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L-Topological Derived Internal (resp. Enclosed) Relation Spaces

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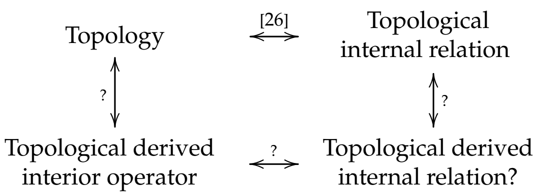
Abstract. In this paper, notions of L-topological derived internal relation space, L-topological derived interior operator space, L-topological derived enclosed relation space and L-topological derived closure operator space are introduced. It is proved that all of these spaces are categorically isomorphic to L-topological space, L-topological internal relation space and L-topological enclosed relation space.

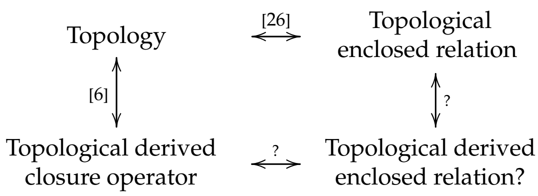
1. Introduction

Since Zadeh introduce the concept of fuzzy set [38], fuzzy set theory has been combined with many mathematical branches, such as fuzzy topology [1, 11, 36, 37, 39], fuzzy convergence [5, 7–9, 12, 13, 17, 35], fuzzy matroid [4, 21, 22] and fuzzy convexities [10, 14–20, 27–31, 33, 34] and so on.

Derived operator is an important tool to characterized many mathematical structures such as topological spaces, M-fuzzifying matroids and M-fuzzifying convex spaces [2, 6, 18, 32, 40]. Among many characterizations of L-topological spaces [3, 23], Shi et al characterized L-topological spaces by L-topological enclosed relation spaces and L-topological internal relation spaces [26]. Then a natural question arises: is there any topological derived internal relation or topological derived enclosed relation which can be used to characterize topologies? That is, is there any topological derived internal relation or any topological derived enclosed relation such that the following diagrams communicate? Do them hold in L-fuzzy setting?



Problem 1.



Problem 2.

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Keywords. L-topological derived internal space, L-topological derived enclosed space, L-topological derived interior space, L-topological derived closure space

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The aim of this paper is to accomplish the above diagrams in L-fuzzy setting. The arrangement of this paper is as follows. In Section 2, we recall some basic concepts, denotations and results. In Section 3, we introduce L-topological derived internal relation spaces and L-topological derived interior spaces by which we obtain two characterizations of L-topological spaces. In Section 4, we introduce L-topological derived enclosed relation spaces and L-topological derived closure spaces by which we obtain two other characterizations of L-topological spaces.

2. Preliminaries

In this paper, X and Y are nonempty sets. The power set of X is denoted by 2^X. L is a completely distributive lattice with an inverse involution $'$. The smallest (resp. largest) element in L is denoted by $\bot$ (resp. $\top$). An element $a \in L$ is called a co-prime, if for all $b, c \in L$, $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. The set of all co-primes in $L\backslash\{\bot\}$ is denoted by $J(L)$. For any $a \in L$, there is an $L_1 \in J(L)$ such that $a = \bigvee_{b \in L_1} b$. A binary relation $\prec$ on L is defined by $a \prec b$ iff for each $L_1 \subseteq L$, $b \leq \bigvee L_1$ implies some $d \in L_1$ such that $a \leq d$. The mapping $\beta : L \to 2^L$, defined by $\beta(a) = \{b : b \prec a\}$, satisfies $\beta(\bigvee_{i \in I} a_i) = \bigcup_{i \in I} \beta(a_i)$ for any $\{a_i\}_{i \in I} \subseteq L$. For any $a \in L$, $\beta(a)$ and $\beta^*(a) = \beta(a) \cap J(L)$ satisfies $a = \bigvee \beta(a) = \bigvee \beta^*(a)$ [23].

An L-fuzzy set on X is a mapping $A : X \to L$. The set of all L-fuzzy sets on X is denoted by L^X. The smallest (resp, largest) element in L^X is denoted by $\underline{\bot}$ (resp. $\underline{\top}$). For a mapping $f : X \to Y$, the L-fuzzy mapping $f_L^{\to} : L^X \to L^Y$ is defined by $f_L^{\to}(A)(y) = \bigvee\{A(x) : f(x) = y\}$ for $A \in L^X$ and $y \in Y$, and the mapping $f_L^{\leftarrow} : L^Y \to L^X$ is defined by $f_L^{\leftarrow}(B)(x) = B(f(x))$ for $B \in L^Y$ and $x \in X$ [23].

Definition 2.1. ([23]) A subset $\mathcal{T} \subseteq L^X$ is called an L-topology on X and $(X, \mathcal{T})$ is called an L-topological space if

(LT1) $\underline{\top}, \underline{\bot} \in \mathcal{T}$;

(LT2) $\bigvee_{i \in I} A_i \in \mathcal{T}$ for any subset $\{A_i\}_{i \in I} \subseteq \mathcal{T}$;

(LT3) $A \vee B \in \mathcal{T}$ for all $A, B \in \mathcal{T}$.

Theorem 2.2. ([23]) *Let $(X, \mathcal{T})$ be an L-topological space.*

(1) The L-topological closure operator $Cl_{\mathcal{T}} : L^X \to L^X$ of $\mathcal{T}$ is defined by $Cl_{\mathcal{T}}(A) = \bigwedge\{B \in L^X : A \leq B, B' \in \mathcal{T}\}$ for any $A \in L^X$. It satisfies

(LCl1) $Cl_{\mathcal{T}}(\underline{\bot}) = \underline{\bot}$;

(LCl2) $A \leq Cl_{\mathcal{T}}(A)$;

(LCl3) $Cl_{\mathcal{T}}(Cl_{\mathcal{T}}(A)) = Cl_{\mathcal{T}}(A)$;

(LCl4) $Cl_{\mathcal{T}}(A \vee B) = Cl_{\mathcal{T}}(A) \vee Cl_{\mathcal{T}}(B)$.

Conversely, if an operator $Cl : L^X \to L^X$ satisfies (LCl1)–(LCl4), then the set $\mathcal{T}_{Cl} = \{A \in L^X : Cl(A') = A'\}$ is an L-topology satisfying $Cl_{\mathcal{T}_{Cl}} = Cl$.

(2) The L-topological interior operator $Int_{\mathcal{T}} : L^X \to L^X$ of $\mathcal{T}$ is defined by $Int_{\mathcal{T}}(A) = \bigvee\{B \in \mathcal{T} : B \leq A\}$ for any $A \in L^X$. It satisfies

(LInt1) $Int_{\mathcal{T}}(\underline{\top}) = \underline{\top}$;

(LInt2) $Int_{\mathcal{T}}(A) \leq A$;

(LInt3) $Int_{\mathcal{T}}(Int_{\mathcal{T}}(A)) = Int_{\mathcal{T}}(A)$;

(LInt4) $Int_{\mathcal{T}}(A \wedge B) = Int_{\mathcal{T}}(A) \wedge Int_{\mathcal{T}}(B)$.

Conversely, if an operator $Int : L^X \to L^X$ satisfies (LInt1)–(LInt4), then the set $\mathcal{T}_{Int} = \{A \in L^X : Int(A) = A\}$ is an L-topology satisfying $Int_{\mathcal{T}_{Int}} = Int$.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be L-topological spaces. A mapping $f : X \to Y$ is an L-continuous mapping, if $f_L^{\leftarrow}(A) \in \mathcal{T}_X$ for any $A \in \mathcal{T}_Y$. It is proved that a mapping $f : X \to Y$ is an L-continuous mapping iff $f_L^{\to}(Cl_{\mathcal{T}_X}(A)) \leq Cl_{\mathcal{T}_Y}(f_L^{\to}(A))$ for any $A \in L^X$, or alternatively, $f_L^{\leftarrow}(Int_{\mathcal{T}_Y}(B)) \leq Int_{\mathcal{T}_X}(f_L^{\leftarrow}(B))$ for any $B \in L^Y$. The category of L-topological spaces and L-continuous mappings is denoted by L-**TOP** [23].

Definition 2.3. ([26]) A binary relation $\preccurlyeq$ on L^X is called an L-topological enclosed relation and the pair $(X, \preccurlyeq)$ is called an L-topological enclosed relation space, if $\preccurlyeq$ satisfies

(LTER1) $\underline{\perp} \lessdot \underline{\perp}$;
(LTER2) $A \lessdot B$ implies $A \le B$;
(LTER3) $A \lessdot \bigwedge_{i \in I} B_i$ iff $A \lessdot B_i$ for all $i \in I$;
(LTER4) $A \lessdot B$ implies some $C \in L^X$ with $A \lessdot C \lessdot B$;
(LTER5) $A \vee B \lessdot C$ iff $A \lessdot C$ and $B \lessdot C$.

Let $(X, \lessdot_X)$ and $(Y, \lessdot_Y)$ be L-topological enclosed relation spaces. A mapping $f : X \to Y$ is called an L-topological enclosed relation preserving mapping, if $f_L^{\leftarrow}(A) \lessdot_X f_L^{\leftarrow}(B)$ for all $A, B \in L^Y$ with $A \lessdot_Y B$. The category of L-topological enclosed relation spaces and L-topological enclosed relation preserving mappings is denoted by L-**TERS** [26].

Theorem 2.4. ([26]) *(1) For an L-topological enclosed relation space $(X, \lessdot)$, the operator $Cl_{\lessdot} : L^X \to L^X$, defined by $Cl_{\lessdot}(A) = \bigwedge\{B \in L^X : A \lessdot B\}$ for any $A \in L^X$, is an L-topological closure operator of some L-topology $\mathcal{T}_{\lessdot}$.*

(2) For an L-topological space $(X, \mathcal{T})$, the binary operator $\lessdot_{\mathcal{T}}$, defined by $A \lessdot_{\mathcal{T}} B$ iff $Cl_{\mathcal{T}}(A) \le B$ for all $A, B \in L^X$, is an L-topological enclosed relation.

*(3) L-**TOP** is isomorphic to L-**TERS**.*

Definition 2.5. ([26]) A binary relation $\preccurlyeq$ on L^X is called an L-topological internal relation and the pair $(X, \preccurlyeq)$ is called an L-topological internal relation space, if $\preccurlyeq$ satisfies

(LTIR1) $\underline{\top} \preccurlyeq \underline{\top}$;
(LTIR2) $A \preccurlyeq B$ implies $A \le B$;
(LTIR3) $\bigvee_{i \in I} A_i \preccurlyeq B$ iff $A_i \preccurlyeq B$ for all $i \in I$;
(LTIR4) $A \preccurlyeq B$ implies some $C \in L^X$ with $A \preccurlyeq C \preccurlyeq B$;
(LTIR5) $A \preccurlyeq B \wedge C$ iff $A \preccurlyeq B$ and $A \preccurlyeq C$.

Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-topological internal relation spaces. A mapping $f : X \to Y$ is called an L-topological internal relation preserving mapping, if $f_L^{\leftarrow}(A) \preccurlyeq_X f_L^{\leftarrow}(B)$ for all $A, B \in L^Y$ with $A \preccurlyeq_Y B$. The category of L-topological internal relation spaces and L-topological internal relation preserving mappings is denoted by L-**TIRS** [26].

Theorem 2.6. ([26]) *(1) For an L-topological internal relation space $(X, \preccurlyeq)$, the operator $Int_{\preccurlyeq} : L^X \to L^X$, defined by $Int_{\preccurlyeq}(A) = \bigvee\{B \in L^X : B \preccurlyeq A\}$ for any $A \in L^X$, is an L-topological interior operator of some L-topology $\mathcal{T}_{\preccurlyeq}$.*

(2) For an L-topological space $(X, \mathcal{T})$, the binary operator $\preccurlyeq_{\mathcal{T}}$, defined by $A \preccurlyeq_{\mathcal{T}} B$ iff $A \le Int_{\mathcal{T}}(B)$ for all $A, B \in L^X$, is an L-topological internal relation.

*(3) L-**TOP** is isomorphic to L-**TIRS**.*

Definition 2.7. ([24]) (1) A mapping $\varphi : J(L^X) \to L^X$ is called a remote-neighborhood mapping, if $x_\lambda \not\le \varphi(x_\lambda)$ for any $x_\lambda \in J(L^X)$. The set of all remote-neighborhood mappings is denoted by $\mathcal{R}(L^X)$. For $\varphi, \psi \in \mathcal{R}(L^X)$, $\varphi \odot \psi \in \mathcal{R}(L^X)$ is defined by $\varphi \odot \psi(x_\lambda) = \bigwedge\{\varphi(y_\mu) : y_\mu \not\le \psi(x_\lambda)\}$ for any $x_\lambda \in J(L^X)$.

(2) A pointwise L-quasi-uniformity on X is a subset $\mathcal{U} \subseteq \mathcal{R}(L^X)$ satisfying the following conditions:

(LU1) $\varphi \in \mathcal{R}(L^X)$, $\psi \in \mathcal{U}$ and $\varphi \le \psi$ implies $\varphi \in \mathcal{U}$;
(LU2) $\varphi, \psi \in \mathcal{U}$ implies $\varphi \vee \psi \in \mathcal{U}$;
(LU3) $\varphi \in \mathcal{U}$ implies an $\psi \in \mathcal{U}$ such that $\varphi \le \psi \odot \psi$.

Theorem 2.8. ([26]) *Let $\mathcal{U}$ be a pointwise L-quasi-uniform on X.*

(1) Define a binary relation $\preccurlyeq_{\mathcal{U}}$ on X by $A \preccurlyeq_{\mathcal{U}} B$ iff there is $\varphi \in \mathcal{U}$ such that $B' \le \bigwedge_{x_\lambda \not\le A'} \varphi(x_\lambda)$ for all $A, B \in L^X$. Then $(X, \preccurlyeq_{\mathcal{U}})$ is an L-topological internal relation space.

(2) Define an another binary relation $\lessdot_{\mathcal{U}}$ on X by $A \lessdot_{\mathcal{U}} B$ iff there is $\varphi \in \mathcal{U}$ such that $A \le \bigwedge_{y_\mu \not\le B} \varphi(y_\mu)$ for all $A, B \in L^X$. Then $(X, \lessdot_{\mathcal{U}})$ is an L-topological enclosed relation space.

Definition 2.9. ([25]) A pointwise S-proximate on X is a mapping $\delta : J(L^X) \times L^X \to \{\perp, \top\}$ satisfying

(SP1) $\delta(x_\lambda, \underline{\perp}) = \perp$ for any $x_\lambda \in J(L^X)$;
(SP2) $\delta(x_\lambda, B) = \perp$ implies $x_\lambda \not\le B$;
(SP3) $\delta(x_\lambda, A \vee B) = \delta(x_\lambda, A) \vee \delta(x_\lambda, B)$;
(SP4) $\delta(x_\lambda, B) = \perp$ implies some $C \in L^X$ such that $\delta(x_\lambda, C) = \perp$ and $\delta(y_\mu, C) = \perp$ for any $y_\mu \not\le C$.

Theorem 2.10. ([26]) *Let δ be a pointwise S-proximate on X.*

(1) Define a binary relation $\preccurlyeq_\delta$ on L^X by $A \preccurlyeq_\delta B$ iff $\delta(x_\lambda, B') = \bot$ for all $A, B \in L^X$ and $x_\lambda \in J(L^X)$ with $x_\lambda \not\leq A'$. Then $(X, \preccurlyeq_\delta)$ is an L-topological internal relation space.

(2) Define an another binary relation $\preccurlyeq_\delta$ on L^X by $A \preccurlyeq_\delta B$ iff $\delta(y_\mu, A) = \bot$ for all $A, B \in L^X$ and $y_\mu \in J(L^X)$ with $x_\lambda \not\leq B$. Then $(X, \preccurlyeq_\delta)$ is an L-topological enclosed relation space.

3. *L*-Topological Derived Internal Relation Spaces

In this section, we introduce notions of *L*-topological derived internal relation space and *L*-topological derived interior space by which we characterize *L*-topological internal relation spaces and the category of *L*-topological spaces.

Definition 3.1. A binary operator $\preccurlyeq$ on L^X is called an *L*-topological derived internal relation and the pair $(X, \preccurlyeq)$ is called an *L*-topological derived internal relation space, if for all $A, B, C \in L^X$ and $x_\lambda \in \beta^*(\underline{\top})$,

(LTDIR1) $\underline{\top} \preccurlyeq \underline{\top}$;

(LTDIR2) $A \preccurlyeq B$ iff $x_\lambda \preccurlyeq B \vee x_\lambda$ for any $x_\lambda \in \beta^*(A)$;

(LTDIR3) $\bigvee_{i\in I} A_i \preccurlyeq B$ if and only if $A_i \preccurlyeq B$ for any $i \in I$;

(LTDIR4) $A \preccurlyeq B$ implies $A \wedge B \preccurlyeq C \preccurlyeq B$ for some $C \in L^X$ with $A \wedge B \leq C$;

(LTDIR5) $A \preccurlyeq B \wedge C$ if and only if $A \preccurlyeq B$ and $A \preccurlyeq C$.

It directly follows from (LTDIR3) and (LTDIR5) that $A \preccurlyeq B$ for all $A, B, C, D \in L^X$ with $A \leq C \preccurlyeq D \leq B$.

Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be *L*-topological derived internal relation spaces. A mapping $f : X \to Y$ is called an *L*-topological derived internal relation preserving mapping, if for all $A, B \in L^Y$,

$$A \preccurlyeq_Y B \quad \text{implies} \quad f_L^{\leftarrow}(A \wedge B) \preccurlyeq_X f_L^{\leftarrow}(B).$$

The category of *L*-topological derived interval relation spaces and *L*-topological derived interval relation preserving mappings is denoted by *L*-**TDIRS**. Next, we discuss the relations between *L*-topological derived internal relation spaces and *L*-topological internal relation spaces.

Theorem 3.2. *Let $(X, \preccurlyeq)$ be an L-topological derived internal relation space. Define a binary relation $\preccurlyeq_\preccurlyeq$ on L^X by*

$$\forall A, B \in L^X, \quad A \preccurlyeq_\preccurlyeq B \;\Leftrightarrow\; \exists C \in L^X,\ C \preccurlyeq B,\ A = B \wedge C.$$

Then $(X, \preccurlyeq_\preccurlyeq)$ is an L-topological internal relation space.

Proof. We check that $\preccurlyeq_\preccurlyeq$ satisfies (LTIR1)–(LTIR5).

(LTIR1). We have $\underline{\top} \preccurlyeq \underline{\top}$ and $\underline{\top} \wedge \underline{\top} = \underline{\top}$ by (LTDIR1). Thus $\underline{\top} \preccurlyeq_\preccurlyeq \underline{\top}$.

(LTIR2). It directly follows from the definition.

(LTIR3). Let $\bigvee_{i\in I} A_i \preccurlyeq_\preccurlyeq B$. Then there is $C \in L^X$ such that $C \preccurlyeq B$ and $\bigvee_{i\in I} A_i = B \wedge C$. For any $i \in I$, we have $A_i \leq \bigvee_{i\in I} A_i \preccurlyeq B$. Thus $A_i \preccurlyeq B$ and $A_i = B \wedge A_i$. Hence $A_i \preccurlyeq_\preccurlyeq B$ for any $i \in I$.

Conversely, assume that $A_i \preccurlyeq_\preccurlyeq B$ for any $i \in I$. For any $i \in I$, there is a $C_i \in L^X$ such that $C_i \preccurlyeq B$ and $A_i = B \wedge C_i$. Thus $\bigvee_{i\in I} C_i \preccurlyeq B$ by (LTDIR3). Further, we have

$$\bigvee_{i\in I} A_i = \bigvee_{i\in I} (B \wedge C_i) = B \wedge \bigvee_{i\in I} C_i.$$

Hence $\bigvee_{i\in I} A_i \preccurlyeq_\preccurlyeq B$.

(LTIR4). Let $A \preccurlyeq_\preccurlyeq B$. Then there is a $D \in L^X$ such that $D \preccurlyeq B$ and $A = B \wedge D$. By $D \preccurlyeq B$ and (LTDIR4), there is a $C \in L^X$ such that $A = D \wedge B \preccurlyeq C \preccurlyeq B$ and $A \leq C$. Let $E = B \wedge C$. Then $E \preccurlyeq B$ and $A \leq E \leq B$. Thus $E \preccurlyeq_\preccurlyeq B$. Further, by $A \leq C \preccurlyeq B$, we have $A \preccurlyeq B$. Thus $A \preccurlyeq B \wedge C = E$ by (LTDER5). Hence $A \preccurlyeq_\preccurlyeq E$. Therefore we conclude that $A \preccurlyeq_\preccurlyeq E \preccurlyeq_\preccurlyeq B$ and $A \wedge B \leq E$ as desired.

(LTIR5). Let $A \preccurlyeq_\preccurlyeq B \wedge C$. Then there is a $D \in L^X$ such that $D \preccurlyeq B \wedge C$ and $A = (B \wedge C) \wedge D$. Let $E = C \wedge D$. Then $E \leq D \preccurlyeq B \wedge C \leq B$. Thus $E \preccurlyeq B$ and $A = B \wedge E$. Hence $A \preccurlyeq_\preccurlyeq B$. Similarly, let $F = B \wedge D$. Then $F \leq D \preccurlyeq B \wedge C \leq C$. Thus $F \preccurlyeq C$ and $A = C \wedge F$. Hence $A \preccurlyeq_\preccurlyeq C$.

Conversely, assume that $A \lessdot_{\preccurlyeq} B$ and $A \lessdot_{\preccurlyeq} C$. Then there are $D, E \in L^X$ such that $D \preccurlyeq B$, $E \preccurlyeq C$ and $A = B \wedge D = C \wedge E$. Thus $D \wedge E \preccurlyeq B$ and $D \wedge E \preccurlyeq C$. Hence $A = (B \wedge C) \wedge (D \wedge E)$ and $D \wedge E \preccurlyeq B \wedge C$ by (LTDIR5). Therefore $A \lessdot_{\preccurlyeq} B \wedge C$. □

Theorem 3.3. *Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-topological derived internal relation spaces. If $f : X \to Y$ is an L-topological derived internal relation preserving mapping, then $f : (X, \lessdot_{\preccurlyeq_X}) \to (Y, \lessdot_{\preccurlyeq_Y})$ is an L-topological internal relation preserving mapping.*

Proof. If $A \lessdot_{\preccurlyeq_Y} B$, then there is a $C \in L^Y$ such that $C \preccurlyeq_Y B$ and $A = B \wedge C$. Thus $f_L^{\leftarrow}(C) \wedge f_L^{\leftarrow}(B) \preccurlyeq_X f_L^{\leftarrow}(B)$ and $f_L^{\leftarrow}(A) = f_L^{\leftarrow}(C) \wedge f_L^{\leftarrow}(B)$. Hence $f_L^{\leftarrow}(A) \lessdot_{\preccurlyeq_X} f_L^{\leftarrow}(B)$. Therefore f is an L-topological internal relation preserving mapping. □

Theorem 3.4. *Let $(X, \preccurlyeq)$ be an L-topological internal relation space. Define a binary operator $\lessdot_{\preccurlyeq}$ on L^X by*

$$\forall A, B \in L^X,\ \ A \lessdot_{\preccurlyeq} B \ \ \Leftrightarrow\ \ \forall x_\lambda \in \beta^*(A),\ x_\lambda \preccurlyeq B \vee x_\lambda.$$

Then $(X, \lessdot_{\preccurlyeq})$ is an L-topological derived internal relation space.

Proof. It is clear that $A \lessdot_{\preccurlyeq} B$ for any $A, B, C, D \in L^X$ with $A \leq C \lessdot_{\preccurlyeq} D \leq B$. Next, we check that $\lessdot_{\preccurlyeq}$ satisfies (LTDIR1)–(LTDIR5).

(LTDIR1). If $x_\lambda \in \beta^*(\underline{\top})$, then $x_\lambda \leq \underline{\top} \preccurlyeq \underline{\top} = \underline{\top} \vee x_\lambda$ by (LTIR1). Thus $x_\lambda \preccurlyeq \underline{\top}$. Hence $\underline{\top} \lessdot_{\preccurlyeq} \underline{\top}$.

(LTDIR2). Let $A \lessdot_{\preccurlyeq} B$ and let $x_\lambda \in \beta^*(A)$. To prove that $x_\lambda \lessdot_{\preccurlyeq} B \vee x_\lambda$, let $x_\eta \in \beta^*(x_\lambda)$. Then $x_\eta \in \beta^*(A)$. By $A \lessdot_{\preccurlyeq} B$, we have

$$x_\eta \preccurlyeq B \vee x_\eta \leq (B \vee x_\lambda) \vee x_\eta.$$

Thus $x_\eta \preccurlyeq (B \vee x_\lambda) \vee x_\eta$. Hence $x_\lambda \lessdot_{\preccurlyeq} B \vee x_\lambda$.

Conversely, assume that $x_\lambda \lessdot_{\preccurlyeq} B \vee x_\lambda$ for any $x_\lambda \in \beta^*(A)$. Let $x_\lambda \in \beta^*(A)$. We check that $x_\lambda \preccurlyeq B \vee x_\lambda$.

By $x_\lambda \lessdot_{\preccurlyeq} B \vee x_\lambda$, we have $x_\eta \preccurlyeq B \vee x_\lambda \vee x_\eta = B \vee x_\lambda$ for any $x_\eta \in \beta^*(x_\lambda)$. Thus $x_\lambda = \bigvee_{x_\eta \in \beta^*(x_\lambda)} \preccurlyeq B \vee x_\lambda$ by (LTIR3). Hence $A \lessdot_{\preccurlyeq} B$.

(LTDIR3). Let $\bigvee_{i \in I} A_i \lessdot_{\preccurlyeq} B$. It is clear that $A_i \lessdot_{\preccurlyeq} B$ for any $i \in I$. Conversely, assume that $A_i \lessdot_{\preccurlyeq} B$ for any $i \in I$. To prove that $\bigvee_{i \in I} A_i \lessdot_{\preccurlyeq} B$, let $x_\lambda \in \beta^*(\bigvee_{i \in I} A_i)$. Then there is an $i \in I$ such that $x_\lambda \in \beta^*(A_i)$. By $A_i \lessdot_{\preccurlyeq} B$, we have $x_\lambda \preccurlyeq B \vee x_\lambda$. Therefore $\bigvee_{i \in I} A_i \lessdot_{\preccurlyeq} B$.

(LTDIR4). Let $A \lessdot_{\preccurlyeq} B$. We need to find some $E \in L^X$ such that $A \wedge B \lessdot_{\preccurlyeq} E \lessdot_{\preccurlyeq} B$ and $A \wedge B \leq E$.

If $A \wedge B = \underline{\perp}$, then it is easy to check that $E = A$ satisfies the requirement. Assume that $A \wedge B \neq \underline{\perp}$. Let

$$D = \bigvee \{F \in L^X : F \lessdot_{\preccurlyeq} B\}$$

and let $E = D \wedge B$. Then $A \wedge B \leq E$. In addition, $D \lessdot_{\preccurlyeq} B$ by (LTDIR3). Thus $E \leq D \lessdot_{\preccurlyeq} B$ and so $E \lessdot_{\preccurlyeq} B$. To prove that $A \wedge B \lessdot_{\preccurlyeq} E$, we check that $y_\eta \preccurlyeq E \vee y_\eta$ for any $y_\eta \in \beta^*(A \wedge B)$.

Let $y_\eta \in \beta^*(A \wedge B)$. By $A \lessdot_{\preccurlyeq} B$, we have $y_\eta \preccurlyeq B \vee y_\eta = B$. By (LTIR4), there is a $C \in L^X$ such that $y_\eta \preccurlyeq C \preccurlyeq B$. Thus $y_\eta \leq C \leq B$ by (LTIR2). For any $z_\theta \in \beta^*(C)$, we have $z_\theta \leq C \preccurlyeq B \leq B \vee z_\theta$ which implies that $z_\theta \preccurlyeq B \vee z_\theta$. Hence $C \lessdot_{\preccurlyeq} B$ and so $C \leq D$. Further, we have $y_\eta \preccurlyeq D$ and $y_\eta \leq D$ by $y_\eta \preccurlyeq C$. Notice that $y_\eta \preccurlyeq B$ and $y_\eta \preccurlyeq D$. By (LTIR5), we have $y_\eta \preccurlyeq E = E \vee y_\eta$. By the arbitrariness of $y_\eta \in \beta^*(A \wedge B)$, we have $A \wedge B \lessdot_{\preccurlyeq} E$. Therefore $A \wedge B \lessdot_{\preccurlyeq} E \lessdot_{\preccurlyeq} B$ and $A \wedge B \leq E$ as desired.

(LTDIR5). Let $A \lessdot_{\preccurlyeq} B \wedge C$. For any $x_\lambda \in \beta^*(A)$, we have

$$x_\lambda \preccurlyeq (B \wedge C) \vee x_\lambda = (B \vee x_\lambda) \wedge (C \vee x_\lambda).$$

Thus $x_\lambda \preccurlyeq B \vee x_\lambda$ and $x_\lambda \preccurlyeq C \vee x_\lambda$ by (LTIR5). Hence $A \lessdot_{\preccurlyeq} B$ and $A \lessdot_{\preccurlyeq} C$.

Conversely, let $A \lessdot_{\preccurlyeq} B$ and $A \lessdot_{\preccurlyeq} C$. If$x_\lambda \in \beta^*(A)$, then $x_\lambda \preccurlyeq B \vee x_\lambda$ and $x_\lambda \preccurlyeq C \vee x_\lambda$. By (LTIR5), we have

$$x_\lambda \preccurlyeq (B \vee x_\lambda) \wedge (C \vee x_\lambda) = (B \wedge C) \vee x_\lambda.$$

Therefore $A \lessdot_{\preccurlyeq} B \wedge C$. □

Theorem 3.5. *Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-topological internal relation spaces. If $f : X \to Y$ is an L-topological internal relation preserving mapping, then $f : (X, \preccurlyeq_{\preccurlyeq_X}) \to (Y, \preccurlyeq_{\preccurlyeq_Y})$ is an L-topological derived internal relation preserving mapping.*

Proof. Let $A \preccurlyeq_{\preccurlyeq_Y} B$. If $f_L^{\leftarrow}(A \wedge B) = \underline{\perp}$, then $f_L^{\leftarrow}(A \wedge B) \preccurlyeq_{\preccurlyeq_X} f_L^{\leftarrow}(B)$ is trivial. Assume that $f_L^{\leftarrow}(A \wedge B) \neq \underline{\perp}$. If $x_\lambda \in \beta^*(f_L^{\leftarrow}(A \wedge B))$, then $f_L^{\rightarrow}(x_\lambda) \in \beta^*(A \wedge B)$. Thus $f_L^{\rightarrow}(x_\lambda) \preccurlyeq_Y B \vee f_L^{\rightarrow}(x_\lambda)$ and

$$x_\lambda \leq f_L^{\leftarrow}(f_L^{\rightarrow}(x_\lambda)) \preccurlyeq_X f_L^{\leftarrow}(B) \vee f_L^{\leftarrow}(f_L^{\rightarrow}(x_\lambda)) \leq f_L^{\leftarrow}(B) \vee f_L^{\leftarrow}(A \wedge B) = f_L^{\leftarrow}(B).$$

Hence $x_\lambda \preccurlyeq_X f_L^{\leftarrow}(B)$ and so $f_L^{\leftarrow}(A \wedge B) \preccurlyeq_X f_L^{\leftarrow}(B)$ by (LTIR3). Therefore f is an L-topological derived internal relation preserving mapping. □

Theorem 3.6. *We have $\preccurlyeq_{\preccurlyeq_{\preccurlyeq}} = \preccurlyeq$ for any L-topological interval relation space $(X, \preccurlyeq)$ and $\preccurlyeq_{\preccurlyeq_{\preccurlyeq}} = \preccurlyeq$ for any L-topological derived internal relation space $(X, \preccurlyeq)$.*

Proof. Let $(X, \preccurlyeq)$ be an L-topological internal relation space. If $A \preccurlyeq_{\preccurlyeq_{\preccurlyeq}} B$, then $A \leq B$ by (LTIR2). In addition, there is a $C \in L^X$ such that $C \preccurlyeq_{\preccurlyeq} B$ and $A = B \wedge C$. Thus $A \preccurlyeq_{\preccurlyeq} B$. By $A \preccurlyeq_{\preccurlyeq} B$, we have $x_\lambda \preccurlyeq B \vee x_\lambda = B$ for any $x_\lambda \in \beta^*(A)$. By (LTIR3), we have $A = \bigvee_{x_\lambda \in \beta^*(A)} x_\lambda \preccurlyeq B$.

Conversely, if $A \preccurlyeq B$ then $A \leq B$ by (LTIR2). For any $x_\lambda \in \beta^*(A)$, we have $x_\lambda \leq A \preccurlyeq B$. Thus $x_\lambda \preccurlyeq B = B \vee x_\lambda$. Hence $A \preccurlyeq_{\preccurlyeq} B$. Since $A \wedge B = A$, we have $A \preccurlyeq_{\preccurlyeq_{\preccurlyeq}} B$.

In conclusion, for all $A, B \in L^X$, we have $A \preccurlyeq_{\preccurlyeq_{\preccurlyeq}} B$ iff $A \preccurlyeq B$. That is, $\preccurlyeq_{\preccurlyeq_{\preccurlyeq}} = \preccurlyeq$.

Let $(X, \preccurlyeq)$ be an L-topological derived internal relation space. Let $A \preccurlyeq_{\preccurlyeq_{\preccurlyeq}} B$ and let $x_\lambda \in \beta^*(A)$. By $A \preccurlyeq_{\preccurlyeq_{\preccurlyeq}} B$, we have $x_\lambda \preccurlyeq_{\preccurlyeq} B \vee x_\lambda$. Thus there is a $C^{x_\lambda} \in L^X$ such that $C^{x_\lambda} \preccurlyeq B$ and $x_\lambda = (B \vee x_\lambda) \wedge C^{x_\lambda}$. Hence, by (LTDIR3),

$$A = \bigvee_{x_\lambda \in \beta^*(A)} x_\lambda = \bigvee_{x_\lambda \in \beta^*(A)} [(B \vee x_\lambda) \wedge C^{x_\lambda}] \leq \bigvee_{x_\lambda \in \beta^*(A)} [(B \vee A) \wedge C^{x_\lambda}] = (B \vee A) \wedge \bigvee_{x_\lambda \in \beta^*(A)} C^{x_\lambda} \leq \bigvee_{x_\lambda \in \beta^*(A)} C^{x_\lambda} \preccurlyeq B.$$

From this result, we conclude that $A \preccurlyeq B$.

Conversely, assume that $A \preccurlyeq B$. If $x_\lambda \in \beta^*(A)$, then $x_\lambda \leq A \preccurlyeq B \leq B \vee x_\lambda$. Thus $x_\lambda \preccurlyeq B \vee x_\lambda$. By this result and $x_\lambda = x_\lambda \wedge (B \vee x_\lambda)$, we have $x_\lambda \preccurlyeq_{\preccurlyeq} B \vee x_\lambda$. Hence $A \preccurlyeq_{\preccurlyeq_{\preccurlyeq}} B$.

In conclusion, for all $A, B \in L^X$, we have $A \preccurlyeq_{\preccurlyeq_{\preccurlyeq}} B$ iff $A \preccurlyeq B$. That is, $\preccurlyeq_{\preccurlyeq_{\preccurlyeq}} = \preccurlyeq$. □

Based on Theorems 3.2 and 3.3, we obtain a functor $\mathbb{U} : L$-**TIERS**$\to L$-**TIRS** defined by

$$\mathbb{U}((X, \preccurlyeq)) = (X, \preccurlyeq_{\preccurlyeq}), \quad \mathbb{U}(f) = f.$$

Based on Theorems 3.2–3.6, $\mathbb{U}$ is an isomorphic functor. Thus we have the following conclusion.

Theorem 3.7. *The category L-***TDIRS** *is isomorphic to the category L-***TIRS***.*

To simply characterize L-**TDIRS**, we introduce L-topological derived interior space as follows.

Definition 3.8. A subset $\mathcal{I} \subseteq L^X$ is called an L-topological derived interior operator on X and the pair $(X, \mathcal{I})$ is called an L-topological derived interior space if for all $A, B \in L^X$ and any $x_\lambda \in \beta^*(\underline{\top})$,

(LTDInt1) $\mathcal{I}(\underline{\top}) = \underline{\top}$;

(LTDInt2) $A \leq \mathcal{I}(B)$ if and only if $x_\lambda \leq \mathcal{I}(B \vee x_\lambda)$ for any $x_\lambda \in \beta^*(A)$;

(LTDInt3) $A \wedge \mathcal{I}(A) \leq \mathcal{I}(\mathcal{I}(A))$;

(LTDInt4) $\mathcal{I}(A \wedge B) = \mathcal{I}(A) \wedge \mathcal{I}(B)$.

Let $(X, \mathcal{I}_X)$ and $(Y, \mathcal{I}_Y)$ be L-topological derived interior spaces. A mapping $f : X \to Y$ is called an L-topological derived interior preserving mapping, if $f_L^{\leftarrow}(\mathcal{I}_Y(B) \wedge B) \leq \mathcal{I}_X(f_L^{\leftarrow}(B))$ for any $B \in L^Y$.

The category of L-topological derived interior spaces and L-topological derived interior preserving mappings is denoted by L-**TDINTS**.

Theorem 3.9. *Let $(X, \mathcal{I})$ be an L-topological derived internal space. Define a binary operator $\leqslant_{\mathcal{I}}$ on L^X by*

$$\forall A, B \in L^X,\ \ A \leqslant_{\mathcal{I}} B \ \ \Leftrightarrow\ \ A \leq \mathcal{I}(B).$$

Then $(X, \leqslant_{\mathcal{I}})$ is an L-topological derived internal relation space.

Proof. We check that $\leqslant_{\mathcal{I}}$ satisfies (LTDIR1)–(LTDIR5).

(LTDIR1). We have $\mathcal{I}(\underline{\top}) = \underline{\top}$ by (LTDInt1). Thus $\underline{\top} \leqslant_{\mathcal{I}} \underline{\top}$.

(LTDIR2). It directly follows from (LTDInt2).

(LTDIR3). If $\bigvee_{i\in I} A_i \leqslant_{\mathcal{I}} B$ then $A_j \leq \bigvee_{i\in I} A_i \leq \mathcal{I}(B)$ for any $j \in I$. Thus $A_j \leqslant_{\mathcal{I}} B$ for any $j \in I$. Conversely, assume that $A_i \leqslant_{\mathcal{I}} B$ for any $i \in I$. Then $A_i \leq \mathcal{I}(B)$ for any $i \in I$. Hence $\bigvee_{i\in I} A_i \leq \mathcal{I}(B)$. Therefore $\bigvee_{i\in I} A_i \leqslant_{\mathcal{I}} B$.

(LTDIR4). Let $A \leqslant_{\mathcal{I}} B$ and let $E = B \wedge \mathcal{I}(B)$. We have $E \leqslant_{\mathcal{I}} E$ by (LTDInt3) and (LTDInt4). Also, we have $A \leq \mathcal{I}(B)$ by $A \leqslant_{\mathcal{I}} B$. Thus $A \wedge B \leq E \leq \mathcal{I}(E)$. Hence $A \wedge B \leqslant_{\mathcal{I}} E$. Further, we have $E \leqslant_{\mathcal{I}} B$ by $E \leq \mathcal{I}(B)$. Therefore $A \wedge B \leqslant_{\mathcal{I}} E \leqslant_{\mathcal{I}} B$ and $A \wedge B \leq E$ as desired.

(LTDIR5). By (LTDInt4), we have $\mathcal{I}(A \wedge B) = \mathcal{I}(A) \wedge \mathcal{I}(B)$. Thus the desired result is clear. □

Theorem 3.10. *Let $(X, \mathcal{I}_X)$ and $(Y, \mathcal{I}_Y)$ be L-topological derived interior spaces. If $f : X \to Y$ is an L-topological derived interior preserving mapping, then $f : (X, \leqslant_{\mathcal{I}_X}) \to (Y, \leqslant_{\mathcal{I}_Y})$ is an L-topological derived internal relation preserving mapping.*

Proof. If $A \leqslant_{\mathcal{I}_Y} B$ then $A \leq \mathcal{I}_Y(B)$. Thus $f_L^{\leftarrow}(A) \leq f_L^{\leftarrow}(\mathcal{I}_Y(B))$ and

$$f_L^{\leftarrow}(A \wedge B) = f_L^{\leftarrow}(A) \wedge f_L^{\leftarrow}(B) \leq f_L^{\leftarrow}(\mathcal{I}_Y(B)) \wedge f_L^{\leftarrow}(B) = f_L^{\leftarrow}(\mathcal{I}_Y(B) \wedge B) \leq \mathcal{I}_X(f_L^{\leftarrow}(B)).$$

So $f_L^{\leftarrow}(A \wedge B) \leqslant_{\mathcal{I}_X} f_L^{\leftarrow}(B)$. Hence f is an L-topological derived internal relation preserving mapping. □

Theorem 3.11. *Let $(X, \leqslant)$ be an L-topological derived internal relation space. Define an operator $\mathcal{I}_{\leqslant} : L^X \to L^X$ by*

$$\forall A \in L^X,\ \ \mathcal{I}_{\leqslant}(A) = \bigvee\{B \in L^X : B \leqslant A\}.$$

Then $(X, \mathcal{I}_{\leqslant})$ is an L-topological derived interior space.

Proof. (LTDInt1). We have $\underline{\top} \leq \mathcal{I}_{\leqslant}(\underline{\top})$ by (LTDIR1). Thus $\mathcal{I}_{\leqslant}(\underline{\top}) = \underline{\top}$.

(LTDInt2). Let $A \leq \mathcal{I}_{\leqslant}(B)$. If $x_\lambda \in \beta^*(A)$, then $x_\lambda \prec \mathcal{I}_{\leqslant}(B)$. Thus there is a $D \in L^X$ such that $x_\lambda \leq D$ and $D \leqslant B$. Hence $x_\lambda \leq D \leqslant B \leq B \vee x_\lambda$ followed by $x_\lambda \leqslant B \vee x_\lambda$. Therefore $x_\lambda \leq \mathcal{I}_{\leqslant}(B \vee x_\lambda)$.

Conversely, let $x_\lambda \leq \mathcal{I}_{\leqslant}(B \vee x_\lambda)$ for any $x_\lambda \in \beta^*(A)$. To prove that $A \leq \mathcal{I}_{\leqslant}(B)$, let $x_\lambda \in \beta^*(A)$. Then $x_\lambda \leq \mathcal{I}_{\leqslant}(B \vee x_\lambda)$. For any $x_\eta \in \beta^*(x_\lambda)$, we have $x_\eta \prec \mathcal{I}_{\leqslant}(B \vee x_\lambda)$. Thus there is a $D \in L^X$ such that $x_\eta \prec D \leqslant B \vee x_\lambda$. Hence $x_\eta \leqslant B \vee x_\lambda$ followed by $x_\lambda \leqslant B \vee x_\lambda$. By (LTDIR2), we have $A \leqslant B$. So $A \leq \mathcal{I}_{\leqslant}(B)$.

(LTDInt3). Let $x_\lambda \in \beta^*(A \wedge \mathcal{I}_{\leqslant}(A))$. By (LTDIR3), we have $\mathcal{I}_{\leqslant}(A) \leqslant A$. By $x_\lambda \prec \mathcal{I}_{\leqslant}(A)$, there is a $D \in L^X$ such that $x_\lambda \leq D \leqslant A$. By $D \leqslant A$, there is a $C \in L^X$ such that $D \wedge A \leqslant C \leqslant A$ and $x_\lambda \leq D \wedge A \leq C$. Further, since $C \leqslant A$, we have $C \leq \mathcal{I}_{\leqslant}(A)$. Thus $D \wedge A \leqslant \mathcal{I}_{\leqslant}(A)$ which implies that $x_\lambda \leq D \wedge A \leq \mathcal{I}_{\leqslant}(\mathcal{I}_{\leqslant}(A))$. Therefore $A \wedge \mathcal{I}_{\leqslant}(A) \leq \mathcal{I}_{\leqslant}(\mathcal{I}_{\leqslant}(A))$.

(LTDInt4). Clearly, $\mathcal{I}_{\leqslant}(A \wedge B) \leq \mathcal{I}_{\leqslant}(A) \wedge \mathcal{I}_{\leqslant}(B)$. Conversely, let $x_\lambda \in \beta^*(\mathcal{I}_{\leqslant}(A) \wedge \mathcal{I}_{\leqslant}(B))$. By $x_\lambda \in \beta^*(\mathcal{I}_{\leqslant}(A))$, there is a $C \in L^X$ such that $x_\lambda \leq C \leqslant A$. Similarly, by $x_\lambda \in \beta^*(\mathcal{I}_{\leqslant}(B))$, there is a $D \in L^X$ such that $x_\lambda \leq D \leqslant B$. Thus $x_\lambda \leq C \wedge D$. By (LTDIR5), we have $x_\lambda \leq C \wedge D \leqslant A \wedge B$. Hence $x_\lambda \leq C \wedge D \leq \mathcal{I}_{\leqslant}(A \wedge B)$. Therefore $\mathcal{I}_{\leqslant}(A) \wedge \mathcal{I}_{\leqslant}(B) \leq \mathcal{I}_{\leqslant}(A \wedge B)$. □

Theorem 3.12. *Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be L-topological derived internal relation spaces. If $f : X \to Y$ is an L-topological derived internal relation preserving mapping, then $f : (X, \mathcal{I}_{\leqslant_X}) \to (Y, \mathcal{I}_{\leqslant_Y})$ is an L-topological derived interior preserving mapping.*

Proof. Let $B \in L^Y$. To prove that $f_L^{\leftarrow}(\mathcal{I}_{\leqslant_Y}(B) \wedge B) \leq \mathcal{I}_{\leqslant_X}(f_L^{\leftarrow}(B))$, let $x_\lambda \in \beta^*(f_L^{\leftarrow}(\mathcal{I}_{\leqslant_Y}(B) \wedge B))$. Then $f_L^{\rightarrow}(x_\lambda) \prec \mathcal{I}_{\leqslant_Y}(B) \wedge B$. By $f_L^{\rightarrow}(x_\lambda) \prec \mathcal{I}_{\leqslant_Y}(B)$, there is a $D \in L^X$ such that $f_L^{\rightarrow}(x_\lambda) \leq D \leqslant_Y B$. Thus

$$x_\lambda \leq f_L^{\leftarrow}(D) \wedge f_L^{\leftarrow}(B) = f_L^{\leftarrow}(D \wedge B) \leqslant_X f_L^{\leftarrow}(B).$$

Hence $x_\lambda \leq \mathcal{I}_X(f_L^{\leftarrow}(B))$ and so $f_L^{\leftarrow}(\mathcal{I}_{\leqslant_Y}(B) \wedge B) \leq \mathcal{I}_{\leqslant_X}(f_L^{\leftarrow}(B))$. Therefore f is an L-topological derived interior preserving mapping. □

Theorem 3.13. *We have $\mathcal{I}_{\preccurlyeq_{\mathcal{I}}} = \mathcal{I}$ for any L-topological derived interior space $(X, \mathcal{I})$ and $\preccurlyeq_{\mathcal{I}_{\preccurlyeq}} = \preccurlyeq$ for any L-topological derived internal relation space $(X, \preccurlyeq)$.*

Proof. Let $(X, \mathcal{I})$ be an L-topological derived interior space and $A \in L^X$. For any $D \in L^X$ with $D \preccurlyeq_{\mathcal{I}} A$, we have $D \leq \mathcal{I}(A)$. Thus

$$\mathcal{I}_{\preccurlyeq_{\mathcal{I}}}(A) = \bigvee\{D \in L^X : D \preccurlyeq_{\mathcal{I}} A\} \leq \mathcal{I}(A).$$

Conversely, $\mathcal{I}(A) \preccurlyeq_{\mathcal{I}} A$ by $\mathcal{I}(A) \leq \mathcal{I}(A)$. Thus $\mathcal{I}(A) \leq \mathcal{I}_{\preccurlyeq_{\mathcal{I}}}(A)$. Hence $\mathcal{I}_{\preccurlyeq_{\mathcal{I}}}(A) = \mathcal{I}(A)$ followed by $\mathcal{I}_{\preccurlyeq_{\mathcal{I}}} = \mathcal{I}$.

Let $(X, \preccurlyeq)$ be an L-topological derived internal relation space. If $A \preccurlyeq B$ then $A \leq \mathcal{I}_{\preccurlyeq}(B)$. Thus $A \preccurlyeq_{\mathcal{I}_{\preccurlyeq}} B$. Conversely, if $A \preccurlyeq_{\mathcal{I}_{\preccurlyeq}} B$, then $A \leq \mathcal{I}_{\preccurlyeq}(B)$. For any $x_\lambda \in \beta^*(A)$, we have $x_\lambda \prec \mathcal{I}_{\preccurlyeq}(B)$. Thus there is an $E \in L^X$ such that $x_\lambda \leq E \preccurlyeq B$. Hence $x_\lambda \preccurlyeq B \vee x_\lambda$. By (LTDIR2), we have $A \preccurlyeq B$.

In conclusion, we have $A \preccurlyeq B$ if and only if $A \preccurlyeq_{\mathcal{I}_{\preccurlyeq}} B$. That is, $\preccurlyeq_{\mathcal{I}_{\preccurlyeq}} = \preccurlyeq$. □

Based on Theorems 3.11 and 3.12, we obtain a functor $\mathbb{W} : L$-**TDIRS**$\to L$-**TDINTS** by

$$\mathbb{W}((X, \preccurlyeq)) = (X, \mathcal{I}_{\preccurlyeq}), \quad \mathbb{W}(f) = f.$$

Based on Theorems 3.9–3.13, $\mathbb{W}$ is an isomorphic functor. Thus we have the following result.

Theorem 3.14. *The category L-***TDIRS** *is isomorphic to the category L-***TDINTS**.

Now, we characterize L-topological spaces by L-topological derived internal relation spaces.

Theorem 3.15. *Let $(X, \mathcal{T})$ be an L-topological space. Define a binary relation $\preccurlyeq_{\mathcal{T}}$ on L^X by*

$$\forall A, B \in L^X,\ \ A \preccurlyeq_{\mathcal{T}} B \ \ \Leftrightarrow\ \ \forall x_\lambda \in \beta^*(A),\ x_\lambda \leq Int_{\mathcal{T}}(B \vee x_\lambda).$$

Then $(X, \preccurlyeq_{\mathcal{T}})$ is an L-topological derived internal relation space.

Proof. By definition, it is clear that $A \preccurlyeq_{\mathcal{T}} B$ for all $A, B, C, D \in L^X$ with $A \leq C \preccurlyeq_{\mathcal{T}} D \leq B$. Next, we check that $\preccurlyeq_{\mathcal{T}}$ satisfies (LTDIR1)–(LTDIR5).

(LTDIR1). For any $x_\lambda \in \beta^*(\underline{\top})$, we have $x_\lambda \leq \underline{\top} = Int_{\mathcal{T}}(\underline{\top} \vee x_\lambda)$ by (LInt1). Thus $\underline{\top} \preccurlyeq_{\mathcal{T}} \underline{\top}$.

(LTDIR2). Let $A \preccurlyeq_{\mathcal{T}} B$ and $x_\lambda \in \beta^*(A)$. To prove that $x_\lambda \preccurlyeq_{\mathcal{T}} B \vee x_\lambda$, let $x_\eta \in \beta^*(x_\lambda)$. Then $x_\eta \in \beta^*(A)$. By $A \preccurlyeq_{\mathcal{T}} B$, we have

$$x_\eta \leq Int_{\mathcal{T}}(B \vee x_\eta) \leq Int_{\mathcal{T}}((B \vee x_\lambda) \vee x_\eta).$$

Thus $x_\lambda \preccurlyeq_{\mathcal{T}} B \vee x_\lambda$. Conversely, let $x_\lambda \preccurlyeq_{\mathcal{T}} B \vee x_\lambda$ for any $x_\lambda \in \beta^*(A)$. If $x_\lambda \in \beta^*(A)$, then $x_\lambda \preccurlyeq_{\mathcal{T}} B \vee x_\lambda$. Hence

$$x_\lambda = \bigvee_{x_\eta \in \beta^*(x_\lambda)} x_\eta \leq \bigvee_{x_\eta \in \beta^*(x_\lambda)} Int_{\mathcal{T}}((B \vee x_\lambda) \vee x_\eta) = Int_{\mathcal{T}}(B \vee x_\lambda).$$

Therefore $A \preccurlyeq_{\mathcal{T}} B$.

(LTDIR3). Let $\bigvee_{i\in I} A_i \preccurlyeq_{\mathcal{T}} B$. It is clear that $A_i \preccurlyeq_{\mathcal{T}} B$. Conversely, assume that $A_i \preccurlyeq_{\mathcal{T}} B$ for any $i \in I$. For any $y_\mu \in \beta^*(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} \beta^*(A_i)$, there is an $i \in I$ such that $x_\lambda \in \beta^*(A_i)$. Hence $x_\lambda \leq Int_{\mathcal{T}}(B \vee x_\lambda)$ by $A_i \preccurlyeq_{\mathcal{T}} B$. Therefore $\bigvee_{i\in I} A_i \preccurlyeq_{\mathcal{T}} B$.

(LTDIR4). Let $A \preccurlyeq_{\mathcal{T}} B$ and let $C = Int_{\mathcal{T}}(B)$. For any $x_\lambda \in \beta^*(C)$, we have $x_\lambda \leq C \leq Int_{\mathcal{T}}(B \vee x_\lambda)$. Thus $C \preccurlyeq_{\mathcal{T}} B$. For any $y_\mu \in \beta^*(A \wedge B)$, we have $y_\mu \in \beta^*(A)$. By $A \preccurlyeq_{\mathcal{T}} B$, we have

$$y_\mu \leq Int_{\mathcal{T}}(B \vee y_\mu) = Int_{\mathcal{T}}(B) = Int_{\mathcal{T}}(Int_{\mathcal{T}}(B)) = Int_{\mathcal{T}}(C) \leq Int_{\mathcal{T}}(C \vee y_\mu).$$

Hence $A \wedge B \preccurlyeq_{\mathcal{T}} C \preccurlyeq_{\mathcal{T}} B$. In addition, since $y_\mu \leq Int_{\mathcal{T}}(C) \leq C$ for any $y_\mu \in \beta^*(A \wedge B)$, we have $A \wedge B \leq C$. Therefore $C = Int_{\mathcal{T}}(B)$ satisfies the requirement.

(LTDER5). If $A \preccurlyeq_{\mathcal{T}} B \wedge C$, then it is clear that $A \preccurlyeq_{\mathcal{T}} B$ and $A \preccurlyeq_{\mathcal{T}} C$. Conversely, assume that $A \preccurlyeq_{\mathcal{T}} B$ and $A \preccurlyeq_{\mathcal{T}} C$. For any $x_\lambda \in \beta^*(A)$, we have

$$x_\lambda \leq Int_{\mathcal{T}}(B \vee x_\lambda) \wedge Int_{\mathcal{T}}(C \vee x_\lambda) = Int_{\mathcal{T}}((B \vee x_\lambda) \wedge (C \vee x_\lambda)) = Int_{\mathcal{T}}((B \wedge C) \vee x_\lambda).$$

Thus $A \preccurlyeq_{\mathcal{T}} B \wedge C$. □

Theorem 3.16. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be L-topological spaces. If $f : X \to Y$ is an L-continuous mapping, then $f : (X, \preccurlyeq_{\mathcal{T}_X}) \to (Y, \preccurlyeq_{\mathcal{T}_Y})$ is an L-topological derived internal relation preserving mapping.*

Proof. Let $A \preccurlyeq_{\mathcal{T}_Y} B$. To prove that $f_L^{\leftarrow}(A \wedge B) \preccurlyeq_{\mathcal{T}_X} f_L^{\leftarrow}(B)$, let $x_\lambda \in \beta^*(f_L^{\leftarrow}(A \wedge B))$. Then $f_L^{\rightarrow}(x_\lambda) \in \beta^*(A \wedge B)$. By $A \preccurlyeq_{\mathcal{T}_Y} B$, we have $f_L^{\rightarrow}(x_\lambda) \leq Int_{\mathcal{T}_Y}(B \vee f_L^{\rightarrow}(x_\lambda)) \leq Int_{\mathcal{T}_Y}(B)$. Thus

$$x_\lambda \leq f_L^{\leftarrow}(Int_{\mathcal{T}_Y}(B)) \leq Int_{\mathcal{T}_X}(f_L^{\leftarrow}(B)) = Int_{\mathcal{T}_X}(f_L^{\leftarrow}(B) \vee x_\lambda).$$

Hence $f_L^{\leftarrow}(A \wedge B) \preccurlyeq_{\mathcal{T}_X} f_L^{\leftarrow}(B)$. So f is an L-topological derived internal relation preserving mapping. □

Theorem 3.17. *Let $(X, \preccurlyeq)$ be an L-topological derived internal relation space. Define an operator $Int_{\preccurlyeq} : L^X \to L^X$ by*

$$\forall A \in 2^X,\ \ Int_{\preccurlyeq}(A) = A \wedge \bigvee\{B \in L^X : B \preccurlyeq A\}.$$

Then $Int_{\preccurlyeq}$ is an L-topological interior operator which induces an L-topology denoted by $\mathcal{T}_{\preccurlyeq}$.

Proof. We check that $Int_{\preccurlyeq}$ satisfies (LInt1)–(LInt4).

(LInt1). We have $\underline{\top} \preccurlyeq \underline{\top}$ by (LTDIR1). Thus $\underline{\top} \wedge \underline{\top} = \underline{\top} \leq Int_{\preccurlyeq}(\underline{\top})$ which shows that $Int_{\preccurlyeq}(\underline{\top}) = \underline{\top}$.

(LInt2). It is clear that $Int_{\preccurlyeq}(A) \leq A$.

(LInt3). Clearly, $Int_{\preccurlyeq}(Int_{\preccurlyeq}(A)) \leq Int_{\preccurlyeq}(A)$. To prove that $Int_{\preccurlyeq}(A) \leq Int_{\preccurlyeq}(Int_{\preccurlyeq}(A))$, let $x_\lambda \in \beta^*(Int_{\preccurlyeq}(A))$. Then $x_\lambda \in \beta^*(A)$ and there is a $B \in L^X$ such that $x_\lambda \prec B \preccurlyeq A$. By (LTDIR4), there is a $C \in L^X$ such that $A \wedge B \preccurlyeq C \preccurlyeq A$ and $A \wedge B \leq C$. Thus $A \wedge C \leq Int_{\preccurlyeq}(A)$. Further, by $A \wedge B \leq C \preccurlyeq A$, we have $A \wedge B \preccurlyeq A$. Hence $A \wedge B \preccurlyeq A \wedge C \leq Int_{\preccurlyeq}(A)$ by (LTDIR5). So $A \wedge B \preccurlyeq Int_{\preccurlyeq}(A)$ followed by

$$x_\lambda \leq A \wedge B \leq Int_{\preccurlyeq}(Int_{\preccurlyeq}(A)).$$

This shows that $Int_{\preccurlyeq}(A) \leq Int_{\preccurlyeq}(Int_{\preccurlyeq}(A))$. Therefore $Int_{\preccurlyeq}(Int_{\preccurlyeq}(A)) = Int_{\preccurlyeq}(A)$.

(LInt4). It is clear that $Int_{\preccurlyeq}(A \wedge B) \leq Int_{\preccurlyeq}(A) \wedge Int_{\preccurlyeq}(B)$. Conversely, let $x_\lambda \prec Int_{\preccurlyeq}(A) \wedge Int_{\preccurlyeq}(B)$. By $x_\lambda \prec Int_{\preccurlyeq}(A))$, there is a $C \in L^X$ such that $x_\lambda \leq C \preccurlyeq A$. Similarly, by $x_\lambda \leq Int_{\preccurlyeq}(B)$, there is a $D \in L^X$ such that $x_\lambda \leq D \preccurlyeq B$. Thus $C \wedge D \preccurlyeq A$ and $C \wedge D \preccurlyeq B$. Hence $C \wedge D \preccurlyeq A \wedge B$ by (LTDIR5). Hence

$$x_\lambda \leq A \wedge C \wedge D \leq Int_{\preccurlyeq}(A \wedge B).$$

Therefore $Int_{\preccurlyeq}(A) \wedge Int_{\preccurlyeq}(B) \leq Int_{\preccurlyeq}(A \wedge B)$. □

Theorem 3.18. *Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-topological derived internal relation spaces. If $f : X \to Y$ is an L-topological derived internal relation preserving mapping, then $f : (X, \mathcal{T}_{\preccurlyeq_X}) \to (Y, \mathcal{T}_{\preccurlyeq_Y})$ is an L-continuous mapping.*

Proof. If $B \in \mathcal{T}_{\preccurlyeq_Y}$ then $B = Int_{\preccurlyeq_Y}(B)$. To prove the desired result, we verify that $f_L^{\leftarrow}(B) = Int_{\preccurlyeq_Y}(f_L^{\leftarrow}(B))$.

It is clear that $Int_{\preccurlyeq_Y}(f_L^{\leftarrow}(B)) \leq f_L^{\leftarrow}(B)$. To prove that $f_L^{\leftarrow}(B) \leq Int_{\preccurlyeq_Y}(f_L^{\leftarrow}(B))$, let $x_\lambda \in \beta^*(f_L^{\leftarrow}(B))$. Then $f_L^{\rightarrow}(x_\lambda) \prec B = Int_{\preccurlyeq_Y}(B)$. Thus there is a $D \in L^X$ such that $f_L^{\rightarrow}(x_\lambda) \prec D \preccurlyeq_Y B$. Hence $x_\lambda \leq f_L^{\leftarrow}(D \wedge B) \preccurlyeq_X f_L^{\leftarrow}(B)$ which implies that $x_\lambda \leq Int_{\mathcal{T}_X}(f_L^{\leftarrow}(B))$. Thus $f_L^{\leftarrow}(B) \leq Int_{\mathcal{T}_X}(f_L^{\leftarrow}(B))$ and so $f_L^{\leftarrow}(B) = Int_{\mathcal{T}_X}(f_L^{\leftarrow}(B))$. This implies that $f_L^{\leftarrow}(B) \in \mathcal{T}_{\mathcal{T}_X}$. Therefore f is an L-continuous mapping. □

Theorem 3.19. *We have $\mathcal{T}_{\preccurlyeq_{\mathcal{T}}} = \mathcal{T}$ for any L-topological space $(X, \mathcal{T})$ and $\preccurlyeq_{\mathcal{T}_{\preccurlyeq}} = \preccurlyeq$ for any L-topological derived internal relation space $(X, \preccurlyeq)$.*

Proof. Let $(X, \mathcal{T})$ be an L-topological space. If $A \in \mathcal{T}_{\preccurlyeq_{\mathcal{T}}}$ then $A = Int_{\preccurlyeq_{\mathcal{T}}}(A)$. Let

$$D = \bigvee\{B \in L^X : B \preccurlyeq_{\mathcal{T}} A\}.$$

Then $A = Int_{\preccurlyeq_{\mathcal{T}}}(A) = A \wedge D$ and $A \leq D \preccurlyeq_{\mathcal{T}} A$ by (LTDIR3). For any $x_\lambda \in \beta^*(A)$, we have $x_\lambda \in \beta^*(D)$. In addition, $x_\lambda \leq Int_{\mathcal{T}}(A \vee x_\lambda) = Int_{\mathcal{T}}(A)$ by $D \preccurlyeq_{\mathcal{T}} A$. Thus $A \leq Int_{\mathcal{T}}(A)$ which implies that $A = Int_{\mathcal{T}}(A)$. Hence $A \in \mathcal{T}$. Therefore $\mathcal{T}_{\preccurlyeq_{\mathcal{T}}} \subseteq \mathcal{T}$.

Conversely, let $A \in \mathcal{T}$. If $x_\lambda \in \beta^*(A)$, then $x_\lambda \leq A = Int_{\mathcal{T}}(A) = Int_{\mathcal{T}}(A \vee x_\lambda)$. Thus $A \preccurlyeq_{\mathcal{T}} A$ followed by

$$A \leq A \wedge \bigvee\{B \in L^X : B \preccurlyeq_{\mathcal{T}} A\} = Int_{\preccurlyeq_{\mathcal{T}}}(A).$$

Hence $A = Int_{\leqslant_{\mathcal{T}}}(A)$ which shows that $A \in \mathcal{T}_{\leqslant_{\mathcal{T}}}$. Therefore $\mathcal{T} \subseteq \mathcal{T}_{\leqslant_{\mathcal{T}}}$. In conclusion, we have $\mathcal{T}_{\leqslant_{\mathcal{T}}} = \mathcal{T}$.

Let $(X, \leqslant)$ be an L-topological derived internal relation space. Let $A \leqslant_{\mathcal{T}_{\leqslant}} B$. For any $x_\lambda \in \beta^*(A)$, we have

$$x_\lambda \leq Int_{\mathcal{T}_{\leqslant}}(B \vee x_\lambda) = Int_{\leqslant}(B \vee x_\lambda) \leq \bigvee\{D \in L^X : D \leqslant B \vee x_\lambda\}.$$

For any $x_\eta \in \beta^*(x_\lambda)$, there is $D \in L^X$ such that $x_\eta \prec D \leqslant B \vee x_\lambda$. Hence $x_\eta \leqslant B \vee x_\lambda$ and $x_\lambda = \bigvee_{x_\eta \in \beta^*(x_\lambda)} x_\eta \leqslant B \vee x_\lambda$ by (LTDIR3). Therefore $A \leqslant B$ by (LTDIR2).

Conversely, let $A \leqslant B$. For any $x \in \beta^*(A)$, we have $x_\lambda \leq A \leqslant B \leq B \vee x_\lambda$. Thus $x_\lambda \leqslant B \vee x_\lambda$ and

$$x_\lambda \leq (B \vee x_\lambda) \wedge \bigvee\{D \in L^X : D \leqslant B \vee x_\lambda\} = Int_{\leqslant}(B \vee x_\lambda) = Int_{\mathcal{T}_{\leqslant}}(B \vee x_\lambda).$$

Hence $A \leqslant_{\mathcal{T}_{\leqslant}} B$.

In conclusion, for any $A, B \in L^X$, we have $A \leqslant_{\mathcal{T}_{\leqslant}} B$ iff $A \leqslant B$. That is, $\leqslant_{\mathcal{T}_{\leqslant}} = \leqslant$. □

Based on Theorems 3.17 and 3.18, we obtain a functor $\mathbb{V} : L$-**TDIRS**→ L-**TOP** defined by

$$\mathbb{V}((X, \leqslant)) = (X, \mathcal{T}_{\leqslant}), \quad \mathbb{V}(f) = f.$$

Based on Theorems 3.15–3.19, $\mathbb{V}$ is an isomorphic functor. Thus we have the following conclusion.

Theorem 3.20. *The category L-**TDIRS** is isomorphic to the category L-**TOP**.*

Based on Theorems 3.9–3.19, relations between L-topological derived interior spaces and L-topological spaces can be presented as follows.

Corollary 3.21. *(1) Let $(X, \mathcal{I})$ be an L-topological derived interior space. Define an operator $Int_{\mathcal{I}} : L^X \to L^X$ by*

$$\forall A \in L^X, \ \ Int_{\mathcal{I}}(A) = A \wedge \mathcal{I}(A).$$

Then $Int_{\mathcal{I}}$ is an L-topological interior operator of an L-topological space $(X, \mathcal{T}_{\mathcal{I}})$;

(2) Let $(X, \mathcal{T})$ be an L-concave space. Define an operator $\mathcal{I}_{\mathcal{T}} : L^X \to L^X$ by

$$\forall A \in L^X, \ \ \mathcal{I}_{\mathcal{T}}(A) = \bigvee\{B \in L^X : \forall x_\lambda \in \beta^*(B), x_\lambda \leq Int_{\mathcal{T}}(A \vee x_\lambda)\}.$$

Then $(X, \mathcal{I}_{\mathcal{T}})$ is an L-topological derived interior space;

*(3) The category L-**TDINTS** is isomorphic to the category L-**TOP**.*

At the end of this section, by Theorem 3.4, we present two examples to show that an L-quasi-uniform space or an L-S-quasi-proximate space generates an L-topological derived internal relation space.

Example 3.22. Let $(X, \mathcal{U})$ be an L-quasi-uniform space. Define a binary relation $\leqslant_{\mathcal{U}}$ on L^X by

$$\forall A, B \in 2^X, \ A \leqslant_{\mathcal{U}} B \ \ \Leftrightarrow \ \ \forall x_\lambda \in \beta^*(A), \ \exists \varphi \in \mathcal{U}, \ (B \vee x_\lambda)' \leq \bigwedge_{\mu \not\leq \lambda'} \varphi(x_\mu).$$

For all $A, B \in L^X$, it is easy to check that

$$A \leqslant_{\mathcal{U}} B \Leftrightarrow \forall x_\lambda \in \beta^*(A), \ x_\lambda \leqslant_{\mathcal{U}} B \vee x_\lambda \Leftrightarrow A \leqslant_{\leqslant_{\mathcal{U}}} B.$$

Thus $\leqslant_{\mathcal{U}} = \leqslant_{\leqslant_{\mathcal{U}}}$. Hence $(X, \leqslant_{\mathcal{U}})$ is an L-topological derived internal relation space.

Example 3.23. Let (X, δ) be a S-quasi-proximate space. Define a binary relation $\leqslant_\delta$ on L^X by

$$\forall A, B \in L^X, \ \ A \leqslant_\delta B \ \ \Leftrightarrow \ \ \forall x_\lambda \in \beta^*(A), \ \bigvee_{\mu \not\leq \lambda'} \delta(x_\mu, (B \vee x_\lambda)') = \perp.$$

For all $A, B \in L^X$, it is easy to check that

$$A \leqslant_\delta B \Leftrightarrow \forall x_\lambda \in \beta^*(A), \ x_\lambda \leqslant_\delta B \vee x_\lambda \Leftrightarrow A \leqslant_{\leqslant_\delta} B.$$

Thus $\leqslant_\delta = \leqslant_{\leqslant_\delta}$. Hence $(X, \leqslant_\delta)$ is an L-topological derived internal relation space.

4. *L*-Topological Derived Enclosed Relation Spaces

In this section, we introduce the notion of L-topological derived enclosed relations by which we characterize the category of L-topological enclosed relation spaces and the category L-topological spaces. For this, we introduce the following notions.

For $A \in L^X$ and $x_\lambda \in \beta^*(\underline{\top})$, we denote $A_{x_\lambda} = \bigvee\{y_\mu \in \beta^*(A) : x_\lambda \not\leq y_\mu\}$ and $\beta^*_\lambda(L) = \{\mu \in \beta^*(\top) : \lambda \in \beta^*(\mu)\}$. For convenience, we denote $y_\eta \not\leq^* A$ for any $y_\eta \in \beta^*(\underline{\top})$ with $y_\eta \not\leq A$. We have the following results.

Proposition 4.1. *For all $x_\lambda, y_\eta \in \beta^*(\underline{\top})$, $A \in L^X$ and $\{A_i\}_{i\in I} \subseteq L^X$, we have*

(1) $x_\lambda \not\leq^* A$ implies $A_{x_\lambda} = A$;
(2) $A \leq B$ implies $A_{x_\lambda} \leq B_{x_\lambda}$;
(3) $(A_{x_\lambda})_{x_\lambda} = A_{x_\lambda}$;
(4) $\mu \in \beta^*_\lambda(L)$ implies $A_{x_\lambda} \leq A_{x_\mu}$ and $(A_{x_\mu})_{x_\lambda} = (A_{x_\lambda})_{x_\mu} = A_{x_\lambda}$;
(5) $y_\eta \not\leq \underline{\top}_{x_\lambda}$ iff $x = y$ and $\eta \in \beta^*_\lambda(L)$;
(6) $A = \bigwedge_{x_\lambda \not\leq^* A} \underline{\top}_{x_\lambda}$;
(7) $(\bigvee_{i\in I} A_i)_{x_\lambda} = \bigvee_{i\in I}(A_i)_{x_\lambda}$.

Proof. (1) and (2) are direct.

(3) We have $(A_{x_\lambda})_{x_\lambda} \leq A_{x_\lambda}$ by (2). Conversely, for any $z_\nu \in \beta^*(\underline{\top})$ with $z_\nu \prec A_{x_\lambda}$, there is a $z_\mu \in \beta^*(A)$ such that $x_\lambda \not\leq z_\mu$ and $z_\nu \prec z_\mu$. Thus $z_\nu \prec z_\mu \leq A_{x_\lambda}$ which implies that $z_\nu \in \beta^*(A_{x_\lambda})$. By $x_\lambda \not\leq z_\nu$, we have $z_\nu \leq (A_{x_\lambda})_{x_\lambda}$. Hence $A_{x_\lambda} \leq (A_{x_\lambda})_{x_\lambda}$. Therefore $(A_{x_\lambda})_{x_\lambda} = A_{x_\lambda}$.

(4) For any $\mu \in \beta^*_\lambda(L)$, it is clear that $A_{x_\lambda} \leq A_{x_\mu}$. Further, by (2) and (3), we have

$$A_{x_\lambda} = (A_{x_\lambda})_{x_\lambda} \leq (A_{x_\lambda})_{x_\mu} \leq A_{x_\lambda}.$$

Thus $(A_{x_\lambda})_{x_\mu} = A_{x_\lambda}$. Similarly, we have $A_{x_\lambda} = (A_{x_\lambda})_{x_\lambda} \leq (A_{x_\mu})_{x_\lambda} \leq A_{x_\lambda}$. Therefore $(A_{x_\mu})_{x_\lambda} = A_{x_\lambda}$.

(5) Assume that $y_\eta \not\leq \underline{\top}_{x_\lambda}$. Then there is a $\nu \in \beta^*(\eta)$ such that $y_\nu \not\leq \underline{\top}_{x_\lambda}$. Thus $x_\lambda \leq y_\nu$. Hence $x = y$ and $\lambda \leq \nu \prec \eta$. Therefore $\eta \in \beta^*_\lambda(L)$. Conversely, assume that $\eta \in \beta^*_\lambda(L)$ and $x = y$. Suppose that $y_\eta \leq \underline{\top}_{x_\lambda}$. Then $x_\lambda \prec \underline{\top}_{x_\lambda}$. Thus there is an $x_\theta \in \beta^*(\underline{\top})$ such that $x_\lambda \not\leq x_\theta$ and $x_\lambda \prec x_\theta$. It is a contradiction. Therefore $y_\eta \not\leq \underline{\top}_{x_\lambda}$.

(6) For any $z_\mu \in \beta^*(\underline{\top})$ with $z_\mu \not\leq \bigwedge_{x_\lambda \not\leq^* A} \underline{\top}_{x_\lambda}$, we have $z_\mu \not\leq \underline{\top}_{x_\lambda}$ for some $x_\lambda \not\leq^* A$. Since $z_\mu \not\leq \underline{\top}_{x_\lambda}$, we have $z = x$ and $\mu \in \beta^*_\lambda(L)$. Thus $z_\mu \not\leq A$. Hence $A \leq \bigwedge_{x_\lambda \not\leq^* A} \underline{\top}_{x_\lambda}$.

Conversely, suppose that $\bigwedge_{x_\lambda \not\leq^* A} \underline{\top}_{x_\lambda} \not\leq A$. Then there is a $z_\nu \in \beta^*(\underline{\top})$ such that $z_\nu \not\leq^* A$ and $z_\nu \leq \bigwedge_{x_\lambda \not\leq^* A} \underline{\top}_{x_\lambda}$. By $z_\nu \not\leq A$, there is a $\theta \in \beta^*(\eta)$ such that $z_\theta \not\leq^* A$. Thus $z_\theta \prec z_\nu \leq \bigwedge_{x_\lambda \not\leq^* A} \underline{\top}_{x_\lambda} \leq \underline{\top}_{z_\theta}$. It is a contradiction. Therefore $\bigwedge_{x_\lambda \not\leq^* A} \underline{\top}_{x_\lambda} \leq A$.

(7) We have $(\bigvee_{i\in I} A_i)_{x_\lambda} = \bigvee\{y_\mu \in \bigcup_{i\in I} \beta^*(A_i) : x_\lambda \not\leq y_\mu\} = \bigvee_{i\in I}(A_i)_{x_\lambda}$. □

Definition 4.2. A binary operator $\lessdot$ on L^X is called an L-topological derived enclosed relation and the pair $(X, \lessdot)$ is called an L-topological derived enclosed relation space, if for all $A, B, C \in L^X$ and $x_\lambda \in \beta^*(\underline{\top})$,

(LTDER1) $\underline{\bot} \lessdot \underline{\bot}$;
(LTDER2) $A \lessdot B$ iff $A_{x_\mu} \lessdot \underline{\top}_{x_\lambda}$ and $A_{x_\mu} \leq \underline{\top}_{x_\lambda}$ for any $x_\lambda \not\leq^* B$ and any $\mu \in \beta^*_\lambda(L)$;
(LTDER3) $A \lessdot \bigwedge_{i\in I} B_i$ iff $A \lessdot B_i$ for any $i \in I$;
(LTDER4) $A \lessdot B$ implies $A \lessdot C \lessdot A \vee B$ for some $C \leq A \vee B$;
(LTDER5) $A \vee B \lessdot C$ iff $A \lessdot C$ and $B \lessdot C$.

It directly follows from (LTDER3) and (LTDER5) that $C \lessdot D$ for all $A, B, C, D \in L^X$ with $C \leq A \lessdot B \leq D$.

Let $(X, \lessdot_X)$ and $(Y, \lessdot_Y)$ be L-topological derived enclosed relation spaces. A mapping $f : X \to Y$ is called an L-topological derived enclosed relation preserving mapping if

$$A \lessdot_Y B \quad \text{implies} \quad f^{\leftarrow}_L(A) \lessdot_X f^{\leftarrow}_L(B) \vee f^{\leftarrow}_L(A)$$

The category of L-topological derived enclosed relation spaces and L-topological derived enclosed relation preserving mappings is denoted by L-**TDERS**.

Now, we consider the relations between L-**TDERS** and L-**TERS**.

Theorem 4.3. *Let $(X, \preccurlyeq)$ be an L-topological derived enclosed relation space. Define a binary relation $\preccurlyeq_{\preccurlyeq}$ on L^X by*

$$\forall A, B \in L^X,\ A \preccurlyeq_{\preccurlyeq} B \ \Leftrightarrow\ \exists C \in L^X,\ A \preccurlyeq C,\ A \vee C = B.$$

Then $(X, \preccurlyeq_{\preccurlyeq})$ is an L-topological enclosed relation space.

Proof. We check that $\preccurlyeq_{\preccurlyeq}$ satisfies (LTER1)–(LTER5).

(LTER1). We have $\underline{\bot} \preccurlyeq \underline{\bot}$ and $\underline{\bot} \vee \underline{\bot} = \underline{\bot}$. Thus $\underline{\bot} \preccurlyeq_{\preccurlyeq} \underline{\bot}$.

(LTER2). It directly follows from the definition.

(LTER3). If $A \preccurlyeq_{\preccurlyeq} \bigwedge_{i\in I} B_i$, then there is a $C \in L^X$ such that $A \preccurlyeq C$ and $A \vee C = \bigwedge_{i\in I} B_i$. Then $A \vee C \leq B_i$ for any $i \in I$. Thus $A \preccurlyeq B_i$ and $A \vee B_i = B_i$. That is, $A \preccurlyeq_{\preccurlyeq} B_i$ for any $i \in I$. Conversely, assume that $A \preccurlyeq_{\preccurlyeq} B$ for any $i \in I$. Then there is a $C_i \in L^X$ such that $A \preccurlyeq C_i$ and $A \vee C_i = B_i$ for any $i \in I$. By (LTDER3), we have $A \preccurlyeq \bigwedge_{i\in I} C_i$. In addition, we have

$$A \vee \bigwedge_{i\in I} C_i = \bigwedge_{i\in I} (A \vee C_i) = \bigwedge_{i\in I} B_i.$$

Thus $A \preccurlyeq_{\preccurlyeq} \bigwedge_{i\in I} B_i$.

(LTER4). Let $A \preccurlyeq_{\preccurlyeq} B$. Then there is a $D \in L^X$ such that $A \preccurlyeq D$ and $A \vee D = B$. By (LTDER4), there is a $C \leq A \vee D$ such that $A \preccurlyeq C \preccurlyeq A \vee D$. Let $E = A \vee C$. Then $A \preccurlyeq E$ and $A \vee E = E$. Thus $A \preccurlyeq_{\preccurlyeq} E$. Further, from $A \preccurlyeq A \vee D$ and $C \preccurlyeq A \vee D$, we have $E \preccurlyeq A \vee D$ by (LTDER5). In addition, $E \vee A \vee D = B$. Thus $E \preccurlyeq_{\preccurlyeq} B$. Therefore E satisfies the requirement.

(LTER5). Let $A \vee B \preccurlyeq_{\preccurlyeq} C$. Then there is a $D \in L^X$ such that $A \vee B \preccurlyeq D$ and $(A \vee B) \vee D = C$. Thus $A \preccurlyeq B \vee D$ and $A \vee B \vee D = C$. Hence $A \preccurlyeq_{\preccurlyeq} C$. Similarly, we have $B \preccurlyeq A \vee D$ and $B \vee A \vee D = C$. Therefore $B \preccurlyeq_{\preccurlyeq} C$. Conversely, assume that $A \preccurlyeq_{\preccurlyeq} C$ and $B \preccurlyeq_{\preccurlyeq} C$. Then there are $D, E \in L^X$ such that $A \preccurlyeq D$, $B \preccurlyeq E$, $A \vee D = C$ and $B \vee E = C$. Thus $(A \vee B) \vee (D \vee E) = C$. In addition, we have $A \preccurlyeq D \vee E$ and $B \preccurlyeq D \vee E$. Hence $A \vee B \preccurlyeq D \vee E$ by (LTDER5). Therefore $A \vee B \preccurlyeq_{\preccurlyeq} C$. □

Theorem 4.4. *Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-topological derived enclosed relation spaces. If $f : X \to Y$ is an L-topological derived enclosed relation preserving mapping, then $f : (X, \preccurlyeq_{\preccurlyeq_X}) \to (Y, \preccurlyeq_{\preccurlyeq_Y})$ is an L-topological enclosed relation preserving mapping.*

Proof. Let $A \preccurlyeq_{\preccurlyeq_Y} B$. Then there is a $C \in L^Y$ such that $A \preccurlyeq_Y C$ and $A \vee C = B$. Thus $f_L^{\leftarrow}(A) \preccurlyeq_X f_L^{\leftarrow}(A \vee C)$ and

$$f_L^{\leftarrow}(A) \vee f_L^{\leftarrow}(A \vee C) = f_L^{\leftarrow}(A \vee B) = f_L^{\leftarrow}(C).$$

Hence $f_L^{\leftarrow}(A) \preccurlyeq_{\preccurlyeq_Y} f_L^{\leftarrow}(B)$. Therefore f is an L-topological enclosed relation preserving mapping. □

Theorem 4.5. *Let $(X, \preccurlyeq)$ be an L-topological enclosed relation space. Define a binary relation $\preccurlyeq_{\preccurlyeq}$ on L^X by*

$$\forall A, B \in L^X,\ A \preccurlyeq_{\preccurlyeq} B \ \Leftrightarrow\ \forall x_\lambda \not\leq^* B, \forall \mu \in \beta^*_\lambda(L),\ A_{x_\mu} \preccurlyeq \underline{\top}_{x_\lambda},$$

where $x_\lambda \not\leq^* B$ implies that $x_\lambda \in \beta^*(\underline{\top})$ and $x_\lambda \not\leq B$. Then $(X, \preccurlyeq_{\preccurlyeq})$ is an L-topological derived enclosed relation space.

Proof. It is easy to check that $A \preccurlyeq_{\preccurlyeq} B$ for any $A, B, C, D \in L^X$ with $A \leq C \preccurlyeq_{\preccurlyeq} D \leq B$. To prove the result, we need to check that $\preccurlyeq_{\preccurlyeq}$ satisfies (LTDER1)–(LTDER5).

(LTDER1). It directly follows from (LTER1) of $\preccurlyeq$.

(LTDER2). Let $A \preccurlyeq_{\preccurlyeq} B$, $x_\lambda \not\leq^* B$ and $\mu \in \beta^*_\lambda(L)$. We have $B \leq \underline{\top}_{x_\lambda}$ by $x_\lambda \not\leq B$. Thus $A_{x_\mu} \leq A \preccurlyeq_{\preccurlyeq} B \leq \underline{\top}_{x_\lambda}$ which implies that $A_{x_\mu} \preccurlyeq_{\preccurlyeq} \underline{\top}_{x_\lambda}$. Further, by $A \preccurlyeq_{\preccurlyeq} B$, we have $A_{x_\mu} \preccurlyeq \underline{\top}_{x_\lambda}$. Hence $A_{x_\mu} \leq \underline{\top}_{x_\lambda}$ by (LTER2).

Conversely, assume that $A_{x_\mu} \preccurlyeq_{\preccurlyeq} \underline{\top}_{x_\lambda}$ and $A_{x_\mu} \leq \underline{\top}_{x_\lambda}$ for any $x_\lambda \not\leq B$ and any $\mu \in \beta^*_\lambda(L)$. Suppose that $A \not\preccurlyeq_{\preccurlyeq} B$. Then there are $x_\lambda \not\leq B$ and $\mu \in \beta^*_\lambda(L)$ such that $A_{x_\mu} \not\preccurlyeq \underline{\top}_{x_\lambda}$. Since $\mu \in \beta^*_\lambda(L)$, we have $x_\mu \not\leq \underline{\top}_{x_\lambda}$. Further, by $A_{x_\mu} \preccurlyeq_{\preccurlyeq} \underline{\top}_{x_\lambda}$, we have $A_{x_\mu} = (A_{x_\mu})_{x_\mu} \preccurlyeq (\underline{\top}_{x_\lambda})_{x_\mu} = \underline{\top}_{x_\lambda}$. It is a contradiction. Therefore $A \preccurlyeq_{\preccurlyeq} B$.

(LTDER3). If $A \preccurlyeq_{\preccurlyeq} \bigwedge_{i\in I} B_i$, then it is clear that $A \preccurlyeq_{\preccurlyeq} B_i$ for any $i \in I$. Conversely, assume that $A \preccurlyeq_{\preccurlyeq} B_i$ for any $i \in I$. For any $x_\lambda \not\leq^* \bigwedge_{i\in I} B_i$, there is an $i \in I$ such that $x_\lambda \not\leq^* B_i$. By $A \preccurlyeq_{\preccurlyeq} B_i$, we have $A_{x_\mu} \preccurlyeq \underline{\top}_{x_\lambda}$ for any $\mu \in \beta^*_\lambda(L)$. Therefore $A \preccurlyeq_{\preccurlyeq} \bigwedge_{i\in I} B_i$.

(LTDER4). Let $A \preccurlyeq_{\preccurlyeq} B$. We need to find some $E \in L^X$ such that $A \preccurlyeq_{\preccurlyeq} E \preccurlyeq_{\preccurlyeq} (A \vee B)$ and $E \leq A \vee B$.

Let $D = \bigwedge\{F \in L^X : A \lll_{\lll} F\}$ and let $E = A \vee D$. Then $E \leq A \vee B$ and $A \lll_{\lll} D$ by (LTDER3). This further implies that $A \lll_{\lll} E$. To prove that $E \lll_{\lll} A \vee B$, let $x_\lambda \not\leq^* A \vee B$. We prove that $E_{x_\mu} \lll \underline{\top}_{x_\lambda}$ for any $\mu \in \beta^*_\lambda(L)$.

By $A \lll_{\lll} B$, we have $A_{x_\mu} \lll \underline{\top}_{x_\lambda}$. By (LTER4), there is a $C \in L^X$ such that $A_{x_\mu} \lll C \lll \underline{\top}_{x_\lambda}$. Thus $A_{x_\mu} \leq C \leq \underline{\top}_{x_\lambda}$ by (LTER2). For any $z_\eta \not\leq C$ and any $\theta \in \beta^*_\eta(L)$, we have

$$A_{z_\theta} = A \lll C = C_{z_\eta} \leq \underline{\top}_{z_\eta}.$$

Hence $A \lll_{\lll} C$ and $D \leq C \lll \underline{\top}_{x_\lambda}$ followed by $D \lll \underline{\top}_{x_\lambda}$. By (LTER5), we have

$$E_{x_\mu} = A_{x_\mu} \vee D_{x_\mu} \leq A_{x_\mu} \vee D \lll \underline{\top}_{x_\lambda}$$

This implies that $E_{x_\mu} \lll \underline{\top}_{x_\lambda}$. Therefore $E \lll_{\lll} A \vee B$. That is, we have E satisfies the requirement.

(LTDER5). If $A \vee B \lll_{\lll} C$ then $A \lll_{\lll} C$ and $B \lll_{\lll} C$ are clear. Conversely, assume that $A \lll_{\lll} C$ and $B \lll_{\lll} C$. Let $x_\lambda \not\leq^* C$ and $\mu \in \beta^*_\lambda(L)$. By $A \lll_{\lll} C$ and $B \lll_{\lll} C$, we have $A_{x_\mu} \lll \underline{\top}_{x_\lambda}$ and $B_{x_\mu} \lll \underline{\top}_{x_\lambda}$. By (LTER5), we have

$$A \vee B_{x_\mu} = A_{x_\mu} \vee B_{x_\mu} = (A \vee B)_{x_\mu} \lll \underline{\top}_{x_\lambda}.$$

Hence $A \vee B \lll_{\lll} C$. □

Theorem 4.6. *Let $(X, \lll_X)$ and $(Y, \lll_Y)$ be L-topological enclosed relation spaces. If $f : X \to Y$ is an L-topological enclosed relation preserving mapping, then $f : (X, \lll_{\lll_X}) \to (Y, \lll_{\lll_Y})$ is an L-topological derived enclosed relation preserving mapping.*

Proof. Let $A \lll_{\lll_Y} B$. To prove that $f_L^{\leftarrow}(A) \lll_{\lll_X} f_L^{\leftarrow}(A\vee B)$, let $x_\lambda \not\leq^* f_L^{\leftarrow}(A\vee B)$ and $\mu \in \beta^*_\lambda(L)$. Then $f(x)_\mu \not\leq^* A\vee B$. By $A \lll_{\lll_Y} B$, we have $A = A_{f(x)_\mu} \lll_Y \underline{\top}_{f(x)_\lambda}$. Thus

$$f_L^{\leftarrow}(A)_{x_\mu} = f_L^{\leftarrow}(A) \lll_X f_L^{\leftarrow}(\underline{\top}_{f(x)_\lambda}) \leq \underline{\top}_{x_\lambda}.$$

Hence $f_L^{\leftarrow}(A) \lll_{\lll_X} f_L^{\leftarrow}(A \vee B)$. So f is an L-topological derived enclosed relation preserving mapping. □

Theorem 4.7. *We have $\lll_{\lll_{\lll}} = \lll$ for any L-topological derived enclosed relation space $(X, \lll)$ and $\lll_{\lll_{\lll}} = \lll$ for any L-topological enclosed relation space $(X, \lll)$.*

Proof. Let $(X, \lll)$ be an L-topological enclosed relation space. If $A \lll_{\lll_{\lll}} B$, then there is a $C \in L^X$ such that $A \lll_{\lll} C$ and $A \vee C = B$. Thus $A \lll_{\lll} B$ and $A \leq B$. By $A \lll_{\lll} B$, we have $A = A_{x_\mu} \lll \underline{\top}_{x_\lambda}$ for any $x_\lambda \not\leq^* B$ and $\mu \in \beta^*_\lambda(L)$. Hence $A \lll \bigwedge_{x_\lambda \not\leq^* B} \underline{\top}_{x_\lambda} = B$ by (LTER3). That is, $A \lll B$ holds.

Conversely, if $A \lll B$ then $A \leq B$ by (LTER2). For any $x_\lambda \not\leq^* B$ and $\mu \in \beta^*_\lambda(L)$, we have $A_{x_\mu} = A \lll B \leq \underline{\top}_{x_\lambda}$. Thus $A_{x_\mu} \lll \underline{\top}_{x_\lambda}$. Hence $A \lll_{\lll} B$ by (LTDER2). Further, by $A \lll_{\lll} B$ and $A \vee B = B$, we have $A \lll_{\lll_{\lll}} B$.

In conclusion, for all $A, B \in L^X$, we have $A \lll_{\lll_{\lll}} B$ if and only if $A \lll B$. That is, $\lll_{\lll_{\lll}} = \lll$.

Let $(X, \lll)$ be an L-topological derived enclosed relation space. Let $A \lll_{\lll_{\lll}} B$. If $x_\lambda \not\leq^* B$ and $\mu \in \beta^*_\lambda(L)$, we have $A_{x_\mu} \lll_{\lll} \underline{\top}_{x_\lambda}$. Thus there is a $C \in L^X$ such that $A_{x_\mu} \lll C$ and $A_{x_\mu} \vee C = \underline{\top}_{x_\lambda}$. Hence $A_{x_\mu} \lll \underline{\top}_{x_\lambda}$ and $A_{x_\mu} \leq \underline{\top}_{x_\lambda}$. Therefore $A \lll B$ by (LTDER2).

Conversely, let $A \lll B$. To prove that $A \lll_{\lll_{\lll}} B$, let $x_\lambda \not\leq^* B$ and $\mu \in \beta^*_\lambda(L)$. We need to prove that $A_{x_\mu} \lll_{\lll} \underline{\top}_{x_\lambda}$.

Actually, by $A \lll B \leq \underline{\top}_{x_\lambda}$, we have $A_{x_\mu} \lll \underline{\top}_{x_\lambda}$ and $A_{x_\mu} \leq \underline{\top}_{x_\lambda}$ by (LTDER2). In addition, by $A_{x_\mu} \vee \underline{\top}_{x_\lambda} = \underline{\top}_{x_\lambda}$, we have $A_{x_\mu} \lll_{\lll} \underline{\top}_{x_\lambda}$. Therefore $A \lll_{\lll_{\lll}} B$

In conclusion, for all $A, B \in 2^X$, we have $A \lll_{\lll_{\lll}} B$ iff $A \lll B$. That is, $\lll_{\lll_{\lll}} = \lll$. □

Based on Theorems 4.3 and 4.4, we obtain a functor $\mathbb{F} : L$-**TDERS**$\to L$-**TERS** defined by

$$\mathbb{F}((X, \lll)) = (X, \lll_{\lll}), \quad \mathbb{F}(f) = f.$$

Based on Theorems 4.3–4.7, we find that $\mathbb{F}$ is an isomorphic functor. Thus we have the following conclusion.

Theorem 4.8. *The category L-**TDERS** is isomorphic to the category L-**TERS**.*

To simply characterize L-topological derived enclosed relation spaces, we introduce the following notion.

Definition 4.9. An operator $\mathcal{D} : L^X \to L^X$ is called an L-topological derived closure operator on X and the pair $(X, \mathcal{D})$ is called an L-topological derived closure space if for all $A, B \in L^X$ and any $x_\lambda \in \beta^*(\underline{\top})$,

(LTDCl1) $\mathcal{D}(\underline{\bot}) = \underline{\bot}$;

(LTDCl2) $\mathcal{D}(A) \le B$ iff $\bigvee_{\mu\in\beta^*_\lambda(L)}(\mathcal{D}(A_{x_\mu}) \vee A_{x_\mu}) \le \underline{\top}_{x_\lambda}$ for any $x_\lambda \not\le^* B$;

(LTDCl3) $\mathcal{D}(\mathcal{D}(A)) \le \mathcal{D}(A) \vee A$;

(LTDCl4) $\mathcal{D}(A \vee B) = \mathcal{D}(A) \vee \mathcal{D}(B)$.

Let $(X, \mathcal{D}_X)$ and $(Y, \mathcal{D}_Y)$ be L-topological derived closure spaces. A mapping $f : X \to Y$ is called an L-topological derived closure preserving mapping, if $f_L^{\to}(\mathcal{D}_X(A)) \le \mathcal{D}_Y(f_L^{\to}(A)) \vee f_L^{\to}(A)$ for any $A \in L^X$.

The category of L-topological derived closure spaces and L-topological derived closure preserving mappings is denoted by L-**TDCLS**.

Theorem 4.10. *Let $(X, \mathcal{D})$ be an L-topological derived closure operator space. Define a binary operator $\leqslant_{\mathcal{D}}$ on X by*

$$\forall A, B \in L^X,\ \ A \leqslant_{\mathcal{D}} B \ \ \Leftrightarrow\ \ \mathcal{D}(A) \le B.$$

Then $(X, \leqslant_{\mathcal{D}})$ is an L-topological derived enclosed relation space.

Proof. (LTDER1). We have $\mathcal{D}(\underline{\bot}) = \underline{\bot}$ by (LTDCl1). Thus $\underline{\bot} \leqslant_{\mathcal{D}} \underline{\bot}$.

(LTDER2). It directly follows from (LTDCl2).

(LTDER3). If $A \leqslant_{\mathcal{D}} \bigwedge_{i\in I} B_i$ then $\mathcal{D}(A) \le \bigwedge_{i\in I} B_i \le B_j$ for any $j \in I$. Thus $A \leqslant_{\mathcal{D}} B_j$ for any $j \in I$. Conversely, if $A \leqslant_{\mathcal{D}} B_i$ for any $i \in I$, then $\mathcal{D}(A) \le B_i$. Thus $\mathcal{D}(A) \le \bigwedge_{i\in I} B_i$ which implies that $A \leqslant_{\mathcal{D}} \bigwedge_{i\in I} B_i$.

(LTDER4). Let $A \leqslant_{\mathcal{D}} B$ and let $E = \mathcal{D}(A) \vee A$. We have $E \leqslant_{\mathcal{D}} E$ by (LTDCl3) and (LTDCl4). Also, we have $\mathcal{D}(A) \le B$ and $E \le A \vee B$ by $A \leqslant_{\mathcal{D}} B$. In addition, $A \leqslant_{\mathcal{D}} E$ by $\mathcal{D}(A) \le E$. Therefore $A \leqslant_{\mathcal{D}} E \leqslant_{\mathcal{D}} A \vee B$ and $E \le A \vee B$ as desired.

(LTDER5). By (LTDCl4), we have $\mathcal{D}(A \vee B) = \mathcal{D}(A) \vee \mathcal{D}(B)$. Thus (LTDER5) holds trivially for $\leqslant_{\mathcal{D}}$. □

Theorem 4.11. *Let $(X, \mathcal{D}_X)$ and $(Y, \mathcal{D}_Y)$ be L-topological derived closure spaces. If $f : X \to Y$ is an L-topological derived preserving mapping, then $f : (X, \leqslant_{\mathcal{D}_X}) \to (Y, \leqslant_{\mathcal{D}_Y})$ is an L-topological derived enclosed relation preserving mapping.*

Proof. If $A \leqslant_{\mathcal{D}_Y} B$ then $\mathcal{D}_Y(A) \le B$. Thus

$$f_L^{\to}(\mathcal{D}_X(f_L^{\leftarrow}(A))) \le f_L^{\to}(f_L^{\leftarrow}(A)) \vee \mathcal{D}_Y(f_L^{\to}(f_L^{\leftarrow}(A))) \le A \vee \mathcal{D}_Y(A) \le A \vee B.$$

Hence $\mathcal{D}_X(f_L^{\leftarrow}(A)) \le f_L^{\leftarrow}(A \vee B) = f_L^{\leftarrow}(A) \vee f_L^{\leftarrow}(B)$ followed by $f_L^{\leftarrow}(A) \leqslant_{\mathcal{D}_X} f_L^{\leftarrow}(A) \vee f_L^{\leftarrow}(B)$. Therefore f is an L-topological derived enclosed relation preserving mapping. □

Theorem 4.12. *Let $(X, \leqslant)$ be an L-topological derived enclosed relation space. Define an operator $\mathcal{D}_\leqslant : L^X \to L^X$ by*

$$\forall A \in L^X,\ \ \mathcal{D}_\leqslant(A) = \bigwedge\{B \in L^X : A \leqslant B\}.$$

Then $(X, \mathcal{D}_\leqslant)$ is an L-topological derived closure space.

Proof. (LTDCl1). We have $\mathcal{D}_\leqslant(\underline{\bot}) \le \underline{\bot}$ by (LTDER1). Thus $\mathcal{D}_\leqslant(\underline{\bot}) = \underline{\bot}$.

(LTDCl2). If $\mathcal{D}_\leqslant(A) \le B$ then $A \leqslant \mathcal{D}_\leqslant(A) \le B$ which implies $A \leqslant B$. By (LTDER2), we have $A_{x_\mu} \le \underline{\top}_{x_\lambda}$ and $A_{x_\mu} \leqslant \underline{\top}_{x_\lambda}$ for any $x_\lambda \not\le^* B$ and any $\mu \in \beta^*_\lambda(L)$. Thus $\bigvee_{\mu\in\beta^*(L)}(\mathcal{D}_\leqslant(A_{x_\mu}) \vee A_{x_\mu}) \le \underline{\top}_{x_\lambda}$.

Conversely, assume that $\bigvee_{\mu\in\beta^*(L)}(\mathcal{D}_\leqslant(A_{x_\mu}) \vee A_{x_\mu}) \le \underline{\top}_{x_\lambda}$ for any $x_\lambda \not\le^* B$. By (LTDER3), we have $A_{x_\mu} \leqslant \mathcal{D}_\leqslant(A_{x_\mu})$ for all $x_\lambda \not\le^* B$ and $\mu \in \beta^*_\lambda(L)$. Thus $A_{x_\mu} \leqslant \underline{\top}_{x_\lambda}$ and $A_{x_\mu} \le \underline{\top}_{x_\lambda}$. Hence $A \leqslant \underline{\top}_{x_\lambda}$ by (LTDER2) and (5) of Proposition 4.1. Therefore $\mathcal{D}_\leqslant(A) \le \bigwedge_{x_\lambda \not\le^* B} \underline{\top}_{x_\lambda} = B$ by (6) of Proposition 4.1.

(LTDCl3). Let $x_\lambda \in \beta^*(\underline{\top})$ with $x_\lambda \not\le \mathcal{D}_\leqslant(A) \vee A$. Then $x_\lambda \not\le A$ and $x_\lambda \not\le \mathcal{D}_\leqslant(A)$. By $x_\lambda \not\le \mathcal{D}_\leqslant(A)$, there is $B \in L^X$ such that $x_\lambda \not\le B$ and $A \leqslant B$. By (LTDER4), there is $E \in L^X$ such that $A \leqslant E \le B \vee A$. By $A \leqslant E$ and (LTDER3), we have $\mathcal{D}_\leqslant(A) \leqslant E$. By (LTDER5), we have $\mathcal{D}_\leqslant(A) \vee A \leqslant E$. Thus

$$\mathcal{D}_\leqslant(\mathcal{D}_\leqslant(A) \vee A) \le E \le (A \vee B) \not\ge x_\lambda.$$

Hence $x_\lambda \nleq \mathcal{D}_\leqslant(\mathcal{D}_\leqslant(A) \vee A)$. Therefore, $\mathcal{D}_\leqslant(\mathcal{D}_\leqslant(A) \vee A) \leq \mathcal{D}_\leqslant(A) \vee A$.

(LTDCl4). Clearly, $\mathcal{D}_\leqslant(A) \vee \mathcal{D}_\leqslant(B) \leq \mathcal{D}_\leqslant(A \vee B)$. Conversely, let $x_\lambda \in J(L^X)$ with $x_\lambda \nleq \mathcal{D}_\leqslant(A) \vee \mathcal{D}_\leqslant(B)$. By $x_\lambda \nleq \mathcal{D}_\leqslant(A)$, there is $C \in L^X$ such that $x_\lambda \nleq C$ and $A \leqslant C$. Similarly, by $x_\lambda \nleq \mathcal{D}_\leqslant(B)$, there is $D \in L^X$ such that $x_\lambda \nleq D$ and $B \leqslant D$. Thus $x_\lambda \nleq C \vee D$ and $A \vee B \leqslant C \vee D$ by (LTDER5). Hence $\mathcal{D}_\leqslant(A \vee B) \leq C \vee D$ and $x_\lambda \nleq \mathcal{D}_\leqslant(A \vee B)$. Therefore $\mathcal{D}_\leqslant(A \vee B) \leq \mathcal{D}_\leqslant(A) \vee \mathcal{D}_\leqslant(B)$. □

Theorem 4.13. *Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be L-topological derived enclosed relation spaces. If $f : X \to Y$ is an L-topological derived enclosed relation preserving mapping, then $f : (X, \mathcal{D}_{\leqslant_X}) \to (Y, \mathcal{D}_{\leqslant_Y})$ is an L-topological derived closure preserving mapping.*

Proof. Let $A \in L^X$ and let $x_\lambda \in J(L^X)$ with $x_\lambda \nleq f_L^{\leftarrow}(\mathcal{D}_{\leqslant_Y}(f_L^{\rightarrow}(A))) \vee f_L^{\leftarrow}(f_L^{\rightarrow}(A))$. Then $f_L^{\rightarrow}(x_\lambda) \nleq \mathcal{D}_{\leqslant_Y}(f_L^{\rightarrow}(A)) \vee f_L^{\rightarrow}(A)$. By $f_L^{\rightarrow}(A) \nleq \mathcal{D}_{\leqslant_Y}(f_L^{\rightarrow}(A))$, there is $B \in L^X$ such that $f_L^{\rightarrow}(x_\lambda) \nleq B$ and $f_L^{\rightarrow}(A) \leqslant_Y B$. Thus $x_\lambda \nleq f_L^{\leftarrow}(B)$ and $A \leq f_L^{\leftarrow}(f_L^{\rightarrow}(A)) \leqslant_X f_L^{\leftarrow}(B)$. Hence $A \leqslant_X f_L^{\leftarrow}(B)$ and $x_\lambda \nleq \mathcal{D}_{\leqslant_X}(A)$. Therefore

$$\mathcal{D}_{\leqslant_X}(A) \leq f_L^{\leftarrow}(\mathcal{D}_{\leqslant_Y}(f_L^{\rightarrow}(A))) \vee f_L^{\leftarrow}(f_L^{\rightarrow}(A))$$

and $f_L^{\rightarrow}(\mathcal{D}_{\leqslant_X}(A)) \leq \mathcal{D}_{\leqslant_Y}(f_L^{\rightarrow}(A)) \vee f_L^{\rightarrow}(A)$. So f is an L-topological derived closure preserving mapping. □

Theorem 4.14. *We have $\mathcal{D}_{\leqslant_\mathcal{D}} = \mathcal{D}$ for any L-topological derived closure space $(X, \mathcal{D})$ and $\leqslant_{\mathcal{D}_\leqslant} = \leqslant$ for any L-topological derived enclosed relation space $(X, \leqslant)$.*

Proof. Let $(X, \mathcal{D})$ be an L-topological derived closure space and $A \in L^X$. We have

$$\mathcal{D}(A) \leq \bigwedge\{B \in L^X : A \leqslant_\mathcal{D} B\} = \mathcal{D}_{\leqslant_\mathcal{D}}(A).$$

Conversely, for any $x_\lambda \in J(L^X)$ with $x_\lambda \nleq \mathcal{D}(A)$, we have $\mathcal{D}(A) \leq \underline{\top}_{x_\lambda}$. Thus $A \leqslant_\mathcal{D} \underline{\top}_{x_\lambda}$ and $\mathcal{D}_{\leqslant_\mathcal{D}}(A) \leq \underline{\top}_{x_\lambda}$. So $\mathcal{D}_{\leqslant_\mathcal{D}}(A) \leq \bigwedge_{x_\lambda \nleq \mathcal{D}(A)} \underline{\top}_{x_\lambda} = \mathcal{D}(A)$. Hence $\mathcal{D}_{\leqslant_\mathcal{D}}(A) = \mathcal{D}(A)$ which shows that $\mathcal{D}_{\leqslant_\mathcal{D}} = \mathcal{D}$.

Let $(X, \leqslant)$ be an L-topological derived enclosed relation space. If $A \leqslant B$ then $\mathcal{D}_\leqslant(A) \leq B$ and so $A \leqslant_{\mathcal{D}_\leqslant} B$. Conversely, if $A \leqslant_{\mathcal{D}_\leqslant} B$, then $\mathcal{D}_\leqslant(A) \leq B$ and $A \leqslant \mathcal{D}_\leqslant(A)$ by (LTDER3). Thus $A \leqslant B$. In conclusion, we have $A \leqslant B$ iff $A \leqslant_{\mathcal{D}_\leqslant} B$. That is, $\leqslant_{\mathcal{D}_\leqslant} = \leqslant$. □

Based on Theorems 4.12 and 4.13, we obtain a functor $\mathbb{G} : L$-**TDERS**$\to L$-**TDCLS** by

$$\mathbb{G}((X, \leqslant)) = (X, \mathcal{D}_\leqslant), \quad \mathbb{G}(f) = f.$$

Based on Theorems 4.10–4.14, $\mathbb{G}$ is an isomorphic functor. Thus we have the following result.

Theorem 4.15. *The category L-***TDERS** *is isomorphic to the category L-***TDCLS***.*

Now, we characterize L-topological spaces by L-topological derived enclosed relation spaces

Theorem 4.16. *Let $(X, \mathcal{T})$ be an L-topological space. Define a binary operator $\leqslant_\mathcal{T}$ on X by*

$$\forall A, B \in L^X,\ A \leqslant_\mathcal{T} B \ \Leftrightarrow\ \forall x_\lambda \nleq^* B, \forall \mu \in \beta^*_\lambda(L),\ Cl_\mathcal{T}(A_{x_\mu}) \leq \underline{\top}_{x_\lambda}.$$

Then $(X, \leqslant_\mathcal{T})$ is an L-topological derived enclosed relation space.

Proof. Clearly, we have $A \leqslant_\mathcal{T} B$ for any $A, B, C, D \in L^X$ with $A \leq C \leqslant_\mathcal{T} D \leq B$. Next, we check that $\leqslant_\mathcal{T}$ satisfies (LTDER1)–(LTDER5).

(LTDER1). If $x_\lambda \in \beta^*(\underline{\top})$ and $\mu \in \beta^*_\lambda(L)$, then $Cl_\mathcal{T}(\underline{\bot}_{x_\mu}) = Cl_\mathcal{T}(\underline{\bot}) = \underline{\bot} \leq \underline{\top}_{x_\lambda}$ by (LCL1). Thus $\underline{\bot} \leqslant_\mathcal{T} \underline{\bot}$.

(LTDER2). Let $A \leqslant_\mathcal{T} B$. We have $A_{x_\mu} \leq Cl_\mathcal{T}(A_{x_\mu}) \leq \underline{\top}_{x_\lambda}$ for all $x_\lambda \nleq^* B$ and $\mu \in \beta^*_\lambda(L)$. To prove that $A_{x_\mu} \leqslant_\mathcal{T} \underline{\top}_{x_\lambda}$, let $y_\eta \nleq^* \underline{\top}_{x_\lambda}$ and $\theta \in \beta^*_\eta(L)$. By (5) of Proposition 4.1, we have $y = x$ and $\eta \in \beta^*_\lambda(L)$. Thus

$$Cl_\mathcal{T}((A_{x_\mu})_{x_\theta}) \leq Cl_\mathcal{T}(A_{x_\mu}) \leq \underline{\top}_{x_\lambda} \leq \underline{\top}_{x_\eta}.$$

Hence $A_{x_\mu} \leqslant_\mathcal{T} \underline{\top}_{x_\lambda}$.

Conversely, assume that $A_{x_\mu} \lessdot_{\mathcal{T}} \underline{\top}_{x_\lambda}$ and $A_{x_\mu} \le \underline{\top}_{x_\lambda}$ for all $x_\lambda \not\le^* B$ and $\mu \in \beta^*_\lambda(L)$. To prove that $A \lessdot_{\mathcal{T}} B$, let $x_\lambda \not\le^* B$ and $\mu \in \beta^*_\lambda(L)$. We have to prove that $Cl_{\mathcal{T}}(A_{x_\mu}) \le \underline{\top}_{x_\lambda}$.

Since $x_\lambda \not\le^* B$ and $\mu \in \beta^*_\lambda(L)$, we have $A_{x_\mu} \lessdot_{\mathcal{T}} \underline{\top}_{x_\lambda}$ and $A_{x_\mu} \le \underline{\top}_{x_\lambda}$. For any $y_\eta \not\le^* \underline{\top}_{x_\lambda}$, we have $y_\eta \not\le A_{x_\mu}$. In addition, $x = y$ and $\eta \in \beta^*_\lambda(L)$ by (5) of Proposition 4.1. Further, since $A_{x_\mu} \lessdot_{\mathcal{T}} \underline{\top}_{x_\lambda}$, we have $Cl_{\mathcal{T}}(A_{x_\mu}) = Cl_{\mathcal{T}}((A_{x_\mu})_{x_\theta}) \le \underline{\top}_{x_\eta}$ for any $\theta \in \beta^*_\eta(L)$. Thus $Cl_{\mathcal{T}}(A_{x_\mu}) = \bigwedge_{y_\eta \not\le \underline{\top}_{x_\lambda}} \underline{\top}_{y_\eta} = \underline{\top}_{x_\lambda}$. Therefore $A \lessdot_{\mathcal{T}} B$.

(LTDER3). Let $A \lessdot_{\mathcal{T}} \bigwedge_{i\in I} B_i$. Then it is clear that $A \lessdot_{\mathcal{T}} B_i$ for any $i \in I$. Conversely, assume that $A \lessdot_{\mathcal{T}} B_i$ for any $i \in I$. Let $x_\lambda \not\le^* \bigwedge_{i\in I} B_i$. Then there is $i \in I$ such that $x_\lambda \not\le^* B_i$. By $A \lessdot_{\mathcal{T}} B_i$, we have $Cl_{\mathcal{T}}(A_{x_\mu}) \le \underline{\top}_{x_\lambda}$ for any $\mu \in \beta^*_\lambda(L)$. From this result, we conclude that $A \lessdot_{\mathcal{T}} \bigwedge_{i\in I} B_i$.

(LTDER4). If $A \lessdot_{\mathcal{T}} B$, then $Cl_{\mathcal{T}}(A_{x_\mu}) \le \underline{\top}_{x_\lambda}$ for all $x_\lambda \not\le^* B$ and $\mu \in \beta^*_\lambda(L)$. Let $C = Cl_{\mathcal{T}}(A)$. We have $A \lessdot_{\mathcal{T}} C$ since $Cl_{\mathcal{T}}(A_{y_\theta}) \le C \le \underline{\top}_{y_\eta}$ for all $y_\eta \not\le^* C$ and $\theta \in \beta^*_\eta(L)$. Next, we prove that $C \lessdot_{\mathcal{T}} A \vee B$ and $C \le A \vee B$.

Let $y_\theta \not\le^* (A \vee B)$ and $\mu \in \beta^*_\theta(L)$. By $A \lessdot_{\mathcal{T}} B$, we have

$$Cl_{\mathcal{T}}(C_{y_\mu}) \le Cl_{\mathcal{T}}(C) = C = Cl_{\mathcal{T}}(A) = Cl_{\mathcal{T}}(A_{y_\mu}) \le \underline{\top}_{y_\theta}.$$

Thus $C \lessdot_{\mathcal{T}} (A \vee B)$ and $C \le \underline{\top}_{y_\theta}$. Hence $C \le \bigwedge_{y_\theta \not\le^* A\vee B} \underline{\top}_{y_\theta} = A \vee B$. So $A \lessdot_{\mathcal{T}} C \lessdot_{\mathcal{T}} A \vee B$ and $C \le A \vee B$.

(LTDER5). If $A \vee B \lessdot_{\mathcal{T}} C$, then it is clear that $A \lessdot_{\mathcal{T}} C$ and $B \lessdot_{\mathcal{T}} C$. Conversely, let $A \lessdot_{\mathcal{T}} C$ and $B \lessdot_{\mathcal{T}} C$. For any $x_\lambda \not\le^* C$ and any $\mu \in \beta^*(L)$, we have

$$Cl_{\mathcal{T}}((A \vee B)_{x_\mu}) = Cl_{\mathcal{T}}(A_{x_\mu} \vee B_{x_\mu}) = Cl_{\mathcal{T}}(A_{x_\mu}) \vee Cl_{\mathcal{T}}(B_{x_\mu}) \le \underline{\top}_{x_\lambda}.$$

Therefore $A \vee B \lessdot_{\mathcal{T}} C$. □

Theorem 4.17. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be L-topological spaces. If $f : X \to Y$ is an L-continuous mapping, then $f : (X, \lessdot_{\mathcal{T}_X}) \to (Y, \lessdot_{\mathcal{T}_Y})$ is an L-topological derived enclosed relation preserving mapping.*

Proof. Let $A \lessdot_{\mathcal{T}_Y} B$. To prove that $f_L^{\leftarrow}(A) \lessdot_{\mathcal{T}_X} f_L^{\leftarrow}(A \vee B)$, let $x_\lambda \not\le^* f_L^{\leftarrow}(A \vee B)$ and $\mu \in \beta^*_\lambda(L)$. Then $f_L^{\rightarrow}(x_\mu) \not\le^* A \vee B$ and $x_\mu \not\le^* f_L^{\leftarrow}(A \vee B)$. Further, by $A \lessdot_{\mathcal{T}_Y} B$, we have

$$f_L^{\rightarrow}(Cl_{\mathcal{T}_X}(f_L^{\leftarrow}(A)_{x_\mu})) = f_L^{\rightarrow}(Cl_{\mathcal{T}_X}(f_L^{\leftarrow}(A))) \le Cl_{\mathcal{T}_Y}(f_L^{\rightarrow}(f_L^{\leftarrow}(A))) \le Cl_{\mathcal{T}_Y}(A) = Cl_{\mathcal{T}_Y}(A_{f_L^{\rightarrow}(x_\mu)}) \le \underline{\top}_{f_L^{\rightarrow}(x_\lambda)}.$$

Thus $Cl_{\mathcal{T}_X}(f_L^{\leftarrow}(A)_{x_\mu}) \le \underline{\top}_{x_\mu}$ which implies that $f_L^{\leftarrow}(A) \lessdot_{\mathcal{T}_X} f_L^{\leftarrow}(A \vee B)$. Therefore f is an L-topological derived enclosed relation preserving mapping. □

Theorem 4.18. *Let $(X, \lessdot)$ be an L-topological derived enclosed relation space. Define an operator $Cl_{\lessdot} : L^X \to L^X$ by*

$$\forall A \in L^X,\ \ Cl_{\lessdot}(A) = A \vee \bigwedge\{B \in L^X : A \lessdot B\}.$$

Then $Cl_{\lessdot}$ is an L-topological closure operator which induces an L-topology denoted by $\mathcal{T}_{\lessdot}$.

Proof. (LCL1) and (LCL2) are direct.

(LCL3). It is clear that $Cl_{\lessdot}(A) \le Cl_{\lessdot}(Cl_{\lessdot}(A))$. Conversely, to prove that $Cl_{\lessdot}(Cl_{\lessdot}(A)) \le Cl_{\lessdot}(A)$, let $x_\lambda \not\le Cl_{\lessdot}(A)$. Then $x_\lambda \not\le A$ and there is some $B \in L^X$ such that $x_\lambda \not\le B$ and $A \lessdot B$. By (LTDER4), there is $C \in L^X$ such that $A \lessdot C \lessdot A \vee B$ and $C \le A \vee B$. Thus $Cl_{\lessdot}(A) \le A \vee C$ and $A \lessdot A \vee B$. Since $Cl_{\lessdot}(A) \le A \vee C \lessdot A \vee B$ by (LTDER5), we have $Cl_{\lessdot}(A) \lessdot A \vee B$ and

$$Cl_{\lessdot}(Cl_{\lessdot}(A)) \le Cl_{\lessdot}(A) \vee (A \vee B) = A \vee B \not\ge x_\lambda.$$

This implies that $x_\lambda \not\le Cl_{\lessdot}(Cl_{\lessdot}(A))$. Hence $Cl_{\lessdot}(Cl_{\lessdot}(A)) \le Cl_{\lessdot}(A)$. Therefore $Cl_{\lessdot}(Cl_{\lessdot}(A)) = Cl_{\lessdot}(A)$.

(LCL4). Clearly, we have $Cl_{\lessdot}(A) \vee Cl_{\lessdot}(B) \le Cl_{\lessdot}(A \vee B)$. Conversely, let $x_\lambda \not\le Cl_{\lessdot}(A) \vee Cl_{\lessdot}(B)$. By $x_\lambda \not\le Cl_{\lessdot}(A)$, there is $C \in L^X$ such that $A \lessdot C$ and $x_\lambda \not\le C$. Similarly, by $x_\lambda \not\le Cl_{\lessdot}(B)$, there is $D \in L^X$ such that $B \lessdot D$ and $x_\lambda \not\le D$. Thus $x_\lambda \not\le C \vee D$ and $(A \vee B) \lessdot (C \vee D)$ by (LTDER5). Hence $Cl_{\lessdot}(A \vee B) \le C \vee D$ which shows that $x_\lambda \not\le Cl_{\lessdot}(A \vee B)$. Therefore $Cl_{\lessdot}(A \vee B) \le Cl_{\lessdot}(A) \vee Cl_{\lessdot}(B)$. □

Theorem 4.19. *Let $(X, \lessdot_X)$ and $(Y, \lessdot_Y)$ be L-topological derived enclosed relation spaces. If $f : X \to Y$ is an L-topological derived enclosed relation preserving mapping, then $f : (X, \mathcal{T}_{\lessdot_X}) \to (Y, \mathcal{T}_{\lessdot_Y})$ is an L-continuous mapping.*

Proof. Let $A \in L^X$. To prove that $f_L^{\rightarrow}(Cl_{\leqslant_X}(A)) \leq Cl_{\leqslant_Y}(f_L^{\rightarrow}(A))$, we prove that $Cl_{\leqslant_X}(A)) \leq f_L^{\leftarrow}(Cl_{\leqslant_Y}(f_L^{\rightarrow}(A)))$.

For any $x_\lambda \not\leq f_L^{\leftarrow}(Cl_{\leqslant_Y}(f_L^{\rightarrow}(A)))$, we have $f_L^{\rightarrow}(x_\lambda) \not\leq Cl_{\leqslant_Y}(f_L^{\rightarrow}(A))$. Thus $f_L^{\rightarrow}(x_\lambda) \not\leq f_L^{\rightarrow}(A)$ and there is $B \in L^Y$ such that $f_L^{\rightarrow}(x_\lambda) \not\leq B$ and $f_L^{\rightarrow}(A) \leqslant_Y B$. Thus $A \leq f_L^{\leftarrow}(f_L^{\rightarrow}(A)) \leqslant_X f_L^{\leftarrow}(A \vee B)$ which shows that $A \leqslant_X f_L^{\leftarrow}(A \vee B)$. Since $x_\lambda \not\leq f_L^{\leftarrow}(A \vee B)$, we have $x_\lambda \not\leq Cl_{\leqslant_X}(A)$. Hence $Cl_{\leqslant_X}(A) \leq f_L^{\leftarrow}(Cl_{\leqslant_Y}(f_L^{\rightarrow}(A)))$. Therefore $f_L^{\rightarrow}(Cl_{\leqslant_X}(A)) \leq Cl_{\leqslant_Y}(f_L^{\rightarrow}(A))$. So f is an L-continuous mapping. □

Theorem 4.20. *We have $\mathcal{T}_{\leqslant_\mathcal{T}} = \mathcal{T}$ for any L-topological space $(X, \mathcal{T})$ and $\leqslant_{\mathcal{T}_\leqslant} = \leqslant$ for any L-topological derived enclosed relation space $(X, \leqslant)$.*

Proof. Let $(X, \mathcal{T})$ be an L-topological space. To prove $\mathcal{T}_{\leqslant_\mathcal{T}} = \mathcal{T}$, it is sufficient to prove that $Cl_{\leqslant_\mathcal{T}} = Cl_\mathcal{T}$.

Let $A \in L^X$. To check that $Cl_{\leqslant_\mathcal{T}}(A) \leq Cl_\mathcal{T}(A)$, we firstly check that $A \leqslant_\mathcal{T} \underline{\top}_{x_\lambda}$ for any $x_\lambda \not\leq^* Cl_\mathcal{T}(A)$.

Actually, for any $y_\eta \not\leq \underline{\top}_{x_\lambda}$, we have $x = y$ and $\eta \in \beta^*_\lambda(L)$. For any $\mu \in \beta^*_\eta(L)$, we have $x_\lambda \not\leq Cl_\mathcal{T}(A) \geq Cl_\mathcal{T}(A_{x_\mu})$. Thus $x_\lambda \not\leq Cl_\mathcal{T}(A_{x_\mu})$ and so $Cl_\mathcal{T}(A_{x_\mu}) \leq \underline{\top}_{x_\lambda} \leq \underline{\top}_{x_\mu}$. Hence $A \leqslant_\mathcal{T} \underline{\top}_{x_\lambda}$.

Further, by $A \leqslant_\mathcal{T} \underline{\top}_{x_\lambda}$, we have $Cl_{\leqslant_\mathcal{T}}(A) \leq A \vee \underline{\top}_{x_\lambda}$. Therefore

$$Cl_{\leqslant_\mathcal{T}}(A) \leq \bigwedge_{x_\lambda \not\leq^* Cl_\mathcal{T}(A)} (A \vee \underline{\top}_{x_\lambda}) = A \vee \bigwedge_{x_\lambda \not\leq^* Cl_\mathcal{T}(A)} \underline{\top}_{x_\lambda} = A \vee Cl_\mathcal{T}(A) = Cl_\mathcal{T}(A).$$

Conversely, to prove that $Cl_\mathcal{T}(A) \leq Cl_{\leqslant_\mathcal{T}}(A)$, let $z_\theta \not\leq^* Cl_{\leqslant_\mathcal{T}}(A)$. Then $z_\theta \not\leq A$ and there is $\eta \in \beta^*(\theta)$ such that $z_\eta \not\leq^* Cl_{\leqslant_\mathcal{T}}(A)$. Thus there is $B \in L^X$ such that $A \leqslant_\mathcal{T} B$ and $z_\eta \not\leq^* B$. Hence $Cl_\mathcal{T}(A_{z_\theta}) \leq \underline{\top}_{z_\eta}$ and so

$$Cl_\mathcal{T}(A) = \bigwedge_{z_\theta \not\leq^* Cl_{\leqslant_\mathcal{T}}(A)} Cl_\mathcal{T}(A_{z_\theta}) \leq \bigwedge_{z_\theta \not\leq^* Cl_{\leqslant_\mathcal{T}}(A)} \underline{\top}_{z_\eta} \leq \bigwedge_{z_\theta \not\leq^* Cl_{\leqslant_\mathcal{T}}(A)} \underline{\top}_{z_\theta} = Cl_{\leqslant_\mathcal{T}}(A).$$

Therefore $Cl_\mathcal{T}(A) = Cl_{\leqslant_\mathcal{T}}(A)$ which shows that $\mathcal{T}_{\leqslant_\mathcal{T}} = \mathcal{T}$.

Let $(X, \leqslant)$ be an L-topological derived enclosed relation space. Let $A \leqslant B$. To prove that $A \leqslant_{\mathcal{T}_\leqslant} B$, we firstly check that $A \leqslant_{\mathcal{T}_\leqslant} \underline{\top}_{x_\lambda}$ for any $x_\lambda \not\leq^* B$.

Let $x_\lambda \not\leq^* B$. To prove that $A \leqslant_{\mathcal{T}_\leqslant} \underline{\top}_{x_\lambda}$, let $y_\eta \not\leq^* \underline{\top}_{x_\lambda}$ and $\mu \in \beta^*_\eta(L)$. We need to prove that $Cl_{\mathcal{T}_\leqslant}(A_{y_\mu}) \leq \underline{\top}_{y_\eta}$. By $y_\eta \not\leq^* \underline{\top}_{x_\lambda}$, we have $x = y$ and $\lambda \leq \eta$. Since $A \leqslant B$, we have $A_{x_\mu} \leq \underline{\top}_{x_\eta}$ by (LTDER2). Further, since $A_{x_\mu} \leq A \leqslant B \leq \underline{\top}_{x_\lambda} \leq \underline{\top}_{x_\eta}$, we have $A_{x_\mu} \leqslant \underline{\top}_{x_\eta}$ and

$$Cl_{\mathcal{T}_\leqslant}(A_{x_\mu}) = Cl_\leqslant(A_{x_\mu}) \leq A_{x_\mu} \vee \underline{\top}_{x_\eta} = \underline{\top}_{x_\eta}.$$

Hence $A \leqslant_{\mathcal{T}_\leqslant} \underline{\top}_{x_\lambda}$. Therefore $A \leqslant_{\mathcal{T}_\leqslant} \bigwedge_{x_\lambda \not\leq^* B} \underline{\top}_{x_\lambda} = B$ by (LTDER3).

Conversely, let $A \leqslant_{\mathcal{T}_\leqslant} B$. To prove $A \leqslant B$, let $x_\lambda \not\leq^* B$ and $\mu \in \beta^*_\lambda(L)$. By $A \leqslant_{\mathcal{T}_\leqslant} B$ and (LTDER3), we have

$$A_{x_\mu} \leqslant \bigwedge \{D \in L^X : A_{x_\mu} \leqslant D\} \leq Cl_\leqslant(A_{x_\mu}) = Cl_{\mathcal{T}_\leqslant}(A_{x_\mu}) \leq \underline{\top}_{x_\lambda}.$$

Thus $A_{x_\mu} \leqslant \underline{\top}_{x_\lambda}$ and $A_{x_\mu} \leq \underline{\top}_{x_\lambda}$. Hence $A \leqslant B$ by (LTDER2).

In conclusion, for all $A, B \in L^X$, we have $A \leqslant_{\mathcal{T}_\leqslant} B$ iff $A \leqslant B$. Therefore $\leqslant_{\mathcal{T}_\leqslant} = \leqslant$. □

Based on Theorems 4.18 and 4.19, we obtain a functor $\mathbb{H} : L$-**TDERS**$\rightarrow L$-**TOP** by

$$\mathbb{H}((X, \leqslant)) = (X, \mathcal{T}_\leqslant), \quad \mathbb{H}(f) = f.$$

Based on Theorems 4.16–4.20, $\mathbb{H}$ is an isomorphic functor. Thus we have the following conclusion.

Theorem 4.21. *The category L-***DERS** *is isomorphic to the category L-***TOP***.*

Based on Theorems 4.10–4.20, relations between L-topological derived enclosed relation spaces and L-topological spaces are presented as follows.

Corollary 4.22. *(1) Let $(X, \mathcal{D})$ be an L-topological derived closure space. Define an operator $co_{\mathcal{D}} : L^X \to L^X$ by*

$$\forall A \in L^X,\ \ Cl_{\mathcal{D}}(A) = \mathcal{D}(A) \vee A.$$

Then $Cl_{\mathcal{D}}$ is the L-topological closure operator of an L-topological space $(X, \mathcal{T}_{\mathcal{D}})$.

(2) Let $(X, \mathcal{T})$ be an L-topological space. Define an operator $\mathcal{D}_{\mathcal{T}} : L^X \to L^X$ by

$$\forall A \in L^X,\ \ \mathcal{D}_{\mathcal{T}}(A) = \bigvee \{x_\lambda \in \beta^*(\underline{\top}) : \forall \mu \in \beta^*_\lambda(L), x_\mu \leq Cl_{\mathcal{T}}(A_{x_\mu})\}.$$

Then $(X, \mathcal{D}_{\mathcal{T}})$ is an L-topological derived closure space.

*(3) The category L-**TDCLS** is isomorphic to the category L-**TOP**.*

At the end of this section, by Theorem 4.8, we present two examples to show that an L-quasi-uniform space or an L-S-quasi-proximate space generates an L-topological derived enclosed relation space.

Example 4.23. Let $(X, \mathcal{U})$ be an L-quasi-uniform space. Define a binary relation $\leqslant_{\mathcal{U}}$ on X by

$$\forall A, B \in L^X,\ \ A \leqslant_{\mathcal{U}} B \ \ \Leftrightarrow \ \ \forall x_\lambda \not\leq^* B, \forall \mu \in \beta^*_\lambda(L),\ \exists \varphi \in \mathcal{U},\ A_{x_\mu} \leq \bigwedge_{\eta \in \beta^*_\lambda(L)} \varphi(x_\eta).$$

For all $A, B \in L^X$, it is easy to check that

$$A \leqslant_{\mathcal{U}} B \Leftrightarrow \forall x_\lambda \not\leq^* B, \forall \mu \in \beta^*_\lambda(L),\ A_{x_\mu} \leqslant_{\mathcal{U}} \underline{\top}_{x_\lambda} \Leftrightarrow A \leqslant_{\leqslant_{\mathcal{U}}} B.$$

Thus $\leqslant_{\mathcal{U}} = \leqslant_{\leqslant_{\mathcal{U}}}$. Hence $(X, \leqslant_{\mathcal{U}})$ is an L-topological derived enclosed relation space.

Example 4.24. Let (X, δ) be an L-S-quasi-proximate space. Define a binary relation $\leqslant_\delta$ on X by

$$\forall A, B \in L^X,\ \ A \leqslant_\delta B \ \ \Leftrightarrow \ \ \forall x_\lambda \in \beta^*(A), \forall \mu \in \beta^*_\lambda(L), \bigvee_{\eta \in \beta^*_\lambda(L)} \delta(x_\eta, A_{x_\mu}) = \bot.$$

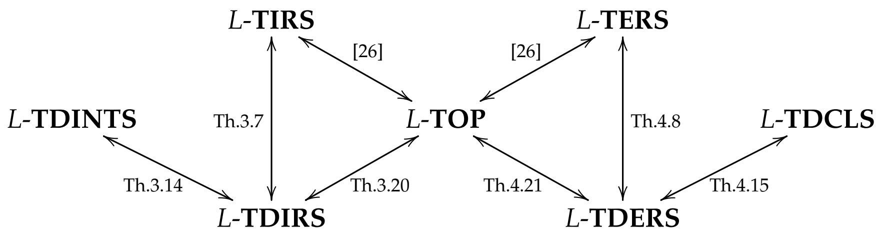
For all $A, B \in L^X$, it is easy to check that

$$A \leqslant_\delta B \Leftrightarrow \forall x_\lambda \in \beta^*(A), \forall \mu \in \beta^*_\lambda(L),\ A_{x_\mu} \leqslant_\delta \underline{\top}_{x_\lambda} \Leftrightarrow A \leqslant_{\leqslant_\delta} B.$$

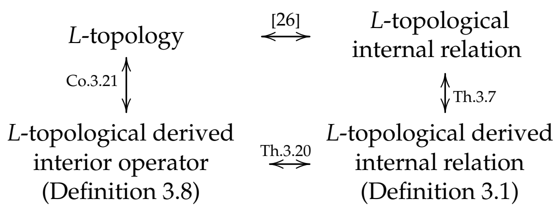
Thus $\leqslant_\delta = \leqslant_{\leqslant_\delta}$. Hence $(X, \leqslant_\delta)$ is an L-topological derived enclosed relation space.

5. Conclusions

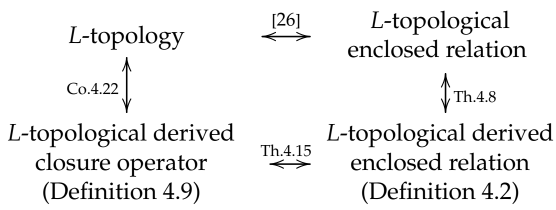
(1) In this paper, we introduce notions of L-topological derived internal relation spaces, L-topological derived enclosed relation spaces, L-topological derived interior spaces and L-topological derived closure spaces. We prove that all these spaces are categorically isomorphic to L-topological spaces. Relations among categories mentioned in this paper can be showed by the following diagram.



(2) The following diagrams give a solution of the problems presented in Introduction in L-fuzzy setting.



Solution 1.



Solution 2.

(3) Relations among L-topological spaces, L-topological derived internal relations and L-topological derived enclosed relations may provide some alternative ways in discussing separation axioms of L-topological spaces and relations among L-topological spaces, L-matroids, L-convex spaces and L-convergence spaces.

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