Filomat 35:8 (2021), 2565–2572
https://doi.org/10.2298/FIL2108565V

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Variants of Shadowing Properties for Iterated Function Systems on Uniform Spaces

Radhika Vasisht[a], Mohammad Salman[a], Ruchi Das[a]

[a]*Department of Mathematics, University of Delhi, Delhi-110007, India*

Abstract. In this paper, the notions of topological shadowing, topological ergodic shadowing, topological chain transitivity and topological chain mixing are introduced and studied for an iterated function system (IFS) on uniform spaces. It is proved that if an IFS has topological shadowing property and is topological chain mixing, then it has topological ergodic shadowing and it is topological mixing. Moreover, if an IFS has topological shadowing property and is topological chain transitive, then it is topologically ergodic and hence topologically transitive. Also, these notions are studied for the product IFS on uniform spaces.

1. Introduction

An iterated function system (IFS), introduced by Hutchinson [10] and later popularized by Barnsley [4], is the semigroup action generated by a finite collection of continuous self maps on a metric space. In the recent years many researchers have worked on IFS depicting its applications in various fields including image processing, stochastic growth models, description of fractals and in the theory of random dynamical systems [8, 9]. Recently, shadowing property, average shadowing property and other variants of shadowing property on iterated function systems have been explored by many authors on metric spaces [3, 14, 17, 26–28].

The study of uniform spaces was initiated by A. Weil in order to provide certain metric structures including Cauchy sequences, completeness and uniform continuity with topological properties [25]. The study of shadowing property began with the works of Anosov and Bowen [2, 5]. In 2016, Shah et al. obtained that a dynamical system on a totally bounded uniform space which is topologically shadowing, mixing, and topologically expansive has the topological specification property [20]. Later in 2018, Das and Das studied variants of topological shadowing and specification property on uniform spaces [7]. Pseudo-orbits play an important role in detecting mixing and recurrent behaviours of a dynamical system which may not be evident from the actual orbits. One of the useful applications of pseudo-orbits is in neuroscience. The theory of shadowing or pseudo-orbit tracing has importance in qualitative theory and has various applications

2020 *Mathematics Subject Classification*. Primary 37B05; Secondary 37B02
Keywords. Iterated function system, uniform space, topological shadowing, topological transitivity
Received: 23 June 2020; Revised: 16 December 2020; Accepted: 02 January 2021
Communicated by Ljubiša D.R. Kočinac
Corresponding author: Mohammad Salman
The first author is funded by Government of India, Ministry of Science and Technology No: DST/INSPIRE Fellowship/[IF160750]. The second author is supported by Ministry of Minority Affairs, Government of India, Maulana Azad National Fellowship (F.No. 61-1/2019(SA-III)).
Email addresses: radhika.vasisht92@gmail.com (Radhika Vasisht), salman25july@gmail.com (Mohammad Salman), rdasmsu@gmail.com (Ruchi Das)

in numerical analysis [16]. Shadowing and its variants have been studied by various researchers for both autonomous and non-autonomous dynamical systems [1, 6, 12, 13, 15, 18, 21–23]. In this paper, we initiate the study of various types of shadowing properties for IFS on non-metrizable uniform spaces.

The paper is organized as follows. In Section 2, we give the prerequisites required for the remaining sections of the paper. In Section 3, we introduce the notions of topological shadowing, topological ergodic shadowing, topological chain transitivity and topological chain mixing for an iterated function system (IFS) on uniform spaces. We show that topological shadowing (topological ergodic shadowing) on an IFS is iteration invariant. It is proved that if an IFS has topological shadowing property and is topological chain mixing, then it has topological ergodic shadowing and is topological mixing. Whereas if an IFS has topological shadowing property and is topological chain transitive, then it is topologically ergodic and hence topologically transitive. In Section 4, we study above shadowing properties on product of iterated function systems on uniform spaces. It is proved that IFS $\mathcal{F}$ and $\mathcal{G}$ have topological shadowing property (topological ergodic shadowing) if and only if $\mathcal{F}\times\mathcal{G}$ has so. Moreover, we show that $\mathcal{F}$ and $\mathcal{G}$ are topological chain mixing if and only if $\mathcal{F}\times\mathcal{G}$ is topological chain mixing.

2. Preliminaries

Throughout the paper, let $\mathbb{Z}^+ = \{0,1,2,\ldots\}$ and $\mathbb{N} = \{1,2,3,\ldots\}$. Let X be a nonempty set and $\Delta_X = \{(x,x) : x \in X\}$ be the diagonal of $X\times X$. Let $E \subseteq X\times X$, then E is *symmetric* if $E = E^{-1}$, where $E^{-1} = \{(x_2,x_1) : (x_1,x_2) \in E\}$ is the *inverse* of E. For any $E, F \subseteq X\times X$, we consider the *composite* $(E \circ F)$ which is defined as $\{(x_1,x_2) \in X\times X :$ there exists an $x_3 \in X$ satisfying $(x_1,x_3) \in E$ and $(x_3,x_2) \in F\}$.

Definition 2.1. ([11]) A *uniform structure* on X is a non empty collection $\mathscr{D}$ of subsets of $X\times X$ satisfying the following:

(1) If $D \in \mathscr{D}$, then $D^{-1} \in \mathscr{D}$ and $\Delta_X \subseteq D$;

(2) If $D \in \mathscr{D}$ and $D \subseteq E \subseteq X\times X$, then $E \in \mathscr{D}$;

(3) If $D, E \in \mathscr{D}$, then $D\cap E \in \mathscr{D}$;

(4) For any $D \in \mathscr{D}$, there exists an $E \in \mathscr{D}$ such that $E\circ E \subseteq D$.

The pair $(X,\mathscr{D})$ is said to be a *uniform space* and the members of $\mathscr{U}$ are called as *entourages*. For a uniformity $\mathscr{D}$, there exists a *uniform topology* which is denoted by $|\mathscr{D}|$ on X and characterized by the neighborhoods of any $x \in X$ consisting of the sets $E[x_1] = \{x_2 \in X : (x_1,x_2) \in E\}$, where $E \in \mathscr{D}$ varies through all entourages of $(X,\mathscr{D})$. For a uniform space $(X,\mathscr{D})$, $f : X \to X$ is called *uniformly continuous* if $(f\times f)^{-1}(\mathscr{D}) \subseteq \mathscr{D}$.

Let $(X,\mathscr{D})$ be a nontrivial uniform space and $f_\mu : X \to X$ be a uniformly continuous map, for each $\mu \in \Gamma$. If Γ is a nonempty finite set, then $\mathcal{F} = \{(X,\mathscr{D}); f_\mu : \mu \in \Gamma\}$ is said to be an *iterated function system* (IFS). For $\Sigma = \Gamma^{\mathbb{Z}^+}$, $\sigma = \sigma_0\sigma_1\sigma_2\ldots \in \Sigma$ and any $n \in \mathbb{N}$, we consider

$$f_\sigma^{(n)}(x) = f^{(n)}_{\sigma_0\ldots\sigma_{n-1}} = f_{\sigma_{n-1}} \circ f_\sigma^{(n-1)} \text{ and } f_\sigma^{(0)} := id,$$

for any $x \in X$. The *orbit* of a point $x_0 \in X$ under the IFS $\mathcal{F}$ is the set,

$$O_{\mathcal{F}}(x_0) = \{f_\sigma^{(m)}(x_0) : m \in \mathbb{Z}^+, \text{ for some } \sigma \in \Sigma\}.$$

For any $k \in \mathbb{N}$, we consider the k-th iterate of the IFS $\mathcal{F}$ as follows:

$$\mathcal{F}^k = \{(X,\mathscr{D}); f^{(k)}_{\sigma_0\ldots\sigma_{k-1}} = f_{\sigma_{k-1}} \circ f_\sigma^{(k-1)} : \sigma_0,\ldots,\sigma_{k-1} \in \Gamma\}.$$

Definition 2.2. ([2]) An IFS $\mathcal{F} = \{(X,\mathscr{D}); f_\mu : \mu \in \Gamma\}$ is said to be *topologically transitive*, if for any pair of nonempty open sets $\dot{U}_1, \dot{U}_2 \subseteq X$, there exists a $\sigma \in \Sigma$ and an $n \in \mathbb{N}$ satisfying $f_\sigma^{(n)}(\dot{U}_1) \cap \dot{U}_2 \neq \emptyset$.

For any two nonempty open sets $\dot{U}_1$, $\dot{U}_2 \subseteq X$, we shall denote

$$N_{\mathcal{F}}(\dot{U}_1, \dot{U}_2) = \{m \in \mathbb{N} : f_\sigma^{(m)}(\dot{U}_1) \cap \dot{U}_2 \neq \emptyset, \text{ for some } \sigma \in \Sigma\}.$$

For any set $A \subseteq \mathbb{Z}^+$, the *upper density* and *lower density* of A, respectively are given by

$$\overline{d}(A) := \limsup_{m\to\infty} \frac{1}{m} |A \cap \{0, 1, \ldots, m-1\}|, \ \underline{d}(A) := \liminf_{m\to\infty} \frac{1}{m} |A \cap \{0, 1, \ldots, m-1\}|.$$

If $\overline{d}(A) = \underline{d}(A) = d(A)$, then $d(A)$ is called the *density* of A.

Definition 2.3. ([27]) An IFS $\mathcal{F} = \{(X, \mathscr{D}); f_\mu : \mu \in \Gamma\}$ is

1. *topologically ergodic*, if for any nonempty open sets $\dot{U}_1$, $\dot{U}_2 \subseteq X$, $N_{\mathcal{F}}(\dot{U}_1, \dot{U}_2)$ has positive upper density;
2. *topologically mixing*, if for any nonempty open sets $\dot{U}_1$, $\dot{U}_2 \subseteq X$, $N_{\mathcal{F}}(\dot{U}, \dot{V})$ is cofinite.

Clearly, we have

$$\text{Topological mixing} \implies \text{Topological ergodicity} \implies \text{Topological transitivity.}$$

3. Shadowing Properties for IFS

In this section, we introduce the notions of topological shadowing, topological ergodic shadowing, topological chain transitivity and topological chain mixing for an iterated function system (IFS) on uniform spaces. We show that topological shadowing (topological ergodic shadowing) on an IFS on uniform spaces is iteration invariant. It is proved that if an IFS on uniform spaces has topological shadowing property and is topological chain mixing, then it has topological ergodic shadowing and is topologically mixing. Whereas if an IFS on uniform spaces has topological shadowing property and is topological chain transitive, then it is topologically ergodic and hence is topologically transitive.

Let $\mathcal{F} = \{(X, \mathscr{D}); f_\mu : \mu \in \Gamma\}$ be an IFS, $A, B \in \mathscr{D}$ and $\sigma = \sigma_0\sigma_1\sigma_2 \ldots \in \Sigma$. A sequence $\{x_k\}_{k=0}^{\infty} \in X$ is an *A-σ-pseudo-orbit* of $\mathcal{F}$ if $(f_{\sigma_k}(x_k), x_{k+1}) \in A$, for any $k \geq 0$. An A-σ-pseudo orbit is said to be *B-traced* by a point $y \in X$ if $(f_\sigma^{(k)}(y), x_k) \in B$, for all $k \in \mathbb{Z}^+$.

Definition 3.1. An IFS $\mathcal{F} = \{(X, \mathscr{D}); f_\mu : \mu \in \Gamma\}$ is said to have *topological shadowing property* if for any entourage $B \in \mathscr{D}$ of $X \times X$, there is an entourage $A \in \mathscr{D}$ such that every A-σ-pseudo orbit is B-traced by some $y \in X$, for some $\sigma = \sigma_0\sigma_1\sigma_2 \ldots \in \Sigma$.

Let $\mathcal{F} = \{(X, \mathscr{D}); f_\mu : \mu \in \Gamma\}$ be an IFS, $\sigma = \sigma_0\sigma_1\sigma_2 \ldots \in \Sigma$ and $\zeta = \{x_k\}_{k=0}^{\infty}$ be any sequence in X, then for any entourage $B \in \mathscr{D}$ and $y \in X$, we use the notations as follows:

$$\wedge(\zeta, \mathcal{F}, B) = \{k \in \mathbb{Z}^+ | (f_{\sigma_k}(x_k), x_{k+1}) \in B\}, \ \wedge^c(\zeta, \mathcal{F}, B) = \{k \in \mathbb{Z}^+ | (f_{\sigma_k}(x_k), x_{k+1}) \notin B\};$$

$$\wedge(\zeta, y, \mathcal{F}, B) = \{k \in \mathbb{Z}^+ | (f_\sigma^{(k)}(y), x_k) \in B\}, \ \wedge^c(\zeta, y, \mathcal{F}, B) = \{k \in \mathbb{Z}^+\} | (f_\sigma^{(k)}(y), x_k) \notin B\}.$$

Let $\mathcal{F} = \{(X, \mathscr{D}); f_\mu : \mu \in \Gamma\}$ be an IFS, A, $B \in \mathscr{D}$ and $\sigma = \sigma_0\sigma_1\sigma_2 \ldots \in \Sigma$. A sequence $\zeta = \{x_k\}_{k=0}^{\infty}$ is said to be *ergodic A-σ-pseudo-orbit* provided that $\wedge^c(\zeta, \mathcal{F}, B)$ has zero density. An ergodic A-σ-pseudo-orbit ζ is *ergodically B-traced* by $y \in X$ if $\wedge^c(\zeta, y, \mathcal{F}, B)$ has zero density.

Definition 3.2. A uniform IFS $\mathcal{F} = \{(X, \mathscr{D}); f_\mu : \mu \in \Gamma\}$ is said to have *topological ergodic shadowing property* if for any entourage $B \in \mathscr{D}$ there is an $A \in \mathscr{D}$ and a $\sigma = \sigma_0\sigma_1 \ldots \in \Sigma$ satisfying that every ergodic A-σ-pseudo orbit is B-traced by some point of X.

For $A \in \mathscr{D}$ and uniform IFS $\mathcal{F} = \{(X, \mathscr{D}); f_\mu : \mu \in \Gamma\}$, we say that there is an *A-chain* from u to v in X if there is a finite sequence $u_0 = u, \ldots, u_m = v$, such that for every $k \in \{0, \ldots, m-1\}$, there exists a $\sigma_k \in \Gamma$ with $(f_{\sigma_k}(u_k), u_{k+1}) \in A$.

Definition 3.3. An IFS $\mathcal{F} = \{(X, \mathscr{D}); f_\mu : \mu \in \Gamma\}$ is said to be

(1) *topologically chain transitive*, if for any $A \in \mathscr{D}$ and for any $u, v \in X$ there is an A-chain $\{u_k\}_{k=0}^m$ from u to v;

(2) *topologically chain mixing*, if for any $A \in \mathscr{D}$ and for any $u, v \in X$ there exists an $M \in \mathbb{N}$ such that for any $m \geq M$, there exists an A-chain $\{u_k\}_{k=0}^m$ from u to v.

The following theorem is the topological version of [24, Proposition 3.2] done for metric spaces.

Theorem 3.1. *Let $\mathcal{F} = \{(X, \mathscr{D}); f_\mu : \mu \in \Gamma\}$ be a uniform IFS, then the IFS $\mathcal{F}$ has topological shadowing property (topological ergodic shadowing property) if and only if $\mathcal{F}^m$ has topological shadowing property (topological ergodic shadowing property), for every $m \in \mathbb{N}$.*

Proof. Let $\mathcal{F}$ have topological shadowing property and $E \in \mathscr{D}$ be arbitrary. Therefore, by topological shadowing property of $\mathcal{F}$ there exist an $A \in \mathscr{D}$ and a $\sigma = \sigma_0\sigma_1\sigma_2 \ldots \in \Sigma$ such that every A-σ-pseudo-orbit $\{x_k\}_{k=0}^\infty$ is B-traced by some $y \in X$. Let $m \in \mathbb{N}$ and $\{y_k\}_{k=0}^\infty$ be any A-σ-pseudo-orbit for $\mathcal{F}^m$, then for any $k \geq 0$, there exists a $g_{\sigma_k} \in \mathcal{F}^m$ such that $(g_{\sigma_k}(y_k), y_{k+1}) \in B$, where $g_{\sigma_k} = f_{\sigma^k_{m-1}} \circ \cdots \circ f_{\sigma^k_0}$. Put $x_{mk} = y_k$ and $x_{mk+j} = f_{\sigma^k_{j-1}} \circ \cdots \circ f_{\sigma^k_0}(y_k)$, for every $k \geq 0$ and for every $0 < j < m$. Therefore, using the fact that $(x, x) \in A$, for every $x \in X$, we get that

$$\{x_k\}_{k=0}^\infty = \{y_0, f_{\sigma_0}(y_0), \ldots, f_{\sigma_{k-1}} \circ \cdots \circ f_{\sigma_0}(y_0), y_1, f_{\sigma_0}(y_1), \ldots, f_{\sigma_{m-1}} \circ \cdots \circ f_{\sigma_0}(y_1), \ldots\}$$

is an A-σ-pseudo-orbit for $\mathcal{F}$ and hence there exists a $y \in X$ such that $(f_\sigma^{(j)}(y), x_j) \in B$, for every $j \geq 0$. In particular, for $j = km$, we get that $(f_\sigma^{(km)}(y), x_{km}) \in B$, that is, $(f_\sigma^{(km)}(y), y_k) \in B$, for every $k \geq 0$. Thus, $\mathcal{F}^m$ has topological shadowing property, for any $m \in \mathbb{N}$. Conversely, if $\mathcal{F}^m$ has topological shadowing property, for every $m \in \mathbb{N}$, then in particular for $m = 1$, we get that $\mathcal{F}^1 = \mathcal{F}$ has topological shadowing property.

Next suppose that $\mathcal{F}$ has topological ergodic shadowing property and $B' \in \mathscr{D}$ be arbitrary. Therefore, by topological ergodic shadowing property of the uniform IFS $\mathcal{F}$ there exist an $A' \in \mathscr{D}$ and a $\sigma = \sigma_0\sigma_1\sigma_2 \ldots \in \Sigma$ such that every ergodic A'-σ-pseudo-orbit $\zeta = \{u_k\}_{k=0}^\infty$ is ergodically B'-traced by some $w \in X$. Let $m \in \mathbb{N}$ and if $\eta = \{v_k\}_{k=0}^\infty$ is an ergodic A'-σ-pseudo-orbit for $\mathcal{F}^m$, that is, density of $\wedge^c(\eta, \mathcal{F}^m, B')$ is zero, then as above we get that

$$\zeta = \{v_0, f_{\sigma_0}(v_0), \ldots, f_{\sigma_{m-1}} \circ \cdots \circ f_{\sigma_0}(v_0), v_1, f_{\sigma_0}(v_1), \ldots, f_{\sigma_{m-1}} \circ \cdots \circ f_{\sigma_0}(v_1), \ldots\}$$

is an ergodic A'-σ-pseudo-orbit for $\mathcal{F}$ which is ergodically B'-traced by $w \in X$. Thus, density of $\wedge^c(\zeta, w, \mathcal{F}, B')$ is zero and proceeding as above we get that $\wedge^c(\eta, w, \mathcal{F}^m, B')$ has zero density. Therefore, $\mathcal{F}^m$ has topological ergodic shadowing property, for any $m \in \mathbb{N}$. Conversely, if $\mathcal{F}^m$ has topological ergodic shadowing property, for any $m \in \mathbb{N}$, then in particular for $m = 1$, we get that $\mathcal{F}^1 = \mathcal{F}$ has topological ergodic shadowing property. □

The following theorem generalizes [24, Theorem 1.2] from metric spaces to non-metrizable uniform spaces.

Theorem 3.2. *Let $\mathcal{F} = \{(X, \mathscr{D}); f_\mu : \mu \in \Gamma\}$ be a uniform IFS. If $\mathcal{F}$ is topologically chain mixing and has topological shadowing property, then*

(1) $\mathcal{F}$ has topological ergodic shadowing property;

(2) $\mathcal{F}$ is topologically mixing.

Proof. (1) Let $\mathcal{F} = \{(X, \mathscr{D}); f_\mu : \mu \in \Gamma\}$ be a uniform IFS with topological shadowing property. Let B be any symmetric neighbourhood of Δ_X and A be an entourage corresponding to B as in the definition of topological shadowing. Let

$$\zeta = \{x_0, x_1, x_2, x_3 \ldots, x_{m_0}, x_{m_0+1}, x_{m_0+2}, \ldots, x_{m_1}, x_{m_1+1}, \ldots, x_{m_2} \ldots\}$$

be an A-ergodic pseudo-orbit for $\mathcal{F}$. By topological chain mixing of $\mathcal{F}$, we have a natural number M such that for any $x, y \in X$ there exists a A-chain $\{y_i\}_{i=0}^{n}$ with $y_0 = x$ and $y_n = y$, with $(f_{\sigma_i}(x_i), x_{i+1}) \in A$, for any $0 \leq i \leq n-1$ and for any $n \geq M$. So for any $k \in \mathbb{N}$, we can consider an A-chain from x_{m_k} to x_{m_k+N-1} and replace $\{x_{m_k}, x_{m_k+1}, \ldots, x_{m_k+N-1}\}$ in ζ by this new A-chain. Therefore, one can note that in this way, ζ becomes a A-pseudo-orbit of $\mathcal{F}$ and hence by topological shadowing property of $\mathcal{F}$, η is B-shadowed by some $y \in X$. Since $\{m_i\}_{i\in\mathbb{N}}$ has zero density, the sequence $\bigcup_{i=1}^{\infty}\{m_i, m_{i+1}, m_{i+2}, \ldots, m_{i+N-1}\}$ also has zero density. Thus, ζ is ergodically shadowed by y.

(2) Let $u, v \in X$ be arbitrary and $\dot{U}, \dot{V}$ be any two nonempty subsets of X such that $u \in \dot{U}$ and $v \in \dot{V}$. Let $B \in \mathscr{D}$ be any entourage such that $B[u] \subseteq \dot{U}$ and $B[v] \subseteq \dot{V}$. Then by topological shadowing property of $\mathcal{F}$, we get an existence of an entourage $A \in \mathscr{D}$ such that every A-σ-pseudo-orbit is B-traced by some $x \in X$, for some $\sigma = \sigma_0\sigma_1\sigma_2\ldots \in \Sigma$. In particular, for $x = v$ and for very large $m \in \mathbb{N}$, by using topological chain mixing, we get an A-σ-chain of length m from u to v, say $\{u_k\}_{k=0}^{m}$ with $u_0 = u$ and $u_m = v$. Therefore, $(f_{\sigma_k}(u_k), u_{k+1}) \in A$, for $k \in \{0, 1, \ldots, m-1\}$, where m is sufficiently large and hence $\{u_k\}_{k=0}^{m}$ is B-traced by v, that is, $(f_\sigma^{(k)}(v), u_k) \in E$, for every $k \in \{0, 1, \ldots, m\}$. Consequently, we get that $v \in A[u] \subseteq \dot{U}$ and $f_\sigma^{(m)}(v) \in A[v] \subseteq \dot{V}$. Thus, $f_\sigma^{(m)}(\dot{U}) \cap \dot{V} \neq \emptyset$, for very large $m \in \mathbb{N}$ and some $\sigma \in \Sigma$ which implies that the IFS $\mathcal{F} = \{(X, \mathscr{D}); f_\mu : \mu \in \Gamma\}$ is topological mixing. □

In [3, Theorem 2.3] authors have proved that if an IFS on a metric space is chain transitive and has the shadowing property, then it is transitive. We generalize this result to non-metrizable uniform spaces and prove its strong version.

Theorem 3.3. *Let $\mathcal{F} = \{(X, \mathscr{D}); f_\mu : \mu \in \Gamma\}$ be a uniform IFS. If $\mathcal{F}$ is topologically chain transitive and has topological shadowing property, then it is topologically ergodic.*

Proof. Let $u, v \in X$ be arbitrary and $\dot{U}, \dot{V}$ be any two nonempty sets such that $u \in \dot{U}$ and $v \in \dot{V}$. Let $B \in \mathscr{D}$ be any entourage such that $B[u] \subseteq \dot{U}$ and $B[v] \subseteq \dot{V}$. Then by topological shadowing property of $\mathcal{F}$, there exists an entourage $A \in \mathscr{D}$ such that every A-σ-pseudo-orbit is B-traced by some $x \in X$, for some $\sigma = \sigma_0\sigma_1\sigma_2\ldots \in \Sigma$. Let $\{u_i\}_{i=0}^{m}$ and $\{v_j\}_{j=0}^{n}$ be two A-chains between u and v with $u_0 = u$, $u_m = v$ and $v_0 = v$ and $v_n = u$. Then

$$\{y_k\}_{k=0}^{\infty} = \{u, u_1, u_2, \ldots, u_{m-1}, v, v_1, v_2, \ldots, u, u_1, u_2, \ldots, v, \ldots\}$$

is an A-σ'-pseudo-orbit, for some $\sigma' \in \Sigma$ (dependent on σ) which is B-traced by $x \in X$. Thus, $(f_{\sigma'}^{(k)}(x), y_k) \in B$, for all $k \geq 0$ and consequently we get that $(x, u) \in B$, $(f_{\sigma'}^{(m)}(x), v) \in B$, $(f_{\sigma'}^{(m+n)}(x), u) \in B, \ldots, (f_{\sigma'}^{(k(m+n)+m)}(x), v) \in B$, for every $k \in \{1, 2, 3, \ldots\}$. Thus, $N_{\mathcal{F}}(\dot{U}, \dot{V}) \geq (m+n)\mathbb{N} + m$ and hence

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{|N_{\mathcal{F}}(\dot{U}, \dot{V}) \cap \{0, 1, \ldots, t-1\}|}{t} \\
&\geq \limsup_{t\to\infty} \frac{|((m+n)\mathbb{N} + m) \cap \{0, 1, \ldots, (m+n)t + m - 1\}|}{(m+n)t + m} \\
&\geq \limsup_{t\to\infty} \frac{t}{(m+n)t + m} > 0.
\end{aligned}$$

Therefore, the upper density of $N_{\mathcal{F}}(\dot{U}, \dot{V})$ is positive implying that the IFS $\mathcal{F}$ is topologically ergodic. Since topological ergodicity implies topological transitivity, therefore $\mathcal{F}$ is topologically transitive. Also, it can be directly seen that topological shadowing property and topological chain transitivity imply topological transitivity.

Let $\dot{U}'$ and $\dot{V}'$ be two nonempty open subsets of X containing points y_1 and y_2 respectively. Choose $B' \in \mathscr{D}$ such that $B'[y_1] \subseteq \dot{U}'$ and $B'[y_2] \subseteq \dot{V}'$. Since $\mathcal{F}$ is chain transitive, therefore there exists a finite A-chain $\{x_k\}_{k=0}^{n}$ with $x_0 = y_1$ and $x_n = y_2$.

Define $\{z_k\}_{k=0}^{\infty}$ as

$$\{z_k\}_{k=0}^{\infty} = \begin{cases} z_k = y_1, & \text{for } k = 0 \\ z_k = x_k, & \text{for } 0 < k < n \\ z_k = y_2, & \text{for } k \geq n. \end{cases}$$

Then clearly z_k is an A-σ''-pseudo-orbit for some $\sigma'' \in \Sigma$ (dependent on σ) for $\mathcal{F}$, so by topological shadowing property of $\mathcal{F}$, there exists a $z \in X$ such that $(f^{(k)}_{\sigma''}(z), z_k) \in B'$, for every $k \geq 0$ which implies that $(f^{(0)}_{\sigma''}(z), y_1) \in B'$. Therefore, we have $z = f^{(0)}_{\sigma''}(z) \in A[y_1] \subseteq \dot{U}'$ and $f^{(n)}_{\sigma''}(z) \in A[y_2] \subseteq \dot{V}'$ implying that $f^{(n)}_{\sigma''}(\dot{U}') \cap \dot{V}' \neq \emptyset$. Thus, the IFS $\mathcal{F} = \{(X, \mathscr{D}); f_\mu : \mu \in \Gamma\}$ is topologically transitive. □

4. Shadowing Properties on Product of IFS

In this section, we study shadowing properties on product of iterated function systems on uniform spaces. It is proved that IFS $\mathcal{F}$ and $\mathcal{G}$ have topological shadowing property (topological ergodic shadowing) if and only if $\mathcal{F} \times \mathcal{G}$ has so. Moreover, it is shown that $\mathcal{F}$ and $\mathcal{G}$ are topologically chain mixing if and only if $\mathcal{F} \times \mathcal{G}$ is topologically chain mixing.

In [3], authors have proved that if an IFS on a metric space; consisting of two maps has topological shadowing property, then their step skew product also has topological shadowing property and vice versa. We generalize this result for iterated function systems on uniform spaces.

Theorem 4.1. *Let $\mathcal{F} = \{(X, \mathscr{D}_1); f_\mu : \mu \in \Gamma\}$ and $\mathcal{G} = \{(Y, \mathscr{D}_2); g_\mu : \mu \in \Gamma\}$ be two uniform IFS. Then IFS $\mathcal{F}$ and $\mathcal{G}$ have topological shadowing property if and only if $\mathcal{F} \times \mathcal{G}$ has topological shadowing property.*

Proof. Suppose the IFS $\mathcal{F}$ and $\mathcal{G}$ have topological shadowing property and let $\mathcal{H} = \mathcal{F} \times \mathcal{G}$. Let B be a symmetric neighbourhood of the diagonal $\Delta_{(X\times Y)}$. Let $B_1 = \{(x_1, x_2) \in X \times X$ such that $(x_1, y_1, x_2, y_2) \in B$ for some $(y_1, y_2) \in Y\}$ and $B_2 = \{(y_1, y_2) \in Y \times Y$ such that $(x_1, y_1, x_2, y_2) \in B$ for some $(x_1, x_2) \in X\}$. Clearly B_1 and B_2 are symmetric neighbourhoods of Δ_X and Δ_Y respectively. As $\mathcal{F}$ and $\mathcal{G}$ have the topological shadowing property there exists an entourage A_1 of $\mathscr{D}_1$ such that every A_1-pseudo orbit is B_1-traced by some point of X and similarly there exists an entourage A_2 of $\mathscr{D}_2$ such that every A_2-pseudo orbit is B_2-traced by some point of Y.
Fix $A = \{(x_1, y_1, x_2, y_2) \in (X \times Y) \times (X \times Y)$ such that $(x_1, x_2) \in A_1$ and $(y_1, y_2) \in A_2$ for some $(x_1, x_2) \in X$ and $(y_1, y_2) \in Y\}$. Let $\{(x_i, y_i)\}_{i=0}^{\infty}$ be an A-pseudo orbit for $\mathcal{F} \times \mathcal{G}$. Then by definition, we have $(f_{\sigma_{i+1}} \times g_{\sigma_{i+1}}(x_i, y_i), (x_{i+1}, y_{i+1})) \in A$ which implies $(f_{\sigma_{i+1}}(x_i), x_{i+1}) \in A_1$ and $(g_{\sigma_{i+1}}(y_i), y_{i+1}) \in A_2$. Since the IFS $\mathcal{F}$ and $\mathcal{G}$ have topological shadowing property, therefore there exist $x \in X$ and $y \in Y$ such that $(f^{(i)}_\sigma(x), x_i) \in B_1$ and $(g^{(i)}_\sigma(y), y_i) \in B_2$ and hence we have $(f^{(i)}_\sigma \times g^{(i)}_\sigma(x, y), (x_i, y_i)) \in B$. Thus, every A-pseudo orbit for $\mathcal{F} \times \mathcal{G}$ is B-traced.

Conversely, suppose $\mathcal{F} \times \mathcal{G}$ has shadowing property. Let B_1 and B_2 be symmetric neighbourhood of Δ_X and Δ_Y respectively. Consider the open set $B = \{(x_1, y_1, x_2, y_2) \in (X \times Y) \times (X \times Y)$ such that $(x_1, x_2) \in B_1$ and $(y_1, y_2) \in B_2\}$ containing the diagonal of $(X \times Y) \times (X \times Y)$. As $\mathcal{F} \times \mathcal{G}$ has topological shadowing property, there exists an entourage A of $\mathscr{D}_1 \times \mathscr{D}_2$ such that every A-pseudo orbit is B-traced by some point of $X \times Y$. Set $A_1 = \{(x, x') \in X \times X$ such that $(x, y, x', y') \in A$ for some $(y, y') \in Y \times Y\}$ and $A_2 = \{(y, y') \in Y \times Y$ such that $(x, y, x', y') \in A$ for some $(x, x') \in X \times X\}$. Let $\{x_i\}_{i=0}^{\infty}$ be an A_1-pseudo-orbit for $\mathcal{F}$ and $\{y_i\}_{i=0}^{\infty}$ be an A_2-pseudo orbit for $\mathcal{G}$. It can be noted that $\{(x_i, y_i)_{i=0}^{\infty}\}$ is an A-pseudo orbit for $\mathcal{F} \times \mathcal{G}$ and therefore by topological shadowing property of $\mathcal{F} \times \mathcal{G}$, there exists a point $(x, y) \in X \times Y$ such that $(f^{(i)}_\sigma \times g^{(i)}_\sigma(x, y), (x_i, y_i)) \in B$. Hence, we have $x \in X$ and $y \in Y$ such that $(f^{(i)}_\sigma(x), x_i) \in B_1$ and $(g^{(i)}_\sigma(y), y_i) \in B_2$. Thus, the IFS $\mathcal{F}$ and $\mathcal{G}$ have topological shadowing property. □

In [19], the author has proved that if an IFS on a metric space has ergodic shadowing property, then its product also has ergodic shadowing property. We generalize this result and also prove its converse for an IFS on uniform spaces.

Theorem 4.2. *Let $\mathcal{F} = \{(X, \mathscr{D}_1); f_\mu : \mu \in \Gamma\}$ and $\mathcal{G} = \{(Y, \mathscr{D}_2); g_\mu : \mu \in \Gamma\}$ be two uniform IFS. The IFS $\mathcal{F}$ and $\mathcal{G}$ have topological ergodic shadowing property if and only if $\mathcal{F} \times \mathcal{G}$ has topological ergodic shadowing property.*

Proof. Suppose the IFS $\mathcal{F}$ and $\mathcal{G}$ have topological ergodic shadowing property and let $\mathcal{H} = \mathcal{F} \times \mathcal{G}$. Let B be a symmetric neighbourhood of the diagonal $\Delta_{(X\times Y)}$. Let $B_1 = \{(x_1, x_2) \in X \times X$ such that $(x_1, y_1, x_2, y_2) \in B$ for some $(y_1, y_2) \in Y\}$ and $B_2 = \{(y_1, y_2) \in Y \times Y$ such that $(x_1, y_1, x_2, y_2) \in B$ for some $(x_1, x_2) \in X\}$. Clearly B_1 and B_2 are symmetric neighbourhoods of Δ_X and Δ_Y respectively. As $\mathcal{F}$ and $\mathcal{G}$ have the topological ergodic shadowing property, there exists an entourage A_1 of $\mathscr{D}_1$ such that every ergodic A_1-pseudo orbit is ergodically B_1-traced by some point of X and similarly there exists an entourage A_2 of $\mathscr{D}_2$ such that every ergodic A_2-pseudo orbit is ergodically B_2-traced by some point of Y.
Fix $A = \{(x_1, y_1, x_2, y_2) \in (X \times Y) \times (X \times Y)$ such that $(x_1, x_2) \in A_1$ and $(y_1, y_2) \in A_2$ for some $(x_1, x_2) \in X$ and $(y_1, y_2) \in Y\}$. Let $\eta = \{(x_i, y_i)\}_{i=0}^{\infty}$ be an ergodic A-pseudo orbit for $\mathcal{F} \times \mathcal{G}$. Then by definition, we have $\wedge^c(\eta, \mathcal{F} \times \mathcal{G}, A)$ has density zero which implies $\wedge^c(\eta_1, \mathcal{F}, A_1)$ and $\wedge^c(\eta_2, \mathcal{G}, A_2)$ both have zero density, where $\eta_1 = \{x_i\}_{i=0}^{\infty}$ and $\eta_2 = \{y_i\}_{i=0}^{\infty}$. Since the IFS $\mathcal{F}$ and $\mathcal{G}$ have topological ergodic shadowing property, therefore there exist $x \in X$ and $y \in Y$ such that $\wedge^c(\eta_1, x, \mathcal{F}, B_1)$ and $\wedge^c(\eta_2, y, \mathcal{G}, B_2)$ both have zero density and hence we have $\wedge^c(\eta, (x, y), \mathcal{F} \times \mathcal{G}, B)$ has density zero. Thus, every ergodic A-pseudo orbit for $\mathcal{F} \times \mathcal{G}$ is ergodically B-traced.

Conversely, suppose $\mathcal{F} \times \mathcal{G}$ has topological ergodic shadowing property. Let B_1 and B_2 be symmetric neighbourhood of Δ_X and Δ_Y respectively. Consider the open set $B = \{(x_1, y_1, x_2, y_2) \in (X \times Y) \times (X \times Y)$ such that $(x_1, x_2) \in B_1$ and $(y_1, y_2) \in B_2\}$ containing the diagonal of $(X \times Y) \times (X \times Y)$. As $\mathcal{F} \times \mathcal{G}$ has topological ergodic shadowing property, there exists an entourage A of $\mathscr{D}_1 \times \mathscr{D}_2$ such that every ergodic A-pseudo orbit is ergodically B-traced by some point of $X\times Y$. Set $A_1 = \{(x, x') \in X\times X$ such that $(x, y, x', y') \in A$ for some $(y, y') \in Y \times Y\}$ and $A_2 = \{(y, y') \in Y \times Y$ such that $(x, y, x', y') \in A$ for some $(x, x') \in X \times X\}$. Let $\{x_i\}_{i=0}^{\infty}$ be an ergodic A_1-pseudo-orbit for $\mathcal{F}$ and $\{y_i\}_{i=0}^{\infty}$ be an ergodic A_2-pseudo orbit for $\mathcal{G}$.
It can be noted that $\{(x_i, y_i)_{i=0}^{\infty}\}$ is an ergodic A-pseudo orbit for $\mathcal{F} \times \mathcal{G}$ and therefore by topological ergodic shadowing property of $\mathcal{F} \times \mathcal{G}$, there exists a point $(x, y) \in X \times Y$ such that $\wedge^c(\eta, (x, y), \mathcal{F} \times \mathcal{G}, B)$ has density zero. Hence, we have $x \in X$ and $y \in Y$ such that $\wedge^c(\eta_1, x, \mathcal{F}, B_1)$ and $\wedge^c(\eta_2, y, \mathcal{G}, B_2)$ both have zero density. Thus, the IFS $\mathcal{F}$ and $\mathcal{G}$ have topological ergodic shadowing property. □

In [27], authors have proved that an IFS on a metric space has topological chain mixing if and only if the corresponding step skew product has topological chain mixing. We generalize this result for an IFS and its product on uniform spaces.

Theorem 4.3. *Let $\mathcal{F} = \{(X, \mathscr{D}_1); f_\mu : \mu \in \Gamma\}$ and $\mathcal{G} = \{(Y, \mathscr{D}_2); g_\mu : \mu \in \Gamma\}$ be two uniform IFS. The IFS $\mathcal{F}$ and $\mathcal{G}$ are topological chain mixing if and only if $\mathcal{F} \times \mathcal{G}$ is topological chain mixing.*

Proof. Suppose the IFS $\mathcal{F}$ and $\mathcal{G}$ have chain mixing and let $\mathcal{H} = \mathcal{F} \times \mathcal{G}$. Let A be a symmetric neighbourhood of the diagonal $\Delta_{(X\times Y)}$. Let $A_1 = \{(x_1, x_2) \in X \times X$ such that $(x_1, y_1, x_2, y_2) \in A$ for some $(y_1, y_2) \in Y\}$ and $A_2 = \{(y_1, y_2) \in Y \times Y$ such that $(x_1, y_1, x_2, y_2) \in A$ for some $(x_1, x_2) \in X\}$. Clearly A_1 and A_2 are symmetric neighbourhoods of Δ_X and Δ_Y respectively. Let $(x, y, x', y') \in A$, then $(x, x') \in A_1$ and $(y, y') \in A_2$. Since $\mathcal{F}$ is chain mixing, there exists a natural number m_1 such that for any $n \geq m_1$ there is a A_1-chain $\{x_i\}_{i=0}^{n} \subseteq X$ such that $x_0 = x$, $x_n = x'$ and $(f_{\sigma_i}(x_i), x_{i+1}) \in A_1$ for some $\sigma_i \in \Gamma$. Similarly, since $(Y, \mathcal{G})$ has chain mixing, there exists a natural number m_2 such that for any $n \geq m_2$ there is a A_2-chain $\{y_i\}_{i=0}^{n} \subseteq Y$ such that $y_0 = y$, $y_n = y'$ and $(f_{\sigma_i}(y_i), y_{i+1}) \in A_2$ for some $\sigma_i \in \Gamma$. Choose $m = \max\{m_1, m_2\}$, then for every $n \geq m$, $\{(x_i, y_i)\}_{i=0}^{n}$ is A-chain with $(x_0, y_0) = (x, y)$ and $(x_n, y_n) = (x', y')$. Thus, $\mathcal{F} \times \mathcal{G}$ is topological chain mixing.

Conversely, let $\mathcal{F} \times \mathcal{G}$ have shadowing property. Let A_1 and A_2 be symmetric neighbourhood of Δ_X and Δ_Y respectively. Consider the open set $A = \{(x_1, y_1, x_2, y_2) \in (X \times Y) \times (X \times Y)$ such that $(x_1, x_2) \in A_1$ and $(y_1, y_2) \in A_2\}$ containing the diagonal of $(X \times Y) \times (X \times Y)$. Let $(x, x') \in A_1$ and $(y, y') \in A_2$, then $(x, y, x', y') \in A$. Since $(X\times Y, \mathcal{F} \times \mathcal{G})$ is chain mixing, there exists a natural number m such that for any $n \geq m$, there is A-chain $\{(x_i, y_i)\}_{i=0}^{n}$ such that $(x_0, y_0) = (x, y)$, $(x_n, y_n) = (x', y')$ and $((f_{\sigma_i} \times g_{\sigma_i})(x_i, y_i), (x_{i+1}, y_{i+1})) \in A$ for some $\sigma_i \in \Gamma$. Thus, for any $n \geq m$, we get a A_1-chain $\{x_i\}_{i=0}^{n} \subseteq X$ such that $x_0 = x$, $x_n = x'$ and $(f_{\sigma_i}(x_i), x_{i+1}) \in A_1$ for some $\sigma_i \in \Gamma$. Similarly, we get a A_2-chain $\{y_i\}_{i=0}^{n} \subseteq Y$ such that $y_0 = y$, $y_n = y'$ and $(f_{\sigma_i}(y_i), y_{i+1}) \in A_2$ for some $\sigma_i \in \Gamma$. Thus, $\mathcal{F}$ and $\mathcal{G}$ both are topological chain mixing. □

Acknowledgements

The authors are grateful to the anonymous reviewer for careful reading and valuable suggestions.

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