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Starlikeness, Convexity and Landau Type Theorem of the Real Kernel α−Harmonic Mappings

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Abstract. In [26], Olofsson introduced a kind of second order homogeneous partial differential equation. We call the solution of this equation real kernel α−harmonic mappings. In this paper, we study some geometric properties of this real kernel α−harmonic mappings. We give univalence criteria and sufficient coefficient conditions for real kernel α−harmonic mappings that are fully starlike or fully convex of order $\gamma, \gamma \in [0, 1)$. Furthermore, we establish a Landau type theorem for real kernel α−harmonic mappings.

1. Introduction

Let $\mathbb{C}$ be the complex plane and $\mathbb{D}_\rho = \{z : |z| < \rho\}$. In particular, $\mathbb{D}$ denotes the open unit disk $\mathbb{D}_1$. For $\alpha \in \mathbb{R}$ and $z \in \mathbb{D}$, let

$$T_\alpha = -\frac{\alpha^2}{4}(1-|z|^2)^{-\alpha-1} + \frac{\alpha}{2}(1-|z|^2)^{-\alpha-1}\left(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right) + (1-|z|^2)^{-\alpha}\triangle$$

be the second order elliptic partial differential operator, where $\triangle$ is the usual complex Laplacian operator

$$\triangle := 4\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad z = x + yi.$$

The corresponding partial differential equation is

$$T_\alpha(u) = 0 \quad \text{in } \mathbb{D} \tag{1.1}$$

and its associated the Dirichlet boundary value problem are as follows

$$\begin{cases} T_\alpha(u) = 0 & \text{in } \mathbb{D}, \\ u = u^* & \text{on } \partial\mathbb{D}. \end{cases} \tag{1.2}$$

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Here, the boundary data $u^* \in \mathfrak{D}'(\partial\mathbb{D})$ is a distribution on the boundary $\partial\mathbb{D}$ of $\mathbb{D}$, and the boundary condition in (1.2) is interpreted in the distributional sense that $u_r \to u^*$ in $\mathfrak{D}'(\partial\mathbb{D})$ as $r \to 1^-$, where

$$u_r(e^{i\theta}) = u(re^{i\theta}), \quad e^{i\theta} \in \partial\mathbb{D},$$

for $r \in [0,1)$. In [26], Olofsson proved that, for parameter $\alpha > -1$, if a function $u \in C^2(\mathbb{D})$ satisfies (1.1) with $\lim_{r\to 1^-} u_r = u^* \in \mathfrak{D}'(\partial\mathbb{D})$, then it has the form of Poisson type integral

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} K_\alpha(ze^{-i\tau})u^*(e^{i\tau})d\tau, \quad \text{for } z \in \mathbb{D}, \tag{1.3}$$

where

$$K_\alpha(z) = c_\alpha \frac{(1-|z|^2)^{\alpha+1}}{|1-z|^{\alpha+2}}, \tag{1.4}$$

$c_\alpha = \Gamma^2(\alpha/2+1)/\Gamma(1+\alpha)$ and $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$ for $s > 0$ is the standard Gamma function. In fact, by Proposition 3.2 of [26], temperate growth of the solution is equivalent to distributional boundary value for a solution of (1.1) when $\alpha > -1$.

If we take $\alpha = 2(p-1)$, then u is polyharmonic (or $p-$harmonic), where $p \in \{1, 2, ...\}$. For related study of polyharmonic mappings, see [1, 3, 7, 8, 11, 25, 30]). In particular, if $\alpha = 0$, then u is harmonic. Thus, u is a kind of generalization of classical harmonic mappings. Actually, by [27], we know that it is related to standard weighted harmonic mappings. Furthermore, since the kernel K_α in (1.4) is a real-valued function, we can call u of (1.3) real kernel $\alpha-$harmonic mappings. For the related discussion on standard weighted harmonic mappings, see [9, 12, 16, 17, 19, 24].

The Gauss hypergeometric function is defined by the series

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}$$

for $|x| < 1$, and by continuation elsewhere, where $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ for $n = 1, 2, ...$ are the Pochhammer symbols. Obviously, for $n = 0, 1, 2, ...$, $(a)_n = \Gamma(a+n)/\Gamma(a)$. It is easily verified that

$$\frac{d}{dx}F(a,b;c;x) = \frac{ab}{c}F(a+1,b+1;c+1;x). \tag{1.5}$$

Furthermore, for $Re(c-a-b) > 0$, we have (cf.[4],Theorem 2.2.2)

$$F(a,b;c;1) = \lim_{x\to 1} F(a,b;c;x) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{1.6}$$

The following Lemma involves the determination of monotonicity of Gauss hypergeometric functions.

Lemma 1.1. *[26] Let $c > 0$, $a \le c$, $b \le c$ and $ab \le 0$ ($ab \ge 0$). Then the function $F(a,b;c;x)$ is decreasing (increasing) on $x \in (0,1)$.*

Gauss hypergeometric function as an analytic function in the complex domain itself is widely and deeply studied [28, 29, 31–33]. Recently, the research on harmonic mapping constructed by Gauss hypergeometric function has also aroused people's interest [5].

The following result of [26] is the homogeneous expansion of solutions of (1.1).

Theorem 1.2. *[26] Let $\alpha \in \mathbb{R}$ and $u \in C^2(\mathbb{D})$. Then u satisfies (1.1) if and only if it has a series expansion of the form*

$$u(z) = \sum_{k=0}^{\infty} c_k F(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; |z|^2) z^k + \sum_{k=1}^{\infty} c_{-k} F(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; |z|^2) \bar{z}^k, \quad z \in \mathbb{D}, \tag{1.7}$$

for some sequence $\{c_k\}_{-\infty}^{\infty}$ of complex number satisfying

$$\lim_{|k|\to\infty} \sup |c_k|^{\frac{1}{|k|}} \leq 1. \tag{1.8}$$

In particular, the expansion (1.7), subject to (1.8), converges in $C^\infty(\mathbb{D})$, and every solution u of (1.1) is C^∞−smooth in $\mathbb{D}$.

In [26], the author pointed out that if $\alpha \leq -1$, $u \in C^2(\mathbb{D})$ satisfies (1.1), and the boundary limit $u^* = \lim_{r\to 1^-} u_r$ exists in $\mathfrak{D}'(\partial\mathbb{D})$, then $u(z) = 0$ for all $z \in \mathbb{D}$. So, in this paper, we always assume that $\alpha > -1$.

Definition 1.3. *Suppose $\alpha > -1$, $u(z)$ have the expansion of (1.7). We call $u(z)$ real kernel α−harmonic mapping.*

Definition 1.4. *A univalent and sense-preserving real kernel α−harmonic mapping u, with $u(0) = 0$, is said to be fully starlike of order γ, $\gamma \in [0, 1)$, in $\mathbb{D}$ if*

$$\frac{\partial(\arg u(re^{i\theta}))}{\partial\theta} = \Re\left(\frac{\mathcal{D}u}{u}\right) > \gamma \tag{1.9}$$

for all $z \neq 0$ and $r \in (0, 1)$, where $\mathcal{D}$ is a linear operator defined by

$$\mathcal{D} = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}.$$

In particular, when $\gamma = 0$ in (1.9), u is said to be fully starlike.

Definition 1.5. *A univalent and sense-preserving real kernel α−harmonic mapping u with $u(0) = 0$ is said to be fully convex of order γ, $\gamma \in [0, 1)$, in $\mathbb{D}$ if*

$$\frac{\partial(\arg \frac{\partial}{\partial\theta} u(re^{i\theta}))}{\partial\theta} = \Re\left(\frac{\mathcal{D}^2 u}{\mathcal{D}u}\right) > \gamma \tag{1.10}$$

for all $z \neq 0$ and $r \in (0, 1)$, where $\mathcal{D}^2 = \mathcal{D}(\mathcal{D})$ is the composition of $\mathcal{D}$ and itself. In particular, when $\gamma = 0$ in (1.10), u is said to be fully convex.

The starlikeness and convexity with order γ of functions are widely and deeply studied in analytic functions, harmonic functions and polyharmonic functions, see [13–15, 21–23].

The classical Landau's theorem states that if f is an analytic function on the unit disk $\mathbb{D}$ with $f(0) = f'(0) - 1 = 0$ and $|f(z)| < M$ for $z \in \mathbb{D}$, then f is univalent in the $\mathbb{D}_{r_0} = \{z||z| < r_0\}$ with

$$r_0 = \frac{1}{M + \sqrt{M^2 - 1}},$$

and $f(\mathbb{D}_{r_0})$ contains a disk $|\omega| < R_0$ with $R_0 = Mr_0^2$. This result is sharp, with the extremal function $f_0(z) = Mz\frac{1-Mz}{M-z}$. For other types of functions, Landau-type theorem were also studied. See [20] for harmonic mappings, [1, 6, 10, 11] for polyharmonic mappings, [2] for logharmonic mappings, [18] for log-p-harmonic mappings, [9] for weighted harmonic mappings.

The main purpose of this paper is to study the properties of the real kernel α−harmonic mappings. In section 2, for the real kernel α−harmonic mappings, we give a necessary and sufficient condition for the relationship between full starlikeness and full convexity. Furthermore, We give univalence criteria and sufficient coefficient conditions for real kernel α−harmonic mappings that are starlike or convex of order γ, $\gamma \in [0, 1)$. In section 3, we get a Landau type theorem for real kernel α−harmonic mappings.

2. Starlikeness and convexity

In the rest of this paper, we use the following denotations.
Let $z = re^{i\theta}$,

$$t = |z|^2 = r^2, \tag{2.1}$$

$$F = F_k = F(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; t), \tag{2.2}$$

and

$$F_t = F_{k,t}(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; t) = \frac{dF_k}{dt} = \frac{dF}{dt}. \tag{2.3}$$

Proposition 2.1. *The operator $\mathcal{D}$ is a real kernel α-harmonic mapping preserving operator.*

Proof. Let u be a real kernel α-harmonic mapping with the series expansion of (1.7). Then by direct computation, we have

$$\begin{aligned}\mathcal{D}u &= zu_z - \bar{z}u_{\bar{z}}\\ &= z\left[\sum_{k=1}^{\infty} c_k(F_t\bar{z}z^k + Fkz^{k-1}) + \sum_{k=1}^{\infty} c_{-k}F_t\bar{z}^{k+1}\right] - \bar{z}\left[\sum_{k=1}^{\infty} c_kF_tz^{k+1} + \sum_{k=1}^{\infty} c_{-k}(F_tz\bar{z}^k + Fk\bar{z}^{k-1})\right]\\ &= \sum_{k=1}^{\infty} kc_kFz^k - \sum_{k=1}^{\infty} kc_{-k}F\bar{z}^k.\end{aligned}$$

Furthermore, for sequence $\{c_k\}_{-\infty}^{\infty}$ if (1.8) holds, then

$$\limsup_{k\to\infty} |kc_k|^{\frac{1}{|k|}} \le 1$$

and

$$\limsup_{k\to\infty} |-kc_{-k}|^{\frac{1}{|k|}} \le 1.$$

Therefore, by Theorem 1.2, we get that $\mathcal{D}u$ is a real kernel α-harmonic mapping. □

Theorem 2.2. *Suppose real kernel α-harmonic mappings*

$$u(z) = \sum_{k=1}^{\infty} c_kFz^k + \sum_{k=1}^{\infty} c_{-k}F\bar{z}^k$$

and

$$v(z) = \sum_{k=1}^{\infty} \frac{1}{k}c_kFz^k - \sum_{k=1}^{\infty} \frac{1}{k}c_{-k}F\bar{z}^k$$

are univalent in $\mathbb{D}$. Then $u(z)$ is fully starlike of order γ if and only if $v(z)$ is fully convex of order γ.

Proof. Direct computation leads to

$$\frac{\mathcal{D}u}{u} = \frac{\sum_{k=1}^{\infty} kc_kFz^k - \sum_{k=1}^{\infty} kc_{-k}F\bar{z}^k}{\sum_{k=1}^{\infty} c_kFz^k + \sum_{k=1}^{\infty} c_{-k}F\bar{z}^k}$$

and

$$\frac{\mathcal{D}^2 v}{\mathcal{D} v} = \frac{\mathcal{D}(\sum_{k=1}^{\infty} c_k F z^k + \sum_{k=1}^{\infty} c_{-k} F \bar{z}^k)}{\sum_{k=1}^{\infty} c_k F z^k + \sum_{k=1}^{\infty} c_{-k} F \bar{z}^k} = \frac{\sum_{k=1}^{\infty} k c_k F z^k - \sum_{k=1}^{\infty} k c_{-k} F \bar{z}^k}{\sum_{k=1}^{\infty} c_k F z^k + \sum_{k=1}^{\infty} c_{-k} F \bar{z}^k}.$$

It follows that

$$\frac{\mathcal{D} u}{u} = \frac{\mathcal{D}^2 v}{\mathcal{D} v}.$$

Therefore, $\Re\left(\frac{\mathcal{D}u}{u}\right) > \gamma$ is equivalent to $\Re\left(\frac{\mathcal{D}^2 v}{\mathcal{D} v}\right) > \gamma$. The proof is completed. □

Lemma 2.3. *[29, 34] Let r_n and s_n $(n = 0, 1, 2, ...)$ be real numbers, and let the power series*

$$R(x) = \sum_{n=0}^{\infty} r_n x^n \quad \text{and} \quad S(x) = \sum_{n=0}^{\infty} s_n x^n$$

be convergent for $|x| < r$, $(r > 0)$ with $s_n > 0$ for all n. If the non-constant sequence $\{r_n/s_n\}$ is increasing (decreasing) for all n, then the function $x \mapsto R(x)/S(x)$ is strictly increasing (resp. decreasing) on $(0, r)$.

Lemma 2.4. *Let $\frac{\alpha}{2} \in (0, 1]$. Then it holds that*

(1) $\dfrac{F_k}{F_1} \le 1$ *for $k = 2, 3, ...$ and $t \in [0, 1)$;*

(2) $\dfrac{|F_t|}{F_1} \le \dfrac{(k - \frac{\alpha}{2})\Gamma(k+1)\Gamma(2 + \frac{\alpha}{2})}{2\Gamma(k + 1 + \frac{\alpha}{2})}$ *for $k = 1, 2, ...$and $t \in (0, 1)$.*

Proof. (1) We divide it into two subcases to discuss.

If $\frac{\alpha}{2} = 1$, then we can get $F_1 \equiv 1$ and F_k is decreasing for $t \in [0, 1)$ and $k = 2, 3, ...$ by Lemma 1.1. Thus, $\frac{F_k}{F_1}$ is decreasing for $t \in [0, 1)$.

If $0 < \frac{\alpha}{2} < 1$, let

$$A_n = -\frac{(-\frac{\alpha}{2})_n (k - \frac{\alpha}{2})_n}{(k+1)_n n!}, \qquad B_n = -\frac{(-\frac{\alpha}{2})_n (1 - \frac{\alpha}{2})_n}{(2)_n n!}. \tag{2.4}$$

Then it follows that $B_n > 0$ for $n = 1, 2, ...$ and

$$\frac{A_n}{B_n} = \frac{(-\frac{\alpha}{2})_n (k - \frac{\alpha}{2})_n}{(k+1)_n n!} \frac{(2)_n n!}{(-\frac{\alpha}{2})_n (1 - \frac{\alpha}{2})_n} = \frac{(k - \frac{\alpha}{2})_n (2)_n}{(k+1)_n (1 - \frac{\alpha}{2})_n}.$$

It can be verified that

$$\frac{\frac{A_{n+1}}{B_{n+1}}}{\frac{A_n}{B_n}} = \frac{(k - \frac{\alpha}{2} + n)(2 + n)}{(k + 1 + n)(1 - \frac{\alpha}{2} + n)} > 1$$

for $k = 2, 3, ...$ Thus $\frac{A_n}{B_n}$ is strictly increasing for all $n = 1, 2, ...$ By Lemma 2.3, we get that

$$f(t) = \frac{\sum_{n=1}^{\infty} A_n t^n}{\sum_{n=1}^{\infty} B_n t^n}$$

is strictly increasing for $t \in (0, 1)$. Furthermore, we have

$$f(0) = \lim_{t \to 0} f(t) = \frac{A_1}{B_1} = \frac{2(k - \frac{\alpha}{2})}{(k+1)(1 - \frac{\alpha}{2})} > 1$$

for $k = 2, 3, \ldots$. It follows that

$$f(t) > 1 \tag{2.5}$$

for $t \in [0, 1)$. Observe that

$$\frac{F_k}{F_1} = \frac{1 + \sum_{n=1}^{\infty} \frac{(-\frac{\alpha}{2})_n (k-\frac{\alpha}{2})_n}{(k+1)_n} \frac{t^n}{n!}}{1 + \sum_{n=1}^{\infty} \frac{(-\frac{\alpha}{2})_n (1-\frac{\alpha}{2})_n}{(2)_n} \frac{t^n}{n!}} = \frac{1 - \sum_{n=1}^{\infty} A_n t^n}{1 - \sum_{n=1}^{\infty} B_n t^n} = \frac{1 - f(t)\sum_{n=1}^{\infty} B_n t^n}{1 - \sum_{n=1}^{\infty} B_n t^n}.$$

Noting that F_1 is positive and considering the monotonicity of f and (2.5), we can get

$$\frac{d}{dt}\left(\frac{F_k}{F_1}\right) = \frac{-(\sum_{n=1}^{\infty} B_n t^n) f'(t) F_1 - (f(t) - 1)(\sum_{n=1}^{\infty} B_n t^n)'}{F_1^2} < 0.$$

Thus, $\frac{F_k}{F_1}$ is strictly decreasing for $t \in (0, 1)$.

Therefore, for $\frac{\alpha}{2} \in (0, 1]$ we have

$$\frac{F_k}{F_1} \le \lim_{t\to 0} \frac{F_k}{F_1} = 1.$$

(2) If $0 < \frac{\alpha}{2} < 1$, then F_t is negative for $t \in (0, 1)$. By (1.5), we have

$$|F_t| = -F_t = \frac{\frac{\alpha}{2}(k - \frac{\alpha}{2})}{k+1} F(1 - \frac{\alpha}{2}, k + 1 - \frac{\alpha}{2}; k + 2; t).$$

Observe that both $F(1 - \frac{\alpha}{2}, k + 1 - \frac{\alpha}{2}; k + 2; t)$ and F_1 are positive. Furthermore, $F(1 - \frac{\alpha}{2}, k + 1 - \frac{\alpha}{2}; k + 2; t)$ is increasing with respect to $t \in [0, 1)$ as well as F_1 is decreasing with respect to $t \in [0, 1)$. Therefore, we have

$$\begin{aligned}\frac{|F_t|}{F_1} &< \lim_{r\to 1^-} \frac{\frac{\frac{\alpha}{2}(k-\frac{\alpha}{2})}{k+1} F(1 - \frac{\alpha}{2}, k + 1 - \frac{\alpha}{2}; k + 2; t)}{F_1} \\ &= \frac{\frac{\alpha}{2}(k - \frac{\alpha}{2})}{k+1} \frac{\Gamma(k + 2)\Gamma(\alpha)}{\Gamma(k + 1 + \frac{\alpha}{2})\Gamma(1 + \frac{\alpha}{2})} \frac{\Gamma(2 + \frac{\alpha}{2})\Gamma(1 + \frac{\alpha}{2})}{\Gamma(2)\Gamma(1 + \alpha)} \\ &= \frac{(k - \frac{\alpha}{2})\Gamma(k + 1)\Gamma(2 + \frac{\alpha}{2})}{2\Gamma(k + 1 + \frac{\alpha}{2})}.\end{aligned} \tag{2.6}$$

The above first equality holds because of (1.6).

If $\frac{\alpha}{2} = 1$, then $F_1 \equiv 1$ and $F_k = 1 - \frac{k-1}{k+1}t$. Then it is easy to see that the equality of Lemma 2.4 (2) holds. □

Theorem 2.5. *Suppose $\gamma \in [0, 1)$, $\frac{\alpha}{2} \in (0, 1]$ and $u(z)$ be a real kernel α-harmonic mapping that has series expansion of (1.7) with $c_1 = 1$, $c_0 = 0$ and $|c_{-1}| < \min\{\frac{1-\gamma}{1+\gamma}, \frac{\alpha}{4-\alpha}\}$. Let*

$$\sum_{k=2}^{\infty} (A_k|c_k| + B_k|c_{-k}|) \le C, \tag{2.7}$$

where

$$A_k = \frac{(k - \frac{\alpha}{2})\Gamma(k + 1)\Gamma(2 + \frac{\alpha}{2})}{\Gamma(k + 1 + \frac{\alpha}{2})} \frac{1}{1 - |c_{-1}|} + \frac{k - \gamma}{(1 - \gamma) - (1 + \gamma)|c_{-1}|},$$

$$B_k = \frac{(k - \frac{\alpha}{2})\Gamma(k + 1)\Gamma(2 + \frac{\alpha}{2})}{\Gamma(k + 1 + \frac{\alpha}{2})} \frac{1}{1 - |c_{-1}|} + \frac{k + \gamma}{(1 - \gamma) - (1 + \gamma)|c_{-1}|},$$

and

$$C = 1 - (1 - \frac{\alpha}{2})\frac{1 + |c_{-1}|}{1 - |c_{-1}|}.$$

Then $u(z)$ is fully starlike of order γ in $\mathbb{D}$. Furthermore, the coefficient bound given by (2.7) is sharp.

Proof. Before proving this theorem, we first point out that the constraint condition $|c_{-1}| < \min\{\frac{1-\gamma}{1+\gamma}, \frac{\alpha}{4-\alpha}\}$ is to ensure that the denominators in the expression of the above A_k and B_k are positive and C itself is positive.

Observe that inequality (2.7) is equivalent to

$$\sum_{k=1}^{\infty} \frac{(k-\frac{\alpha}{2})\Gamma(k+1)\Gamma(2+\frac{\alpha}{2})}{\Gamma(k+1+\frac{\alpha}{2})} \frac{|c_k|+|c_{-k}|}{1-|c_{-1}|} + \sum_{k=2}^{\infty} \frac{(k-\gamma)|c_k|+(k+\gamma)|c_{-k}|}{(1-\gamma)-(1+\gamma)|c_{-1}|} \leq 1. \tag{2.8}$$

So, in the following proof process, we often replace (2.7) with (2.8).

First we prove $u(z)$ is sense-preserving in $\mathbb{D}$. By (1.7), direct computation leads to

$$u_z = \sum_{k=1}^{\infty} c_k(F_t\bar{z}z^k + kFz^{k-1}) + \sum_{k=1}^{\infty} c_{-k}F_t\bar{z}^{k+1},$$

and

$$u_{\bar{z}} = \sum_{k=1}^{\infty} c_kF_tz^{k+1} + \sum_{k=1}^{\infty} c_{-k}(F_tz\bar{z}^k + kF\bar{z}^{k-1}).$$

It follows that

$$\begin{aligned}
&|u_z| - |u_{\bar{z}}| \\
&= \left|\sum_{k=1}^{\infty} c_k(F_t\bar{z}z^k + kFz^{k-1}) + \sum_{k=1}^{\infty} c_{-k}F_t\bar{z}^{k+1}\right| - \left|\sum_{k=1}^{\infty} c_kF_tz^{k+1} + \sum_{k=1}^{\infty} c_{-k}(F_tz\bar{z}^k + kF\bar{z}^{k-1})\right| \\
&\geq F_1 - \sum_{k=1}^{\infty} |c_k||F_t|r^{k+1} - \sum_{k=2}^{\infty} |c_k|Fkr^{k-1} - \sum_{k=1}^{\infty} |c_{-k}||F_t|r^{k+1} - \sum_{k=1}^{\infty} |c_k||F_t|r^{k+1} \\
&\quad - \sum_{k=1}^{\infty} |c_{-k}|(|F_t|r^{k+1} + kFr^{k-1}) \\
&= (1-|c_{-1}|)F_1 - 2\sum_{k=1}^{\infty}(|c_k|+|c_{-k}|)|F_t|r^{k+1} - \sum_{k=2}^{\infty} k(|c_k|+|c_{-k}|)Fr^{k-1} \\
&> (1-|c_{-1}|)F_1 - 2\sum_{k=1}^{\infty}(|c_k|+|c_{-k}|)|F_t| - \sum_{k=2}^{\infty} k(|c_k|+|c_{-k}|)F.
\end{aligned} \tag{2.9}$$

If

$$2\sum_{k=1}^{\infty} \frac{|c_k|+|c_{-k}|}{1-|c_{-1}|} \frac{|F_t|}{F_1} + \sum_{k=2}^{\infty} k\frac{|c_k|+|c_{-k}|}{1-|c_{-1}|} \frac{F}{F_1} \leq 1, \tag{2.10}$$

then inequality (2.9) implies $|u_z| > |u_{\bar{z}}|$, that is to say u is sense-preserving. By Lemma 2.4, it is enough for us to prove

$$\sum_{k=1}^{\infty} \frac{(k-\frac{\alpha}{2})\Gamma(k+1)\Gamma(2+\frac{\alpha}{2})}{\Gamma(k+1+\frac{\alpha}{2})} \frac{|c_k|+|c_{-k}|}{1-|c_{-1}|} + \sum_{k=2}^{\infty} k\frac{|c_k|+|c_{-k}|}{1-|c_{-1}|} \leq 1. \tag{2.11}$$

It can be directly verified that

$$\frac{(k-\gamma)|c_k|+(k+\gamma)|c_{-k}|}{(1-\gamma)-(1+\gamma)|c_{-1}|} \geq k\frac{|c_k|+|c_{-k}|}{1-|c_{-1}|} \tag{2.12}$$

for $\gamma \in [0,1)$, $|c_{-1}| < \frac{1-\gamma}{1+\gamma}$ and $k = 2, 3, \ldots$. Thus if inequality (2.8) holds, then (2.11) follows from (2.12).

To show that $u(z)$ is univalent in $\mathbb{D}$ we need to show that $u(z_1) \neq u(z_2)$ when $z_1 \neq z_2$. Suppose $z_1, z_2 \in \mathbb{D}$ so that $z_1 \neq z_2$. Since $\mathbb{D}$ is simply connected and convex, we have $z(s) = (1-s)z_1 + sz_2 \in \mathbb{D}$, where $s \in [0,1]$. Then we can write

$$u(z_2) - u(z_1) = \int_0^1 \left[(z_2 - z_1) u_z(z(s)) + \overline{(z_2 - z_1)} u_{\bar{z}}(z(s)) \right] ds.$$

Dividing the above equation by $z_2 - z_1 \neq 0$ and taking the real parts we obtain

$$\begin{aligned} \Re \frac{u(z_2) - u(z_1)}{z_2 - z_1} &= \int_0^1 \Re \left[u_z(z(s)) + \frac{\overline{z_2 - z_1}}{z_2 - z_1} u_{\bar{z}}(z(s)) \right] ds \\ &\geq \int_0^1 \left(\Re u_z(z(s)) - |u_{\bar{z}}(z(s))| \right) ds. \end{aligned} \tag{2.13}$$

On the other hand

$$\begin{aligned} &\Re u_z(z) - |u_{\bar{z}}(z)| \\ &> F_1 - \sum_{k=1}^{\infty} |c_k||F_t| - \sum_{k=2}^{\infty} |c_k| Fk - \sum_{k=1}^{\infty} |c_{-k}||F_t| - \sum_{k=1}^{\infty} |c_k||F_t| - \sum_{k=1}^{\infty} |c_{-k}|(|F_t| + kF) \\ &= (1 - |c_{-1}|)F_1 - 2\sum_{k=1}^{\infty} (|c_k| + |c_{-k}|)|F_t| - \sum_{k=1}^{\infty} k(|c_k| + |c_{-k}|)F \\ &\geq (1 - |c_{-1}|)F_1 - 2\sum_{k=1}^{\infty} (|c_k| + |c_{-k}|)|F_t| - \sum_{k=1}^{\infty} \frac{(k-\gamma)|c_k| + (k+\gamma)|c_{-k}|}{(1-\gamma)|c_1| - (1+\gamma)|c_{-1}|}(1 - |c_{-1}|)F \\ &\geq 0 \end{aligned}$$

by inequality (2.12), Lemma 2.4 and inequality (2.8) in order. This in conjunction with the inequality (2.13) leads to the univalence of u.

Now we show that the inequality (1.9) holds. Direct computation yields

$$\begin{aligned} &\left| \mathcal{D}u + (1-\gamma)u \right| - \left| \mathcal{D}u - (1+\gamma)u \right| \\ &= \left| \sum_{k=1}^{\infty} c_k k F z^k - \sum_{k=1}^{\infty} c_{-k} k F \bar{z}^k + (1-\gamma)\left(\sum_{k=1}^{\infty} c_k F z^k + \sum_{k=1}^{\infty} c_{-k} F \bar{z}^k \right) \right| \\ &\quad - \left| \sum_{k=1}^{\infty} c_k k F z^k - \sum_{k=1}^{\infty} c_{-k} k F \bar{z}^k - (1+\gamma)\left(\sum_{k=1}^{\infty} c_k F z^k + \sum_{k=1}^{\infty} c_{-k} F \bar{z}^k \right) \right| \\ &= \left| \sum_{k=1}^{\infty} (k+1-\gamma) c_k F z^k + \sum_{k=1}^{\infty} (1-\gamma-k) c_{-k} F \bar{z}^k \right| - \left| \sum_{k=1}^{\infty} (k-1-\gamma) c_k F z^k - \sum_{k=1}^{\infty} (k+1+\gamma) c_{-k} F \bar{z}^k \right| \\ &\geq (2-\gamma) F_1 r - \sum_{k=2}^{\infty} (k+1-\gamma)|c_k| F r^k - \gamma |c_{-1}| F_1 r - \sum_{k=2}^{\infty} (k-1+\gamma)|c_{-k}| F r^k \\ &\quad - \gamma F_1 r - \sum_{k=2}^{\infty} (k-1-\gamma)|c_k| F r^k - (2+\gamma)|c_{-1}| F_1 r - \sum_{k=2}^{\infty} (k+1+\gamma)|c_{-k}| F r^k \\ &= 2[(1-\gamma) - (1+\gamma)|c_{-1}|] F_1 r - 2\sum_{k=2}^{\infty} [(k-\gamma)|c_k| + (k+\gamma)|c_{-k}|] F r^k > 0 \end{aligned}$$

for $r \in (0,1)$ by Lemma 2.4 and inequality (2.8). Furthermore, we observe that for $u \neq 0$, it holds that

$$\begin{aligned} & \left|\mathcal{D}u + (1-\gamma)u\right| - \left|\mathcal{D}u - (1+\gamma)u\right| > 0. \\ \Leftrightarrow & \left|\frac{\mathcal{D}u}{u} - \gamma - 1\right| < \left|\frac{\mathcal{D}u}{u} - \gamma + 1\right| \\ \Leftrightarrow & \ \Re\left(\frac{\mathcal{D}u}{u} - \gamma\right) > 0 \\ \Leftrightarrow & \ \Re\left(\frac{\mathcal{D}u}{u}\right) > \gamma. \end{aligned}$$

That is to say that if (2.8) holds then (1.9) holds.

The real kernel α-harmonic mapping

$$u(z) = F_1 z + \sum_{k=2}^{\infty} \frac{1}{A_k} x_k F_k z^k + c_{-1} F_1 \bar{z} + \sum_{k=2}^{\infty} \frac{1}{B_k} y_k F_k \bar{z}^k, \tag{2.14}$$

where

$$\sum_{k=2}^{\infty} (|x_k| + |y_k|) = C, \tag{2.15}$$

show that coefficient bound given by (2.8) is sharp. That is to say, the mapping represented by (2.14) is the corresponding extremal function of Theorem 2.5. □

Now we have a look about a special case of Theorem 2.5.

Example 2.6. *If $\frac{\alpha}{2} = 1$, then $F(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; t) = 1 - \frac{k-1}{k+1}|z|^2$ and the corresponding extremal function (2.14) deduce to*

$$u(z) = z + \sum_{k=2}^{\infty} \frac{1}{A_k} x_k \left(1 - \frac{k-1}{k+1}|z|^2\right) z^k + c_{-1}\bar{z} + \sum_{k=2}^{\infty} \frac{1}{B_k} y_k \left(1 - \frac{k-1}{k+1}|z|^2\right) \bar{z}^k,$$

where

$$\begin{aligned} A_k &= \frac{2(k-1)}{k+1} \frac{1}{1-|c_{-1}|} + \frac{k-\gamma}{(1-\gamma)-(1+\gamma)|c_{-1}|}, \\ B_k &= \frac{2(k-1)}{k+1} \frac{1}{1-|c_{-1}|} + \frac{k+\gamma}{(1-\gamma)-(1+\gamma)|c_{-1}|}, \end{aligned}$$

and

$$\sum_{k=2}^{\infty} (|x_k| + |y_k|) = 1.$$

Actually, the above $u(z)$ is biharmonic and can be rewritten as

$$u(z) = |z|^2 H(z) + G(z),$$

where

$$H(z) = -\sum_{k=2}^{\infty} \frac{1}{A_k} x_k \frac{k-1}{k+1} z^k - \sum_{k=2}^{\infty} \frac{1}{B_k} y_k \frac{k-1}{k+1} \bar{z}^k,$$

and

$$G(z) = z + \sum_{k=2}^{\infty} \frac{1}{A_k} x_k z^k + c_{-1}\bar{z} + \sum_{k=2}^{\infty} \frac{1}{B_k} y_k \bar{z}^k.$$

Theorem 2.7. *Suppose* $\gamma \in [0,1)$, $\frac{\alpha}{2} \in (0,1]$ *and* $u(z)$ *be a real kernel* α*-harmonic mapping that has series expansion of (1.7) with* $c_1 = 1$, $c_0 = 0$ *and* $|c_{-1}| < \min\{\frac{1-\gamma}{1+\gamma}, \frac{\alpha}{4-\alpha}\}$. *Let*

$$\sum_{k=2}^{\infty}(M_k|c_k| + N_k|c_{-k}|) \leq C, \tag{2.16}$$

where

$$\begin{aligned} M_k &= \frac{(k-\frac{\alpha}{2})\Gamma(k+1)\Gamma(2+\frac{\alpha}{2})}{\Gamma(k+1+\frac{\alpha}{2})}\frac{1}{1-|c_{-1}|} + \frac{k(k-\gamma)}{(1-\gamma)-(1+\gamma)|c_{-1}|},\\ N_k &= \frac{(k-\frac{\alpha}{2})\Gamma(k+1)\Gamma(2+\frac{\alpha}{2})}{\Gamma(k+1+\frac{\alpha}{2})}\frac{1}{1-|c_{-1}|} + \frac{k(k+\gamma)}{(1-\gamma)-(1+\gamma)|c_{-1}|}, \end{aligned}$$

and

$$C = 1 - (1-\frac{\alpha}{2})\frac{1+|c_{-1}|}{1-|c_{-1}|}.$$

Then $u(z)$ *is fully convex of order* γ *in* $\mathbb{D}$. *Furthermore, the coefficient bound given by (2.16) is sharp.*

Proof. Before proving this theorem, we first point out that the constraint condition $|c_{-1}| < \min\{\frac{1-\gamma}{1+\gamma}, \frac{\alpha}{4-\alpha}\}$ is to ensure that the denominators in the expression of M_k and N_k are positive and the above C is positive.

Observe that inequality (2.16) is equivalent to

$$\sum_{k=1}^{\infty}\frac{(k-\frac{\alpha}{2})\Gamma(k+1)\Gamma(2+\frac{\alpha}{2})}{\Gamma(k+1+\frac{\alpha}{2})}\frac{|c_k|+|c_{-k}|}{1-|c_{-1}|} + \sum_{k=2}^{\infty}k\frac{(k-\gamma)|c_k| + (k+\gamma)|c_{-k}|}{(1-\gamma)-(1+\gamma)|c_{-1}|} \leq 1. \tag{2.17}$$

Because inequality (2.17) implies inequality (2.8), by Theorem 2.5, we can know that $u(z)$ is sense-preserving and univalent.

Now we just need to prove (1.10) holds. Firstly, we observe that for $\mathcal{D}u \neq 0$. Actually, if $\mathcal{D}u = 0$, then $zu_z = \bar{z}u_{\bar{z}}$. It follows that $|u_z| = |u_{\bar{z}}|$. This is in contradiction with the fact that u is sense-preserving. Thus $\mathcal{D}u \neq 0$. We have

$$\begin{aligned} &\Re\frac{\mathcal{D}^2u}{\mathcal{D}u} > \gamma\\ \Leftrightarrow\ &\Re\left(\frac{\mathcal{D}^2u}{\mathcal{D}u} - \gamma\right) > 0\\ \Leftrightarrow\ &\left|\frac{\mathcal{D}^2u}{\mathcal{D}u} - \gamma - 1\right| < \left|\frac{\mathcal{D}^2u}{\mathcal{D}u} - \gamma + 1\right|\\ \Leftrightarrow\ &\left|\mathcal{D}^2u + (1-\gamma)\mathcal{D}u\right| - \left|\mathcal{D}^2u - (1+\gamma)\mathcal{D}u\right| > 0. \end{aligned}$$

Secondly, direct computation leads to

$$\begin{aligned}
&\left|\mathcal{D}^2u+(1-\gamma)\mathcal{D}u\right|-\left|\mathcal{D}^2u-(1+\gamma)\mathcal{D}u\right|\\
=&\left|\sum_{k=1}^{\infty}c_kk^2Fz^k+\sum_{k=1}^{\infty}c_{-k}k^2F\bar{z}^k+(1-\gamma)\left(\sum_{k=1}^{\infty}kc_kFz^k-\sum_{k=1}^{\infty}kc_{-k}F\bar{z}^k\right)\right|\\
&-\left|\sum_{k=1}^{\infty}c_kk^2Fz^k+\sum_{k=1}^{\infty}c_{-k}k^2F\bar{z}^k-(1+\gamma)\left(\sum_{k=1}^{\infty}kc_kFz^k-\sum_{k=1}^{\infty}kc_{-k}F\bar{z}^k\right)\right|\\
=&\left|\sum_{k=1}^{\infty}(k^2+k(1-\gamma))c_kFz^k+\sum_{k=1}^{\infty}(k^2-k(1-\gamma))c_{-k}F\bar{z}^k\right|\\
&-\left|\sum_{k=1}^{\infty}(k^2-k(1+\gamma))c_kFz^k+\sum_{k=1}^{\infty}(k^2+k(1+\gamma))c_{-k}F\bar{z}^k\right|\\
\geq&(2-\gamma)F_1r-\sum_{k=2}^{\infty}(k^2+k(1-\gamma))|c_k|Fr^k-\gamma|c_{-1}|F_1r-\sum_{k=2}^{\infty}(k^2-k(1-\gamma))|c_{-k}|Fr^k\\
&-\gamma F_1r-\sum_{k=2}^{\infty}(k^2-k(1+\gamma))|c_k|Fr^k-(2+\gamma)|c_{-1}|F_1r-\sum_{k=2}^{\infty}(k^2+k(1+\gamma))|c_{-k}|Fr^k\\
=&2[(1-\gamma)-(1+\gamma)|c_{-1}|]F_1r-2\sum_{k=2}^{\infty}k[(k-\gamma)|c_k|+(k+\gamma)|c_{-k}|]Fr^k>0
\end{aligned}$$

for $r\in(0,1)$ by Lemma 2.4 and inequality (2.17). Thus (1.10) holds.

The real kernel $\alpha-$harmonic mapping

$$u(z)=F_1z+\sum_{k=2}^{\infty}\frac{1}{M_k}x_kF_kz^k+c_{-1}F_1\bar{z}+\sum_{k=2}^{\infty}\frac{1}{N_k}y_kF_k\bar{z}^k, \tag{2.18}$$

where

$$\sum_{k=2}^{\infty}(|x_k|+|y_k|)=C, \tag{2.19}$$

show that coefficient bound given by (2.16) is sharp. That is to say, the function represented by (2.18) is the corresponding extremal function of Theorem 2.7. The proof is completed. □

Example 2.8. *If $\frac{\alpha}{2}=1$, by (2.18), we have that the corresponding extremal function of Theorem 2.7 deduces to*

$$u(z)=z+\sum_{k=2}^{\infty}\frac{1}{M_k}x_k(1-\frac{k-1}{k+1}|z|^2)z^k+c_{-1}\bar{z}+\sum_{k=2}^{\infty}\frac{1}{N_k}y_k(1-\frac{k-1}{k+1}|z|^2)\bar{z}^k,$$

where

$$\begin{aligned}
M_k&=\frac{2(k-1)}{k+1}\frac{1}{1-|c_{-1}|}+\frac{k(k-\gamma)}{(1-\gamma)-(1+\gamma)|c_{-1}|},\\
N_k&=\frac{2(k-1)}{k+1}\frac{1}{1-|c_{-1}|}+\frac{k(k+\gamma)}{(1-\gamma)-(1+\gamma)|c_{-1}|},
\end{aligned}$$

and

$$\sum_{k=2}^{\infty}(|x_k|+|y_k|)=1.$$

Actually, the above $u(z)$ is biharmonic and can be rewritten as

$$u(z) = |z|^2 A(z) + B(z),$$

where

$$A(z) = -\sum_{k=2}^{\infty} \frac{1}{M_k} x_k \frac{k-1}{k+1} z^k - \sum_{k=2}^{\infty} \frac{1}{N_k} y_k \frac{k-1}{k+1} \bar{z}^k,$$

and

$$B(z) = z + \sum_{k=2}^{\infty} \frac{1}{M_k} x_k z^k + c_{-1}\bar{z} + \sum_{k=2}^{\infty} \frac{1}{N_k} y_k \bar{z}^k.$$

3. The Landau type theorem

In [9], Chen and Vuorinen obtained the Landau type theorem for real kernel α−harmoinc mappings when $\alpha \in (-1, 0)$. In this section, we explore the Landau type theorem for real kernel α−harmoinc mappings for $\alpha \in (0, 2)$. We need the following Lemma 3.1 at first.

Lemma 3.1. *For $r \in [0, 1)$, let*

$$\varphi(r) = \frac{\beta}{M(2+\alpha)} - \frac{8M}{\pi}\left[\frac{a}{(1-r)^4(1+r)^2} - a + \frac{(2a-1)r^2 + ar}{(1-r)^3(1+r)^2} + \frac{2a-1}{2}\frac{r^2}{1-r^2}\right],$$

where $\alpha \in (0, 2)$, $\beta > 0$ and $M > 0$ are constants, $a = \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1+\alpha)}$. Then φ is strictly decreasing and there is an unique $\rho_0 \in (0, 1)$ such that $\varphi(\rho_0) = 0$.

Proof. We observe that $\frac{\Gamma(2)}{\Gamma(3)} = \frac{1}{2}$, $\lim_{\alpha\to 0^+} \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1+\alpha)} = 1$ and

$$\frac{d \log \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1+\alpha)}}{d\alpha} = \frac{1}{2}\psi(1+\frac{\alpha}{2}) - \psi(1+\alpha) < 0$$

for $\alpha \in (0, 2)$. Then we have $\frac{1}{2} < a < 1$ for $\alpha \in (0, 2)$. Here, ψ is the digamma function. It is defined by $\psi(x) = \Gamma'(x)/\Gamma(x)$ and it is well-known (cf.[4]) that $\psi(x)$ is strictly increasing on $(0, +\infty)$. Let

$$h_1(r) = \frac{a}{(1-r)^4(1+r)^2} - a,$$
$$h_2(r) = \frac{(2a-1)r^2 + ar}{(1-r)^3(1+r)^2},$$

and

$$h_3(r) = \frac{2a-1}{2}\frac{r^2}{1-r^2}.$$

Then

$$\varphi(r) = \frac{\beta}{M(2+\alpha)} - \frac{8M}{\pi}(h_1(r) + h_2(r) + h_3(r)).$$

It is easy to verify that $h_1'(r) > 0$, $h_2'(r) > 0$ and $h_3'(r) > 0$ for any $r \in (0, 1)$ and $\alpha \in (0, 2)$. It follows that $\varphi'(r) < 0$ for $r \in (0, 1)$. Furthermore, we can observe that $\lim_{r\to 0} \varphi(r) = \frac{\beta}{M(2+\alpha)} > 0$ and $\lim_{r\to 1^-} \varphi(r) = -\infty$. Therefore, the proof is completed. □

Theorem 3.2. *For $\alpha \in (0,2)$, let $a = \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1+\alpha)}$, $u \in C^2(\mathbb{D})$ be a solution to (1.1) satisfying $u(0) = |J_u(0)| - \beta = 0$ and $\sup_{z\in\mathbb{D}} |u(z)| \leq M$, where M and β are positive constants and J_u is the Jacobian of u. Then u is univalent in $\mathbb{D}_{\rho_0}$, where ρ_0 satisfies the following equation*

$$\frac{\beta}{M(2+\alpha)} - \frac{8M}{\pi}\left[\frac{a}{(1-\rho_0)^4(1+\rho_0)^2} - a + \frac{(2a-1)\rho_0^2 + a\rho_0}{(1-\rho_0)^3(1+\rho_0)^2} + \frac{2a-1}{2}\frac{\rho_0^2}{1-\rho_0^2}\right] = 0. \tag{3.1}$$

Moreover, $u(\mathbb{D}_{\rho_0})$ contains an univalent disk $\mathbb{D}_{R_0}$ with

$$R_0 \geq \rho_0\left[\frac{\beta}{M(2+\alpha)} - \frac{8M}{\pi}\left(\frac{a}{(1-\rho_0)^4(1+\rho_0)^2} - a + \frac{((2a-1)\rho_0 + a)\rho_0}{2(1-\rho_0)^3(1+\rho_0)^2} + \frac{(2a-1)\rho_0^2}{6(1-\rho_0^2)}\right)\right].$$

Proof. We still adopt the notations (2.1)-(2.3). For $\alpha \in (0,2)$ and $k \in \{1,2,...\}$, we observe that

$$F_t = F_t(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; t) = \frac{(-\frac{\alpha}{2})(k-\frac{\alpha}{2})}{k+1}\sum_{n=0}^{\infty}\frac{(1-\frac{\alpha}{2})_n(k+1-\frac{\alpha}{2})_n}{(k+2)_n}\frac{t^n}{n!} < 0.$$

That is to say that $F(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; t)$ is decreasing on $t \in [0,1)$. So,

$$F(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; 1) < F(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; t) \leq F(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; 0) = 1. \tag{3.2}$$

It follows from equations (1.6) and (3.2) that

$$\frac{\Gamma(k+1)\Gamma(1+\alpha)}{\Gamma(k+1+\frac{\alpha}{2})\Gamma(1+\frac{\alpha}{2})} < 1 + \sum_{n=1}^{\infty}\frac{(-\frac{\alpha}{2})_n(k-\frac{\alpha}{2})_n}{(k+1)_n}\frac{t^n}{n!},$$

$$-\sum_{n=1}^{\infty}\frac{(-\frac{\alpha}{2})_n(k-\frac{\alpha}{2})_n}{(k+1)_n}\frac{t^n}{n!} < 1 - \frac{\Gamma(k+1)\Gamma(1+\alpha)}{\Gamma(k+1+\frac{\alpha}{2})\Gamma(1+\frac{\alpha}{2})}.$$

Thus,

$$-\sum_{n=1}^{\infty}\frac{(-\frac{\alpha}{2})_n(k-\frac{\alpha}{2})_n}{(k+1)_n}\frac{1}{n!} \leq 1 - \frac{\Gamma(k+1)\Gamma(1+\alpha)}{\Gamma(k+1+\frac{\alpha}{2})\Gamma(1+\frac{\alpha}{2})}.$$

Notice that the left side of the above inequality is an infinite sum of terms, where each term is a positive for $\alpha \in (0,2)$. Therefore, for $\alpha \in (0,2)$ and $k \in \{1,2,...\}$, we have

$$\left|\frac{(-\frac{\alpha}{2})_n(k-\frac{\alpha}{2})_n}{(k+1)_n}\frac{1}{n!}\right| \leq 1 - \frac{\Gamma(k+1)\Gamma(1+\alpha)}{\Gamma(k+1+\frac{\alpha}{2})\Gamma(1+\frac{\alpha}{2})}. \tag{3.3}$$

Corollary 1.1 of [9] shows that for $k \in \{1,2,...\}$, it holds that

$$(|c_k| + |c_{-k}|) \leq \frac{4M}{\pi}\frac{\Gamma(k+1+\frac{\alpha}{2})\Gamma(1+\frac{\alpha}{2})}{\Gamma(k+1)\Gamma(1+\alpha)}. \tag{3.4}$$

Therefore, inequalities (3.3) and (3.4) imply that

$$(|c_k| + |c_{-k}|)\left|\frac{(-\frac{\alpha}{2})_n(k-\frac{\alpha}{2})_n}{(k+1)_n}\frac{1}{n!}\right| \leq \frac{4M}{\pi}[\frac{\Gamma(k+1+\frac{\alpha}{2})\Gamma(1+\frac{\alpha}{2})}{\Gamma(k+1)\Gamma(1+\alpha)} - 1] < \frac{4M}{\pi}[(k+1)\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1+\alpha)} - 1] \tag{3.5}$$

for $\alpha \in (0,2)$, $k \in \{1,2,...\}$, and $n \in \{1,2,...\}$.

Since $c_0 = u(0) = 0$, considering (1.7), we compute that

$$u_z(z) - u_z(0) = \sum_{k=2}^{\infty} kc_k F z^{k-1} + \sum_{k=1}^{\infty} c_k F_t \bar{z} z^k + \sum_{k=1}^{\infty} c_{-k} F_t \bar{z}^{k+1} + c_1 \left(F(-\frac{\alpha}{2}, 1-\frac{\alpha}{2}; 2; |z|^2) - 1\right), \tag{3.6}$$

$$u_{\bar{z}}(z) - u_{\bar{z}}(0) = \sum_{k=2}^{\infty} kc_{-k} F \bar{z}^{k-1} + \sum_{k=1}^{\infty} c_k F_t z^{k+1} + \sum_{k=1}^{\infty} c_{-k} F_t \bar{z}^k z + c_{-1} \left(F(-\frac{\alpha}{2}, 1-\frac{\alpha}{2}; 2; |z|^2) - 1\right). \tag{3.7}$$

Applying (3.6), (3.7), (3.2), (3.4) and (3.5) in turn, we obtain

$$\begin{aligned}
&|u_z(z) - u_z(0)| + |u_{\bar{z}}(z) - u_{\bar{z}}(0)| \\
&\le \sum_{k=2}^{\infty} k(|c_k| + |c_{-k}|) F r^{k-1} + 2\sum_{k=1}^{\infty} (|c_k| + |c_{-k}|)|F_t| r^{k+1} + (|c_1| + |c_{-1}|) \left|F(-\frac{\alpha}{2}, 1-\frac{\alpha}{2}; 2; r^2) - 1\right| \\
&< \sum_{k=2}^{\infty} k \frac{4M}{\pi} \frac{\Gamma(k+1+\frac{\alpha}{2})\Gamma(1+\frac{\alpha}{2})}{\Gamma(k+1)\Gamma(1+\alpha)} r^{k-1} + 2\sum_{k=1}^{\infty} \left[\frac{4M}{\pi}\left((k+1)\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1+\alpha)} - 1\right)\sum_{n=1}^{\infty} n r^{2(n-1)}\right] r^{k+1} \\
&\quad + \frac{4M}{\pi}\left(2\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1+\alpha)} - 1\right)\sum_{n=1}^{\infty} r^{2n} \\
&< \frac{4M}{\pi} a \sum_{k=2}^{\infty} k(k+1) r^{k-1} + \frac{8M}{\pi}\sum_{k=1}^{\infty}((k+1)a - 1)\frac{r^{k+1}}{(1-r^2)^2} + \frac{4M}{\pi}(2a-1)\frac{r^2}{1-r^2} \\
&= \frac{8M}{\pi}\left[\frac{ar(3-3r+r^2)}{(1-r)^3} + \frac{ar^2(2-r)}{(1-r)^4(1+r)^2} - \frac{r^2}{(1-r)^3(1+r)^2} + \frac{2a-1}{2}\frac{r^2}{1-r^2}\right] \\
&= \frac{8M}{\pi}\left[\frac{a}{(1-r)^4(1+r)^2} - a + \frac{(2a-1)r^2 + ar}{(1-r)^3(1+r)^2} + \frac{2a-1}{2}\frac{r^2}{1-r^2}\right]
\end{aligned} \tag{3.8}$$

where $a = \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1+\alpha)}$.

The inequality (3.6) of [9] shows that

$$l(D_u(0)) := \|u_z(0)| - |u_{\bar{z}}(0)\| \ge \frac{\beta}{M(2+\alpha)}. \tag{3.9}$$

In order to prove the univalence of u in $\mathbb{D}_{\rho_0}$, we choose two distinct points $z_1, z_2 \in \mathbb{D}_{\rho_0}$ and let $[z_1, z_2]$ denote the segment from z_1 to z_2 with the endpoints z_1 and z_2, where ρ_0 satisfies equation (3.1). By inequalities (3.8), (3.9) and Lemma 3.1, we have

$$\begin{aligned}
&|u(z_2) - u(z_1)| = \left|\int_{[z_1,z_2]} u_z(z)dz + u_{\bar{z}}(z)d\bar{z}\right| \\
&\ge \left|\int_{[z_1,z_2]} u_z(0)dz + u_{\bar{z}}(0)d\bar{z}\right| - \left|\int_{[z_1,z_2]} (u_z(z) - u_z(0))dz + (u_{\bar{z}}(z) - u_{\bar{z}}(0))d\bar{z}\right| \\
&\ge l(D_u)(0)|z_2 - z_1| - \left|\int_{[z_1,z_2]} (|u_z(z) - u_z(0)| + |u_{\bar{z}}(z) - u_{\bar{z}}(0)|)d|z|\right| \\
&> |z_2 - z_1|\left[\frac{\beta}{M(2+\alpha)} - \frac{8M}{\pi}\left(\frac{a}{(1-\rho_0)^4(1+\rho_0)^2} - a + \frac{(2a-1)\rho_0^2 + a\rho_0}{(1-\rho_0)^3(1+\rho_0)^2} + \frac{2a-1}{2}\frac{\rho_0^2}{1-\rho_0^2}\right)\right] \\
&= 0.
\end{aligned}$$

Thus, $u(z_1) \ne u(z_2)$. The univalence of u follows from the arbitrariness of z_1 and z_2. This implies that u is univalent in $\mathbb{D}_{\rho_0}$.

Now, for any $\zeta = \rho_0 e^{i\theta} \in \partial\mathbb{D}_{\rho_0}$, we obtain that

$$
\begin{aligned}
&|u(\zeta) - u(0)| = \left|\int_{[0,\zeta]} u_z(z)dz + u_{\bar{z}}(z)d\bar{z}\right| \\
&\geq \left|\int_{[0,\zeta]} u_z(0)dz + u_{\bar{z}}(0)d\bar{z}\right| - \left|\int_{[0,\zeta]} (u_z(z) - u_z(0))dz + (u_{\bar{z}}(z) - u_{\bar{z}}(0))d\bar{z}\right| \\
&\geq l(D_u)(0)\rho_0 - \int_{[0,\zeta]} (|u_z(z) - u_z(0)| + |u_{\bar{z}}(z) - u_{\bar{z}}(0)|)d|z| \\
&\geq \frac{\beta\rho_0}{M(2+\alpha)} - \frac{8M}{\pi}\int_0^{\rho_0}\left(\frac{a}{(1-r)^4(1+r)^2} - a + \frac{(2a-1)r^2 + ar}{(1-r)^3(1+r)^2} + \frac{2a-1}{2}\frac{r^2}{1-r^2}\right)dr \\
&\geq \frac{\beta\rho_0}{M(2+\alpha)} - \frac{8M}{\pi}\left(\frac{a\rho_0}{(1-\rho_0)^4(1+\rho_0)^2} - a\rho_0 + \frac{(2a-1)\rho_0 + a}{(1-\rho_0)^3(1+\rho_0)^2}\int_0^{\rho_0} rdr + \frac{2a-1}{2}\frac{1}{1-\rho_0^2}\int_0^{\rho_0} r^2dr\right) \\
&= \rho_0\left[\frac{\beta}{M(2+\alpha)} - \frac{8M}{\pi}\left(\frac{a}{(1-\rho_0)^4(1+\rho_0)^2} - a + \frac{((2a-1)\rho_0 + a)\rho_0}{2(1-\rho_0)^3(1+\rho_0)^2} + \frac{(2a-1)\rho_0^2}{6(1-\rho_0^2)}\right)\right].
\end{aligned}
$$

Hence $U(\mathbb{D}_{\rho_0})$ contains an univalent disk $\mathbb{D}_{R_0}$ with

$$
R_0 \geq \rho_0\left[\frac{\beta}{M(2+\alpha)} - \frac{8M}{\pi}\left(\frac{a}{(1-\rho_0)^4(1+\rho_0)^2} - a + \frac{((2a-1)\rho_0 + a)\rho_0}{2(1-\rho_0)^3(1+\rho_0)^2} + \frac{(2a-1)\rho_0^2}{6(1-\rho_0^2)}\right)\right].
$$

The proof is complete. □

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