for the matrix $P^T A_1 P$ (the result of Kolotilina for Nekrasov matrices), and our bound from Theorem 2.2 applied directly to matrix A_1 and applied to $P^T A_1 P$.

Exact value for the max-norm of the inverse matrix is $\|A_1^{-1}\|_\infty = 0.387$. Max-norm bound of the inverse matrix obtained applying Theorem 1.2 from [16] to the matrix $P^T A_1 P$ is 126. Our bound from Theorem 2.2 applied directly to matrix A_1 gives 2.81354. Also, our bound applied to $P^T A_1 P$ gives 2.15904.

Example 2. Consider the following matrix :

$$A_2 = \begin{pmatrix} -4 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -5 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -5 & 4 & 1 & -0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1 & 2 & -1.2757 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 12 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & -0.4 & 14 \end{pmatrix}$$

The matrix A_2 is not lower-semi Nekrasov matrix, but, for the partition $\pi = \{0, 5, 10\}$ this matrix is a block $\pi-$lower-semi Nekrasov matrix of type I and also a block $\pi-$ Nekrasov matrix of type II. For this matrix, the block-case gives results for the max-norm bound of the inverse, as follows.

The exact value of the norm of the inverse is $\|A_2^{-1}\|_\infty = 1.47504$. The bound for type I obtained as in Theorem 3.3 gives 5.43943. The bound for type II obtained as in Theorem 3.4 gives 4.00648.

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