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Generalized Cline's Formula for g-Drazin Inverse in a Ring

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Abstract. In this paper, we give a generalized Cline's formula for the generalized Drazin inverse. Let R be a ring, and let $a, b, c, d \in R$ satisfying

$$(ac)^2 = (db)(ac), (db)^2 = (ac)(db);$$
$$b(ac)a = b(db)a, c(ac)d = c(db)d.$$

Then $ac \in R^d$ if and only if $bd \in R^d$. In this case, $(bd)^d = b((ac)^d)^2 d$. We also present generalized Cline's formulas for Drazin and group inverses. Some weaker conditions in a Banach algebra are also investigated. These extend the main results of Cline's formula on g-Drazin inverse of Liao, Chen and Cui (Bull. Malays. Math. Soc., **37**(2014), 37-42), Lian and Zeng (Turk. J. Math., **40**(2016), 161-165) and Miller and Zguitti (Rend. Circ. Mat. Palermo, II. Ser., **67**(2018), 105-114). As an application, new common spectral property of bounded linear operators over Banach spaces is obtained.

1. Introduction

Let R be an associative ring with an identity. The commutant of $a \in R$ is defined by $comm(a) = \{x \in R \mid xa = ax\}$. The double commutant of $a \in R$ is defined by $comm^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in comm(a)\}$. An element $a \in R$ has g-Drazin inverse (i.e., generalized Drazin inverse) in the case that there exists $b \in R$ such that

$$b = bab, b \in comm^2(a), a - a^2 b \in R^{qnil}.$$

The preceding b is unique if it exists and denoted by a^d. Here, $R^{qnil} = \{a \in R \mid 1 + ax \in R^{-1} \text{ for every } x \in comm(a)\}$. For a Banach algebra $\mathcal{A}$ it is well known that

$$a \in \mathcal{A}^{qnil} \Leftrightarrow \lim_{n \to \infty} \| a^n \|^{\frac{1}{n}} = 0 \Leftrightarrow 1 - \lambda a \in \mathcal{A}^{-1} \text{ for any scarlar } \lambda.$$

Let $a, b \in R$. The Cline's formula for g-Drazin inverse stated that $ab \in R^d$ if and only if $ba \in R^d$. In this case, $(ba)^d = b[(ab)^d]^2 a$ (see [6, Theorem 2.1]). Here, $R^d = \{x \in R \mid x \text{ has g-Drazin inverse }\}$. Cline's formula plays an important role in the generalized inverse of matrix and operator theory ([2, 4, 7, 11, 13–15]). In

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[5, Theorem 2.3], Lian and Zeng proved the generalized Cline's formula to the case when $aba = aca$. In [7, Theorem 3.2], Miller and Zguitti generalized the Cline's formula under the condition $acd = dbd$ and $dba = aca$. In [10, Theorem 3.2], Mosić investigated the generalized Cline's formula under the condition $bac = bdb$ and $cdb = cac$. The motivation of this paper is to extend the Cline's formula for g-Drazin inverse to a wider case.

In Section 2, we present a new generalized Cline's formular for g-Draziin inverse. We also prove generalized Cline's formulas for Drazin and group inverses. Let R be a ring, and let $a, b, c, d \in R$ satisfying

$$(ac)^2 = (db)(ac), (db)^2 = (ac)(db);$$
$$b(ac)a = b(db)a, c(ac)d = c(db)d.$$

Then $ac \in R^d$ if and only if $bd \in R^d$. In this case, $(bd)^d = b((ac)^d)^2 d$. This improves the main results of Cline's formula on g-Drazin inverse of Liao, Chen and Cui ([6, Theorem 2.1]), Lian and Zeng ([5, Theorem 2.3]) and Miller and Zguitti ([7, Theorem 3.2]). In Section 3, we investigate some weaker conditions in a Banach algebra under which the generalized Cline's formula holds. We prove that the preceding condition "$b(ac)a = b(db)a, c(ac)d = c(db)d$" can be dropped in a Banach algebra. Finally, in Section 4, we apply the generalized Cline's formula to common spectral property of bounded linear operators in a Banach space.

Throughout the paper, all rings are associative with an identity and all Banach algebras are complex. We use $\mathcal{A}^{rad}$ to denote the Jacobson radical of $\mathcal{A}$. The notations $\mathcal{A}^d, \mathcal{A}^D, \mathcal{A}^\#$ and $\mathcal{A}^\ddagger$ stand for the sets of all g-Drazin, Drazin, group and p-Drazin invertible elements, respectively.

2. Generalized Cline's Formula

For any elements a, b in a ring R, it is well known that $ab \in R^{qnil}$ if and only if $ba \in R^{qnil}$ (see [5, Lemma 2.2]). We start with the following generalization.

Lemma 2.1. *Let R be a ring, and let $a, b, c, d \in R$ satisfying*

$$(ac)^2 = (db)(ac), (db)^2 = (ac)(db);$$
$$b(ac)a = b(db)a, c(ac)d = c(db)d.$$

Then $ac \in R^{qnil}$ if and only if $bd \in R^{qnil}$.

Proof. $\Longrightarrow$ Let $x \in comm(bd)$. Then we check that

$$\begin{aligned}
(dbdx^5bdbac)ac &= dbdx^5bd(baca)c \\
&= dbdx^5bd(bdba)c \\
&= dbdx^5b(dbdba)c \\
&= (dbdbd)x^5bdbac \\
&= (dbdb)dx^5bdbac \\
&= (acdb)dx^5bdbac \\
&= (acdbd)x^5bdbac \\
&= ac(dbdx^5bdbac)
\end{aligned}$$

Hence, $dbdx^5bdbac \in comm(ac)$, and so $1 - dbd(x^5bdbacac) = 1 - (dbdx^5bdbac)ac \in R^{-1}$. By using Jacobson's Lemma (see [5, Lemma 2.1]), we see that

$$\begin{aligned}
1 - x^5bdbdbdbdbd &= 1 - (x^5bdbdb(dbdb)d \\
&= 1 - (x^5bdbdb(acdb)d \\
&= 1 - (x^5bdbdbac)dbd \\
&= 1 - (x^5bdbacac)dbd \\
&\in R^{-1}.
\end{aligned}$$

Then

$$\begin{array}{rl} & (1-xbd)(1+xbd+x^2bdbd+x^3bdbdbd+x^4bdbdbdbd) \\ = & (1+xbd+x^2bdbd+x^3bdbdbd+x^4bdbdbdbd)(1-xbd) \\ = & 1-x^5bdbdbdbdbd \\ \in & R^{-1}, \end{array}$$

and so $bd \in R^{qnil}$.

$\Longleftarrow$ Since $bd \in R^{qnil}$, by [5, Lemma 2.2], $db \in R^{qnil}$. Applying the preceding discussion, we see that $ca \in R^{qnil}$. By using [5, Lemma 2.2] again, we have $ac \in R^{qnil}$, as desired. □

We are now ready to prove:

Theorem 2.2. *Let R be a ring, and let $a, b, c, d \in R$ satisfying*

$$(ac)^2 = (db)(ac), (db)^2 = (ac)(db);$$
$$b(ac)a = b(db)a, c(ac)d = c(db)d.$$

Then $ac \in R^d$ if and only if $bd \in R^d$. In this case,

$$(bd)^d = b((ac)^d)^2d.$$

Proof. $\Longrightarrow$ Let $(ac)^d = h$ and $e = bh^2d$. We shall prove that e is the g-Drazin inverse of bd.

Step 1. Let $t \in comm(bd)$. Then we check that

$$\begin{array}{rcl} ac(dtbdbdbac) & = & (acdbdbd)(tbac) \\ & = & (dbdbdbd)(tbac) \\ & = & dtbdb(dbdba)c \\ & = & (dtbdbdbac)ac. \end{array}$$

Thus $dtbdbdbac \in comm(ac)$, and so $(dtbdbdbac)h = h(dtbdbdbac)$. It is easy to verify that $acdbd = a(cdbd) = a(cacd) = (ac)^2d = (dbac)d = dbacd$, and so $(bacd)(bd) = (bd)(bacd)$. Then we compute that

$$\begin{array}{rcl} et & = & (bh^6(ac)^4d)t = bh^6(db)^3acdt = bh^6dbdbd(bacd)t \\ & = & bh^6d(bacd)bdbdt = bh^6db(acdb)dbdt = bh^6(dbdbdbdbd)t \\ & = & bh^6dtbdb(dbdb)d = bh^6dtbdba(cdbd) = bh^6dtbdb(acac)d \\ & = & bh^6(dtbdbdbac)d = b(dtbdbdbac)h^6d = tbdbdbdbach^6d \\ & = & tb(ac)^4h^6d = tbh^2d = te, \end{array}$$

and then $e \in comm^2(bd)$.

Step 2. We directly verify that

$$\begin{array}{rcl} e(bd)e & = & bh^2dbdbh^2d = bh^2acdbh^2d \\ & = & bh^2acdb(ac)h^3d = bh^2(ac)^3h^3d = b(ac)^3h^5d = e. \end{array}$$

Step 3. Let $p = 1-(ac)h$. Then $(pa)c = ac-(ac)^2h \in R^{qnil}$. One easily checks that

$$\begin{array}{rcl} bd-(bd)^2e & = & bd-bdbdbh^2d \\ & = & bd-b(dbdb)ach^3d \\ & = & bd-b(acdb)ach^3d \\ & = & bd-bac(dbac)h^3d \\ & = & bd-b(ac)^3h^3d \\ & = & b(1-ach)d \\ & = & b(pd). \end{array}$$

We directly compute that

$$\begin{array}{rcl}
(pac)^2 & = & p(ac)^2p = p(dbac)p = (pdb)(pac),\\
(pdb)^2 & = & pdbpdb = pdb[1-(1-p)]db\\
& = & p(db)^2 - pdbac(ac)^d db\\
& = & p(db)^2 - p(ac)^2(ac)^d db\\
& = & p(db)^2 = pacdb = (pac)(pdb);\\
b(pac)pa & = & bacpa - b(ac)^d(ac)^2pa\\
& = & bacpa - b(ac)^d d(bacpa) = bdbpa - b(ac)^d d(bdbpa)\\
& = & bdbpa - b(ac)^d(acdb)pa = b(pdb)pa,\\
c(pac)pd & = & cacd - c(ac)^d(ac)^2d = cdbd - c(ac)^d(ac)^2d\\
& - & [c(ac)^d a(cdbd) - c(ac)^d a(cacd)] = cdbd - cac(ac)^d acd\\
& - & [c(ac)^d acdbd - c(ac)^d acdbac(ac)^d d]\\
& = & cdbd - cac(ac)^d acd - c(ac)^d acdb[1-ac(ac)^d]d\\
& = & cdbd - cdb(ac)^d acd - c(ac)^d acdbpd\\
& = & [cdb - c(ac)^d acdb]pd = c(pdb)pd.
\end{array}$$

Since $(pa)c = ac - (ac)^2(ac)^d \in R^{qnil}$. In view of Lemma 2.1, $b(pd) \in R^{qnil}$. Therefore bd has g-Drazin inverse e and $e = bh^2a = (bd)^d$.

$\Longleftarrow$ In view of [6, Theorem 2.2], $db \in R^d$. Applying the preceding discussion, we have $ca \in R^d$. By using [6, Theorem 2.2] again, $ac \in R^d$. This completes the proof. □

In the case that $c = b$ and $d = a$, we recover the Cline's formula for g-Drazin inverse . In [5, Theorem 2.3], Lian and Zeng concerned Cline's formula under the condition $aba = aca$. We now derive

Corollary 2.3. *Let R be a ring, and let $a, b, c \in R$ satisfying*

$$(aba)b = (aca)b, b(aba) = b(aca),$$
$$(aba)c = (aca)c, c(aba) = c(aca).$$

Then $ac \in R^d$ if and only if $ba \in R^d$. In this case, $(ba)^d = b((ac)^d)^2a$.

Proof. Choosing $d = a$ in Theorem 3.2, we obtain the result. □

Corollary 2.4. *([7, Theorem 3.2]) Let R be a ring, and let $a, b, c \in R$ satisfying*

$$acd = dbd, dba = aca.$$

Then $ac \in R^d$ if and only if $bd \in R^d$. In this case, $(bd)^d = b((ac)^d)^2d$.

Proof. By hypothesis, we easily check that

$$(ac)^2 = (db)(ac), (db)^2 = (ac)(db);$$
$$b(ac)a = b(db)a, c(ac)d = c(db)d.$$

This completes the proof by Theorem 2.2. □

An element $a \in R$ has Drazin inverse in the case that there exists $b \in R$ such that

$$b = bab, ab = ba, a^k = a^{k+1}b$$

for some $k \in \mathbb{N}$ The preceding b is unique if it exists. It is denoted by a^D. The smallest k satisfying the preceding condition is called the Drazin index of a. It is denoted by $i(a)$.

Theorem 2.5. *Let R be a ring, and let $a, b, c, d \in R$ satisfying*

$$(ac)^2 = (db)(ac), (db)^2 = (ac)(db);$$
$$b(ac)a = b(db)a, c(ac)d = c(db)d.$$

Then $ac \in R^D$ if and only if $bd \in R^D$. In this case,

$$(bd)^D = b((ac)^D)^2 d,$$
$$i(bd) \leq i(ac) + 2.$$

Proof. $\Longrightarrow$ Since $ac \in R^D$, we see that $ac \in R^d$. It follows by Theorem 2.2 that $bd \in R^d$ and $(bd)^d = b((ac)^d)^2 d$. Then $bd(bd)^d = (bd)^d bd$ and $(bd)^d = (bd)^d (bd)(bd)^d$. Clearly, we have

$$\begin{aligned} 1 - (db)^2((ac)^d)^2 &= 1 - acdb(ac)((ac)^d)^3 \\ &= 1 - ac(dbac)((ac)^d)^3 \\ &= 1 - ac(ac)^d. \end{aligned}$$

Hence,

$$\begin{aligned} bd - (bd)^2(bd)^d &= bd - (bd)^2 b((ac)^d)^2 d \\ &= b[1 - (db)^2((ac)^d)^2]d \\ &= b[1 - ac(ac)^d]d. \end{aligned}$$

Further, we have

$$\begin{aligned} [1 - (bd)(bd)^d](bd)^3 &= b[1 - ac(ac)^d](dbdb)d \\ &= b[1 - ac(ac)^d](acdb)d \\ &= b[1 - (ac)(ac)^d](ac)dbd. \end{aligned}$$

Write $m = i(ac)$. By induction, we have

$$\begin{aligned} [1 - (bd)(bd)^d](bd)^{m+2} &= b[1 - (ac)(ac)^d](ac)^m dbd \\ &= 0. \end{aligned}$$

Therefore $[bd - (bd)^2(bd)^d]^{m+2} = 0$, and so bd has Drazin inverse. Moreover, we have $i(bd) \leq m + 2 = i(ac) + 2$, as required.

$\Longleftarrow$ This is proved as in Theorem 2.2. □

Corollary 2.6. *Let R be a ring, and let $a, b, c \in R$ satisfying*

$$(aba)b = (aca)b, b(aba) = b(aca),$$
$$(aba)c = (aca)c, c(aba) = c(aca).$$

Then $ac \in R^D$ if and only if $ba \in R^D$. In this case,

$$\begin{aligned} (ba)^D &= b((ac)^D)^2 a, \\ i(ba) &\leq i(ac) + 1. \end{aligned}$$

Proof. We prove that $ac \in R^D$ if and only if $ba \in R^D$ by choosing $d = a$ in Theorem 2.5. Moreover, we check that $ba - (ba)^2(ba)^D = b[1 - ac(ac)^D]a$, and so

$$\begin{aligned} [1 - (ba)(ba)^D](ba)^2 &= b[1 - ac(ac)^D]aba \\ &= baba - bac(ac)^D aba \\ &= baca - bac((ac)^D)^2 a(caba) \\ &= baca - bac((ac)^D)^2 a(caca) \\ &= b[ac - (ac)^2(ac)^D]a \\ &= b[1 - (ac)(ac)^D](ac)a. \end{aligned}$$

By induction, we have $[1 - (ba)(ba)^D](ba)^{m+1} = b[1 - (ac)(ac)^D](ac)^m a = 0$, where $m = i(ac)$. This shows that

$$(ba)^{m+1} - (ba)^{m+2}(ba)^D = [1 - (ba)(ba)^d](ba)^{m+1} = 0.$$

Therefore $i(ba) \leq m + 1 = i(ac) + 1$, as asserted. □

The group inverse of $a \in R$ is the unique element $a^\# \in R$ which satisfies $aa^\# = a^\# a, a = aa^\# a, a^\# = a^\# aa^\#$. We denote the set of all group invertible elements of R by $R^\#$. As is well known, $a \in R^\#$ if and only if $a \in R^D$ and $i(a) = 1$.

Theorem 2.7. *Let R be a ring, and let $a, b, c \in R$ satisfying*

$$(aba)b = (aca)b, b(aba) = b(aca),$$
$$(aba)c = (aca)c, c(aba) = c(aca).$$

If $ac \in R^\#$, then $(ba)^2 \in R^\#$. In this case, $(ac)^\# = a[(ba)^2]^\# c$.

Proof. Since $ac \in \mathcal{A}^\#$, it follows by Corollary 2.6 that $ba \in \mathcal{A}^D$ and $(ba)^D = b[(ac)^2]^D a$. Moreover, we have $i(ba) \leq i(ac) + 1 = 2$. Set $x = (ba)^D$. Then $(ba)^2 = (ba)^3 x = (ba)^2 x^2 (ba)^2, x^2 = x^2 (ba)^2 x^2$. Hence $[(ba)^2]^\# = x^2$. We observe that

$$\begin{aligned} a[(ba)^2]^D c &= a[(ba)^D]^2 c \\ &= ab[(ac)^2]^D ab[(ac)^2]^D ac \\ &= ab[(ac)^2]^D (abac)(ac)^D]^2 \\ &= ab[(ac)^2]^D (ac)^2 [(ac)^D]^2 \\ &= (abac)[(ac)^D]^3 \\ &= (ac)^2 [(ac)^D]^3 \\ &= (ac)^D, \end{aligned}$$

therefore $(ac)^\# = a[(ba)^2]^\# c$, as desired. □

Corollary 2.8. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c \in \mathcal{A}$ satisfying $aba = aca$. If $ac \in \mathcal{A}^\#$, then $(ba)^2 \in \mathcal{A}^\#$. In this case, $(ac)^\# = a[(ba)^2]^\# c$.*

Proof. This is clear from Theorem 2.7. □

3. Extensions in Banach algebras

In this section, we investigate the generalized Cline's formula in a Banach algebra. We observe that the condition "$b(ac)a = b(db)a, c(ac)d = c(db)d$" in Theorem 2.2 can be dropped in Banach algebra.

Theorem 3.1. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c, d \in \mathcal{A}$ satisfying*

$$(ac)^2 = (db)(ac),$$
$$(db)^2 = (ac)(db).$$

Then $ac \in \mathcal{A}^d$ if and only if $bd \in \mathcal{A}^d$. In this case, $(bd)^d = b[(ac)^d]^3 dbd$.

Proof. $\Longrightarrow$ Let $aca = a^{'}, b = b^{'}, c = c^{'}$ and $dbd = d^{'}$. Then we have

$$\begin{aligned} &(a'c')^2 = (ac)^4 = (db)^3 ac = dbdbacac = (d'b')(a'c'), \\ &(d'b')^2 = (db)^4 = (ac)(db)^2(db) = (ac)^2(db)^2 = (a'c')(d'b'), \\ &b'(a'c')a' = b(ac)^3 a = bdbdbaca = b'(d'b')a', \\ &c'(a'c')d' = c(acac)dbd = c(db)^3 d = c'(d'b')d'. \end{aligned}$$

Since $ac \in \mathcal{A}^d$, it follows by [3, Theorem 2.7] that $a'c' = (ac)^2 \in \mathcal{A}^d$, In light of Theorem 2.2, $b'd' = (bd)^2 \in \mathcal{A}^d$. Therefore $bd \in \mathcal{A}^d$ by [3, Theorem 2.7]. Moreover, we have

$$\begin{aligned} (bd)^d &= [(bd)^2]^d bd \\ &= (b'd')^d bd = b'[(a'c')^d]^2 d'bd \\ &= b[(ac)^d]^4 (db)^2 d \\ &= b[(ac)^d]^4 (acdb)d \\ &= b[(ac)^d]^3 dbd, \end{aligned}$$

as required.

$\Longleftarrow$ Since $db \in \mathcal{A}^d$, applying the preceding discussion, we have $ca \in \mathcal{A}^d$. Therefore $ac \in \mathcal{A}^d$, by using the Cline's formula. □

As easy consequences, we now derive

Corollary 3.2. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c \in \mathcal{A}$. If $(aba)b = (aca)b, (aba)c = (aca)c$, then $ac \in \mathcal{A}^d$ if and only if $ba \in \mathcal{A}^d$. In this case, $(ba)^d = b[(ac)^d]^3 aba$.*

Proof. This is obvious by choosing $d = a$ in Theorem 3.1. □

Corollary 3.3. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c, d \in \mathcal{A}$ satisfying*

$$(ac)^2 = (db)(ac),$$
$$(db)^2 = (ac)(db).$$

Then $ac \in \mathcal{A}^D$ if and only if $bd \in \mathcal{A}^D$. In this case, $(bd)^D = b[(ac)^D]^3 dbd$.

Proof. $\Longrightarrow$ Since $ac \in \mathcal{A}^D$, we see that $ac \in \mathcal{A}^d$. By virtue of Theorem 3.1, $bd \in \mathcal{A}^d$ and $(bd)^d = b[(ac)^d]^3 dbd$. Let $m = i(ac)$. Then $[ac - (ac)^2(ac)^D]^m = 0$. One easily checks that

$$\begin{array}{rcl} 1 - (bd)(bd)^d & = & 1 - bdb[(ac)^D]^3 dbd \\ & = & 1 - bdbac[(ac)^D]^4 dbd \\ & = & 1 - b(ac)^2[(ac)^D]^4 dbd \\ & = & 1 - b[(ac)^D]^2 dbd. \end{array}$$

Then

$$\begin{array}{rcl} [1 - (bd)(bd)^d](bd)^3 & = & [1 - b[(ac)^D]^2 dbd](bd)^3 \\ & = & (bd)^3 - b[(ac)^D]^2 d(bd)^4 \\ & = & bacdbd - b[(ac)^D]^2 (ac)^3 dbd \\ & = & b[1 - ac(ac)^D]ac(dbd). \end{array}$$

By induction, we have

$$\begin{array}{rcl} [bd - (bd)^2(bd)^d]^{m+2} & = & [1 - (bd)(bd)^d](bd)^{m+2} \\ & = & b[1 - ac(ac)^D](ac)^m(dbd) \\ & = & b[ac - ac^2(ac)^D]^m(dbd) \\ & = & 0. \end{array}$$

Therefore

$$(bd)^D = (bd)^d = b[(ac)^D]^3 dbd,$$

as required.

$\Longleftarrow$ This is proved as in Theorem 3.1. □

An element a in a Banach algebra $\mathcal{A}$ has p-Drazin inverse provided that there exists $b \in comm(a)$ such that $b = b^2 a, a^k - a^{k+1}b \in \mathcal{A}^{rad}$ for some $k \in \mathbb{N}$. The preceding b is unique if it exists. It is denoted by $a^\ddagger$ (see [12]). We now derive

Theorem 3.4. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c, d \in \mathcal{A}$ satisfying*

$$(ac)^2 = (db)(ac),$$
$$(db)^2 = (ac)(db).$$

Then $ac \in \mathcal{A}^\ddagger$ if and only if $bd \in \mathcal{A}^\ddagger$. In this case, $(bd)^\ddagger = b[(ac)^\ddagger]^3 dbd$.

Proof. $\Longrightarrow$ Since $ac \in \mathcal{A}^{\ddagger}$, we have $ac \in \mathcal{A}^{d}$. In light of Theorem 3.1, $bd \in \mathcal{A}^{d}$ and $(bd)^{d} = b[(ac)^{\ddagger}]^{3}dbd$. Assume that $[ac-(ac)^{2}(ac)^{\ddagger}]^{m} \in \mathcal{A}^{rad}$ for some $m \in \mathbb{N}$. As in the proof of Corollary 3.3, we have

$$\begin{array}{rcl}[bd-(bd)^{2}(bd)^{d}]^{m+2} & = & b[ac-ac^{2}(ac)^{d}]^{m}(dbd)\\ & \in & \mathcal{A}^{rad}.\end{array}$$

Therefore

$$(bd)^{\ddagger} = b[(ac)^{\ddagger}]^{3}dbd,$$

as asserted.

$\Longleftarrow$ By virtue of [12, Theorem 3.6], $db \in \mathcal{A}^{\ddagger}$. Then we have $ca \in \mathcal{A}^{\ddagger}$ by the discussion above. So the theorem is true by [12, Theorem 3.6]. □

Corollary 3.5. *Let $\mathcal{A}$ be a Banach algebra, and let $a,b,c \in \mathcal{A}$. If $(aba)b = (aca)b, (aba)c = (aca)c$, then $ac \in \mathcal{A}^{\ddagger}$ if and only if $ba \in \mathcal{A}^{\ddagger}$. In this case, $(ba)^{\ddagger} = b[(ac)^{\ddagger}]^{3}aba$:*

Proof. This is obvious by choosing $d = a$ in Theorem 3.4. □

The following example illustrates that Theorem 3.4 is not a trivial generalization of [7, Theorem 4.1].

Example 3.6.

Let $\mathcal{A} = M_4(\mathbb{C})$. Choose

$$a=b=c=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}, d=\begin{pmatrix}0&2&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}\in M_4(\mathbb{C}).$$

Then

$$(ac)^{2} = (db)(ac) = 0,$$
$$(db)^{2} = (ac)(db) = 0.$$

We see that ac and bd are nilpotent matrices and so have p-Drazin inverses. In this case,

$$acd=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\neq\begin{pmatrix}0&0&0&2\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}=dbd.$$

4. Applications

Let X be a Banach space, and let $\mathcal{L}(X)$ denote the set of all bounded linear operators from the Banach space X to itself, and let $A \in \mathcal{L}(X)$. The Drazin spectrum $\sigma_D(A)$ and g-Drazin spectrum $\sigma_d(A)$ are defined by

$$\sigma_D(A) = \{\lambda \in \mathbb{C} \mid \lambda I - A \notin \mathcal{L}(X)^{D}\};$$
$$\sigma_d(A) = \{\lambda \in \mathbb{C} \mid \lambda I - A \notin \mathcal{L}(X)^{d}\}.$$

For the further use, we now record the following generalized Jacobson's lemma (see [9]).

Lemma 4.1. *Let R be a ring, and let $a,b,c,d \in R$ satisfying*

$$(ac)^{2} = (db)(ac), (db)^{2} = (ac)(db);$$
$$b(ac)a = b(db)a, c(ac)d = c(db)d.$$

Then $1-ac \in R^{-1}$ if and only if $1-bd \in R^{-1}$. In this case,

$$(1-bd)^{-1} = [1-b(1-ac)^{-1}(acd-dbd)][1+b(1-ac)^{-1}d].$$

Proof. $\Longrightarrow$ Let $s = (1 - ac)^{-1}$. Then $s(1 - ac) = 1 = (1 - ac)s$, and so $1 - s = -sac = -acs$. We check that

$$\begin{array}{rcl}(1 + bsd)(1 - bd) & = & 1 - b(1 - s)d - bsdbd \\ & = & 1 + bsacd - bsdbd \\ & = & 1 + bs(acd - dbd).\end{array}$$

Hence,

$$\begin{array}{rl} & [1 - bs(acd - dbd)](1 + bsd)(1 - bd) \\ = & 1 - bs(acd - dbd)bs(acd - dbd) \\ = & 1.\end{array}$$

Also we check that

$$\begin{array}{rcl}(1 - bd)(1 + bd + bacsd) & = & 1 - bdbd + b(1 - db)acsd \\ & = & 1 - b[db(1 - ac) - (1 - db)ac]sd \\ & = & 1 - b(db - ac)sd;\end{array}$$

hence, we have

$$\begin{array}{rl} & (1 - bd)(1 + bd + bacsd)[1 + b(db - ac)sd] \\ = & 1 - b(db - ac)sdb(db - ac)sd \\ = & 1 - b(db - ac)sdb(db - ac)(1 + acs)d \\ = & 1 - b(db - ac)sdb(db - ac)d \\ = & 1.\end{array}$$

That is, $1 - bd$ is right and left invertible. Obviously, the left and right inverses of $1 - bd$ coincide with each other. Therefore

$$(1 - bd)^{-1} = [1 - bs(acd - dbd)](1 + bsd),$$

as desired.

$\Longleftarrow$ In light of [9, Lemma 1.4], $1 - db \in R^{-1}$. Applying the discussion above, we see that $1 - ca \in R^{-1}$. By using [7, Theorem 2.1] again, $1 - ac \in R^{-1}$, as asserted. □

We have at our disposal all the information necessary to prove the following.

Theorem 4.2. *Let $A, B, C, D \in \mathcal{L}(X)$ such that*

$$(AC)^2 = (DB)(AC), (DB)^2 = (AC)(DB);$$
$$B(AC)A = B(DB)A, C(AC)D = C(DB)D.$$

then

$$\sigma_d(BD) = \sigma_d(AC).$$

Proof. Case 1. $0 \in \sigma_d(BD)$. Then $BD \notin \mathcal{L}(X)^d$. In view of Theorem 2.2, $AC \notin \mathcal{L}(X)^d$. Thus $0 \in \sigma_d(AC)$.

Case 2. $0 \notin \lambda \in \sigma_d(BD)$. Then $\lambda \in acc\sigma(BD)$; hence,

$$\lambda = \lim_{n\to\infty}\{\lambda_n \mid \lambda_n I - BD \notin \mathcal{L}(X)^{-1}\}.$$

Let $\lambda_n \neq 0$. Then $I - B(\frac{1}{\lambda_n}D) \in \mathcal{L}(X)^{-1}$. By hypothesis, we have

$$(\tfrac{1}{\lambda_n}AC)^2 = (\tfrac{1}{\lambda_n}DB)(\tfrac{1}{\lambda_n}AC), (\tfrac{1}{\lambda_n}DB)^2 = (\tfrac{1}{\lambda_n}AC)(\tfrac{1}{\lambda_n}DB);$$
$$B(\tfrac{1}{\lambda_n}AC)\tfrac{1}{\lambda_n}A = B(\tfrac{1}{\lambda_n}DB)\tfrac{1}{\lambda_n}A, C(\tfrac{1}{\lambda_n}AC)\tfrac{1}{\lambda_n}D = C(\tfrac{1}{\lambda_n}DB)\tfrac{1}{\lambda_n}D.$$

In light of Lemma 4.1, we have $I - (\frac{1}{\lambda_n}A)C \notin \mathcal{L}(X)^{-1}$. Then we check that

$$\lambda = \lim_{n\to\infty}\{\lambda_n \mid \lambda_n I - AC \notin \mathcal{L}(X)^{-1}\} \in acc\sigma(AC) = \sigma_d(AC).$$

Therefore $\sigma_d(BD) \subseteq \sigma_d(AC)$. Analogously, we have $\sigma_d(AC) \subseteq \sigma_d(BD)$, the result follows. □

Corollary 4.3. *Let $A, B, C \in \mathcal{L}(X)$. If $(ABA)B = (ACA)B, (AB\,A)C = (ACA)C$, then*

$$\sigma_d(AC) = \sigma_d(BA).$$

Proof. By choosing $D = A$ in Theorem 4.2, we complete the proof. □

For the Drazin spectrum $\sigma_D(a)$, we now derive

Theorem 4.4. *Let $A, B, C, D \in \mathcal{L}(X)$ such that*

$$(AC)^2 = (DB)(AC), (DB)^2 = (AC)(DB);$$
$$B(AC)A = B(DB)A, C(AC)D = C(DB)D.$$

then

$$\sigma_D(BD) = \sigma_D(AC).$$

Proof. By virtue of Theorem 2.5, $AC \in \mathcal{L}(X)^D$ implies that $BD \in \mathcal{L}(X)^D$. This completes the proof by [13, Theorem 3.1]. □

A bounded linear operator $T \in \mathcal{L}(X)$ is Fredholm operator if $dimN(T)$ and $codimR(T)$ are finite, where $N(T)$ and $R(T)$ are the null space and the range of T respectively. For each nonnegative integer n define $T_{|n|}$ to be the restriction of T to $R(T^n)$. If for some n, $R(T^n)$ is closed and $T_{|n|}$ is a Fredholm operator then T is called a B-Fredholm operator. The B-Fredholm spectrum of T are defined by

$$\sigma_{BF}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not a B-Fredholm operator}\}.$$

Corollary 4.5. *Let $A, B, C \in \mathcal{L}(X)$ such that*

$$(ABA)B = (ACA)B, (ABA)C = (ACA)C,$$

then

$$\sigma_{BF}(AC) = \sigma_{BF}(BA).$$

Proof. Let $\pi : \mathcal{L}(X) \to \mathcal{L}(X)/F(X)$ be the canonical map and $F(X)$ be the ideal of finite rank operators in $\mathcal{L}(X)$. As is well known, $T \in \mathcal{L}(X)$ is B-Fredholm if and only if $\pi(T)$ has Drazin inverse. By assumption, we have

$$\pi(A)\pi(B)\pi(A)\pi(B) = \pi(A)\pi(C)\pi(A)\pi(B),$$
$$\pi(A)\pi(B)\pi(A)\pi(C) = \pi(A)\pi(C)\pi(A)\pi(C).$$

The corollary is therefore established by Theorem 4.4. □

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