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Atomic Decompositions in Weighted Bergman Spaces of Analytic Functions on Strictly Pseudoconvex Domains

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Abstract. We construct an atomic decomposition of the weighted Bergman spaces $A^p_\alpha(D)$ $(0 < p \leq 1,$ $\alpha > -1)$ of analytic functions on a bounded strictly pseudoconvex domain D in $\mathbb{C}^n$ with smooth boundary. The atoms used are atoms in the real-variable sense.

1. Introduction

Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Then there is a neighborhood U of $\overline{D}$ and a defining C^∞ function $\rho : U \to (-1, 1)$ for D such that $D = \{z \in U : \rho(z) > 0\}$, $\|\nabla\rho(z)\| \geq c > 0$ for all z in ∂D, $\rho > 0$ on D and $-\rho$ is strictly plurisubharmonic in a neighborhood W of ∂D. Throughout the paper this defining function ρ is fixed. We use it to define weighted Lebesgue measure on D: $d\lambda_\alpha(z) = \rho^\alpha(z)dV(z)$ and weighted Lebesgue spaces $L^p_\alpha = L^p(D, d\lambda_\alpha)$ for $0 < p \leq \infty$. The corresponding (quasi)-norm is denoted by $\|f\|_{p,\alpha}$. The space L^p_α is a Banach space for $p \geq 1$ and a complete metric space with metric $d(f, g) = \|f - g\|^p_{p,\alpha}$ for $0 < p < 1$. Different choices of defining function ρ lead to equivalent (quasi) norms $\|\cdot\|_{p,\alpha}$.

$H(\Omega)$ denotes the space of all holomorphic functions on an open set $\Omega \subset \mathbb{C}^n$. The weighted Bergman space on D is defined by $A^p_\alpha(D) = A^p_\alpha = L^p_\alpha \cap H(D)$. We denote the Bergman kernel for A^p_α by $K_\alpha(z, \zeta)$ and the corresponding Bergman projection by P_α. We will always assume in the following that $\alpha > -1$, otherwise the above A^p_α spaces are trivial.

We are interested in atomic decomposition of A^p_α, $0 < p \leq 1$. There is an extensive literature on atomic decomposition of various spaces, we note a paper [14] where atomic decomposition is obtained for more general spaces, but using analytic atoms. The case of Hardy spaces was investigated in many papers, as an example we mention, among many others, [9] and [8]. Results on atomic decompositions for function spaces (Bloch, Hardy, BMO, weighted Bergman...) on the unit ball in $\mathbb{C}^n$ can be found in [20]. An atomic decomposition result for a given space can be used to prove boundedness of an operator acting on such a space simply by investigating the action of the operator on the atoms. Moreover, such decompositions can be used to describe the dual of the corresponding space and derive properties of small Hankel operators acting on the space. As an example of such applications, in the case of Hardy spaces, see [9].

The present paper is motivated by research of Z. Chen and W. Ouyang, see [3]. They derived atomic decomposition for $A^p_\alpha(D)$ in the case of $D = \mathbb{B}^n$, the unit ball in $\mathbb{C}^n$, in terms of atoms which are compactly

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supported and satisfy certain moment conditions, see Definition 2.1 from [3]. They claimed that the methods employed work also in the case of smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^n$. That claim is indeed true. We use the same general scheme of proof as in [3] to establish an atomic decomposition result in a more general setting. We plan to treat applications mentioned above in a separate publication.

Due to absence of explicit formulae for the Bergman kernel and related metrics, which are available in the case of the unit ball, the general case presents various difficulties. In particular, we need C. Fefferman's estimates of derivatives of the Bergman kernel.

Section 2 contains preliminaries on the geometry of strongly pseudoconvex domains. The main point is that we are working on a space of homogeneous type (D, d_D, λ_α). This enables us to invoke results on maximal functions on such spaces, which generalize classical theorems in $\mathbb{R}^n$. An exposition of this general theory can be found in [17].

Sections 3 and 4 deal with three types of maximal functions: local $M_\delta f$, global $M^T f$ and "grand" $\mathcal{M}_{L,\delta} f$. The main result of these sections is that all of them are bounded in $L^p(d\lambda_\alpha)$ (quasi) norm when acting on A^p_α spaces. These results are valid for all exponents $p > 0$.

In Section 5 we introduce (p,α)-atoms on D, $0 < p \leq 1$, and derive basic properties of these atoms. The main result of that section is Theorem 2 which establishes the boundedness of P_α acting on these atoms.

Section 6 deals with Whitney type decompositions and partitions of unity, it has auxilliary character. The main result of the paper is Theorem 3 in Section 7, which is an atomic decomposition theorem for $A^p_\alpha(D)$.

We use common convention regarding constants: letter C denotes a constant which can change its value from one occurrence to the next one.

2. Preliminary results

Set $D_r = \{z \in U : \rho(z) > r\}$ and $\Gamma_r = \partial D_r$ for $-1 < r < 1$. Also set $W_r^+ = \{z \in D : 0 < \rho(z) < r\}$ and $W_r = \{z \in U : -r < \rho(z) < r\}$ for $0 < r < 1$. There are $r_0 > 0$ and $c_0 > 0$ such that

$$\|\nabla\rho(w)\| \geq c_0 \quad \text{for all} \quad w \in U \quad \text{with} \quad |\rho(w)| \leq r_0. \tag{1}$$

We can assume, by shrinking r_0 if necessary, that $-\rho$ is strictly plurisubharmonic in a neighborhood of $\overline{W_{r_0}}$. Hence, for $-r_0 \leq r \leq r_0$ the domains D_r are also smoothly bounded strictly pseudoconvex domains and $\Gamma_r = \{z \in U : \rho(z) = r\}$.

Let $d(z) = \operatorname{dist}(z, \partial D)$. Note that $d(z) \asymp |\rho(z)|$ for z in $D_{-r_0} \supset \overline{D}$.

The surface measure on ∂D is denoted by $d\sigma$ and the surface measure on Γ_r is denoted by $d\sigma_r$ for $0 \leq r \leq r_0$.

2.1. A flow, a projection map and change of variables

We are going to use the gradient flow corresponding to ρ. Namely, let $\psi(w, r)$ denote the solution of an initial value problem

$$\frac{d}{dr}\psi(w,r) = \nabla\rho(\psi(w,r)), \qquad \psi(w,0) = w. \tag{2}$$

Note that $\psi(w, r)$ is defined for all w in a neighborhood U_0 of $\overline{D}_{-r_0} \supset \overline{D}$ and all $r \geq 0$. Also, as r increases the point $\psi(w, r)$ "flows inside" starting at w.

Using (1) one deduces that for every z in W_{r_0} the integral curve through z intersects ∂D, let us denote the intersection point, clearly unique, by $\pi(z)$. Thus we have a C^∞ map π from W_{r_0} to ∂D which might be called a "projection" from W_{r_0} onto ∂D.

More generally, if $|\rho(w)| < r_0$ and $-r_0 - \rho(w) < r < r_0 - \rho(w)$, then there is a unique point w' on the integral curve of (2) such that $\rho(w') = \rho(w) + r$. Let us denote that point by $\kappa_r(w)$. The map κ is C^∞ in both variables r and w, it has local group transformation property: $\kappa_r \circ \kappa_t(w) = \kappa_{r+t}(w)$ for small r and t, $|\rho(w) < r_0$. Also,

$\kappa_r : \partial D \to \Gamma_r$ is a C^∞ diffeomorphism for $|r| < r_0$. In particular we have the following change of variables formula:

$$\int_{\Gamma_r} f(\eta) a_r(\eta) d\sigma_r(\eta) = \int_{\partial D} f(\kappa_r(\xi)) d\sigma(\xi), \qquad f \in C(\Gamma_r), \tag{3}$$

where $a_r(w)$ is a C^∞ function on W_{r_0} such that $0 < c \le a_r(w) \le C$ for some constants c and C which are independent of r. There is a corresponding change of variables formula for volume integrals:

$$\int_{D_t} f(\kappa_{-t}(w)) dV(w) = \int_D f(z) J(z,t) dV(z) \tag{4}$$

provided $f \in C(D)$, supp $f \subset W^+_{r_0}$, $t \ge 0$ and $\kappa_{-t}(\text{supp } f) \subset D$. Here the Jacobian $J(z,t)$ satisfies $0 < c \le J(z,t) \le C$ for some constants c and C independent of t.

2.2. Coordinate systems and differential operators

We are going to use differential operators acting on smooth functions in D. In order to define the weighted order (adapted to the complex geometry of D) of such an operator we need a special family of coordinate systems on D.

Proposition 1. *Let $\zeta \in \partial D$. Then there is a neighborhood $V_\zeta \subset \mathbb{C}^n$ of ζ contained in W_{r_0} and a C^∞ map $\Psi^\zeta = \Psi : V_\zeta \to (\mathbb{R}^{2n})^{2n} \cong (\mathbb{C}^n)^{2n}$ such that for each $z \in V_\zeta$ the following conditions are satisfied:*

1^o: $\Psi(z) = (\Psi_1(z), \dots, \Psi_{2n}(z))$ is an orthonormal basis of $\mathbb{R}^{2n}$;

2^o: $\Psi_1(z) = -\nabla\rho(z)/\|\nabla\rho(z)\|$;

3^o: $\Psi_{2j}(z) = i\Psi_{2j-1}(z)$ for $j = 1, \dots, n$, where identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ is assumed;

4^o: The vectors $\Psi_{2j-1}(z)$, $j = 2, \dots, n$, form a basis for the complex tangent space $T^{\mathbb{C}}_z(\Gamma_r)$, $r = \rho(z)$.

Proof. Let us note that 4^o follows from the remaining conditions: the vectors $\Psi_k(z)$, $3 \le k \le 2n$ are orthogonal to $\nabla\rho(z)$ and $i\nabla\rho(z)$ and therefore span the complex tangent space $T^{\mathbb{C}}_z(\Gamma_r)$. The vectors Ψ_1 and Ψ_2 are uniquelly determined by 2^o and 3^o. The task is to complete these two vector fields to an orthonomal frame in a C^∞ fashion so that condition 3^o is satisfied.

We begin with locally selecting a C^∞ orthonormal frame $(\Psi_j(\eta))_{j=1}^{2n}$ such that $\Psi_1(\eta) = -\nabla\rho(\eta)/\|\nabla\rho(\eta)\|$ and $\Psi_{2j}(\eta) = i\Psi_{2j-1}(\eta)$ for $1 \le j \le n$. Here η is in a neighborhood $\Omega_\zeta \subset \partial D$ of ζ. The tangent map $T_\eta\kappa_r : T^{\mathbb{R}}_\eta \partial D \to T^{\mathbb{R}}_{\kappa_r(\eta)}\partial\Gamma_r$ maps vectors $\Psi_j(\eta)$, $j \ge 3$ to vectors $\Psi'_j(\kappa_r(\eta))$ tangent to Γ_r at $z = \kappa_r(\eta)$ ($|r| < r_0$). We got a C^∞ frame $(\Psi'_j(z))_{j=3}^{2n}$ in a neighborhood $V'_\zeta = \cup_{|r|<r_0}\kappa_r(\Omega_\zeta) \subset W_{r_0}$ of ζ. Now we define $\Psi_1(z) = -\nabla\rho(z)/\|\nabla\rho(z)\|$, set $\Psi_2(z) = i\Psi_1(z)$ and proceed inductively to define $\Psi_j(z)$ for $j \ge 3$. If j is odd, it is obtained from $\Psi'_j(z)$ (and previously defined vectors) using Gramm-Schmidt method. If j is even we set $\Psi_j(z) = i\Psi_{j-1}$. For points z near ζ the vector $\Psi_2(z)$, which clearly belongs to $T^{\mathbb{R}}_z\Gamma_r$, $r = \rho(z)$, is linearly independent from the vectors $\Psi'_j(z)$, $1 \le 3 \le 2n$. Similarly, for z near ζ, the replacement of $\Psi'_{2j}(z)$ by $i\Psi'_{2j-1}(z)$ does not change the linear span of the first $2j$ vectors. Hence, in a neigborhood $V_\zeta \subset V'_\zeta$ the above procedure yields the desired basis. □

There is a finite cover of ∂D by such neighborhoods $V_{\zeta_1}, \dots, V_{\zeta_l}$, therefore $\overline{W}_{r_1} \subset \cup_{k=1}^l V_{\zeta_k}$ for some $0 < r_1 \le r_0$. Clearly, we can assume without loss of generality that $r_1 = r_0$. This allows us to attach to each $z \in W_{r_0}$ a special affine coordinate system. Namely, choose any $k \in \{1, \dots, l\}$ such that $z \in V_{\zeta_k}$. Then the relation

$$w - z = \sum_{j=1}^{2n} \kappa_j(z,w)\Psi^{\zeta_k}_j(z)$$

which assigns to each $w \in \mathbb{C}^n$ the coordinates $\kappa_1(z,w), \dots, \kappa_{2n}(z,w)$ of $w - z$ with respect to the basis $\Psi^{\zeta_k}(z)$ is an affine map $\kappa_z : \mathbb{C}^n \to \mathbb{R}^{2n}$. We denote its inverse by θ_z.

For $z \in D \setminus W^+_{r_0}$ we define local coordinate system κ_z in an obvious manner: for $w \in \mathbb{C}^n$ we set $\kappa_z(w) = (\Re(w_1 - z_1), \Im(w_1 - z_1), \dots \Re(w_n - z_n), \Im(w_n - z_n))$, again θ_z denotes its inverse.

Since we are going to work with differential operators we use multi-index notation: for $J = (j_1, \dots, j_{2n}) \in \mathbb{Z}_+^{2n}$ we set $|J| = j_1 + \dots + j_{2n}$ (the order of J) and $d(J) = j_1 + j_2 + (j_3 + \dots + j_{2n})/2$ (the weighted order of J). Given a function $f \in C^\infty(D)$ and $z \in D$ we denote by $D_z^J f$ the action of differential operator

$$D^J = \frac{\partial^{|J|}}{\partial^{j_1} w_1 \partial^{j_2} w_2 \cdots \partial^{j_{2n}} w_{2n}}$$

on f in local θ_z coordinates. More precisely: $(D_z^J f)(w) = [D^J(f \circ \theta_z)](\kappa_z(w))$ for $w \in D$.

We also define "polynomials with respect to κ_z". Namely, let $z \in D$ and let $J = (j_1, j_2, \dots, j_{2n})$ be a multi index. Then we define

$$P_z^J(w) = x_1^{j_1} y_1^{j_2} \dots x_n^{j_{2n-1}} y_n^{j_{2n}}, \qquad \text{where} \quad \kappa_z(w) = (x_1, y_1, \dots, x_n, y_n). \tag{5}$$

2.3. Distances and pseudo-distances on D

We denote by d_K the Koranyi type pseudo-hyperbolic distance on ∂D and set $B_K(z, r) = \{w \in \partial D : d_K(z, w) < r\}$, $z \in \partial D$, see [18] for construction and properties of d_K. Such distances were constructed in a more general context of domains of finite type, see for example [13]. Note that $d_K(z, w) = |1 - \langle z, w \rangle|^{1/2}$ in the case D is the unit ball in $\mathbb{C}^n$. We define a quasi-metric d'_D on $\overline{W_{r_0}^+}$ by

$$d'_D(z, w) = |\rho(z) - \rho(w)| + d_K^2(\pi(z), \pi(w)), \qquad z, w \in \overline{W_{r_0}^+}. \tag{6}$$

We note that d'_D, being a quasi-metric, satisfies the following inequality

$$d'_D(z_1, z_3) \le Q(d'_D(z_1, z_2) + d'_D(z_2, z_3)), \qquad z_1, z_2, z_3 \in \overline{W_{r_0}^+}, \tag{7}$$

where constant Q depends only on the domain D, we can assume $Q \ge 2$.

It is convenient to extend this quasi-metric to $\overline{D}$. Let us choose $0 < r'_0 < r_0$ and a C^∞ partition of unity $\phi_1 + \phi_2 = 1$ subordinated to the cover of D by open sets $\Omega_1 = W_{r_0}^+$ and $\Omega_2 = \{z \in D : \rho(z) > r'_0\}$; we can extend ϕ_1 continuously to the boundary of D by setting $\phi_1(\zeta) = 1$ for $\zeta \in \partial D$. Then we set

$$d_D(z, w) = \phi_1(z)\phi_1(w) d_D(z, w) + [1 - \phi_1(z)\phi_1(w)]|z - w|, \qquad z, w \in \overline{D}. \tag{8}$$

It is easy to check that d_D is also a quasi-metric, with the same constant Q as d'_D. However, it is defined on $\overline{D}$ and coincides with d'_D for $z, w \in \overline{W_{r'_0}^+}$. Let R_D denote the diameter of D with respect to quasi-metric d_D.

Also, there are constants Q' (a flow constant) and $r_2 \in (0, r_0)$ such that

$$d_D(\kappa_r(w), w) \le Q' r, \qquad 0 \le r \le r_2, \quad w \in \overline{W_{r_0}^+}. \tag{9}$$

Moreover, there is an $\epsilon_1 > 0$ such that $B_D(z, \epsilon_1) \cap \overline{D}_{r_0} = \emptyset$ for all $z \in W_{r_2}^+$. We fix such ϵ_1.

On the other hand, note that by definition of the map κ we have

$$\rho(\kappa_r(w)) \ge r, \qquad w \in \overline{W}_{r_0}^+, \quad 0 < r \le r_2. \tag{10}$$

We denote by β the Bergman metric on D and the corresponding balls by $B_\beta(z, r) = \{w \in D : \beta(z, w) < r\}$ where $z \in D$ and $r > 0$. Similarly, we set $B_D(z, r) = \{w \in \overline{D} : d_D(z, w) < r\}$ for $z \in \overline{D}$ and $r > 0$. Let us note that, for $\zeta \in \partial D$ and $r > 0$ the set $B_D(\zeta, r) \cap \partial D$ is the ball of radius $\sqrt{r}$ centered at ζ in the d_K metric on ∂D, we denote it by $Q(\zeta, \sqrt{r})$.

We state below five propositions on measures and distances in D, since these results are either known or represent easy generalizations from the case of the unit ball, we omit the proofs.

Proposition 2. *There are constants $c_1 = c_1(D)$ and $c_2 = c_2(D)$ in $[1, +\infty)$ such that*

$$B_K(\pi(z), \sqrt{\rho(z)}) \subset B_D(z, c_1 \rho(z)), \qquad z \in \overline{W_{r_0}^+}, \tag{11}$$

$$B_D(z, r) \subset B(z, c_2 r), \qquad z \in D, \quad r > 0. \tag{12}$$

Proposition 3. *Let $\gamma > 0$. Then there are constants $0 < \sigma < 1$ and $C > 0$ depending only on γ and D such that*

$$D_\beta(w,\gamma) \subset B_D(w, C\rho(w)), \qquad \text{whenever } \ 0 < \rho(w) < \sigma. \tag{13}$$

Proposition 4. *Let $\gamma > 0$. Then there are constants $c_1 = c_1(\gamma, D) > 0$ and $C_1(\gamma, D) < \infty$ such that*

$$c_1\rho(w)^{n+1+\alpha} \leq \lambda_\alpha(B_\beta(w,\gamma)) \leq C_1\rho(w)^{n+1+\alpha}, \qquad w \in D. \tag{14}$$

Proposition 5. *Let $\alpha > -1$. Then there are constants $c = c(\alpha, D) > 0$ and $C = C(\alpha, D)$ such that*

$$cr^{n+1}\left(\max(\rho(z), r)\right)^\alpha \leq \lambda_\alpha(B_D(z,r)) \leq Cr^{n+1}\left(\max(\rho(z), r)\right)^\alpha \tag{15}$$

for $z \in D$, $0 < r \leq R_D$.

The above proposition implies that the measure $d\lambda_\alpha$ satisfies the doubling property with respect to quasi-metric d_D. This is the essential part of the following proposition.

Proposition 6. *(D, d_D, λ_α) is a homogeneous space.*

This proposition enables us to use results from the theory of homogeneous spaces (see [17] and [19] for discussion of homogeneuous spaces).

The following proposition states that $|f(z)|^p$ has subharmonic behavior with respect to (non-isotropic) balls $B_\beta(z,r)$. It will play an important role in proving boundedness in norm of a maximal function operator, see Proposition 9 below.

Proposition 7. *Let $0 < p < \infty$. Then there is a constant C, depending only on p and D, such that for any $f \in H(D)$ we have*

$$|f(z)|^p \leq \frac{C}{\lambda_\alpha(B_\beta(z,r))}\int_{B_\beta(z,r)} |f(w)|^p d\lambda_\alpha(w), \qquad z \in D, \quad r > 0. \tag{16}$$

3. Maximal functions

For a measurable function f on D the corresponding maximal function Mf, with respect to homogeneous space (D, d_D, λ_α), is defined by

$$Mf(z) = \sup_{r>0} \frac{1}{\lambda_\alpha(B_D(z,r))}\int_{B_D(z,r)} |f(w)| d\lambda_\alpha(w), \qquad z \in D. \tag{17}$$

We record here the following classical result, see [17].

Proposition 8. *The maximal function operator maps $L^1_\alpha(D)$ into $L^{1,\infty}_\alpha(D)$, i.e. there is a constant $C = C(D,\alpha)$ such that*

$$\lambda_\alpha(\{z \in D : Mf(z) > \lambda\}) \leq C\frac{\|f\|_{1,\alpha}}{\lambda}, \qquad f \in A^1_\alpha(D), \quad \lambda > 0. \tag{18}$$

For $1 < q \leq \infty$ we have

$$\|Mf\|_{q,\alpha} \leq C\|f\|_{q,\alpha}, \qquad f \in L^q_\alpha(D), \tag{19}$$

where C is a constant depending only on q, $\alpha > -1$ and D.

For $z \in D$ and $\delta > 0$ we define a neighborhood $A_\delta(z)$ of z by

$$A_\delta(z) = \{w \in D : d_D(z, w) < \delta\rho(w)\}. \tag{20}$$

Given a measurable function f on D we define its (local) maximal function

$$(M_\delta f)(z) = \sup_{w \in A_\delta(z)} |f(w)|, \qquad z \in D. \tag{21}$$

Later we will see, in Lemma 2, that the different sizes of apertures $\delta > 0$ lead to equivalent, in terms of $L^p(d\lambda_\alpha)$ (quasi) norms, local maximal functions

We also define, for $T > 0$, the (global) maximal function of a measurable function f on D by

$$(M^T f)(z) = \sup_{w \in D} \left(\frac{\rho(w)}{\rho(w) + d_D(z, w)} \right)^T |f(w)|, \qquad z \in D. \tag{22}$$

Next we prove that these two maximal functions acting on holomorphic functions are bounded sublinear operators in the weighted (quasi) norms $\| \cdot \|_{p,\alpha}$.

Proposition 9. *If $\alpha > -1$, $\delta > 0$ and $0 < p < \infty$ then there is a constant C, depending only on D, p, α and δ, such that*

$$\|M_\delta f\|_{p,\alpha} \le C\|f\|_{p,\alpha}, \qquad f \in A^p_\alpha. \tag{23}$$

Proof. Let us start with the following observation: for any $C > 0$ we have

$$B_D(w, C\rho(w)) \subset B_D(z, Q(C + \delta)\rho(w)), \qquad z \in D, \quad w \in A_\delta(z). \tag{24}$$

Indeed, if $w \in A_\delta(z)$ and $u \in B_D(w, C\rho(w))$, then

$$d_D(u, z) \le Q[d_D(u, w) + d_D(w, z)[< Q[C\rho(w) + \delta\rho(w)] = Q(C + \delta)\rho(w).$$

Next, let us fix $\gamma > 0$. Then by Propositions 7 and 4 we have

$$\begin{aligned} |f(w)|^{p/2} &\le \frac{C_{p,D}}{\lambda_\alpha(D_\beta(w, \gamma))} \int_{D_\beta(w,\gamma)} |f(z)|^{p/2} d\lambda_\alpha(z) \\ &\le \frac{C_{p,D} C_{\gamma,D}}{\rho(w)^{n+1+\alpha}} \int_{D_\beta(w,\gamma)} |f(z)|^{p/2} d\lambda_\alpha(z), \qquad w \in D. \end{aligned}$$

Now we choose constants $0 < \sigma < 1$ and $C > 0$ from Proposition 3. Then the above estimate, combined with Proposition 5 and (24), gives

$$\begin{aligned} |f(w)|^{p/2} &\le \frac{C}{\rho(w)^{n+1+\alpha}} \int_{B_D(w, C\rho(w))} |f|^{p/2} d\lambda_\alpha & (25) \\ &\le \frac{C}{\lambda_\alpha(B_D(z, Q(C + \delta)\rho(w))} \int_{B_D(z, Q(C+\delta)\rho(w))} |f|^{p/2} d\lambda_\alpha & (26) \\ &\le C(M|f|^{p/2})(z), \qquad 0 < \rho(w) < \sigma, \quad w \in A_\delta(z). & (27) \end{aligned}$$

Therefore we obtained the following pointwise estimate:

$$M_\delta f(z) \le C[M(|f|^{p/2})(z)]^{2/p} + \sup_{\rho(w) \ge \sigma} |f(w)|, \qquad z \in D. \tag{28}$$

Now we apply Proposition 8, with $q = 2$, and estimate

$$\sup_{\rho(w) \ge \sigma} |f(w)| \le C(\sigma, p, \alpha, D)\|f\|_{p,\alpha},$$

which is valid even for harmonic functions, to (28) and complete the proof. □

Lemma 1. *Let $N > 1$ and let E be a measurable subset of D. Assume w is in $A_N(z)$ and $B_D(w, \rho(w)) \subset E$. Then there is a constant $\gamma = \gamma(\alpha, D)$ such that*

$$M(\chi_E(z)) \geq \gamma(N+1)^{-n-1-\alpha}. \tag{29}$$

Proof. It is easily seen that

$$B_D(z, (N+1)\rho(w)) \subset B_D(w, Q(2N+1)\rho(w)) \qquad \text{if} \quad w \in A_N(z).$$

Let $c(\alpha, D)$ and $C(\alpha, D)$ be the constants from Proposition 5. Using (29) we obtain

$$\begin{aligned}
(M\chi_E)(z) &\geq \frac{\int_{B_D(z,(N+1)\rho(w))} \chi_E \, d\lambda_\alpha}{\lambda_\alpha(B_D(z, (N+1)\rho(w)))} \\
&\geq \frac{1}{\lambda_\alpha(B_D(z, (N+1)\rho(w)))} \int_{B_D(z,(N+1)\rho(w))} \chi_{B_D(w,\rho(w))} \, d\lambda_\alpha \\
&\geq \frac{\lambda_\alpha(B_D(w, \rho(w)))}{\lambda_\alpha(B_D(w, Q(2N+1)\rho(w)))} \geq \frac{c(\alpha, D)}{C(\alpha, D)} \frac{1}{[Q(2N+1)]^{n+1+\alpha}},
\end{aligned}$$

and we see that one can take $\gamma = c(\alpha, D)/(3Q)^{n+1+\alpha} C(\alpha, D)$. □

Lemma 2. *There is a constant C, depending only on $0 < p < \infty$, $\alpha > -1$ and D such that*

$$\int_D (M_N f)^p d\lambda_\alpha \leq C(N+1)^{n+1+\alpha} \int_D (M_1 f)^p d\lambda_\alpha, \quad f \in A^p_\alpha(D), \quad N \in \mathbb{N}. \tag{30}$$

Proof. Let us prove that for every $t > 0$ we have

$$\{z \in D : M_N f(z) > t\} \subset \left\{ z \in D : M\chi_{E_t}(z) > \frac{\gamma}{(N+1)^{n+1+\alpha}} \right\}, \tag{31}$$

where $E_t = \{u \in D : M_1 f(u) > t\}$. Indeed, if z belongs to the first set in (31), then $|f(w)| > t$ for some $w \in A_N(z)$. This implies that $B_D(w, \rho(w)) \subset E_t$ and Lemma 1 gives (31). Using Proposition 8 and (31) we obtain

$$\begin{aligned}
\lambda_\alpha(\{z \in D : M_N f(z) > t\}) &\leq C\gamma^{-1}(N+1)^{n+1+\alpha} \|E_t\|_{1,\alpha} \\
&\leq C(N+1)^{n+1+\alpha} \lambda_\alpha(\{z \in D : M_1 f(z) > t\})
\end{aligned}$$

This weak type estimate gives L^p_α estimate (30) by a standard argument. □

Proposition 10. *If $\alpha > -1$, $0 < p < \infty$ and $Tp > n + 1 + \alpha$, then there is a constant C, depending only on D, p, α and T such that*

$$\|M^T f\|_{p,\alpha} \leq C\|f\|_{p,\alpha}, \qquad f \in A^p_\alpha. \tag{32}$$

Proof. Let $t > 0$ and assume $M^T f(z) > t$ for some $z \in D$. Set

$$D_0 = \{w \in D : d_D(z, w) < \rho(w)\} = A_1(z)$$

and

$$D_k = \{w \in D : 2^{k-1}\rho(w) \leq d_D(z, w) < 2^k \rho(w)\} = A_{2^k}(z) \setminus A_{2^{k-1}}(z), \qquad k \geq 1.$$

Clearly $D = \cup_{k=0}^\infty D_k$. Since $M^T f(z) > t$ there is a $k \geq 0$ such that $w \in D_k$ and

$$\left(\frac{\rho(w)}{\rho(w) + d_D(z, w)} \right)^T |f(w)| > t.$$

If $k=0$, then $|f(w)|>t$ and therefore $M_1 f(z)>t>2^{-T}t$. If $k\geq 1$, inequality

$$\left(\frac{\rho(w)}{\rho(w)+d_D(z,w)}\right)^T \leq \frac{1}{2^{(k-1)T}}, \qquad w\in D_k$$

gives $M_{2^k}f(z)>2^{(k-1)T}t$. The above argument proves the following inclusion:

$$\{z\in D: M^T f(z)>t\}\subset \bigcup_{k=0}^{\infty}\{z\in D: M_{2^k}f(z)>2^{(k-1)T}t\}, \qquad t>0. \tag{33}$$

Now we use (33) to estimate the L^p_α norm of $M^T f$ in terms of norms of localized maximal functions $M_{2^k}f$:

$$\begin{aligned}
\|M^T f\|^p_{p,\alpha} &= p\int_0^\infty t^{p-1}\lambda_\alpha(\{M^T f>t\})dt \qquad (34)\\
&\leq p\int_0^\infty t^{p-1}\sum_{k=0}^{\infty}\lambda_\alpha(\{M_{2^k}f>2^{(k-1)T}t\})\,dt\\
&= p\sum_{k=0}^{\infty}\int_0^\infty t^{p-1}\lambda_\alpha(\{M_{2^k}f>2^{(k-1)T}t\})\,dt\\
&= \sum_{k=0}^{\infty}\frac{1}{2^{(k-1)Tp}}\|M_{2^k}f\|^p_{p,\alpha}.\\
&\leq 2^{pT}C\sum_{k=0}^{\infty}\frac{2^{(n+1+\alpha)k}}{2^{(k-1)Tp}}\|M_1 f\|^p_{p,\alpha} \qquad (35)\\
&= C\|M_1 f\|^p_{p,\alpha}.
\end{aligned}$$

Note that we used Lemma 2 in deducing (35) and condition $Tp>n+1+\alpha$ in ensuring convergence of the series in (35). Now we apply Proposition 9 and complete the proof. □

4. Grand maximal function

In addition to maximal functions discussed in the previous section, we are going to use so called grand maximal function. In order to define it, we have to consider certain spaces of test functions.

Let us fix a non-negative integer L. Then, for $z_0\in D$ and $0<r_0\leq R_D$ (R_D is the d_D - diameter of D) we set

$$\|g\|_{L;z_0,r_0}=\lambda_\alpha(B_D(z_0,r_0))\sup_{|J|\leq L} r_0^{d(J)}\|D^J_{z_0}g\|_{L^\infty(B_D(z_0,r_0))} \tag{36}$$

for $g\in C^\infty(D)$.

Let us define, for $z\in D$, $0<r_0\leq R_D$ and $0<\delta\leq 1$, the set $\mathcal{D}_{L,\delta,r_0}(z)$ of test functions localized near $z\in D$ as the set of all $g\in C^\infty(D)$ such that there is a point z_0 in D satisfying

$$d_D(z,z_0)<\delta r_0, \qquad \operatorname{supp} g\subset B_D(z_0,r_0), \qquad \|g\|_{L;z_0,r_0}\leq 1. \tag{37}$$

We set, for $z\in D$ and $0<\delta\leq 1$:

$$\mathcal{D}_{L,\delta}(z)=\bigcup_{0<r_0\leq R_D}\mathcal{D}_{L,\delta,r_0}(z).$$

This set of test functions is used in the following definition of the grand maximal function of a measurable function f on D:

$$\mathcal{M}_{L,\delta}f(z)=\sup\left\{\left|\int_D f(w)g(w)d\lambda_\alpha(w)\right| : g\in\mathcal{D}_{L,\delta}(z)\right\}, \qquad z\in D. \tag{38}$$

Proposition 11. *Let $T > 0$ and $\delta > 0$. Assume L is an integer and $L > T$. Then there is a constant $C = C(D, L, T, \delta)$ such that for every holomorphic function f on D we have*

$$\mathcal{M}_{L,\delta} f(z) \le C(M_S f(z) + M^T f(z)), \qquad z \in D, \tag{39}$$

where $S = Q[Q' + (1+\delta)Q]$, Q is the pseudometric constant from (7) and Q' is the flow constant from (9).

Proof. Let us fix $f \in H(D)$ and $z \in D$. Let us choose $g \in \mathcal{D}_{L,\delta,r}(z)$, where $0 < r \le R_D$. This means that for some $z_0 \in D$ the condition (37) is satisfied. We want to estimate the absolute value of $I = \int_D f(w)g(w)d\lambda_\alpha(w)$ in terms of the right hand side of (39). Such an estimate is immediate for z in a compact subset of D and also for r bounded away from zero. Therefore we assume $0 < r < r_2$ and $z \in W^+_{r_0}$. Let us set $f_{r_0}(w) = f(\kappa_{r_0}(w))$, see Section 2. We have

$$\begin{aligned} |I| &\le \left| \int_D [f(w) - f_{r_0}(w)] g(w) d\lambda_\alpha(w) \right| + \left| \int_D f_{r_0}(w) g(w) d\lambda_\alpha(w) \right| \\ &= |I_1| + |I_2|. \end{aligned} \tag{40}$$

Let us estimate $|I_2|$. Since supp $g \subset B_D(z_0, r_0)$ and $d_D(z, z_0) \le \delta r_0$ we have

$$d_D(w, z) \le Q(d_D(w, z_0) + d_D(z_0, z)) < Q(1+\delta) r_0, \qquad w \in \operatorname{supp} g. \tag{41}$$

Therefore, using (9), we obtain

$$d_D(\kappa_{r_0}(w), z) \le Q[d_D(\kappa_{r_0}(w), w) + d_D(w, z)] < Q[Q(1+\delta) + Q'] r_0, \tag{42}$$

for $w \in \operatorname{supp} g$. On the other hand, using (10), we have $\rho(\kappa_{r_0}(w)) \ge r_0$ and we conclude that $w \in A_S(z)$ for all $w \in \operatorname{supp} g$, i.e. $B_D(z_0, r_0) \subset A_S(z)$. Next, taking $J = 0$ in (36), we have $|g(w)| \le \lambda_\alpha(B_D(z_0, r_0))^{-1}$ and this implies

$$|I_2| \le \frac{1}{\lambda_\alpha(B_D(z_0, r_0))} \int_{B_D(z_0, r_0)} |f(w)| d\lambda_\alpha(w) \le M_S f(z). \tag{43}$$

Now we consider I_1. Let us fix a $\chi \in C^\infty(\mathbb{R})$ such that $0 \le \chi(t) \le 1$ for all $t \in \mathbb{R}$, $\chi(t) = 1$ for $t \le 1$, $\chi(t) = 0$ for $t \ge 2$ and set $\chi_r(t) = \chi(t/r)$ for $r > 0$. Note that

$$|\chi^k(t)| \le C_k r^{-k} \qquad \text{for } k \ge 0 \quad \text{and } t \in \mathbb{R}, \tag{44}$$

where constants C_k do not depend on $r > 0$. Also, set $a(w) = \|\nabla \rho(w)\|$, clearly $a \in C^\infty(U)$. We are going to use a substitute for the formula for integration in polar coordinates which is based on Fubini's theorem and (3). Namely, there is a C^∞ density function $s(\xi, r) = s(w)$ ($w = \kappa_r(\xi)$), $0 < c \le s(\xi, r) \le C$ such that for any λ_α - integrable function f supported in $W^+_{r_0}$ we have

$$\int_D f(w) d\lambda_\alpha(w) = \int_0^1 \int_{\partial D} f(\kappa_r(\xi)) s(\xi, r) d\sigma(\xi) r^\alpha dr. \tag{45}$$

Next we transform I_1:

$$\begin{aligned} I_1 &= \int_D g(w)[f(w) - f_{r_0}(w)] d\lambda_\alpha(w) = -\int_D g(w) \int_0^{r_0} \frac{d}{dt} f(\kappa_t(w)) dt\, d\lambda_\alpha(w) \\ &= -\int_D g(w) \int_0^{r_0} \frac{\partial f}{\partial \nu}(\kappa_t(w)) dt\, d\lambda_\alpha(w) \\ &= -\int_0^1 \int_0^{r_0} \int_{\partial D} \frac{\partial f}{\partial \nu}(\kappa_{t+r}(\xi)) g(\kappa_r(\xi)) s(\xi, r) d\sigma(\xi)\, r^\alpha dt\, dr \\ &= -\int_0^1 \int_0^{r_0} J(r, t) r^\alpha dt\, dr. \end{aligned}$$

Here $\nu = \nu(w)$ is the unit outward pointing vector normal to $\Gamma_{\rho(w)}$ at w and

$$J(r,t) = \int_{\partial D} \frac{\partial f}{\partial \nu}(\kappa_{t+r}(\xi))g(\kappa_r(\xi))s(\xi,r)d\sigma(\xi). \tag{46}$$

Let $G_r(w) = g(\kappa_r(w))\chi_r(\rho(w))$. Using (3) and Green's formula we obtain

$$\begin{aligned} J(r,t) &= \int_{\partial D} \frac{\partial f}{\partial \nu}(\kappa_{t+r}(w))G_r(w)s(\kappa_r(w))d\sigma(w) \\ &= \int_{\Gamma_{t+r}} \frac{\partial f}{\partial \nu}(u)G_r(\kappa_{-t-r}(w))s(\kappa_{-t}(w))a(w)d\sigma_{t+r}(w) \\ &= J_1(r,t) - J_2(r,t) + J_3(r,t). \end{aligned}$$

where

$$J_1(r,t) = \int_{\Gamma_{t+r}} f(w)\frac{\partial}{\partial \nu}[G_r(\kappa_{-t-r}(w))s(\kappa_{-t}(w))a(w)]d\sigma(w),$$

$$J_2(r,t) = \int_{D_{t+r}} f(w)\Delta_w[G_r(\kappa_{-t-r}(w))s(\kappa_{-t}(w))a(w)]dV(w),$$

$$J_3(r,t) = \int_{D_{t+r}} [G_r(\kappa_{-t-r}(w))s(\kappa_{-t}(w))a(w)\Delta_w f(w)dV(w).$$

Note that $J_3(r,t) = 0$ since $f(w)$ is harmonic. Next we make a change of variables in $J_1(r,t)$ and $J_2(r,t)$, replacing w by $\kappa_{r+t}(w)$. This gives, using chain rule for differentiation, (3) and (4):

$$J_1(r,t) = \int_{\partial D} f(\kappa_{r+t}(w))A_1[G_r(w)a_1(w,r,t)]\frac{d\sigma(w)}{a(\kappa_{r+t}(w))},$$

$$J_2(r,t) = \int_D f(\kappa_{r+t}(w))A_2[G_r(w)a_2(w,r,t)]J(w,r+t)dV(w),$$

where A_1, resp. A_2, is a first order, resp. second order, linear differential operator with smooth and bounded coefficients depending on w, r and t and a_1 and a_2 are C^∞ functions; all partial derivatives of a_1 and a_2 are bounded.

Let A be a linear partial differential operator in w variable of order l with bounded coefficients depending on w and parameters r and t. Then

$$|Ag(\kappa_r(w))| \leq C \sum_{|J|\leq l} \frac{1}{\lambda_\alpha(B_D(z_0,r_0))r_0^{d(J)}} \leq C \sum_{|J|=l} \frac{1}{\lambda_\alpha(B_D(z_0,r_0))r_0^{d(J)}}.$$

Here we used chain rule, a fact that $\kappa_r(w)$ is a C^∞ map and the assumption $g \in \mathcal{D}_{L,\delta,r}(z)$. This, combined with (44), gives

$$|A[G_r(w)a(w,r,t)]| \leq \frac{C}{\lambda_\alpha(B_D(z_0,r_0))r_0^l}, \tag{47}$$

where $a(w,r,t)$ is any C^∞ function, with bounded partial derivatives up to order l, constant C depends only on the operator A of order l and on a.

Next we write $J_2(r,t) = J_{2,1}(r,t) + J_{2,2}(r,t)$ where

$$J_{2,1}(r,t) = \int_{D\setminus D_{r_0}} f(\kappa_{r+t}(w))A_2[G_r(w)a_2(w,r,t)]J(w,r+t)dV(w),$$

$$J_{2,2}(r,t) = \int_{D_{r_0}} f(\kappa_{r+t}(w))A_2[G_r(w)a_2(w,r,t)]J(w,r+t)dV(w).$$

If $\kappa_r(w) \in \operatorname{supp} g$ then, by (41), we have

$$\begin{aligned} d_D(\kappa_{r+t}(w),z) &\le Q(d_D(\kappa_{r+t}(w),\kappa_r(w)) + d_D(\kappa_r(w),z)) \le Q(t + Q(1+\delta)r_0) \\ &\le Q(1+Q(1+\delta))r_0 < \rho(w) \le Sr_0 < S\rho(w) \end{aligned}$$

for $w \in D_{r_0}$, in other words $w \in A_S(z)$. Therefore, since $J(w,s)$ is bounded:

$$\begin{aligned} |J_{2,2}(r,t)| &\le (M_S f)(z) \int_{D_{r_0}} |A_2[G_r(w)a_2(w,r,t)]J(w,r+t)|dV(w) \\ &\le C(M_S f)(z) \int_{D_{r_0}} |A_2[G_r(w)a_2(w,r,t)]|dV(w). \end{aligned}$$

Setting $I_{1,j} = \int_0^1 \int_0^{r_0} J_j(r,t)dt\, r^\alpha dr$ for $j = 1,2$ we have $I_1 = -I_{1,1} + I_{1,2}$. Using the above estimate and (47) with $l = 2$ we obtain:

$$\begin{aligned} \left| \int_0^1 \int_0^{r_0} J_{2,2}(r,t)dtr^\alpha dr \right| &= \left| \int_{r_0}^{2r_0} \int_0^{r_0} J_{2,2}(r,t)dtr^\alpha dr \right| \le \frac{CM_S f(z)}{\lambda_\alpha(B_D(z_0,r_0))r_0^2} \\ &\times \int_{r_0}^{2r_0} r^\alpha \left(\int_0^{r_0} Vol(\kappa_{-r}(B_D(z_0,r_0)) \cap D)dt \right) dr \\ &\le CM_S f(z). \end{aligned} \tag{48}$$

Therefore $|I_{1,2}| \le |\int_0^1 \int_0^{r_0} J_{2,1}(r,t)r^\alpha dtdr| + CM_S f(z)$. Combining the above estimates with (40) and (43) we deduce:

$$\begin{aligned} |I| &\le \left| \int_0^1 \int_0^{r_0} \int_{W_{r_0}^+} f(\kappa_{r+t}(w))A_2[G_r(w)a_2(w,r,t)]J(w,r+t)dV(w)dtr^\alpha dr \right| \\ &+ \left| \int_0^1 \int_0^{r_0} \int_{\partial D} f(\kappa_{r+t}(w))A_1[G_r(w)a_1(w,r,t)] \frac{d\sigma(w)}{a(\kappa_{r+t}(w))} dt\, r^\alpha dr \right| \\ &+ CM_S f(z). \end{aligned} \tag{49}$$

Now we apply the same methods to the above two integrals. Namely, for fixed $0 < t < r_0$ the double integral with respect to dr and $d\sigma(w)$ in the second one is of the same form as the initial one, but with f replaced by $f \circ \kappa_t$ and we can repeat the same method involving Green's formula. For the first integral we repeat the process which starts by decomposition of $f \circ \kappa_{r+t}$ into a sum of $(f \circ \kappa_{r+t})(\kappa_{r_0}(w))$ and $(f \circ \kappa_{r+t})(w) - (f \circ \kappa_{r+t})(\kappa_{r_0}(w))$. Note that the number of repeated integrals increases by one (one new integration with respect to t) or by two (one new integration with respect to t and one new one dimensional "radial" integration). Moreover, each new integration comes with action of a second order (resp. first order) linear partial differential operator with smooth bounded coefficients. Clearly, we have an iterative procedure and after l iterations we obtain an estimate

$$|I| \le CM_S f(z) + C \sum_{0 \le k \le l} I(k,l),$$

where

$$\begin{aligned} I(k,l) &= \int_0^1 r^\alpha dr \int_0^{r_0} \cdots \int_0^{r_0} ds_k \ldots ds_1 \int_0^{r_0} \int_{t_1}^{r_0} \cdots \int_{t_{l-1}}^{r_0} dt_l \ldots dt_1 \int_{\partial D} d\sigma(\xi) \\ &\quad |f(\kappa_{s_1+\cdots+s_k+t_l}(\xi))A_{k+l}[G_r(\kappa_{s_1+\cdots+s_k}(\xi))a(\xi,r,s_1,\ldots,s_k,t_1,\ldots,t_l)]|, \end{aligned}$$

the order of operator A_{k+l} is $k+l$. If we apply (41) to $w = \kappa_{r+s_1+\cdots s_k}(\xi)$ we conclude that either the integrand is zero or

$$\begin{aligned} d_D(\kappa_{s_1+\cdots+s_k+t_l}(\xi),z) &\le Qd_D(\kappa_{s_1+\cdots s_k+r}(\xi),\kappa_{s_1+\cdots+s_k+t_l}(\xi)) \\ &\quad + Qd_D(\kappa_{s_1+\cdots+s_k}(\xi),z)) \\ &< Qr_0 + Q^2(1+\delta)r_0 < Q^2(2+\delta)r_0. \end{aligned}$$

Since $\rho(\kappa_{s_1+\cdots+s_k+t_l}(\xi) \geq t_l$ we can assume that in the integrand the following estimate holds:

$$\begin{aligned}|f(\kappa_{s_1+\cdots+s_k+t_l}(\xi))| &\leq \left(1+\frac{d_D(z,\kappa_{s_1+\cdots+s_k+t_l}(\xi))}{\rho(\kappa_{s_1+\cdots+s_k+t_l}(\xi))}\right)^T M^T f(z)\\ &\leq \left(1+\frac{Q^2(2+\delta)r_0}{t_l}\right)^T M^T f(z)\\ &\leq C\left(\frac{r_0}{t_l}\right)^T M^T f(z)\end{aligned}$$

Also, if the integrand is nonzero, then we have $d_D(\kappa_{r+s_1+\cdots+s_k}(\xi), z_0) \leq r_0$ and this implies

$$\begin{aligned}d_D(\kappa_r(\xi), z_0) &\leq Q(d_D(z_0, \kappa_{r+s_1+\cdots+s_k}(\xi)) + d_D(\kappa_{r+s_1+\cdots+s_k}(\xi), \kappa_r(\xi)))\\ &\leq Q(r_0+s_1+\cdots+s_k) \leq Q(k+1)r_0.\end{aligned}$$

Therefore in the surface integration appearing in (50) one can restrict domain of integration to $E_r = \{\xi \in \partial D : d_D(\kappa_r(\xi), z_0) \leq Q(k+1)r_0\}$. Hence, using (47), we obtain:

$$\begin{aligned}I(k,l) &\leq C\frac{M^T f(z)}{\lambda_\alpha(B_D(z_0,r_0))r_0^{k+l}}\int_0^1 r^\alpha dr\int_0^{r_0}\cdots\int_0^{r_0} ds_k\ldots ds_1\\ &\quad \int_0^{r_0}\int_{t_1}^{r_0}\cdots\int_{t_{l-1}}^{r_0} dt_l\ldots dt_1\int_{E_r} d\sigma(\xi)\left(\frac{r_0}{t_l}\right)^T\\ &\leq C\frac{M^T f(z) r_0^{T-l}}{\lambda_\alpha(B_D(z_0,r_0))}\int_0^1 r^\alpha\int_0^{r_0}\int_{t_1}^{r_0}\cdots\int_{t_{l-1}}^{r_0}\frac{1}{t_l^T}dt_l\ldots dt_1\int_{E_r} d\sigma(\xi).\end{aligned}$$

The above estimate is valid for any $l \geq 1$. Now we choose $l = L$, which, since $L > T$, ensures convergence of the iterated integrals over t-variables. This gives, for $0 \leq k \leq L$, the following estimate

$$I(k,L) \leq C\frac{M^T f(z)}{\lambda_\alpha(B_D(z_0,r_0))}\int_0^1 r^\alpha \sigma(E_r)dt \leq CM^T f(z), \tag{50}$$

which completes the proof. □

The above proposition combined with Proposition 9 and Proposition 10 gives the main result of this section.

Theorem 1. *If $L > 0$ is an integer such that $Lp > n+1+\alpha$, then*

$$\|\mathcal{M}_{L,\delta} f(z)\|_{p,\alpha} \leq C\|f\|_{p,\alpha}, \qquad f \in A^p_\alpha. \tag{51}$$

5. Atoms in $L^p_\alpha(D)$, $0 < p \leq 1$, $\alpha > -1$

Starting from this section we assume that $0 < p \leq 1$. We set

$$N_{p,\alpha} = \left(\frac{1}{p}-1\right)\max\{2(n+1), 2(n+1+\alpha)\}. \tag{52}$$

We set, for $\phi \in C^\infty(B_D(z,r))$, where $z \in W^+_{r_2}$ and $0 < r < \epsilon_1$ (for the choice of ϵ_1 see a remark after formula (9)), and a non-negative integer N:

$$\|\phi\|_{N,(z,r)} = \sum_{|J|=N} r^{d(J)}\|D^J_z\phi\|_{L^\infty(B_D(z,r))}. \tag{53}$$

Let $N > N_{p,\alpha}$ be an integer. We say that a measurable function a on D is an (p,α,N) atom if there are $z \in W^+_{r_2}$ and $r < \epsilon_1$ such that the following four conditions are satisfied:

1^o. Support: supp $a \subset B_D(z,r)$.
2^o. Size: $\|a\|_{L^\infty(D)} \leq \lambda_\alpha(B_D(z,r))^{-1/p}$.
3^o. Cancelation: $\int_D a d\lambda_\alpha = 0$.
4^o. Moments condition: For any $\phi \in C^\infty(B_D(z,r))$ we have

$$\left| \int_{B_D(z,r)} a(w)\phi(w) d\lambda_\alpha(w) \right| \leq \|\phi\|_{N,(z,r)} \lambda_\alpha(B_D(z,r))^{1-1/p}. \tag{54}$$

Also, any measurable function a such that $\|a\|_{L^\infty(D)} \leq 1$ is considered to be an (p,α,N) atom. This allows us to work near the boundary in questions related to atomic decomposition. When $N = [N_{p,\alpha}] + 1$ we say simply (p,α)-atom.

Strictly speaking, the above atoms should be called (p,∞,N,α) atoms, or (p,∞,α) atoms, indicating that one could consider instead of 2^o condition

$$\|a\|_{q,\alpha} \leq \lambda_\alpha(B_D(z,r))^{1/q-1/p}$$

and work with (p,q,N,α)-atoms. Since we will always have $q = \infty$ we use the above convention.

Lemma 3. *For any (p,α,N) atom a we have $\|a\|_{p,\alpha} \leq 1$.*

Proof. This follows immediately from conditions 1^o and 2^o. □

Let c_k be a sequence in l^p, $0 < p \leq 1$ and let a_k be a sequence of (p,α,N) atoms on D. Then the series $\sum_k c_k a_k$ converges in L^p_α to a function f and we have $\|f\|_{p,\alpha} \leq \sum_k |c_k|^p$. Real variable atomic weighted Bergman space $At^p_\alpha(D) = At^p_\alpha$ is defined as the set of all $f \in L^p_\alpha$ which admit representation

$$f = \sum_{k=1}^\infty c_k a_k, \qquad c = (c_k)_{k=1}^\infty \in l^p, \quad a_k \text{ are } (p,\alpha) \text{ atoms}, \quad k \geq 1. \tag{55}$$

Note that the series in (55) converges unconditionally.

For $f \in At^p_\alpha$ we define its "atomic quasi-norm" by

$$\|f\|^p_{p,\alpha,at} = \inf \left\{ \sum_{k=1}^\infty |c_k|^p : f = \sum_{k=1}^\infty c_k a_k \right\}, \tag{56}$$

where infimum is taken over all representations of f as described above.

The following estimate on the derivatives of the Bergman kernel K_α due to C. Fefferman, see [7], is essential ingredient in the proof of Theorem 2 below.

Lemma 4. *Let $N \geq 0$ be an integer. Then there is a constant C depending only on N, D and α such that for every z_0 and z in $W^+_{r_2}$ satisfying $d_D(z,z_0) \leq \epsilon_1$ we have*

$$|D^J_{z_0}(K_\alpha(z,\cdot))(w)| \leq \frac{C}{d_D(z,w)^{d(J)}\lambda_\alpha(B_D(w,d_D(z,w)))} \quad \textit{if}$$

$$d_D(w,z_0) < \frac{d_D(z,z_0)}{Q^2}.$$

The main result of this section is the following theorem.

Theorem 2. *Let $N > N_{p,\alpha}$ be an integer. Then there is a constant $C = C(D,p,\alpha,N)$ such that*

$$\|P_\alpha(a)\|_{p,\alpha} \leq C \tag{57}$$

for any (p,α,N) atom a.

Proof. Let a be an (p,α,N) atom. Assume $\|a\|_\infty \leq 1$. Since P_α is bounded on L^2_α, $p<2$ and $\lambda_\alpha(D)<\infty$ we have

$$\|P_\alpha(a)\|_{p,\alpha} \leq C\|P_\alpha(a)\|_{2,\alpha} \leq C\|a\|_{2,\alpha} \leq C\lambda_\alpha(D)^{1/2}.$$

Next, let a be an atom supported in $B_D(z_0,r_0)$, $z_0 \in W^+_{r_2}$, $0<r_0<\epsilon_1$ satisfying conditions 1^o - 4^o with z_0 and r_0 in place of z and r. We decompose D into $D_0=\{z\in D : d_D(z,z_0)\leq Q^2r_0\}$ and $D_k=\{z\in D : Q^{k+1}r_0 < d_D(z,z_0)\leq Q^{k+2}r_0\}$, $k\geq 1$. Then, since P_α is bounded on L^2_α, we have

$$\begin{aligned}\int_{D_0}|P_\alpha(a)(z)|^p d\lambda_\alpha(z) &\leq \left(\int_{D_0}|P_\alpha(a)(z)|^2 d\lambda_\alpha(z)\right)^{p/2}\lambda_\alpha(B_D(z_0,Q^2r_0))^{1-2/p}\\ &\leq C\|a\|^p_{2,\alpha}\lambda_\alpha(B_D(z_0,Q^2r_0))^{1-2/p}\leq C.\end{aligned}$$

In the last inequality we used conditions 1^o and 2^o. For every $k\geq 1$ we have, using moments condition 4^o, the following estimate:

$$\begin{aligned}\int_{D_k}|P_\alpha(a)(z)|^p d\lambda_\alpha(z) &= \int_{D_k}\left|\int_D K_\alpha(z,\zeta)a(\zeta)d\lambda_\alpha(\zeta)\right|^p d\lambda_\alpha(z)\\ &= \int_{D_k}\left|\int_{B_D(z_0,r_0)} K_\alpha(z,\zeta)a(\zeta)d\lambda_\alpha(\zeta)\right|^p d\lambda_\alpha(z)\\ &\leq C\int_{D_k}\left(\|K_\alpha(z,\cdot)\|_{N,(z_0,r_0)}\lambda_\alpha(B_D(z_0,r_0))^{1-1/p}\right)^p d\lambda_\alpha(z)\\ &= C\lambda_\alpha(B_D(z_0,r_0))^{p-1}\int_{D_k}\|K_\alpha(z,\cdot)\|^p_{N,(z_0,r_0)}d\lambda_\alpha(z)\end{aligned}$$

In order to estimate the integrand, let us note at first that for $w\in B_D(z_0,r_0)$ and $z\in D_k$ we have $d_D(w,r_0)<r_0$ and $d_D(z_0,z)>Q^{k+1}$; therefore for such z and w we have

$$d_D(w,z_0) < r_0 \leq Q^{k-1}r_0 < \frac{d_D(z,z_0)}{Q^2}, \tag{58}$$

$$d_D(z,w) \geq \frac{1}{Q}(d_D(z,z_0) - Qd(w,z_0)) > Q^k r_0 - r_0 > Q^{k-1}r_0. \tag{59}$$

Also, if $\zeta\in B_D(z_0,Q^{k-1}r_0)$ then

$$d_D(\zeta,w)\leq Q(d_D(\zeta,z_0)+d(z_0,w))\leq Q^k r_0 + Qr_0 \leq Q^{k+1}r_0$$

for every $w\in B_D(z_0,r_0)$. Therefore, using (59), we obtain for $w\in B_D(z_0,r_0)$ and $z\in D_k$ the following inclusions

$$B_D(z_0,Q^{k-1}r_0)\subset B_D(w,Q^{k+1}r_0)\subset B_D(w,Q^2d(z,w)). \tag{60}$$

Now (58) ensures that we can apply the estimates of derivatives of K_α given in Lemma 4 and obtain, using respectively (59), doubling property of λ_α, (60) and inequality $d(J)\geq |J|/2$, the following uniform estimate over $z \in D_k$:

$$\begin{aligned}
\|K_\alpha(z,\cdot)\|_{N,(z_0,r_0)} &= \sum_{|J|=N} r_0^{d(J)} \sup_{w\in B_D(z_0,r_0)} |D^J_{z_0} K_\alpha(z,\cdot)(w)| \\
&\le C\sum_{|J|=N} r_0^{d(J)} \sup_{w\in B_D(z_0,r_0)} \frac{1}{d_D(z,w)^{d(J)}\lambda_\alpha(B_D(w,d_D(z,w)))} \\
&\le C\sum_{|J|=N} r_0^{d(J)} \sup_{w\in B_D(z_0,r_0)} \frac{1}{(Q^{k-1}r_0)^{d(J)}\lambda_\alpha(B_D(w,d_D(z,w)))} \\
&\le C\sum_{|J|=N} (Q^{k-1})^{-d(J)} \sup_{w\in B_D(z_0,r_0)} \frac{1}{\lambda_\alpha(B_D(w,Q^2 d_D(z,w)))} \\
&\le C\sum_{|J|=N} (Q^{k-1})^{-d(J)} \frac{1}{\lambda_\alpha(B_D(z_0,Q^{k-1}r_0))} \\
&\le C(Q^{k-1})^{-N/2}\frac{1}{\lambda_\alpha(B_D(z_0,Q^{k-1}r_0))}.
\end{aligned} \tag{61}$$

Therefore, using doubling property of λ_α, we obtain from the previous estimates

$$\begin{aligned}
\int_{D_k} |P_\alpha(a)(z)|^p d\lambda_\alpha(z) &\le C(Q^{k-1})^{-pN/2}\frac{\lambda_\alpha(D_k)\lambda_\alpha(B_D(z_0,r_0))^{p-1}}{\lambda_\alpha(B_D(z_0,Q^{k-1}r_0))^p} \\
&\le C(Q^{k-1})^{-pN/2}\left(\frac{\lambda_\alpha(B_D(z_0,r_0))}{\lambda_\alpha(B_D(z_0,Q^{k-1}r_0))}\right)^{p-1} = I_k
\end{aligned}$$

Let k_0 be the largest integer k such that $Q^{k-1}r_0 \le \rho(z_0)$. Then using Proposition 5 we obtain

$$I_k \le C\frac{1}{(Q^{k-1})^{\frac{pN}{2}}}\frac{1}{Q^{(k-1)(n+1)(p-1)}} = [Q^{\frac{p}{2}(k-1)}]^{2(n+1)(1/p-1)-N}, \quad k\le k_0 \tag{62}$$

However, since $N > 2(n+1)(1/p-1)$ due to $N \ge N_{p,\alpha}$ we obtain $\sum_{k\le k_0} I_k \le C$, where C depends only on Q and the constants involved in Lemma 4. Similarly we obtain, for $k > k_0$

$$I_k \le C\frac{1}{Q^{(k-1)\frac{pN}{2}}}\frac{1}{Q^{(k-1)(n+1+\alpha)(p-1)}} = [Q^{\frac{p}{2}(k-1)}]^{2(n+1+\alpha)(\frac{1}{p}-1)-N}, \quad k > k_0 \tag{63}$$

Again, since $N > 2(n+1+\alpha)(1/p-1)$ due to $N \ge N_{p,\alpha}$ we obtain $\sum_{k>k_0} I_k \le C$ where $C = C(D,\alpha)$.

Adding up these estimates we complete the proof:

$$\int_D |P_\alpha(a)|^p d\lambda_\alpha \le C + \sum_{k\le k_0} I_k + \sum_{k>k_0} I_k \le C \qquad \square. \tag{64}$$

Corollary 1. *If a_k is a sequence of (p,α) atoms, then for any complex sequence c_k in l^p the series $\sum_k c_k P_\alpha(a_k)(z)$ converges unconditionally in $A^p_\alpha(D)$ to a function $g \in A^p_\alpha(D)$ and we have*

$$\|g\|^p_{p,\alpha} \le C\sum_{k=1}^\infty |c_k|^p, \tag{65}$$

where C is a constant depending only on D, p and α.

An equivalent statement is that the Bergman projection maps $At^p_\alpha(D)$ continuously into $A^p_\alpha(D)$:

$$\|P_\alpha(f)\|_{p,\alpha} \le C\|f\|_{p,\alpha,at}, \qquad f \in At^p_\alpha(D). \tag{66}$$

6. Partitions of unity

We use the following notation: $\mathrm{dist}_D(z, A) = \inf\{d_D(z, a) : a \in A\}$ for z in D and $A \subset D$. The next lemma is a well known Whitney type decomposition for homogeneous spaces, applied to (D, d_D, λ_α), see for example [19], Lemma 2.4.

Lemma 5. *There is an integer $N_0 \geq 1$ and there are constants $0 < \lambda < \nu < 1 < \mu$ such that for every open set $\Omega \subset D$, $\Omega \neq D$ there is a sequence of pseudo-balls $B_D(z_i, r_i)$ in Ω such that*

1^o. *For every i we have $r_i = \frac{1}{2Q}\mathrm{dist}_D(z_i, \Omega^c)$.*

2^o. $\Omega = \bigcup_i B_D(z_i, \nu r_i)$.

3^o. $B_D(z_i, \mu r_i) \cap \Omega^c \neq \emptyset$ *for every i.*

4^o. *The balls $B_D(z_i, \lambda r_i)$ are pairwise disjoint.*

5^o. *No point in Ω lies in more than N_0 of the balls $B_D(z_i, r_i)$.*

The following lemma is proved with help of the above lemma in the same fashion as in [9], Lemma 4.3.

Lemma 6. *Let V be an open subset of D, $V \neq D$. Then there is a sequence $B_D(z_j, r_j)$ of pseudo-balls, a sequence of functions $\phi_j \in C^\infty(D)$ and a constant $\mu > 1$ depending only on D satisfying the following properties:*

1^o. $0 \leq \phi_j(z) \leq 1$, $z \in D$.

2^o. $\sum_j \phi_j = \chi_V$.

3^o. $\mathrm{supp}\, \phi_j \subset B_D(z_j, r_j)$.

4^o. *For any nonnegative integer L there is a constant $c_L > 0$ depending only on L and D such that for each j and any $w_j \in B_D(z_j, \mu r_j) \setminus V$ we have*

$$\frac{c_L}{\|\phi_j\|_{1,\alpha}}\phi_j \in \mathcal{D}_{L,\delta}(w_j).$$

7. Atomic Decomposition

Finally we state the main result of this paper.

Theorem 3. *Let $0 < p \leq 1$ and $\alpha > -1$. Let $N > N_{p,\alpha}$ be an integer. For any $f \in A^p_\alpha(D)$ there is a sequence c_k of complex numbers such that $\sum_k |c_k|^p < \infty$ and a sequence a_k of (p, α, N) atoms such that:*

1^o. $f = \sum_k c_k a_k$ *in the sense of distributions.*

2^o. $f = \sum_k c_k a_k$, *where the series converges unconditionally in $L^p(D, \lambda_\alpha)$.*

3^o. $f = \sum_k c_k P(a_k)$, *where the series converges unconditionally in $A^p_\alpha(D)$.*

4^o. $\sum_k |c_k|^p \leq C\|f\|^p_{p,\alpha}$, *where constant C depends only on D, p, N and α.*

In contrast with the case of Hardy spaces, unconditional convergence in norm is obtained even for $p < 1$. Let us note that Theorem 3 combined with Theorem 2 gives the following proposition.

Proposition 12. *Let $0 < p \leq 1$ and $\alpha > -1$. Then we have*

$$\|f\|^p_{p,\alpha} \asymp \inf\left\{\sum_k |c_k|^p : f = \sum_k c_k P_\alpha(a_k)\right\}, \qquad f \in A^p_\alpha(D), \tag{67}$$

where infimum is taken over all decompositions of f as a sum of series $\sum_k c_k a_k$ as described in the above theorem.

The proof of Theorem 3 is rather long, let us observe that it suffices to prove existence of c_k and a_k such that 1^o and 4^o holds. Indeed, since $\|a\|_{p,\alpha} \leq 1$ by Lemma 3, we obtain 2^o. Then, using $\|P_\alpha(a_k)\|_{p,\alpha} \leq C$ obtained in Theorem 2, we get $f = P_\alpha(\sum_k c_k a_k) = \sum_k c_k P_\alpha(a_k)$, which is part 3^o.

We begin the proof of Theorem 3, following closely arguments from [3]. Let us choose $f \in A^p_\alpha$ and select $N > N_{p,\alpha}$. Let $\mu > 1$ be the constant from Lemma 5. Let us choose $\delta > 0$ and an integer $L \geq \max\{N, \frac{1}{p}(n+1+\alpha)+1\}$. Using Theorem 1 and Proposition 9 we obtain

$$\mathcal{M}_{L,\mu} f + M_\delta f \in L^p_\alpha(D).$$

Let $k_0 \in \mathbb{Z}$ be the smallest integer such that

$$\|\mathcal{M}_{L,\mu} f + M_\delta f\|_{p,\alpha} \leq 2^{k_0} \lambda_\alpha(D)^{1/p}.$$

For $k = 0, 1, 2, \ldots$ we define

$$V_k = \{z \in D : \mathcal{M}_{L,\mu}(z) + M_\delta f(z) > 2^{k_0+k}\}.$$

Then V_k is an open subset of D and, by the choice of k_0, $V_k \neq D$ for any $k \geq 0$. If $V_k = \emptyset$ for some k the theorem is trivial, because we consider any measurable function a such that $\|a\|_{L^\infty(D)} \leq 1$ to be an atom. Therefore we assume $V_k \neq \emptyset$ for all $k \geq 0$.

For each $k \geq 0$ we choose a Whitney type cover $(B_D(z^k_j, r^k_j))_{j=1}^\infty$ of V_k and a partition of unity $(\phi^k_j)_{j=1}^\infty$ for V_k as given in Lemma 6.

For each j, k we have a probability measure

$$d\lambda_{\alpha,\phi^k_j} = \frac{\phi^k_j}{\|\phi^k_j\|_{1,\alpha}} d\lambda_\alpha$$

and the corresponding Hilbert space $L^2(D, d\lambda_{\alpha,\phi^k_j})$ which we denote simply by L^2_{α,ϕ^k_j}; the norm on this space is denoted by $\|\cdot\|_{\alpha,\phi^k_j}$.

Let, for an integer $L \geq 0$, $V^L_{\phi^k_j}$ denote the finite dimensional subspace of L^2_{α,ϕ^k_j} spanned by "polynomials" $P^J_{z^k_j}$, $|J| \leq L$. Let $\pi_J(z)$, $|J| \leq L$, be an orthonormal basis for $V^L_{\phi^k_j}$.

The following lemma is proved in the same manner as Lemma 4.3 from [3], see also [9].

Lemma 7. *For every integer $L \geq 0$ there is a constant $C_L > 0$ depending only on L and D such that*

$$\frac{C_L}{\|\phi^k_j\|_{1,\alpha}} \pi_J \phi^k_j \in \mathcal{D}_{L,\mu}(w^k_j) \tag{68}$$

for all $w^k_j \in B_D(z^k_j, \mu r^k_j) \setminus V_k$.

Let $\mathcal{P}^L_{\phi^k_j}$ be the orthogonal projections of L^2_{α,ϕ^k_j} onto its subspace $V^L_{\phi^k_j}$. Using the above lemma one deduces, as in [8], the following lemma (see also [3]):

Lemma 8. *There is a constant $C > 0$ such that for all $f \in A^p_\alpha$ and all $i, j, k \in \mathbb{N}$ we have*

$$\left| \mathcal{P}^L_{\phi^k_j}(f)(z)\phi^k_j(z) \right| \leq C2^{k_0+k} \tag{69}$$

and

$$\left| \mathcal{P}^L_{\phi^{k+1}_j}\left(\left[f - \mathcal{P}^L_{\phi^{k+1}_j}(f) \right] \phi^k_j \right)(z)\phi^{k+1}_j(z) \right| \leq C2^{k_0+k+1}. \tag{70}$$

Continuing with the proof of Theorem 3 we have, for each $k \geq 1$:

$$f = \left(f - \sum_{j=1}^{\infty} f\phi_j^k \right) + \sum_{j=1}^{\infty} f\phi_j^k = h_k + \sum_{j=1}^{\infty} \left(f - \mathcal{P}^L_{\phi_j^k}(f) \right) \phi_j^k, \tag{71}$$

where

$$h_k = \left(f - \sum_{j=1}^{\infty} f\phi_j^k \right) + \sum_{j=1}^{\infty} \mathcal{P}^L_{\phi_j^k}(f)\phi_j^k. \tag{72}$$

Using (69) and property 5^o from Lemma 5 we obtain

$$\left| \sum_{j=1}^{\infty} \mathcal{P}^L_{\phi_j^k}(f)(z)\phi_j^k(z) \right| \leq C2^{k_0+k}.$$

Since $(\phi_j^k)_{j=1}^{\infty}$ form a partition of unity for V_k we have

$$\operatorname{supp}(f - h_k) = \operatorname{supp}\left(\sum_{j=1}^{\infty} \left[f - \mathcal{P}^L_{\phi_j^k}(f) \right] \phi_j^k \right) \subset V_k, \qquad k \geq 1.$$

Since $\cap_{k=1}^{\infty} V_k = \emptyset$ this means that $\lim_{k\to\infty}(f - h_k)(z) = 0$ a.e. $z \in D$. Therefore

$$f = h_0 + \sum_{k=0}^{\infty} (h_{k+1} - h_k) \qquad \text{a.e.} \tag{73}$$

Now one easily sees that

$$\begin{aligned}
h_{k+1} - h_k &= (f - h_k) - (f - h_{k+1}) \\
&= \sum_{j=1}^{\infty} \left[f - \mathcal{P}^L_{\phi_j^k}(f) \right] \phi_j^k - \sum_{j=1}^{\infty} \left[f - \mathcal{P}^L_{\phi_j^{k+1}}(f) \right] \phi_j^{k+1} \\
&= \sum_{j=1}^{\infty} \left[f - \mathcal{P}^L_{\phi_j^k}(f) \right] \phi_j^k \\
&- \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ \left[f - \mathcal{P}^L_{\phi_i^{k+1}}(f) \right] \phi_j^k - \mathcal{P}^L_{\phi_i^{k+1}} \left(\left[f - \mathcal{P}^L_{\phi_i^{k+1}(f)} \right] \phi_j^k \right) \right\} \phi_j^{k+1} \\
&= \sum_{j=1}^{\infty} b_j^k
\end{aligned}$$

where

$$\begin{aligned}
b_j^k &= \left[f - \mathcal{P}^L_{\phi_j^k}(f) \right] \phi_j^k \\
&- \sum_{i=1}^{\infty} \left\{ \left[f - \mathcal{P}^L_{\phi_i^{k+1}}(f) \right] \phi_j^k - \mathcal{P}^L_{\phi_i^{k+1}} \left(\left[f - \mathcal{P}^L_{\phi_i^{k+1}(f)} \right] \phi_j^k \right) \right\} \phi_j^{k+1}.
\end{aligned}$$

Therefore we have

$$f = h_0 + \sum_{j,k} b_j^k.$$

This gives us desired atomic decomposition after normalization. Namely, we set $a_j^k = 2^{-k}\lambda_\alpha(B_D(z_j^k, r_j^k))^{-1/p} b_j^k$. The conditions on support and size follow from the construction (since $(\phi_j^k)_{j=1}^\infty$ is a partition of unity for V_k) and from Lemma 8. Lemma 7 ensures that a_j^k satisfy the moment and cancellation conditions, Therefore a_j^k are atoms. Next we have, using definition of the sets V_k:

$$\begin{aligned}
\sum_{j,k} 2^{kp}\lambda_\alpha(B_D(z_j^k, r_j^k)) &\leq C\sum_k 2^{kp}\lambda_\alpha(V_k) \\
&\leq C\int_1^\infty t^{p-1}\lambda_\alpha\{z : (\mathcal{M}_{L,\mu} f + M_\delta f)(z) > 2^{k_0+k}\}\, dt \\
&\leq C\|f\|_{p,\alpha}^p.
\end{aligned}$$

Theorem 1 is used in the last inequality and property 5^o from Lemma 5 in the first inequality. This gives 4^o, since we have almost everywhere convergence condition 1^o is also satisfied. □.

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