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Analytic Transient Solutions of a Cylindrical Heat Equation

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Abstract. The principles of superposition and separation of variables are used here in order to investigate the analytical solutions of a certain transient heat conduction equation. The structure of the transient-temperature appropriations and the heat-transfer distributions are summed up for a straight mix of the results by means of the Fourier-Bessel arrangement of the exponential type for the investigated partial differential equation.

1. Introduction and Formulation of the Problem

A number of restrictive assumptions are introduced before studying the transient analysis, some of which are due to Chiang *et al.* [3]. One can apply Fourier's law and Newton's energy-conservation law to form the two-dimensional heat equation, together with the initial condition and the boundary conditions as follows:

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$$\frac{\partial u(r,z,t)}{\partial t} = \frac{\partial^2 u(r,z,t)}{\partial z^2} + \frac{\partial^2 u(r,z,t)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,z,t)}{\partial r} + u(r,z,t). \tag{1}$$

A heat-flow problem with lateral heat loss into an insulated problem, where the term $+u(r,z,t)$ on the right-hand side represents the heat flow across the lateral boundary. Diffusion of heat within the fin is due to the following terms:

$$\frac{\partial^2 u(r,z,t)}{\partial z^2} + \frac{\partial^2 u(r,z,t)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,z,t)}{\partial r}.$$

If there were no diffusion within the fin, then the temperature at each point z would damp exponentially to zero. Initially, the cylindrical fin is in equilibrium with the surrounding fluid, that is,

$$t = 0: \ u(r,z,0) = 0. \tag{2}$$

The boundary conditions are given by

$$t > 0 \ \text{ and } \ r = 0: \ u(0,z,t) = \text{finite}, \tag{3}$$

$$r = 1: \ u_r(1,z,t) = 0, \tag{4}$$

$$z = 0: \ -u_z(r,0,t) + \mathrm{Bi}\cdot u(r,0,t) = \mathrm{Bi} + q \tag{5}$$

and

$$z = \ell: \ u_z(r,\ell,t) + \mathrm{Bi}_\ell \cdot u(r,\ell,t) = 0, \tag{6}$$

where Bi and Bi_ℓ are the Biot numbers on the root surface and the lateral surface, respectively, and q is the constant heat flux at the root of the fin.

2. The Main Set of Analytical Transient Solutions

A very classical method to solve a given initial- and boundary-value problem for partial differential equations is to use the principle of superposition and separation of variables (see [1] and [2]). However, the partial differential equation (1), when directly transformed into ordinary differential equations by the method of separation of variables, does not seem to have been considered in the existing literature on this subject.

While separating variables in the r-direction, one can choose a Bessel function $J_\nu(z)$ of the first kind of order $\nu = 0$ as the characteristic function for the corresponding Bessel equation and it fits the boundary conditions in the equations (3) and (5) automatically. In this situation, α_m denotes the mth positive zero of the following transcendental equation for all of the cases:

$$J_1(\alpha_m) = 0 \qquad (m = 1, 2, 3, \cdots). \tag{7}$$

where, as noted above, $J_\nu(z)$ denotes the familiar Bessel function of the first kind defined by (see, for details, [16])

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{\nu+2n}}{n!\Gamma(\nu+n+1)}.$$

Thus, clearly, a Fourier series can be derived to fit the resulting ordinary differential equation in the z-direction and the revised boundary conditions in the equations (5) and (6). The first-order ordinary differential equation in the time-domain can be solved through integrating factor and satisfying the initial condition, and the results in the solution with different C_{mn} have the following form:

$$u_{mn}(t) = \frac{C_{mn}}{\beta_n^2 + \alpha_m^2}[\beta_n^2 e^{-(\beta_n^2+\alpha_m^2)t} + \alpha_m^2]. \tag{8}$$

Then the solution formed by the product of these chosen functions would satisfy the heat conduction partial differential equation (1) and fit the initial condition and boundary conditions automatically. By definition, the Biot number Bi represents the convection condition between solid and fluid interfaces. For a larger value of the Biot number Bi, more heat convection on the lateral surface and more thermal energy are efficiently transferred into the surrounding environment through the interface. As the Biot number Bi becomes infinitesimal, a constant heat-flux condition is shown. When it approaches infinity, a constant temperature condition is presented.

Remark. Some obviously trivial and inconsequential parametric and argument variations of the above-defined Bessel function $J_\nu(z)$ have regrettably misled many mainly amateurish-type researchers to believe that such variations can actually produce a "generalization" of the celebrated Bessel function $J_\nu(z)$ (see, for details, [10, Section 6, pp. 1512–1514]). As a matter of fact, a truly non-trivial generalization of the widely-investigated Bessel function $J_\nu(z)$ is the Bessel-Wright function $J_\nu^\mu(z)$, which is defined as follows (see, for example, [12, p. 2, Eq. (4)]):

$$J_\nu^\mu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma(\mu n + \nu + 1)} \qquad (z, \nu \in \mathbb{C};\ \mu > 0).$$

This function $J_\nu^\mu(z)$ was introduced by Sir Edward Maitland Wright (1906–2005), with whom the third-named author (H. M. Srivastava) had the privilege to meet and discuss researches emerging from his publications on hypergeometric and related functions during the third-named author's visit to the University of Aberdeen in the year 1976 (see [17]). In fact, there exists a significantly more general function than the Bessel function $J_\nu(z)$ and the Bessel-Wright function $J_\nu^\mu(z)$, that is, the widely- and extensively-investigated Fox-Wright function (see, for details, [7] and [11]).

In this section, we present 12 different solutions relating to various convection conditions.
Case 1. Let Bi_ℓ = constant and Bi = constant.

The corresponding analytical solution is given by

$$u(r,z,t) = \sum_{m=1}^{\infty}\left(zA_m + (\ell - z)B_m + \sum_{n=1}^{\infty} u_{mn}(t)\cdot[\cos(\beta_n z) + \frac{\mathrm{Bi}}{\beta_n}\sin(\beta_n z)]\right)\cdot e^{-t}\, J_0(\alpha_m r). \tag{9}$$

Moreover, the heat-transfer rate in the z-direction is given by

$$Q(r,z,t) = -\int^r 2\pi r \frac{\partial u(r,z,t)}{\partial z} dr. \tag{10}$$

By performing the necessary calculations, we obtain

$$Q(r,z,t) = -2\pi \sum_{m=1}^{\infty}\left(A_m - B_m + \sum_{n=1}^{\infty} u_{mn}(t)[-\beta_n \sin(\beta_n z) + \mathrm{Bi}\cos(\beta_n z)]\right)\frac{re^{-t}}{\alpha_m}\, J_1(\alpha_m r), \tag{11}$$

where

$$A_m = \frac{D_m}{(\mathrm{Bi} + \mathrm{Bi}_\ell + \ell\,\mathrm{Bi}\cdot\mathrm{Bi}_\ell)\ell} \qquad (m = 1, 2, 3, \cdots), \tag{12}$$

$$B_m = \frac{(\ell\,\mathrm{Bi}_\ell + 1)D_m}{(\mathrm{Bi} + \mathrm{Bi}_\ell + \ell\,\mathrm{Bi}\cdot\mathrm{Bi}_\ell)\ell} \qquad (m = 1, 2, 3, \cdots), \tag{13}$$

$$C_{mn} = -\frac{2\Big(\beta_n\cos(\beta_n\ell) + [A_m\beta_n^2\ell + (A_m - B_m)\mathrm{Bi}]\sin(\beta_n\ell) - [A_m - B_m(1 + \mathrm{Bi}_\ell)]\beta_n\Big)}{(\beta_n^2 - \mathrm{Bi}^2)\cos(\beta_n\ell)\sin(\beta_n\ell) - 2Bi\beta_n\cos^2(\beta_n\ell) + [(\beta_n^2 + \mathrm{Bi}^2)\ell + 2Bi]\beta_n}, \tag{14}$$

$$D_m = \begin{cases} (\mathrm{Bi} + q)e^t & (m = 1) \\ \dfrac{2(\mathrm{Bi} + q)e^t J_1(\alpha_m)}{\alpha_m J_0^2(\alpha_m)} & (m = 2, 3, 4, \cdots) \end{cases} \tag{15}$$

and β_n denotes the nth positive root of the following transcendental equation:

$$\tan(\beta_n\ell) = \frac{\beta_n(\mathrm{Bi} + \mathrm{Bi}_\ell)}{\beta_n^2 - \mathrm{Bi}\cdot\mathrm{Bi}_\ell} \qquad (n = 1, 2, 3, \cdots). \tag{16}$$

In Equations (9) and (11), the summation is taken over all eigenvalues. The final linear-series sums of the solution satisfy the heat conduction partial differential equation (1), together with the initial condition (2) and the boundary conditions (3) to (6).

Case 2. Let Bi = constant and $\mathrm{Bi}_\ell = 0$.

The corresponding analytical solution is given by

$$u(r, z, t) = \sum_{m=1}^{\infty}\left(zA_m + (\ell - z)B_m + \sum_{n=1}^{\infty} u_{mn}(t)\frac{\cos[\beta_n(\ell - z)]}{\cos(\beta_n\ell)}\right)\cdot e^{-t}\, J_0(\alpha_m r) \tag{17}$$

Moreover, the heat-transfer rate is given by

$$Q(r, z, t) = -2\pi\sum_{m=1}^{\infty}\left(A_m - B_m + \sum_{n=1}^{\infty} u_{mn}(t)\frac{\beta_n\sin[\beta_n(\ell - z)]}{\cos(\beta_n\ell)}\right)\frac{re^{-t}}{\alpha_m}\, J_1(\alpha_m r), \tag{18}$$

where

$$A_m = \frac{D_m}{\ell\,\mathrm{Bi}} \qquad (m = 1, 2, 3, \cdots), \tag{19}$$

$$B_m = \frac{D_m}{\ell\,\mathrm{Bi}} \qquad (m = 1, 2, 3, \cdots), \tag{20}$$

$$C_{mn} = \frac{2[(A_m - B_m)[\cos(\beta_n\ell) - 1] - B_m\beta_n\ell\sin(\beta_n\ell)]\cos(\beta_n\ell)}{\beta_n[\sin(\beta_n\ell)\cos(\beta_n\ell) + \beta_n\ell]} \tag{21}$$

$$(m, n = 1, 2, 3, \cdots),$$

$$D_m = \begin{cases} (\mathrm{Bi}+q)e^t & (m=1) \\ \dfrac{2(\mathrm{Bi}+q)J_1(\alpha_m)e^t}{\alpha_m J_0^2(\alpha_m)} & (m=2,3,4,\cdots) \end{cases} \tag{22}$$

and β_n denotes the nth positive root of the following transcendental equation:

$$\tan(\beta_n \ell) = \frac{\mathrm{Bi}}{\beta_n} \qquad (n=1,2,3,\cdots). \tag{23}$$

Case 3. Let Bi = constant and $\mathrm{Bi}_\ell \to \infty$.

The corresponding analytical solution is given by

$$u(r,z,t) = \sum_{m=1}^{\infty} \left((\ell - z)B_m + \sum_{n=1}^{\infty} u_{mn}(t) \frac{\sin[\beta_n(\ell - z)]}{\sin(\beta_n \ell)} \right) \cdot e^{-t}\, J_0(\alpha_m r). \tag{24}$$

Moreover, the heat-transfer rate is given by

$$Q(r,z,t) = 2\pi \sum_{m=1}^{\infty} \left(B_m + \sum_{n=1}^{\infty} u_{mn}(t) \frac{\beta_n \cos[\beta_n(\ell - z)]}{\sin(\beta_n \ell)} \right) \frac{re^{-t}}{\alpha_m} J_1(\alpha_m r), \tag{25}$$

where

$$B_m = \frac{D_m}{\ell\,\mathrm{Bi} + 1} \qquad (m=1,2,3,\cdots), \tag{26}$$

$$C_{mn} = -\frac{2B_m[\beta_n \ell \sin(\beta_n \ell)\cos(\beta_n \ell) + \cos^2(\beta_n \ell) - 1]}{\beta_n[\sin(\beta_n \ell)\cos(\beta_n \ell) - \beta_n \ell]}, \tag{27}$$

$$(m,n = 1,2,3,\cdots),$$

$$D_m = \begin{cases} (\mathrm{Bi}+q)e^t & (m=1) \\ \dfrac{2(\mathrm{Bi}+q)J_1(\alpha_m)e^t}{\alpha_m J_0^2(\alpha_m)} & (m=2,3,4,\cdots) \end{cases} \tag{28}$$

and β_n denotes the nth positive root of the following transcendental equation:

$$\cot(\beta_n \ell) = -\frac{\mathrm{Bi}}{\beta_n} \qquad (n=1,2,3,\cdots). \tag{29}$$

Case 4. Let Bi = 0 and $\mathrm{Bi}_\ell = 0$.

The corresponding the analytical solution is given by

$$u(r,z,t) = \sum_{m=1}^{\infty} \left((\ell\, z - \frac{z^2}{2})B_m + C_{m0} + \sum_{n=1}^{\infty} u_{mn}(t) \cos\beta_n z \right) \cdot e^{-t}\, J_0(\alpha_m r) \tag{30}$$

Moreover, the heat-transfer rate is given by

$$Q(r,z,t) = 2\pi \sum_{m=1}^{\infty} \left((z-\ell)B_m + \sum_{n=1}^{\infty} u_{mn}(t)\beta_n \sin\beta_n z \right) \frac{re^{-t}}{\alpha_m} J_1(\alpha_m r), \tag{31}$$

where

$$B_m = -\frac{D_m}{\ell} \qquad (m = 1,2,3,\cdots), \tag{32}$$

$$C_{mn} = \begin{cases} C_{m0} = -\dfrac{B_m \ell^2}{3} \\ C_{mn} = -\dfrac{2B_m \ell^2}{(n\pi)^2} \qquad (n = 1,2,3,\cdots), \end{cases} \tag{33}$$

$$D_m = \begin{cases} q \qquad (m = 1) \\ \dfrac{2qJ_1(\alpha_m)e^t}{\alpha_m J_0^2(\alpha_m)} \qquad (m = 2,3,4,\cdots) \end{cases} \tag{34}$$

and β_n is given by

$$\beta_n = \frac{n\pi}{\ell} \qquad (n = 1,2,3,\cdots). \tag{35}$$

Case 5. Let $\mathrm{Bi} = 0$ and $\mathrm{Bi}_\ell \to \infty$.

The corresponding analytical solution is given by

$$u(r,z,t) = \sum_{m=1}^{\infty} \left((\ell - z)B_m + \sum_{n=1}^{\infty} u_{mn}(t) \cos\beta_n z \right) \cdot e^{-t} J_0(\alpha_m r). \tag{36}$$

The heat-transfer rate is given by

$$Q(r,z,t) = 2\pi \sum_{m=1}^{\infty} \left(B_m + \sum_{n=1}^{\infty} u_{mn}(t)\beta_n \sin\beta_n z \right) \frac{re^{-t}}{\alpha_m} J_1(\alpha_m r), \tag{37}$$

where

$$B_m = D_m \qquad (m = 1,2,3,\cdots), \tag{38}$$

$$C_{mn} = -\frac{8B_m \ell}{[(2n-1)\pi]^2} \qquad (m,n = 1,2,3,\cdots), \tag{39}$$

$$D_m = \begin{cases} qe^t \qquad (m = 1) \\ \dfrac{2qe^t J_1(\alpha_m)}{\alpha_m J_0^2(\alpha_m)} \qquad (m = 2,3,4,\cdots) \end{cases} \tag{40}$$

and β_n is given by

$$\beta_n = \frac{(2n-1)\pi}{2\ell} \qquad (n = 1, 2, 3, \cdots). \tag{41}$$

Case 6. Let $\mathrm{Bi} \to \infty$ and $\mathrm{Bi}_\ell \to \infty$.

The corresponding analytical solution is given by

$$u(r, z, t) = \sum_{m=1}^{\infty} \left((\ell - z) B_m + \sum_{n=1}^{\infty} u_{mn}(t) \sin \beta_n z \right) \cdot e^{-t} \, J_0(\alpha_m r). \tag{42}$$

The heat-transfer rate is given by

$$Q(r, z, t) = 2\pi \sum_{m=1}^{\infty} \left(B_m - \sum_{n=1}^{\infty} u_{mn}(t) \beta_n \cos \beta_n z \right) \frac{r e^{-t}}{\alpha_m} \, J_1(\alpha_m r), \tag{43}$$

where

$$B_m = \frac{D_m}{\ell}, \tag{44}$$

$$C_{mn} = -\frac{2 B_m \ell}{n\pi}, \tag{45}$$

$$D_m = \begin{cases} e^t & (m = 1) \\[2ex] \dfrac{2 J_1(\alpha_m) e^t}{\alpha_m J_0^2(\alpha_m)} & (m = 2, 3, 4, \cdots), \end{cases} \tag{46}$$

and

$$\beta_n = \frac{n\pi}{\ell} \qquad (n = 1, 2, 3, \cdots). \tag{47}$$

3. Solutions with a Different Set of Boundary Conditions

In this section, we first set

$$r = 1 : \; u(1, z, t) = 0 \tag{48}$$

and suppose that the associated α_m denotes the mth positive zero of the following transcendental equation for all of the cases:

$$J_0(\alpha_m) = 0 \qquad (m = 1, 2, 3, \cdots). \tag{49}$$

In this case, by applying the same procedure as described in the preceding sections, the following solutions would easily emerge for the modified problem.

Case 7. $\mathrm{Bi}_\ell = \text{constant}$ and $\mathrm{Bi} = constant$.

The corresponding analytical solution is given by

$$u(r,z,t) = \sum_{m=1}^{\infty}\left[zA_m + (\ell - z)B_m + \sum_{n=1}^{\infty} u_{mn}(t)\left(\cos\beta_n z + \frac{\mathrm{Bi}}{\beta_n}\sin\beta_n z\right)\right]\cdot e^{-t}\, J_0(\alpha_m r). \tag{50}$$

Moreover, the heat-transfer rate is given by

$$Q(r,z,t) = -2\pi\sum_{m=1}^{\infty}\left(A_m - B_m + \sum_{n=1}^{\infty} u_{mn}(t)\Big(-\beta_n\sin\beta_n z + \mathrm{Bi}\cos\beta_n z\Big)\right)\frac{re^{-t}}{\alpha_m}\, J_1(\alpha_m r), \tag{51}$$

where

$$A_m = \frac{D_m}{(\mathrm{Bi} + \mathrm{Bi}_\ell + \ell\,\mathrm{Bi}\cdot\mathrm{Bi}_\ell)\ell} \qquad (m = 1,2,3,\cdots), \tag{52}$$

$$B_m = \frac{(\ell\,\mathrm{Bi}_\ell + 1)D_m}{(\mathrm{Bi} + \mathrm{Bi}_\ell + \ell\,\mathrm{Bi}\cdot\mathrm{Bi}_\ell)\ell} \qquad (m = 1,2,3,\cdots), \tag{53}$$

$$C_{mn} = -\frac{2\Big([A_m(1-\mathrm{Bi}_\ell) - B_m]\beta_n\cos\beta_n\ell + \Xi\Big(\beta_n,\mathrm{Bi}_\ell,A_m,B_m\Big)\Big)}{\Big(\beta_n^2 - \mathrm{Bi}^2\Big)\cos\beta_n\ell\sin\beta_n\ell - 2Bi\beta_n\cos^2\beta_n\ell + \Big[\Big(\beta_n^2 + \mathrm{Bi}^2\Big)\ell + 2Bi\Big]\beta_n}, \tag{54}$$

$$D_m = \frac{2(\mathrm{Bi} + q)e^t}{\alpha_m J_1(\alpha_m)} \qquad (m = 1,2,3,\cdots) \tag{55}$$

and, for convenience,

$$\Xi\Big(\beta_n,\mathrm{Bi}_\ell,A_m,B_m\Big) := [A_m\beta_n^2\ell + (A_m - B_m)\mathrm{Bi}]\sin\beta_n\ell - [A_m - B_m(1+\mathrm{Bi}_\ell)]\beta_n.$$

Furthermore, β_n denotes the nth positive root of the following transcendental equation:

$$\tan(\beta_n\ell) = \frac{\beta_n(\mathrm{Bi} + \mathrm{Bi}_\ell)}{\beta_n^2 - \mathrm{Bi}\cdot\mathrm{Bi}_\ell} \qquad (n = 1,2,3,\cdots). \tag{56}$$

Case 8. $\mathrm{Bi} = \text{constant}$ and $\mathrm{Bi}_\ell = 0$.

The corresponding analytical solution is given by

$$u(r,z,t) = \sum_{m=1}^{\infty}\left(zA_m + (\ell - z)B_m + \sum_{n=1}^{\infty} u_{mn}(t)\frac{\cos[\beta_n(\ell - z)]}{\cos(\beta_n\ell)}\right)\cdot e^{-t}\, J_0(\alpha_m r). \tag{57}$$

Moreover, the heat-transfer rate is given by

$$Q(r,z,t) = -2\pi\sum_{m=1}^{\infty}\left(A_m - B_m + \sum_{n=1}^{\infty} u_{mn}(t)\frac{\beta_n\sin\beta_n(\ell - z)}{\cos(\beta_n\ell)}\right)\frac{re^{-t}}{\alpha_m}\, J_1(\alpha_m r), \tag{58}$$

where

$$A_m = \frac{D_m}{\ell\,\mathrm{Bi}} \qquad (m = 1, 2, 3, \cdots), \tag{59}$$

$$B_m = \frac{D_m}{\ell\,\mathrm{Bi}} \qquad (m = 1, 2, 3, \cdots), \tag{60}$$

$$C_{mn} = \frac{2[(A_m - B_m)[\cos(\beta_n\ell) - 1] - B_m(\beta_n\ell)\sin(\beta_n\ell)]\cos(\beta_n\ell)}{\beta_n[\sin(\beta_n\ell)\cos(\beta_n\ell) + \beta_n\ell]}, \tag{61}$$

$$D_m = \frac{2(\mathrm{Bi} + q)e^t}{\alpha_m J_1(\alpha_m)} \qquad (m = 1, 2, 3, \cdots) \tag{62}$$

and β_n denotes the nth positive root of the following transcendental equation:

$$\tan\beta_n\ell = \frac{\mathrm{Bi}}{\beta_n} \qquad (n = 1, 2, 3, \cdots). \tag{63}$$

Case 9. Let Bi = constant and $\mathrm{Bi}_\ell \to \infty$.

The corresponding analytical solution is given by

$$u(r, z, t) = \sum_{m=1}^{\infty}\left((\ell - z)B_m + \sum_{n=1}^{\infty} u_{mn}(t)\frac{\sin\beta_n(\ell - z)}{\sin\beta_n\ell}\right)\cdot e^{-t}\, J_0(\alpha_m r) \tag{64}$$

Moreover, the heat-transfer rate is given by

$$Q(r, z, t) = 2\pi\sum_{m=1}^{\infty}\left(B_m + \sum_{n=1}^{\infty} u_{mn}(t)\frac{\beta_n\cos\beta_n(\ell - z)}{\sin\beta_n\ell}\right)\frac{re^{-t}}{\alpha_m}\, J_1(\alpha_m r), \tag{65}$$

where

$$B_m = \frac{D_m}{\ell\,\mathrm{Bi} + 1}, \tag{66}$$

$$C_{mn} = -\frac{2B_m[\beta_n\ell\sin(\beta_n\ell)\cos(\beta_n\ell) - \sin^2(\beta_n\ell)]}{\beta_n[\sin(\beta_n\ell)\cos(\beta_n\ell) - \beta_n\ell]}, \tag{67}$$

$$D_m = \frac{2(\mathrm{Bi} + q)e^t}{\alpha_m J_1(\alpha_m)} \qquad (m = 1, 2, 3, \cdots) \tag{68}$$

and β_n denotes the nth positive root of the following transcendental equation:

$$\cot(\beta_n \ell) = -\frac{\mathrm{Bi}}{\beta_n} \qquad (n = 1, 2, 3, \cdots). \tag{69}$$

Case 10. Let $\mathrm{Bi} = 0$ and $\mathrm{Bi}_\ell = 0$.

The corresponding the analytical solution is given by

$$u(r, z, t) = \sum_{m=1}^{\infty} \left((\ell\, z - \frac{z^2}{2}) B_m + C_{m0} + \sum_{n=1}^{\infty} u_{mn}(t) \cos(\beta_n z) \right) \cdot e^{-t}\, J_0(\alpha_m r) \tag{70}$$

Moreover, the heat-transfer rate is given by

$$Q(r, z, t) = 2\pi \sum_{m=1}^{\infty} \left((z - \ell) B_m + \sum_{n=1}^{\infty} u_{mn}(t) \beta_n \sin(\beta_n z) \right) \frac{r e^{-t}}{\alpha_m}\, J_1(\alpha_m r), \tag{71}$$

where

$$B_m = -\frac{D_m}{\ell} \qquad (m = 1, 2, 3, \cdots), \tag{72}$$

$$C_{mn} = \begin{cases} C_{m0} = -\frac{B_m \ell^2}{3} & (n = 0) \\ C_{mn} = -\frac{2 B_m \ell^2}{(n\pi)^2} & (n = 1, 2, 3, \cdots), \end{cases} \tag{73}$$

$$D_m = \frac{2 q e^t}{\alpha_m J_1(\alpha_m)} \qquad (m = 1, 2, 3, \cdots) \tag{74}$$

and β_n is given by

$$\beta_n = \frac{n\pi}{\ell} \qquad (n = 1, 2, 3, \cdots). \tag{75}$$

Case 11. Let $\mathrm{Bi} = 0$ and $\mathrm{Bi}_\ell \to \infty$.

The corresponding analytical solution is given by

$$u(r, z, t) = \sum_{m=1}^{\infty} \left((\ell - z) B_m + \sum_{n=1}^{\infty} u_{mn}(t) \cos(\beta_n z) \right) \cdot e^{-t}\, J_0(\alpha_m r). \tag{76}$$

Moreover, the heat-transfer rate is given by

$$Q(r, z, t) = 2\pi \sum_{m=1}^{\infty} \left(B_m + \sum_{n=1}^{\infty} u_{mn}(t) \beta_n \sin(\beta_n z) \right) \frac{r e^{-t}}{\alpha_m}\, J_1(\alpha_m r), \tag{77}$$

where

$$B_m = D_m \qquad (m = 1, 2, 3, \cdots), \tag{78}$$

$$C_{mn} = -\frac{8B_m \ell}{[(2n-1)\pi]^2} \qquad (m, n = 1, 2, 3, \cdots), \tag{79}$$

$$D_m = \frac{2qe^t}{\alpha_m J_1(\alpha_m)} \qquad (m = 1, 2, 3, \cdots) \tag{80}$$

and β_n is given by

$$\beta_n = \frac{(2n-1)\pi}{2\ell} \qquad (n = 1, 2, 3, \cdots). \tag{81}$$

Case 12. Let $\mathrm{Bi} \to \infty$ and $\mathrm{Bi}_\ell \to \infty$.

The corresponding analytical solution is given by

$$u(r, z, t) = \sum_{m=1}^{\infty} \left((\ell - z)B_m + \sum_{n=1}^{\infty} u_{mn}(t) \sin(\beta_n z) \right) \cdot e^{-t} \, J_0(\alpha_m r). \tag{82}$$

Moreover, the heat-transfer rate is given by

$$Q(r, z, t) = 2\pi \sum_{m=1}^{\infty} \left(B_m - \sum_{n=1}^{\infty} u_{mn}(t) \beta_n \cos(\beta_n z) \right) \frac{re^{-t}}{\alpha_m} \, J_1(\alpha_m r), \tag{83}$$

where

$$B_m = \frac{D_m}{\ell} \qquad (m = 1, 2, 3, \cdots), \tag{84}$$

$$C_{mn} = -\frac{2B_m \ell}{n\pi} \qquad (m, n = 1, 2, 3, \cdots), \tag{85}$$

$$D_m = \frac{2e^t}{\alpha_m J_1(\alpha_m)} \qquad (m = 1, 2, 3, \cdots) \tag{86}$$

and β_n is given by

$$\beta_n = \frac{n\pi}{\ell} \qquad (n = 1, 2, 3, \cdots). \tag{87}$$

4. Concluding Remarks and Observations

Here, in our present investigation, the method of separation of variables is applied to the transient heat-conduction equation, which is subjected to different lateral surface conditions, in order to provide a simplified formulation that can be used to identify the temperature distributions and the heat-transfer rates. The analytical solutions are expressed in terms of the Fourier-Bessel series of the exponential type. Furthermore, the solutions presented in this paper can be used to verify the two- or three-dimensional numerical conduction codes.

For motivating further researches on this subject, we choose to refer the reader to the related earlier works (see, for example, [6], [15] and [18]). In particular, the recent investigation by Zhukovsky and Srivastava [18] presented a detailed description of a broad range of physical problems including the heat-conduction problem by applying operational methods with recourse to inverse derivative operators, integral transformations and operational exponent (see also [5] and [13] for several related developments on differential, integral and integro-differential equations, as well as on fractional differential equations). Several recent developments on the fractional-order modeling and analysis of applied and real-world problems can be found in (for example) [4], [8], [9] and [11]. It should be remarked here that Zhukovsky and Srivastava [18] also considered the evolutional type problems for heat transfer in various heat-conduction models and derived the exact analytical solutions for the Guyer–Krumhansl hyperbolic heat equation and compared these exact analytic solutions with those of the Fourier and Cattaneo equations.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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