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# The Truncated Euler–Maruyama Method for Highly Nonlinear Neutral Stochastic Differential Equations With Time-Dependent Delay

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**Abstract.** The main goal of this paper is to establish the $L^q$-convergence of the truncated Euler–Maruyama method for neutral stochastic differential equations with time-dependent delay under the Khasminskii-type condition. Whole consideration is influenced by the presence of the neutral term and delay function. The main theoretical result is illustrated by an example. Since the main result is related to the $L^q$-convergence of the Euler–Maruyama method under the global Lipschitz condition on the coefficients of the equation under consideration, for completeness of the paper, the appropriate results are given in Appendix.

## 1. Introduction and Preliminary Results

Stochastic differential equations (SDEs) with time-dependent delay are widely known as a useful tool for describing phenomena which do not have an immediate effect from the moment of their occurrence. In general, we can find many systems, in almost any area of science (medicine, physics, ecology, biology, economics, etc.) where it is necessary to include this time lag in the corresponding model which we call the delay function, since it may change with respect to time. We refer the reader to [3, 5–10, 12, 15] and the literature cited therein, where the existence, uniqueness, stability and approximations of solutions are considered, for stochastic differential delay equations, neutral SDEs with time-dependent delay and neutral functional SDEs.

As is already known, explicit solutions of stochastic differential equations are difficult to obtain. As a result, different numerical methods have been developed to produce approximate solutions of the observed equation such as those from the papers [1, 2, 4, 11, 13, 14].

Recently, the truncated Euler–Maruyama method for different types of stochastic differential equations has been attracted the attention of many authors, as one can observe from papers [1, 2, 11, 13], for example. In these papers, authors studied the $L^p$-convergence of the truncated Euler–Maruyama solutions for ordinary SDEs, stochastic differential delay equations and neutral SDEs with constant delay under nonlinear-growth conditions, as well as properties of these solutions. So, the main goal of this paper is to determine the sufficient conditions under which the truncated Euler–Maruyama solutions for neutral SDEs with time-dependent delay converge in the $L^p$-sense to the exact solution of the equation. These conditions include

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the Khasminskii-type condition, as well as certain conditions related to the neutral term and delay function. Theoretical results are illustrated through an example. Since the proof of the main result of this paper, that is, the $L^q$-convergence of the truncated Euler–Maruyama method is based on the $L^p$-convergence of the Euler–Maruyama method under the global Lipschitz condition, in order to complete the analysis, the corresponding $L^p$-convergence result of the second method is given in Appendix.

The initial assumption is that all random variables and processes are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is increasing and right-continuous, and $\mathcal{F}_0$ contains all $P$-null sets). Let $B(t) = (B^1(t), B^2(t), ..., B^m(t))^T, t \geq 0$ be an $m$-dimensional standard Brownian motion, $\mathcal{F}_t$-adapted and independent of $\mathcal{F}_0$. Let the Euclidean norm be denoted by $|\cdot|$ and, for simplicity, trace$[A^T A] = |A|^2$ for matrix $A$, where $A^T$ is the transpose of a vector or a matrix. Moreover, for two real numbers $a$ and $b$, we use $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$.

For a given $\tau > 0$, let $C([-\tau, 0]; R^d)$ be the family of continuous functions $\varphi$ from $[-\tau, 0]$ to $R^d$, equipped with the supremum norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$. Also, denote by $C^b_{\mathcal{F}_0}([-\tau, 0]; R^d)$ the family of bounded, $\mathcal{F}_0$-measurable, $C([-\tau, 0]; R^d)$-valued random variables.

Let $\delta : [0, +\infty) \to [0, \tau]$ be the delay function which is Borel-measurable. We introduce the following autonomous neutral stochastic differential equation with time-dependent delay

$$d[x(t) - u(x(t - \delta(t)))] = f(x(t), x(t - \delta(t)))dt + g(x(t), x(t - \delta(t)))dB(t), \quad t \geq 0, \tag{1}$$

satisfying the initial condition

$$x_0 = \xi = \{\xi(\theta), \theta \in [-\tau, 0]\}, \tag{2}$$

where the functions

$$f : R^d \times R^d \to R^d, \quad g : R^d \times R^d \to R^{d\times m}, \quad u : R^d \to R^d,$$

are all Borel measurable and $x(t)$ is a $d$-dimensional state process. The initial condition $\xi$ is supposed to be a $C^b_{\mathcal{F}_0}([-\tau, 0]; R^d)$-valued random variable.

A $d$-dimensional stochastic process $\{x(t), t \geq -\tau\}$ is said to be a solution to Eq. (1) if it is a.s. continuous, $\mathcal{F}_t$-adapted, $\int_0^\infty |f(x(t), x(t - \delta(t)))|dt < \infty$ a.s., $\int_0^\infty |g(x(t), x(t - \delta(t)))|^2 dt < \infty$, a.s., $x_0 = \xi$ a.s. and for every $t \geq 0$, the integral form of Eq. (1) holds a.s. A solution $\{x(t), t \geq -\tau\}$ is said to be unique if any other solution $\{\tilde{x}(t), t \geq -\tau\}$ is indistinguishable from it, in the sense that $P\{x(t) = \tilde{x}(t), t \geq -\tau\} = 1$.

For the purpose of the following consideration we impose some hypotheses.

$\mathcal{H}_1$ : (The local Lipschitz condition) For each integer $R$, there exists a positive constant $K_R$, such that for all $x, y, \bar{x}, \bar{y} \in R^d$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq R$,

$$|f(x, y) - f(\bar{x}, \bar{y})|^2 \vee |g(x, y) - g(\bar{x}, \bar{y})|^2 \leq K_R\Big(|x - y|^2 + |\bar{x} - \bar{y}|^2\Big).$$

$\mathcal{H}_2$: There exists a constant $k \in (0, 1)$ such that, for all $x, y \in R^d$,

$$|u(x) - u(y)| \leq k|x - y|. \tag{3}$$

Moreover, we suppose that $u(0) = 0$ which, together with (3), implies that

$$|u(x)| \leq k|x|. \tag{4}$$

$\mathcal{H}_3$: The delay function $\delta$ is continuously differentiable and $\delta'(t) \leq \bar{\delta} < 1$.

$\mathcal{H}'_4$ : There are constants $p \geq 2$ and $K > 0$ such that, for all $a \in (0, 1]$ and $x, y \in R^d$,

$$(x - au(y))^T f(x, y) + \frac{p - 1}{2}|g(x, y)|^2 \leq K(1 + |x|^2 + a|y|^2).$$

Clearly, particularly for $a = 1$ we obtain the following condition.

$\mathcal{H}_4$ : (The Khasminski-type condition) There are constants $p \geq 2$ and $K > 0$ such that, for all $x, y \in R^d$,

$$(x - u(y))^T f(x, y) + \frac{p - 1}{2}|g(x, y)|^2 \leq K(1 + |x|^2 + |y|^2).$$

$\mathcal{H}_5$: There exists a positive constant $C_\xi$ such that, for $p \geq 2$,

$$E \sup_{t,s\in[-\tau,0],|s-t|\leq\Delta} |\xi(s) - \xi(t)|^p \leq C_\xi \Delta^{\frac{p}{2}}. \tag{5}$$

$\mathcal{H}_6$: There exists a positive constant $\eta$ such that

$$|\delta(t) - \delta(s)| \leq \eta|t - s|, \qquad t, s \geq 0. \tag{6}$$

At the end of this section we introduce the following existence and uniqueness theorem. The proof of this result can be found in [12] for $V(x) = |x|^p$, $U(x) = 0$ for all $x \in R^d$, $c_1 = c_2 = 1$, $\lambda_1 = 2K[1 + 3(p-2)2^{p-2}]$ and arbitrary $\lambda_2 > 0$ for which $\lambda_2 > \lambda_1/(1-\bar{\delta})$.

**Theorem 1.1.** *Let Assumptions $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ and $\mathcal{H}_4$ be satisfied. Then, for any initial condition $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^d)$ there exists a unique global solution $x = \{x(t), t \geq -\tau\}$ of Eq. (1). Moreover, the solution has the property that for all $T > 0$, we have that*

$$\sup_{-\tau\leq t\leq T} E|x(t)|^p < \infty.$$

## 2. The Truncated Euler–Maruyama Approximate Solution

In this section we will introduce the truncated Euler–Maruyama approximate solution to Eq. (1). To define the truncated Euler–Maruyama numerical solutions, we first choose a strictly increasing continuous function $\mu : R_+ \to R_+$ such that $\mu(r) \to \infty$ when $r \to \infty$ and for all $r \geq 1$

$$\sup_{|x|\vee|y|\leq r} (|f(x, y)| \vee |g(x, y)|) \leq \mu(r). \tag{7}$$

The inverse function of $\mu$ is denoted by $\mu^{-1}$. So, $\mu^{-1} : [\mu(0), \infty) \to R^+$ is a strictly increasing continuous function. We also choose constant $\Delta^* \in (0, 1)$ and a strictly decreasing function $h : (0, \Delta^*] \to (0, \infty)$ such that

$$h(\Delta^*) \geq \mu(1), \quad \lim_{\Delta\to 0} h(\Delta) = \infty, \tag{8}$$

and for all $\Delta \in (0, \Delta^*]$ and some $\epsilon_0 \in (0, 1)$ we have that

$$\Delta^{\frac{1-\epsilon_0}{4}} h(\Delta) \leq 1. \tag{9}$$

Now, for a given step size $\Delta \in (0, \Delta^*]$, we define a mapping $\pi_\Delta(x)$ from $R^d$ to the closed ball $\{x \in R^d : |x| \leq \mu^{-1}(h(\Delta))\}$ by

$$\pi_\Delta(x) = (|x| \wedge \mu^{-1}(h(\Delta)))\frac{x}{|x|}, \tag{10}$$

where we set $x/|x| = 0$ when $x = 0$. Function $\pi_\Delta(x)$ will map $x$ to itself when $|x| \leq \mu^{-1}(h(\Delta))$ and to $\mu^{-1}(h(\Delta))\frac{x}{|x|}$ when $|x| > \mu^{-1}(h(\Delta))$. Now, we can define the truncated functions

$$f_\Delta(x, y) = f(\pi_\Delta(x), \pi_\Delta(y)), \quad g_\Delta(x, y) = g(\pi_\Delta(x), \pi_\Delta(y)), \quad u_\Delta(x) = u(\pi_\Delta(x)),$$

for $x, y \in R^d$. It is easy to see that

$$\left|(|x| \wedge \mu^{-1}(h(\Delta)))\frac{x}{|x|}\right| \leq \mu^{-1}(h(\Delta)),$$

which together with (7) implies that, for all $x, y \in R^d$,

$$|f_\Delta(x, y)| \vee |g_\Delta(x, y)| \leq \mu(\mu^{-1}(h(\Delta))) = h(\Delta). \tag{11}$$

Namely, we have showed that both truncated functions $f_\Delta$ and $g_\Delta$ are bounded although $f$ and $g$ may not be. So, we consider the following stochastic differential equation

$$d[x(t) - u_\Delta(x(t-\delta(t)))] = f_\Delta(x(t), x(t-\delta(t)))dt + g_\Delta(x(t), x(t-\delta(t)))dB(t), \quad t \ge 0,$$

satisfying the initial condition (2).

Now, we will define the truncated Euler–Maruyama approximate solution $x_\Delta$ of Eq. (1) on the finite time interval $[0, T]$, for arbitrary $T > 0$. Without loss of generality, we may assume that $T/\tau$ is a rational number; otherwise we may replace $T$ by a larger number. Let the step size $\Delta \in (0, 1)$ be a fraction of $\tau$ and $T$, namely, $\Delta = \tau/M = T/N$ for some integers $M > \tau$ and $N > T$.

Primarily, we define the discrete-time truncated Euler–Maruyama approximate solution $X_\Delta$ of Eq. (1) on the equidistant partition $t_k = k\Delta$, for $k \in \{-(M+1), -M, \dots, 0, 1, \dots, N\}$. In order for this solution to be well defined, set

$$\delta(-\Delta) = \delta(0), \qquad X_\Delta(-(M+1)\Delta) = \xi(-M\Delta). \tag{12}$$

Then, define

$$X_\Delta(t_k) = \xi(t_k),\ k = -M, -(M-1), \dots, 0, \tag{13}$$

$$\begin{aligned} X_\Delta(t_{k+1}) = {} & X_\Delta(t_k) + u_\Delta(X_\Delta(t_k - I_\Delta[\delta(t_k)]\Delta)) - u_\Delta(X_\Delta(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta)) \\ & + f_\Delta(X_\Delta(t_k), X_\Delta(t_k - I_\Delta[\delta(t_k)]\Delta))\Delta + g_\Delta(X_\Delta(t_k), X_\Delta(t_k - I_\Delta[\delta(t_k)]\Delta))\Delta B_k, \quad k \in \{0, 1, \dots, N\}, \end{aligned} \tag{14}$$

where $\Delta B_k = B(t_{k+1}) - B(t_k)$. Here $I_\Delta[u]$ denotes the integer part of the real number $u/\Delta$, where $u \in [0, \tau]$. Thus, $I_\Delta[\delta(t_k)]$ represents the integer part of $\delta(t_k)/\Delta$. In our analysis, it is more convenient to work with the continuous-time approximations. In that sense, we introduce the step processes

$$\bar{x}_\Delta(t) = \sum_{k=0}^{N-1} X_\Delta(t_k) I_{[t_k, t_{k+1})}(t), \quad \bar{y}_\Delta(t) = \sum_{k=-1}^{N-1} X_\Delta(t_k - I_\Delta[\delta(t_k)]\Delta) I_{[t_k, t_{k+1})}(t), \tag{15}$$

where $I_{[t_k, t_{k+1})}(t)$ is the indicator function of $[t_k, t_{k+1})$. Then, we define the linear combination of $X_\Delta(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta)$ and $X_\Delta(t_k - I_\Delta[\delta(t_k)]\Delta)$ as

$$Z_k(t) = \bar{y}_\Delta(t_{k-1}) + \frac{t - t_k}{\Delta}(\bar{y}_\Delta(t_k) - \bar{y}_\Delta(t_{k-1})), \tag{16}$$

whenever $t \in [t_k, t_{k+1})$, $k = 0, 1, 2, \dots, N-1$. For convenience, denote that

$$Z_k(t) = \left(1 - \frac{t - t_k}{\Delta}\right)\bar{y}_\Delta(t_{k-1}) + \frac{t - t_k}{\Delta}\bar{y}_\Delta(t_k), \tag{17}$$

$$\bar{z}_\Delta(t) = \sum_{k=0}^{N-1} Z_k(t) I_{[t_k, t_{k+1})}(t). \tag{18}$$

Then, we are in a position to define the continuous-time truncated Euler–Maruyama approximate solution $\{x_\Delta(t), t \in [-\tau, T]\}$ such that $x_\Delta(t) = \xi(t)$, $t \in [-\tau, 0]$, while

$$\begin{aligned} x_\Delta(t) = {} & \xi(0) + u_\Delta(\bar{z}_\Delta(t)) - u_\Delta(X_\Delta(-\Delta - I_\Delta[\delta(-\Delta)]\Delta)) \\ & + \int_0^t f_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s))ds + \int_0^t g_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s))dB(s), \quad t \in [0, T]. \end{aligned} \tag{19}$$

Clearly, for $t \in [t_k, t_{k+1})$, Eq. (19) can be written as

$$x_\Delta(t) = X_\Delta(t_k) + u_\Delta(Z_k(t)) - u_\Delta(X_\Delta(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta)) + \int_{t_k}^t f_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s))ds + \int_{t_k}^t g_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s))dB(s). \tag{20}$$

Clearly, $x_\Delta(t_k) = \bar{x}_\Delta(t_k) = X_\Delta(t_k)$ for every $k \geq -M$, that is, the discrete and continuous-time truncated Euler–Maruyama solutions coincide at the grid points $t_k$. Bearing in mind that, for $t \in [0,T]$, there is a unique integer $k \geq 0$ such that $t \in [t_k, t_{k+1})$, from (18), (17) and (15), respectively we see that

$$\begin{aligned}|\bar{z}_\Delta(t)| &= |Z_k(t)| \\ &\leq \left(1 - \frac{t-t_k}{\Delta}\right)|\bar{y}_\Delta(t_{k-1})| + \frac{t-t_k}{\Delta}|\bar{y}_\Delta(t_k)| \\ &\leq \left(1 - \frac{t-t_k}{\Delta}\right)\sup_{-\tau\leq s\leq t}|x_\Delta(s)| + \frac{t-t_k}{\Delta}\sup_{-\tau\leq s\leq t}|x_\Delta(s)| \\ &\leq \sup_{-\tau\leq s\leq t}|x_\Delta(s)|. \end{aligned} \tag{21}$$

In order to complete this section we establish the following lemmas which are essential for obtaining the main results of this paper.

**Lemma 2.1.** *Let Assumption $\mathcal{H}_2$ hold. Then, for $k \in (0, \frac{1}{3})$ and all $x, y \in R^d$,*

$$|u_\Delta(x) - u_\Delta(y)| \leq 3k|x-y|. \tag{22}$$

*Moreover, for all $x \in R^d$,*

$$|u_\Delta(x)| \leq k|x|. \tag{23}$$

*Proof.* Let $x, y \in R^d$ be arbitrary. In following consideration we observe $x, y \in R^d$ such that $x \neq 0$ and $y \neq 0$. For $x, y \in R^d$ with $|x| \leq \mu^{-1}(h(\Delta))$ and $|y| \leq \mu^{-1}(h(\Delta))$, the assertion (22) follows immediately.

On the other hand, for $x, y \in R^d$ with $|x| > \mu^{-1}(h(\Delta))$ and $|y| > \mu^{-1}(h(\Delta))$, on the basis of (3), we have that

$$\begin{aligned}|u_\Delta(x) - u_\Delta(y)| &= |u(\pi_\Delta(x)) - u(\pi_\Delta(y))| \\ &\leq k|\pi_\Delta(x) - \pi_\Delta(y)| \\ &= k\left|\frac{\mu^{-1}(h(\Delta))}{|x|}x - \frac{\mu^{-1}(h(\Delta))}{|y|}y\right| \\ &= k\left|\frac{\mu^{-1}(h(\Delta))|y|x - \mu^{-1}(h(\Delta))|y|y + \mu^{-1}(h(\Delta))|y|y - \mu^{-1}(h(\Delta))|x|y}{|x||y|}\right| \\ &\leq k\left(\mu^{-1}(h(\Delta))|y|\frac{|x-y|}{|x||y|} + \mu^{-1}(h(\Delta))|y|\frac{||y|-|x||}{|x||y|}\right) \\ &\leq k\left(\frac{\mu^{-1}(h(\Delta))}{|x|}|x-y| + \frac{\mu^{-1}(h(\Delta))}{|x|}|x-y|\right). \end{aligned}$$

Using the condition $\mu^{-1}(h(\Delta))/|x| < 1$, we conclude that

$$|u_\Delta(x) - u_\Delta(y)| \leq 2k|x-y|,$$

that is, the inequality (22) holds.

For $x, y \in R^d$ with $|x| > \mu^{-1}(h(\Delta))$ and $|y| \leq \mu^{-1}(h(\Delta))$, bearing in mind (3), we find that

$$\begin{aligned}|u_\Delta(x) - u_\Delta(y)| &= |u(\pi_\Delta(x)) - u(\pi_\Delta(y))| \\ &\leq k|\pi_\Delta(x) - \pi_\Delta(y)| \\ &= k\left|\frac{\mu^{-1}(h(\Delta))}{|x|}x - y\right| \\ &= k\left|\frac{\mu^{-1}(h(\Delta))x|y| - y|x||y|}{|x||y|}\right| \\ &= k\left|\frac{\mu^{-1}(h(\Delta))x|y| - \mu^{-1}(h(\Delta))|x|y + \mu^{-1}(h(\Delta))|x|y - y|y||x|}{|x||y|}\right|. \end{aligned}$$

Thus, we conclude that

$$|u_\Delta(x) - u_\Delta(y)| \le k\left(\frac{\mu^{-1}(h(\Delta))}{|x|}\frac{|x|y| - y|x||}{|y|} + |\mu^{-1}(h(\Delta)) - |y||\right) \le k\left(\frac{|x|y| - y|x||}{|y|} + |\mu^{-1}(h(\Delta)) - |y||\right). \tag{24}$$

Since $|x| > \mu^{-1}(h(\Delta)) \ge |y|$, then

$$|\mu^{-1}(h(\Delta)) - |y|| = \mu^{-1}(h(\Delta)) - |y| \le |x| - |y| = ||x| - |y|| \le |x - y|. \tag{25}$$

Moreover, we have that

$$\frac{|x|y| - y|x||}{|y|} = \frac{|x|y| - y|y| + y|y| - y|x||}{|y|} \le |x - y| + ||y| - |x|| \le 2|x - y|. \tag{26}$$

Substituting (25) and (26) into (24) yields (22). The same way (22) follows for $x, y \in R^d$ with $\mu^{-1}(h(\Delta)) \ge |x|$ and $\mu^{-1}(h(\Delta)) < |y|$.
Now, we suppose that one of the values $x$ and $y$ is equal to null vector, e.g. $y = 0$. Then, from (4), we have that

$$|u_\Delta(x) - u_\Delta(y)| = |u(\pi_\Delta(x))| \le k|\pi_\Delta(x)| \le k|x| = k|x - y|.$$

If $x = 0$ and $y = 0$ then (22) holds. It should be pointed out that (23) holds as well. □

**Lemma 2.2.** *Let Assumption $\mathcal{H}_4'$ hold. Then, for every $\Delta \in (0, \Delta^*]$ and all $x, y \in R^d$, we have that*

$$(x - u_\Delta(y))^T f_\Delta(x, y) + \frac{p-1}{2}|g_\Delta(x, y)|^2 \le \bar{K}(1 + |x|^2 + |y|^2), \tag{27}$$

*where $\bar{K} = \frac{3}{2}K$.*

*Proof.* Fix any $\Delta \in (0, \Delta^*]$. Since the function $h$ is decreasing and function $\mu^{-1}$ is increasing, it follows that, for all $\Delta \in (0, \Delta^*]$,

$$\mu^{-1}(h(\Delta)) \ge \mu^{-1}(h(\Delta^*)). \tag{28}$$

Thus, using the condition (8) we obtain

$$\mu^{-1}(h(\Delta)) \ge \mu^{-1}(\mu(1)) = 1. \tag{29}$$

For $x, y \in R^d$ with $|x| \vee |y| < \mu^{-1}(h(\Delta))$, on the basis of Assumption $\mathcal{H}_4'$, the inequality (27) obviously holds for $a = 1$.

Furtheron, let $|x| \wedge |y| \ge \mu^{-1}(h(\Delta))$. Then, we have that

$$\begin{aligned}
&(x - u_\Delta(y))^T f_\Delta(x, y) + \frac{p-1}{2}|g_\Delta(x, y)|^2 \\
&\quad = (x - u(\pi_\Delta(y)))^T f(\pi_\Delta(x), \pi_\Delta(y)) + \frac{p-1}{2}|g(\pi_\Delta(x), \pi_\Delta(y))|^2 \\
&\quad = \frac{|x|}{\mu^{-1}(h(\Delta))}\left[\left(\pi_\Delta(x) - \frac{\mu^{-1}(h(\Delta))}{|x|}u(\pi_\Delta(y))\right)^T f(\pi_\Delta(x), \pi_\Delta(y)) + \frac{p-1}{2}\frac{\mu^{-1}(h(\Delta))}{|x|}|g(\pi_\Delta(x), \pi_\Delta(y))|^2\right] \\
&\quad \le \frac{|x|}{\mu^{-1}(h(\Delta))}\left[\left(\pi_\Delta(x) - \frac{\mu^{-1}(h(\Delta))}{|x|}u(\pi_\Delta(y))\right)^T f(\pi_\Delta(x), \pi_\Delta(y)) + \frac{p-1}{2}|g(\pi_\Delta(x), \pi_\Delta(y))|^2\right].
\end{aligned}$$

So, on the basis of Assumption $\mathcal{H}_4'$ and (29), we see that

$$\begin{aligned}(x-u_\Delta(y))^T f_\Delta(x,y)+\frac{p-1}{2}|g_\Delta(x,y)|^2 &\le \frac{|x|}{\mu^{-1}(h(\Delta))}K\left[1+|\pi_\Delta(x)|^2+\frac{\mu^{-1}(h(\Delta))}{|x|}|\pi_\Delta(y)|^2\right]\\ &= K\left[\frac{|x|}{\mu^{-1}(h(\Delta))}+\frac{\mu^{-1}(h(\Delta))}{|x|}|x|^2+|\pi_\Delta(y)|^2\right]\\ &\le K\left[|x|+|x|^2+|y|^2\right]\\ &\le \bar{K}\left[1+|x|^2+|y|^2\right],\end{aligned}$$

where $\bar{K}=\frac{3}{2}K$.

On the other hand, if $|x|\ge\mu^{-1}(h(\Delta))$ and $y<\mu^{-1}(h(\Delta))$, then Assumption $\mathcal{H}_4'$ and (29) imply

$$\begin{aligned}&(x-u_\Delta(y))^T f_\Delta(x,y)+\frac{p-1}{2}|g_\Delta(x,y)|^2\\ &= (x-u(y))^T f(\pi_\Delta(x),y)+\frac{p-1}{2}|g(\pi_\Delta(x),y)|^2\\ &\le \frac{|x|}{\mu^{-1}(h(\Delta))}\left[\left(\pi_\Delta(x)-\frac{\mu^{-1}(h(\Delta))}{|x|}u(y)\right)^T f(\pi_\Delta(x),y)+\frac{p-1}{2}|g(\pi_\Delta(x),y)|^2\right]\\ &\le \frac{|x|}{\mu^{-1}(h(\Delta))}K\left[1+|\pi_\Delta(x)|^2+\frac{\mu^{-1}(h(\Delta))}{|x|}|y|^2\right]\\ &= K\left[\frac{|x|}{\mu^{-1}(h(\Delta))}+\frac{\mu^{-1}(h(\Delta))}{|x|}|x|^2+|y|^2\right]\\ &\le K\left[|x|+|x|^2+|y|^2\right]\\ &\le \bar{K}\left[1+|x|^2+|y|^2\right],\end{aligned}$$

that is, the desired assertion (27) holds.

Finally, for $|y|\ge\mu^{-1}(h(\Delta))$ and $x<\mu^{-1}(h(\Delta))$, applying Assumption $\mathcal{H}_4'$ for $a=1$, we obtain

$$\begin{aligned}(x-u_\Delta(y))^T f_\Delta(x,y)+\frac{p-1}{2}|g_\Delta(x,y)|^2 &= (x-u(\pi_\Delta(y)))^T f(x,\pi_\Delta(y))+\frac{p-1}{2}|g(x,\pi_\Delta(y))|^2\\ &\le K\left[1+|x|^2+|\pi_\Delta(y)|^2\right]\\ &\le \bar{K}\left[1+|x|^2+|y|^2\right].\end{aligned}$$

Therefore, the proof is complete. □

## 3. The $L^q$-convergence of the Truncated Euler–Maruyama Solutions

The main aim in this section is to establish the $L^q$ closeness between the truncated Euler–Maruyama solution (19) and exact solution of Eq. (1). In that sense, we first prove several assertions which are essential for proving the main result.

Let $\lfloor\cdot\rfloor$ be the integer part function.

**Lemma 3.1.** *If Assumptions $\mathcal{H}_2$, $\mathcal{H}_3$, $\mathcal{H}_5$ and $\mathcal{H}_6$ hold, together with the condition*

$$k(3+\lfloor\eta\rfloor)^2<1, \tag{30}$$

*then, for any $\Delta \in (0, \Delta^*]$ and for any integer $l > 1$ and $p \geq 2$,*

$$E \sup_{-M \leq k \leq N} |X_\Delta(t_k) - X_\Delta(t_{k-1})|^p \leq \tilde{c}\Delta^{\frac{p}{2}} + \tilde{c}_l \Delta^{\frac{pl-1}{2l}} (h(\Delta))^p, \tag{31}$$

*where constants $\tilde{c} > 0$ and $\tilde{c}_l > 0$ are independent of $\Delta$ and $\tilde{c}_l$ is dependent on $l$.*

*Proof.* Fix any $\Delta \in (0, \Delta^*]$. On the basis of (12), (13) and Assumption $\mathcal{H}_5$, we observe that

$$E \sup_{-M \leq k \leq 0} |X_\Delta(t_k) - X_\Delta(t_{k-1})|^p = E \sup_{-M \leq k \leq 0} |\xi(t_k) - \xi(t_{k-1})|^p \leq C_\xi \Delta^{\frac{p}{2}}. \tag{32}$$

When $k \in \{1, 2, \ldots, N\}$, from (14), we have

$$\begin{aligned} X_\Delta(t_k) - X_\Delta(t_{k-1}) &= u_\Delta(X_\Delta(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta)) - u_\Delta(X_\Delta(t_{k-2} - I_\Delta[\delta(t_{k-2})]\Delta)) \\ &\quad + f_\Delta(X_\Delta(t_{k-1}), X_\Delta(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta))\Delta + g_\Delta(X_\Delta(t_{k-1}), X_\Delta(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta))\Delta B_{k-1}. \end{aligned}$$

Applying the elementary inequality

$$|a + b|^p \leq \left[1 + \epsilon^{\frac{1}{p-1}}\right]^{p-1} \left(|a|^p + \frac{|b|^p}{\epsilon}\right), \quad a, b \in R, \ p > 1, \tag{33}$$

and on the basis of (22), we obtain

$$\begin{aligned} E \sup_{1 \leq k \leq N} |X_\Delta(t_k) - X_\Delta(t_{k-1})|^p &\leq \left[1 + \epsilon^{\frac{1}{p-1}}\right]^{p-1} \frac{(3k)^p}{\epsilon} E \sup_{1 \leq k \leq N} |X_\Delta(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta) - X_\Delta(t_{k-2} - I_\Delta[\delta(t_{k-2})]\Delta)|^p \\ &\quad + 2^{p-1} \left[1 + \epsilon^{\frac{1}{p-1}}\right]^{p-1} E \sup_{1 \leq k \leq N} |f_\Delta(X_\Delta(t_{k-1}), X_\Delta(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta))|^p \Delta^p \\ &\quad + 2^{p-1} \left[1 + \epsilon^{\frac{1}{p-1}}\right]^{p-1} E \sup_{1 \leq k \leq N} |g_\Delta(X_\Delta(t_{k-1}), X_\Delta(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta))\Delta B_{k-1}|^p. \end{aligned} \tag{34}$$

In the sequel, we will deal with these three terms separately. Thus, on the basis of $\mathcal{H}_6$, we have that, for $k = 2, 3, \ldots, N+1$,

$$|t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta - t_{k-2} + I_\Delta[\delta(t_{k-2})]\Delta| \leq \Delta + |I_\Delta[\delta(t_{k-2})] - I_\Delta[\delta(t_{k-1})]|\Delta \leq \Delta + (1 + \eta)\Delta \leq (3 + \lfloor \eta \rfloor)\Delta.$$

For $k = 1$, since $\delta(-\Delta) = \delta(0)$, we find that

$$|t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta - t_{k-2} + I_\Delta[\delta(t_{k-2})]\Delta| = \Delta.$$

So, for $k = 1, 2, \ldots, N+1$ we get

$$|(k-1) - I_\Delta[\delta(t_{k-1})] - (k-2) + I_\Delta[\delta(t_{k-2})]| \leq (3 + \lfloor \eta \rfloor). \tag{35}$$

Since, on the basis of $\mathcal{H}_3$, we have that $(k-1) - I_\Delta[\delta(t_{k-1})] \geq (k-2) - I_\Delta[\delta(t_{k-2})]$ for any $k \in \{1, 2, \ldots, N+1\}$, then, using (35), we obtain

$$\begin{aligned} |X_\Delta(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta) - X_\Delta(t_{k-2} - I_\Delta[\delta(t_{k-2})]\Delta)|^p &\leq (3 + \lfloor \eta \rfloor)^{p-1} \sum_{j=(k-2)-I_\Delta[\delta(t_{k-2})]+1}^{(k-1)-I_\Delta[\delta(t_{k-1})]} |X_\Delta(t_j) - X_\Delta(t_{j-1})|^p \\ &\leq (3 + \lfloor \eta \rfloor)^p \sup_{-M \leq j \leq N} |X_\Delta(t_j) - X_\Delta(t_{j-1})|^p. \end{aligned} \tag{36}$$

Consequently, we find that

$$E \sup_{1 \leq k \leq N+1} |X_\Delta(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta) - X_\Delta(t_{k-2} - I_\Delta[\delta(t_{k-2})]\Delta)|^p \leq (3 + \lfloor \eta \rfloor)^p E \sup_{-M \leq k \leq N} |X_\Delta(t_k) - X_\Delta(t_{k-1})|^p. \tag{37}$$

On the basis of (11), we see that

$$E \sup_{1 \leq k \leq N} |f_\Delta(X_\Delta(t_{k-1}), X_\Delta(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta))|^p \leq (h(\Delta))^p. \tag{38}$$

By (11) and the Hölder inequality, for any integer $l > 1$, we get

$$\begin{aligned} &E \sup_{1\le k\le N} |g_\Delta (X_\Delta(t_{k-1}), X_\Delta (t_{k-1} - I_\Delta [\delta(t_{k-1})]\Delta))\Delta B_{k-1}|^p \\ &\quad \le E\left(\sup_{1\le k\le N} |g_\Delta (X_\Delta(t_{k-1}), X_\Delta (t_{k-1} - I_\Delta [\delta(t_{k-1})]\Delta))|^p \sup_{1\le k\le N} |\Delta B_{k-1}|^p\right) \\ &\quad \le (h(\Delta))^p \left[E\left(\sup_{0\le k\le N-1} |\Delta B_k|^{2pl}\right)\right]^{\frac{1}{2l}} \\ &\quad \le (h(\Delta))^p \left[\sum_{k=0}^{N-1} E|\Delta B_k|^{2pl}\right]^{\frac{1}{2l}}. \end{aligned} \tag{39}$$

Since $B(t) = (B^1(t), B^2(t), \ldots, B^m(t))^T$, $t \ge 0$ is an $m$-dimensional standard Brownian motion, we have

$$E|\Delta B_k|^{2pl} = E\left(|\Delta B_k^1|^2 + |\Delta B_k^2|^2 + \cdots + |\Delta B_k^m|^2\right)^{pl} \le m^{pl-1} \sum_{i=1}^{m} E|\Delta B_k^i|^{2pl} = m^{pl}(2pl-1)!!\Delta^{pl}, \tag{40}$$

where $(2pl-1)!! = 1 \times 3 \times \cdots \times (2pl-1)$. Substituting (40) into (39), we obtain

$$\begin{aligned} E \sup_{1\le k\le N} |g_\Delta (X_\Delta(t_{k-1}), X_\Delta (t_{k-1} - I_\Delta [\delta(t_{k-1})]\Delta))\Delta B_{k-1}|^p &\le (h(\Delta))^p \left[\sum_{k=0}^{N-1} m^{pl}(2pl-1)!!\Delta^{pl}\right]^{\frac{1}{2l}} \\ &\le m^{\frac{p}{2}} (T(2pl-1)!!)^{\frac{1}{2l}} (h(\Delta))^p \Delta^{\frac{pl-1}{2l}}. \end{aligned} \tag{41}$$

Substituting (37), (38) and (41) into (34), bearing in mind that $\Delta \in (0,1)$, we get

$$\begin{aligned} E \sup_{1\le k\le N} |X_\Delta(t_k) - X_\Delta(t_{k-1})|^p &\le \left[1 + \epsilon^{\frac{1}{p-1}}\right]^{p-1} \frac{(3k)^p}{\epsilon} (3 + \lfloor\eta\rfloor)^p E \sup_{-M\le k\le N} |X_\Delta (t_k) - X_\Delta (t_{k-1})|^p \\ &\quad + 2^{p-1}\left[1 + \epsilon^{\frac{1}{p-1}}\right]^{p-1} \left[1 + m^{\frac{p}{2}} (T(2pl-1)!!)^{\frac{1}{2l}}\right] (h(\Delta))^p \Delta^{\frac{pl-1}{2l}}. \end{aligned}$$

So, choosing $\epsilon = \left[\frac{\sqrt{9k}}{1-\sqrt{9k}}\right]^{p-1}$, on the basis of (32) and the previous inequality, the application of (30) yields

$$\begin{aligned} &E \sup_{-M\le k\le N} |X_\Delta(t_k) - X_\Delta(t_{k-1})|^p \\ &\le C_\xi \Delta^{\frac{p}{2}} + 3k^{\frac{p}{2}+\frac{1}{2}} (3 + \lfloor\eta\rfloor)^p E \sup_{-M\le k\le N} |X_\Delta(t_k) - X_\Delta(t_{k-1})|^p + \frac{2^{p-1}}{(1-\sqrt{9k})^{p-1}} \left[1 + m^{\frac{p}{2}} (T(2pl-1)!!)^{\frac{1}{2l}}\right] \Delta^{\frac{pl-1}{2l}} (h(\Delta))^p \\ &\le C_\xi \Delta^{\frac{p}{2}} + \sqrt{9k} E \sup_{-M\le k\le N} |X_\Delta(t_k) - X_\Delta(t_{k-1})|^p + \frac{2^{p-1}}{(1-\sqrt{9k})^{p-1}} \left[1 + m^{\frac{p}{2}} (T(2pl-1)!!)^{\frac{1}{2l}}\right] \Delta^{\frac{pl-1}{2l}} (h(\Delta))^p. \end{aligned}$$

Thus, we find that

$$E \sup_{-M\le k\le N} |X_\Delta(t_k) - X_\Delta(t_{k-1})|^p \le \tilde{c}\Delta^{\frac{p}{2}} + \tilde{c}_l \Delta^{\frac{pl-1}{2l}} (h(\Delta))^p,$$

where

$$\tilde{c} = \frac{1}{1-\sqrt{9k}} C_\xi, \quad \tilde{c}_l = \frac{2^{p-1}}{(1-\sqrt{9k})^p} \left[1 + m^{\frac{p}{2}} (T(2pl-1)!!)^{\frac{1}{2l}}\right],$$

which completes the proof. □

**Corollary 3.2.** *Let the conditions of Lemma 3.1 hold. Then, for any integer $l > 1$ and $p \geq 2$,*

$$E \sup_{-\Delta \leq t \leq T} |\bar{y}_\Delta(t) - \bar{z}_\Delta(t)|^p \leq \bar{c}\Delta^{\frac{p}{2}} + \bar{c}_l \Delta^{\frac{pl-1}{2l}} (h(\Delta))^p, \tag{42}$$

$$E|x_\Delta(t) - \bar{x}_\Delta(t)|^p \leq c'\Delta^{\frac{p}{2}} + c'_l \Delta^{\frac{pl-1}{2l}} (h(\Delta))^p, \quad t \in [0, T], \tag{43}$$

*where constants $\bar{c}, c', \bar{c}_l, c'_l > 0$ are independent of $\Delta$ and $\bar{c}_l, c'_l$ are dependent on $l$.*

*Proof.* Fix any $\Delta \in (0, \Delta^*]$ and $t \in [0, T]$. There is unique $k \in \{0, 1, \ldots, N-1\}$, such that $t \in [t_k, t_{k+1})$.

In order to prove (42), observe that, by (15)–(17) we have

$$\begin{aligned} |\bar{y}_\Delta(t) - \bar{z}_\Delta(t)|^p &= |X_\Delta(t_k - I_\Delta[\delta(t_k)]\Delta) - Z_k(t)|^p \\ &= \left(1 - \frac{s - t_k}{\Delta}\right)^p |X_\Delta(t_k - I_\Delta[\delta(t_k)]\Delta) - X_\Delta(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta)|^p \\ &\leq |X_\Delta(t_k - I_\Delta[\delta(t_k)]\Delta) - X_\Delta(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta)|^p . \end{aligned}$$

Therefore, using (36) we get

$$E \sup_{-\Delta \leq t \leq T} |\bar{y}_\Delta(t) - \bar{z}_\Delta(t)|^p \leq (3 + \lfloor \eta \rfloor)^p E \sup_{-M \leq k \leq N} |X_\Delta(t_k) - X_\Delta(t_{k-1})|^p.$$

So, on the basis of Lemma 3.1, the proof of (42) is completed with

$$\bar{c} = (3 + \lfloor \eta \rfloor)^p \tilde{c}, \quad \bar{c}_l = (3 + \lfloor \eta \rfloor)^p \tilde{c}_l.$$

In order to prove (43), note that, by (20), in a view of (15), we have

$$\begin{aligned} E|x_\Delta(t) - \bar{x}_\Delta(t)|^p \leq{}& 3^{p-1} E|u_\Delta(Z_k(t)) - u_\Delta(X_\Delta(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta))|^p \\ &+ 3^{p-1} \left[ E\left| \int_{t_k}^t f_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s)) ds \right|^p + E\left| \int_{t_k}^t g_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s)) dB(s) \right|^p \right]. \end{aligned} \tag{44}$$

Using (22), (16) and (15), respectively, and then (37) and Lemma 3.1, we find that

$$\begin{aligned} E|u_\Delta(Z_k(t)) - u_\Delta(X_\Delta(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta))|^p &\leq (3k)^p E|Z_k(t) - X_\Delta(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta)|^p \\ &\leq (3k)^p E|X_\Delta(t_k - I_\Delta[\delta(t_k)]\Delta) - X_\Delta(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta)|^p \\ &\leq (3k)^p (3 + \lfloor \eta \rfloor)^p E \sup_{-M \leq k \leq N} |X_\Delta(t_k) - X_\Delta(t_{k-1})|^p \\ &\leq (3k)^p (3 + \lfloor \eta \rfloor)^p \left( \tilde{c}\Delta^{\frac{p}{2}} + \tilde{c}_l \Delta^{\frac{pl-1}{2l}} (h(\Delta))^p \right). \end{aligned} \tag{45}$$

Applying the Hölder and Burkholder–Davis–Gundy inequalities, and then (11), we get

$$\begin{aligned} &E\left| \int_{t_k}^t f_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s)) ds \right|^p + E\left| \int_{t_k}^t g_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s)) dB(s) \right|^p \\ &\quad \leq (t - t_k)^{p-1} \int_{t_k}^t E|f_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s))|^p ds + \left( \frac{p(p-1)}{2} \right)^{\frac{p}{2}} (t - t_k)^{\frac{p}{2}-1} \int_{t_k}^t E\left| g_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s)) \right|^p ds \\ &\quad \leq \Delta^p (h(\Delta))^p + \left( \frac{p(p-1)}{2} \right)^{\frac{p}{2}} \Delta^{\frac{p}{2}} (h(\Delta))^p. \end{aligned} \tag{46}$$

Substituting (45) and (46) into (44), since $\Delta \in (0, 1)$, we obtain that

$$\begin{aligned} E|x_\Delta(t) - \bar{x}_\Delta(t)|^p &\leq 3^{p-1} \left[ (3k)^p (3 + \lfloor \eta \rfloor)^p \left( \tilde{c}\Delta^{\frac{p}{2}} + \tilde{c}_l \Delta^{\frac{pl-1}{2l}} (h(\Delta))^p \right) + \Delta^p (h(\Delta))^p + \left( \frac{p(p-1)}{2} \right)^{\frac{p}{2}} \Delta^{\frac{p}{2}} (h(\Delta))^p \right] \\ &\leq c'\Delta^{\frac{p}{2}} + c'_l \Delta^{\frac{pl-1}{2l}} (h(\Delta))^p, \quad t \in [0, T], \end{aligned}$$

where

$$c' = 3^{p-1}(3k)^p(3+\lfloor\eta\rfloor)^p\tilde{c}, \quad c_l' = 3^{p-1}\left[(3k)^p(3+\lfloor\eta\rfloor)^p\tilde{c}_l + 1 + \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\right].$$

□

**Lemma 3.3.** *Let the assumptions of Lemma 2.1 hold. Then, for all $t \geq 0$ and $p \geq 2$,*

$$\sup_{0\leq s\leq t} E|x_\Delta(s)|^p \leq \frac{k}{1-k}\sup_{-\tau\leq s\leq 0} E|\xi(s)|^p + \frac{1}{(1-k)^p}\sup_{0\leq s\leq t} E|x_\Delta(s) - u_\Delta(\bar{z}_\Delta(s))|^p. \tag{47}$$

*Proof.* Let $\epsilon > 0$ be arbitrary. On the basis of the elementary inequality (33), as well as (23) we have that, for any $t \geq 0$,

$$\begin{aligned}
|x_\Delta(t)|^p &= |x_\Delta(t) - u_\Delta(\bar{z}_\Delta(t)) + u_\Delta(\bar{z}_\Delta(t))|^p \\
&\leq \left[1+\epsilon^{\frac{1}{p-1}}\right]^{p-1}\left(|x_\Delta(t) - u_\Delta(\bar{z}_\Delta(t))|^p + \frac{|u_\Delta(\bar{z}_\Delta(t))|^p}{\epsilon}\right) \\
&\leq \left[1+\epsilon^{\frac{1}{p-1}}\right]^{p-1}\left(|x_\Delta(t) - u_\Delta(\bar{z}_\Delta(t))|^p + k^p\frac{|\bar{z}_\Delta(t)|^p}{\epsilon}\right).
\end{aligned}$$

Letting $\epsilon = \left[\frac{k}{1-k}\right]^{p-1}$, we conclude from (21) that

$$\begin{aligned}
E|x_\Delta(t)|^p &\leq \frac{1}{(1-k)^{p-1}}E|x_\Delta(t) - u_\Delta(\bar{z}_\Delta(t))|^p + kE|\bar{z}_\Delta(t)|^p \\
&\leq \frac{1}{(1-k)^{p-1}}E|x_\Delta(t) - u_\Delta(\bar{z}_\Delta(t))|^p + k\sup_{-\tau\leq s\leq 0}E|\xi(s)|^p + k\sup_{0\leq s\leq t}E|x_\Delta(t)|^p.
\end{aligned}$$

Therefore, we get

$$\sup_{0\leq s\leq t} E|x_\Delta(s)|^p \leq \frac{k}{1-k}\sup_{-\tau\leq s\leq 0} E|\xi(s)|^p + \frac{1}{(1-k)^p}\sup_{0\leq s\leq t} E|x_\Delta(s) - u_\Delta(\bar{z}_\Delta(s))|^p,$$

which completes the proof. □

**Lemma 3.4.** *Let the conditions of Lemma 3.1 hold together with Assumption $\mathcal{H}_4'$. Then, for $p > 0$,*

$$\sup_{0<\Delta\leq\Delta^*}\sup_{0\leq t\leq T} E|x_\Delta(t)|^p \leq C, \tag{48}$$

*where $C$ is a positive real constant dependent on $T, p, l, \bar{K}, k, \xi$, but independent of $\Delta$.*

*Proof.* First, we consider the case when $p \geq 2$. Fix any $\Delta \in (0, \Delta^*]$ and $T \geq 0$. By the Ito formula, we derive from (19) that, for $t \in [0, T]$,

$$\begin{aligned}
E|x_\Delta(t) - u_\Delta(\bar{z}_\Delta(t))|^p &\leq E|\xi(0) - u_\Delta(X_\Delta(-\Delta - I_\Delta[\delta(-\Delta)]\Delta))|^p \\
&\quad + E\int_0^t p|x_\Delta(s) - u_\Delta(\bar{z}_\Delta(s))|^{p-2}(x_\Delta(s) - u_\Delta(\bar{z}_\Delta(s)))^T f_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s))ds \\
&\quad + \frac{p(p-1)}{2}E\int_0^t p|x_\Delta(s) - u_\Delta(\bar{z}_\Delta(s))|^{p-2}|g_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s))|^2 ds \\
&\quad + E\int_0^t p|x_\Delta(s) - u_\Delta(\bar{z}_\Delta(s))|^{p-2}(x_\Delta(s) - u_\Delta(\bar{z}_\Delta(s)))^T g_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s))dB(s).
\end{aligned}$$

Furthermore, for any $t \in [0,T]$, we have that

$$\begin{aligned} & E|x_\Delta(t) - u_\Delta(\bar{z}_\Delta(t))|^p \\ & \leq E|\xi(0) - u_\Delta(X_\Delta(-\Delta - I_\Delta[\delta(-\Delta)]\Delta))|^p \\ & \quad + pE\int_0^t |x_\Delta(s) - u_\Delta(\bar{z}_\Delta(s))|^{p-2}\Big((x_\Delta(s) - u_\Delta(\bar{z}_\Delta(s)))^T f_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s)) + \frac{p-1}{2}|g_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s))|^2\Big)ds \\ & \leq E|\xi(0) - u_\Delta(X_\Delta(-\Delta - I_\Delta[\delta(-\Delta)]\Delta))|^p \\ & \quad + pE\int_0^t |x_\Delta(s) - u_\Delta(\bar{z}_\Delta(s))|^{p-2}\Big((\bar{x}_\Delta(s) - u_\Delta(\bar{y}_\Delta(s)))^T f_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s)) + \frac{p-1}{2}|g_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s))|^2\Big)ds \\ & \quad + pE\int_0^t |x_\Delta(s) - u_\Delta(\bar{z}_\Delta(s))|^{p-2}(x_\Delta(s) - \bar{x}_\Delta(s) - u_\Delta(\bar{z}_\Delta(s)) + u_\Delta(\bar{y}_\Delta(s)))^T f_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s))ds. \end{aligned} \tag{49}$$

In the sequel, we will estimate the summands on the right-hand of the last inequality separately.

Thus, on the basis of (23), we conclude that

$$\begin{aligned} E|\xi(0) - u_\Delta(X_\Delta(-\Delta - I_\Delta[\delta(-\Delta)]\Delta))|^p & \leq E(|\xi(0)| + |u_\Delta(X_\Delta(-\Delta - I_\Delta[\delta(-\Delta)]\Delta))|)^p \\ & \leq 2^{p-1}(E|\xi(0)|^p + k^p E|X_\Delta(-\Delta - I_\Delta[\delta(-\Delta)]\Delta)|^p) \\ & \leq 2^{p-1}(1 + k^p) \sup_{-\tau\leq s\leq 0} E|\xi(s)|^p. \end{aligned} \tag{50}$$

Let $\epsilon, a, b > 0$. Using the elementary inequality

$$a^m b^{1-m} \leq ma + (1-m)b, \quad m \in [0,1], \tag{51}$$

we derive that, for any $\epsilon > 0$,

$$a^{p-2}b = (\epsilon a^p)^{\frac{p-2}{p}}\left(\frac{b^{\frac{p}{2}}}{\epsilon^{(p-2)/2}}\right)^{\frac{2}{p}} \leq \frac{p-2}{p}\epsilon a^p + \frac{2}{p\epsilon^{(p-2)/2}}b^{\frac{p}{2}}. \tag{52}$$

Furtheron, using Assumption $\mathcal{H}_4'$ and (52) for $\epsilon = \frac{1}{4\bar{K}(p-2)T}$, we find that

$$\begin{aligned} & pE\int_0^t |x_\Delta(s) - u_\Delta(\bar{z}_\Delta(s))|^{p-2}\Big((\bar{x}_\Delta(s) - u_\Delta(\bar{y}_\Delta(s)))^T f_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s)) + \frac{p-1}{2}|g_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s))|^2\Big)ds \\ & \leq p\bar{K}E\int_0^t |x_\Delta(s) - u_\Delta(\bar{z}_\Delta(s))|^{p-2}\left(1 + |\bar{x}_\Delta(s)|^2 + |\bar{y}_\Delta(s)|^2\right)ds \\ & \leq \frac{1}{4T}E\int_0^t |x_\Delta(s) - u_\Delta(\bar{z}_\Delta(s))|^p ds + 2\bar{K}[4\bar{K}(p-2)T]^{\frac{p-2}{p}}E\int_0^t (1 + |\bar{x}_\Delta(s)|^2 + |\bar{y}_\Delta(s)|^2)^{\frac{p}{2}}ds \\ & \leq \frac{1}{4T}E\int_0^t |x_\Delta(s) - u_\Delta(\bar{z}_\Delta(s))|^p ds + 2\bar{K}[4\bar{K}(p-2)T]^{\frac{p-2}{p}}3^{\frac{p}{2}-1}E\int_0^t (1 + |\bar{x}_\Delta(s)|^p + |\bar{y}_\Delta(s)|^p)ds. \end{aligned} \tag{53}$$

Moreover, the application of (52) yields

$$\begin{aligned} & pE\int_0^t |x_\Delta(s) - u_\Delta(\bar{z}_\Delta(s))|^{p-2}(x_\Delta(s) - \bar{x}_\Delta(s) - u_\Delta(\bar{z}_\Delta(s)) + u_\Delta(\bar{y}_\Delta(s)))^T f_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s))ds \\ & \leq (p-2)\epsilon E\int_0^t |x_\Delta(s) - u_\Delta(\bar{z}_\Delta(s))|^p ds + \frac{2}{\epsilon^{(p-2)/2}}E\int_0^t |x_\Delta(s) - \bar{x}_\Delta(s) - u_\Delta(\bar{z}_\Delta(s)) + u_\Delta(\bar{y}_\Delta(s))|^{\frac{p}{2}}|f_\Delta(\bar{x}_\Delta(s), \bar{y}_\Delta(s))|^{\frac{p}{2}}ds. \end{aligned} \tag{54}$$

By (11) and the Hölder inequality, we get

$$E\int_0^t |x_\Delta(s)-\bar{x}_\Delta(s)-u_\Delta(\bar{z}_\Delta(s))+u_\Delta(\bar{y}_\Delta(s))|^{\frac{p}{2}}|f_\Delta(\bar{x}_\Delta(s),\bar{y}_\Delta(s))|^{\frac{p}{2}}ds$$
$$\le (h(\Delta))^{\frac{p}{2}}\int_0^t \Big(E|x_\Delta(s)-\bar{x}_\Delta(s)-u_\Delta(\bar{z}_\Delta(s))+u_\Delta(\bar{y}_\Delta(s))|^p\Big)^{\frac{1}{2}}ds$$
$$\le 2^{\frac{p-1}{2}}(h(\Delta))^{\frac{p}{2}}\int_0^t \Big[E|x_\Delta(s)-\bar{x}_\Delta(s)|^p+E|u_\Delta(\bar{z}_\Delta(s))-u_\Delta(\bar{y}_\Delta(s))|^p\Big]^{\frac{1}{2}}ds.$$

Bearing in mind that $p\ge 2$ and applying (43), (22) and (42), respectively, we find that

$$E\int_0^t |x_\Delta(s)-\bar{x}_\Delta(s)-u_\Delta(\bar{z}_\Delta(s))+u_\Delta(\bar{y}_\Delta(s))|^{\frac{p}{2}}|f_\Delta(\bar{x}_\Delta(s),\bar{y}_\Delta(s))|^{\frac{p}{2}}ds$$
$$\le 2^{\frac{p-1}{2}}(h(\Delta))^{\frac{p}{2}}\int_0^t \Big(c'\Delta^{\frac{p}{2}}+c'_l\Delta^{\frac{pl-1}{2l}}(h(\Delta))^p+(3k)^pE|\bar{z}_\Delta(s)-\bar{y}_\Delta(s)|^p\Big)^{\frac{1}{2}}ds$$
$$\le 2^{\frac{p-1}{2}}(h(\Delta))^{\frac{p}{2}}T\Big[\Big(c'+(3k)^p\bar{c}\Big)\Delta^{\frac{p}{2}}+\Big(c'_l+(3k)^p\bar{c}_l\Big)\Delta^{\frac{pl-1}{2l}}(h(\Delta))^p\Big)\Big]^{\frac{1}{2}}$$
$$\le 2^{\frac{p-1}{2}}T\Big[\Big(c'+(3k)^p\bar{c}\Big)^{\frac{1}{2}}\Big(\Delta^{\frac{1}{2}}h(\Delta)\Big)^{\frac{p}{2}}+\Big(c'_l+(3k)^p\bar{c}_l\Big)^{\frac{1}{2}}\Big(\Delta^{\frac{1-1/(pl)}{4}}h(\Delta)\Big)^p\Big)\Big].$$

So, for $\epsilon_0\in(0,1)$, we can find large enough $l>1$ such that $l\ge\frac{1}{p\epsilon_0}$. Then, the application of (9) and previous estimate gives

$$E\int_0^t |x_\Delta(s)-\bar{x}_\Delta(s)-u_\Delta(\bar{z}_\Delta(s))+u_\Delta(\bar{y}_\Delta(s))|^{\frac{p}{2}}|f_\Delta(\bar{x}_\Delta(s),\bar{y}_\Delta(s))|^{\frac{p}{2}}ds\le C_l,$$

where $C_l=2^{\frac{p-1}{2}}T\Big[\Big(c'+(3k)^p\bar{c}\Big)^{\frac{1}{2}}+\Big(c'_l+(3k)^p\bar{c}_l\Big)^{\frac{1}{2}}\Big)\Big]$. Substituting the last inequality into (54) and choosing $\epsilon=\frac{1}{4(p-2)T}$ we get

$$pE\int_0^t |x_\Delta(s)-u_\Delta(\bar{z}_\Delta(s))|^{p-2}(x_\Delta(s)-\bar{x}_\Delta(s)-u_\Delta(\bar{z}_\Delta(s))+u_\Delta(\bar{y}_\Delta(s)))^Tf_\Delta(\bar{x}_\Delta(s),\bar{y}_\Delta(s))ds \tag{55}$$
$$\le\frac{1}{4T}E\int_0^t |x_\Delta(s)-u_\Delta(\bar{z}_\Delta(s))|^pds+2(4(p-2)T)^{\frac{p-2}{2}}C_l.$$

Since $t\in[0,T]$ is arbitrary, substituting (50), (53) and (55) into (49), we find that

$$\sup_{0\le s\le t}E|x_\Delta(s)-u_\Delta(\bar{z}_\Delta(s))|^p\le 2^{p-1}(1+k^p)\sup_{-\tau\le s\le 0}E|\xi(s)|^p+\frac{1}{2}\sup_{0\le s\le t}E|x_\Delta(s)-u_\Delta(\bar{z}_\Delta(s))|^p$$
$$+2\bar{K}[4\bar{K}(p-2)T]^{\frac{p-2}{p}}3^{\frac{p}{2}-1}\int_0^t(1+E|\bar{x}_\Delta(s)|^p+E|\bar{y}_\Delta(s)|^p)ds+2(4(p-2)T)^{\frac{p-2}{2}}C_l.$$

So, on the basis of (15), we conclude that

$$\sup_{0\le s\le t}E|x_\Delta(s)-u_\Delta(\bar{z}_\Delta(s))|^p \tag{56}$$
$$\le 2^p(1+k^p)\sup_{-\tau\le s\le 0}E|\xi(s)|^p+4\bar{K}[4\bar{K}(p-2)T]^{\frac{p-2}{p}}3^{\frac{p}{2}-1}\int_0^t\Big(1+2\sup_{0\le u\le r}E|x_\Delta(u)|^p\Big)dr+4(4(p-2)T)^{\frac{p-2}{2}}C_l$$
$$\le 2^p(1+k^p)\sup_{-\tau\le s\le 0}E|\xi(s)|^p+4\bar{K}[4\bar{K}(p-2)T]^{\frac{p-2}{p}}3^{\frac{p}{2}-1}T+4(4(p-2)T)^{\frac{p-2}{2}}C_l$$
$$+8\bar{K}[4\bar{K}(p-2)T]^{\frac{p-2}{p}}3^{\frac{p}{2}-1}\int_0^t\sup_{0\le u\le r}E|x_\Delta(u)|^pdr.$$

Then, by Lemma 3.3, we have

$$\sup_{0\le s\le t} E|x_\Delta(s)|^p \le \left(\frac{k}{1-k}+\frac{2^p(1+k^p)}{(1-k)^p}\right)\sup_{-\tau\le s\le 0} E|\xi(s)|^p + M_l + \bar{N}\int_0^t \sup_{0\le u\le r} E|x_\Delta(u)|^p dr,$$

where

$$\bar{N} = \frac{8\bar{K}[4\bar{K}(p-2)T]^{\frac{p-2}{p}} 3^{\frac{p}{2}-1}}{(1-k)^p}, \quad M_l = \frac{4\bar{K}[4\bar{K}(p-2)T]^{\frac{p-2}{p}} 3^{\frac{p}{2}-1}T + 4(4(p-2)T)^{\frac{p-2}{2}} C_l}{(1-k)^p}.$$

Therefore, we obtain that

$$\sup_{0\le s\le t} E|x_\Delta(s)|^p \le Q_l + \bar{N}\int_0^t \sup_{0\le u\le r} E|x_\Delta(u)|^p dr,$$

where

$$Q_l = \left(\frac{k}{1-k}+\frac{2^p(1+k^p)}{(1-k)^p}\right)\sup_{-\tau\le s\le 0} E|\xi(s)|^p + M_l.$$

Applying the Gronwall–Bellman lemma we find that

$$\sup_{0\le s\le T} E|x_\Delta(s)|^p \le Q_l e^{\bar{N}T} \equiv S_l. \tag{57}$$

As this holds for any $\Delta \in (0, \Delta^*]$ while $S_l$ is independent of $\Delta$, we see that

$$\sup_{0<\Delta\le\Delta^*}\sup_{0\le s\le T} E|x_\Delta(s)|^p \le S_l.$$

For $p \in (0, 2)$, on the basis of the Hölder inequality and previous part of the proof, we obtain that

$$E|x_\Delta(t)|^p = \left(E|x_\Delta(t)|^2\right)^{\frac{p}{2}} \le S_l^{\frac{p}{2}}, \quad t \in [0, T].$$

Thus, the proof is complete with $C = S_l \vee S_l^{\frac{p}{2}}$. □

**Lemma 3.5.** *Let Assumptions $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ and $\mathcal{H}_4$ hold and let $R_0$ be large enough integer such that*

$$\|x_0\| \vee 1 = \|\xi\| \vee 1 < R_0. \tag{58}$$

*For any real number $R > R_0$, define the stopping time*

$$\tau_R = \inf\{t \ge 0 : |x(t)| \ge R\}, \tag{59}$$

*where* $\inf \emptyset = \infty$. *Then, for any $p \ge 2$,*

$$P\{\tau_R \le T\} \le \frac{C'}{R^p}, \tag{60}$$

*where $C'$ is constant which depends on $p$, $k$, $K$, $T$ and $\xi$.*

*Proof.* By the Ito formula and Assumptions $\mathcal{H}_4$ and $\mathcal{H}_2$, we derive that for $t \in [0, T]$,

$$\begin{aligned}
&E|x(t\wedge\tau_R) - u(x(t\wedge\tau_R - \delta(t\wedge\tau_R)))|^p \\
&\le E|\xi(0) - u(\xi(-\delta(0)))|^p + pE\int_0^{t\wedge\tau_R} |x(s) - u(x(s-\delta(s)))|^{p-2} \\
&\quad\times\left((x(s) - u(x(s-\delta(s))))^T f(x(s), x(s-\delta(s))) + \frac{p-1}{2}|g(x(s), x(s-\delta(s)))|^2\right) ds \\
&\le E|\xi(0) - u(\xi(-\delta(0)))|^p + pKE\int_0^{t\wedge\tau_R} |x(s) - u(x(s-\delta(s)))|^{p-2}(1+|x(s)|^2+|x(s-\delta(s))|^2) ds \\
&\le E|\xi(0) - u(\xi(-\delta(0)))|^p + pK\left(2^{p-3}\vee 1\right)E\int_0^{t\wedge\tau_R}\left[|x(s)|^{p-2} + k^{p-2}|x(s-\delta(s))|^{p-2}\right](1+|x(s)|^2+|x(s-\delta(s))|^2) ds.
\end{aligned}$$

Applying the elementary inequality (51), we obtain

$$
\begin{aligned}
&E|x(t\wedge\tau_R)-u(x(t\wedge\tau_R-\delta(t\wedge\tau_R)))|^p\\
&\le E|\xi(0)-u(\xi(-\delta(0))|^p+K\Big(2^{p-3}\vee 1\Big)2\Big(1+k^{p-2}\Big)T+K\left(2^{p-3}\vee 1\right)\left[3p-4+2k^{p-2}\right]E\int_0^t|x(s\wedge\tau_R)|^p ds\\
&\quad+K\left(2^{p-3}\vee 1\right)\left[2+(3p-4)k^{p-2}\right]E\int_0^t|x(s\wedge\tau_R-\delta(s\wedge\tau_R))|^p ds\\
&\le C_1+C_2\int_0^t\sup_{0\le u\le s}E|x(u\wedge\tau_R)|^p ds,
\end{aligned}
$$

where

$$C_1=E|\xi(0)-u(\xi(-\delta(0))|^p+K\left(2^{p-3}\vee 1\right)T\Big[2+\Big(2+(3p-4)k^{p-2}\Big)\sup_{-\tau\le s\le 0}E|\xi(s)|^p\Big],$$

and $C_2=K\left(2^{p-3}\vee 1\right)\left(3p-2+(3p-2)k^{p-2}\right)$. Therefore, we obtain that

$$\sup_{0\le u\le t}E|x(u\wedge\tau_R)-u(x(u\wedge\tau_R-\delta(u\wedge\tau_R)))|^p\le C_1+C_2\int_0^t\sup_{0\le u\le s}E|x(u\wedge\tau_R)|^p ds. \tag{61}$$

Using the same arguments as in the proof of Lemma 3.3, we get

$$\sup_{0\le u\le t}E|x(u\wedge\tau_R)|^p\le\frac{k}{1-k}\sup_{-\tau\le u\le 0}E|\xi(u)|^p+\frac{1}{(1-k)^p}\sup_{0\le u\le t}E|x(u\wedge\tau_R)-u(x(u\wedge\tau_R-\delta(u\wedge\tau_R)))|^p.$$

Substituting (61) into the previous inequality, we find that

$$\sup_{0\le u\le t}E|x(u\wedge\tau_R)|^p\le\bar{C}_1+\bar{C}_2\int_0^t\sup_{0\le u\le s}E|x(s\wedge\tau_R)|^p ds,$$

where

$$\bar{C}_1=\frac{k}{1-k}\sup_{-\tau\le u\le 0}E|\xi(u)|^p+\frac{C_1}{(1-k)^p},\quad \bar{C}_2=\frac{C_2}{(1-k)^p}.$$

The Gronwall–Bellman inequality yields

$$\sup_{0\le u\le t}E|x(u\wedge\tau_R)|^p\le\bar{C}_1e^{T\bar{C}_2}\equiv C^{'}.$$

In particular, we have $E|x(T\wedge\tau_R)|^p\le C^{'}$, such that $R^pP\{\tau_R\le T\}\le C^{'}$ and the assertion follows. □

**Lemma 3.6.** *Let the conditions of Lemma 3.1 hold, together with Assumption $\mathcal{H}_4'$. For any real number $R>R_0$, where $R_0$ satisfies (58) and $\Delta\in(0,\Delta^*]$, define the stopping time*

$$\rho_{\Delta,R}=\inf\{t\ge 0:|x_\Delta(t)|\ge R\}. \tag{62}$$

*Then, for any $p\ge 2$, we have*

$$P\{\rho_{\Delta,R}\le T\}\le\frac{C^{''}}{R^p}, \tag{63}$$

*where $C^{''}$ is a positive real constant which depends on $p$, $k$, $l$, $\bar{K}$, $T$ and $\xi$.*

*Proof.* Following the procedure of the proof of Lemma 3.4, analogously to the estimate (56), for $t\in[0,T]$, we get

$$
\begin{aligned}
E|x_\Delta(t\wedge\rho_{\Delta,R})-u_\Delta(\bar{z}_\Delta(t\wedge\rho_{\Delta,R}))|^p&\le 2^p(1+k^p)\sup_{-\tau\le s\le 0}E|\xi(s)|^p+4\bar{K}[4\bar{K}(p-2)T]^{\frac{p-2}{p}}3^{\frac{p}{2}-1}T+4(4(p-2)T)^{\frac{p-2}{2}}C_l\\
&\quad+8\bar{K}[4\bar{K}(p-2)T]^{\frac{p-2}{p}}3^{\frac{p}{2}-1}\int_0^t\sup_{0\le u\le r}E|x_\Delta(u\wedge\rho_{\Delta,R})|^p dr.
\end{aligned}
$$

Then, using the same arguments for obtaining the estimate (57), we conclude from the previous inequality that

$$\sup_{0\le t\le T} E|x_\Delta(t\wedge\rho_{\Delta,R})|^p \le S_l \equiv C''.$$

In particular, we have that $E|x_\Delta(T\wedge\rho_{\Delta,R})|^p \le C''$, such that $R^pP\{\rho_{\Delta,R}\le T\}\le C''$ and the desired assertion follows. □

Now, we are in a position to prove the main result. From now on, we will fix an arbitrary $T>0$ and show that

$$\lim_{\Delta\to 0} E|x_\Delta(T)-x(T)|^q = 0,\quad \lim_{\Delta\to 0} E|\bar{x}_\Delta(T)-x(T)|^q = 0,$$

for every $q\ge 2$.

**Theorem 3.7.** *Let Assumptions $\mathcal{H}_1$–$\mathcal{H}_6$ hold, together with condition (30). Then, for any $q\in[2,p)$,*

$$\lim_{\Delta\to 0} E|x_\Delta(T)-x(T)|^q = 0,\quad \lim_{\Delta\to 0} E|\bar{x}_\Delta(T)-x(T)|^q = 0. \tag{64}$$

*Proof.* Let $\tau_R$ and $\rho_{\Delta,R}$ be the stopping times defined by (59) and (62), respectively and define

$$\theta_{\Delta,R}=\tau_R\wedge\rho_{\Delta,R},\quad e_\Delta(T)=x_\Delta(T)-x(T).$$

Obviously, we have that

$$E|e_\Delta(T)|^q = E\left(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R}>T\}}\right)+E\left(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R}\le T\}}\right). \tag{65}$$

Let $\epsilon>0$ be arbitrary, and $a,b>0$. Using the elementary inequality (51), we see that

$$a^q b \le (\epsilon a^p)^{\frac{q}{p}}\left(\frac{b^{\frac{p}{p-q}}}{\epsilon^{\frac{q}{p-q}}}\right)^{1-\frac{q}{p}} \le \frac{q\epsilon}{p}a^p+\frac{p-q}{p\epsilon^{\frac{q}{p-q}}}b^{\frac{p}{p-q}}.$$

So, we can conclude that

$$E\left(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R}\le T\}}\right)\le \frac{q\epsilon}{p}E|e_\Delta(T)|^p+\frac{p-q}{p\epsilon^{\frac{q}{p-q}}}P\{\theta_{\Delta,R}\le T\}. \tag{66}$$

By Theorem 1.1 and Lemma 3.4, there exists a positive constant $\tilde{C}$, such that

$$E|e_\Delta(T)|^p \le 2^{p-1}(E|x_\Delta(T)|^p+E|x(T)|^p)\le \tilde{C}, \tag{67}$$

while by Lemmas 3.5 and 3.6, we obtain

$$P\{\theta_{\Delta,R}\le T\}\le P\{\tau_R\le T\}+P\{\rho_{\Delta,R}\le T\}\le \frac{C'+C''}{R^p}. \tag{68}$$

Hence, on the basis of (65)–(68), we have that

$$E|e_\Delta(T)|^q \le E\left(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R}>T\}}\right)+\frac{\tilde{C}q\epsilon}{p}+\frac{p-q}{p\epsilon^{\frac{q}{p-q}}}\frac{C'+C''}{R^p}. \tag{69}$$

Now, let $\varepsilon>0$ be arbitrary. Choose $\epsilon$ sufficiently small for $\tilde{C}q\epsilon/p\le\varepsilon/3$ and then choose $R$ sufficiently large for

$$\frac{p-q}{p\epsilon^{\frac{q}{p-q}}}\frac{C'+C''}{R^p}\le\frac{\varepsilon}{3}.$$

So, from (69) we see that for this particularly chosen $R$,

$$E|e_\Delta(T)|^q \le E\left(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R}>T\}}\right)+\frac{2\varepsilon}{3}.$$

If we can show that for all sufficiently small $\Delta$,

$$E\left(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R}>T\}}\right) \leq \frac{\varepsilon}{3}, \tag{70}$$

we can conclude that

$$\lim_{\Delta\to 0} E|e_\Delta(T)|^q = 0,$$

and then by (43), we also have that

$$\lim_{\Delta\to 0} E|x(T) - \bar{x}_\Delta(T)|^q = 0,$$

and the proof of the theorem would be complete.

In order to complete the proof, we need to show (70). Therefore, we define the truncated functions

$$F_R(x,y) = f\left((|x|\wedge R)\frac{x}{|x|}, (|y|\wedge R)\frac{y}{|y|}\right),\ \ G_R(x,y) = g\left((|x|\wedge R)\frac{x}{|x|}, (|y|\wedge R)\frac{y}{|y|}\right),\ \ U_R(x,y) = u\left((|y|\wedge R)\frac{y}{|y|}\right),$$

for $x, y \in R^d$. Without loss of any generality, we may assume that $\Delta^*$ is already sufficiently small for $\mu^{-1}(h(\Delta^*)) \geq R$. Hence, for all $\Delta \in (0, \Delta^*]$, we have that

$$f_\Delta(x,y) = F_R(x,y), \quad g_\Delta(x,y) = G_R(x,y), \quad u_\Delta(x,y) = U_R(x,y),$$

for those $x, y \in R^d$ with $|x| \vee |y| \leq R$. Now, we consider the following neutral stochastic differential equation with time-dependent delay

$$d[z(t) - U_R(z(t-\delta(t)))] = F_R(z(t), z(t-\delta(t)))dt + G_R(z(t), z(t-\delta(t)))dB(t), \quad t \geq 0, \tag{71}$$

with the initial data $z(u) = \xi(u)$, $u \in [-\tau, 0]$. By Assumption $\mathcal{H}_1$ we see that both $F_R(x,y)$ and $G_R(x,y)$ are globally Lipschitz continuous with the Lipschitz constant $K_R$, while, on the basis of Lemma 2.1, the function $U_R(y)$ is contractive mapping with constant $3k$ for $k \in \left(0, \frac{1}{3}\right)$. Thus, Eq. (71) has a unique global solution $\{z(t), t \geq -\tau\}$ (see Theorem 4.1 in Appendix, which can be found in [6]). It is straightforward to see that

$$P\{x(t\wedge\tau_R) = z(t\wedge\tau_R),\ t \in [0,T]\} = 1. \tag{72}$$

On the other hand, for each step size $\Delta \in (0, \Delta^*]$, we can apply the classical Euler–Maruyama method to Eq. (71) and we denote by $z_\Delta(t)$ the continuous-time Euler–Maruyama solution. So, we obtain that

$$P\{x_\Delta(t\wedge\rho_{\Delta,R}) = z_\Delta(t\wedge\rho_{\Delta,R}),\ t \in [0,T]\} = 1. \tag{73}$$

However, from Theorem 4.5 (see Appendix), we have

$$E\sup_{-\tau\leq t\leq T} |z(t) - z_\Delta(t)|^q \leq S(l)\Delta^{\frac{ql-1}{2l}},$$

where $S(l)$ is a positive constant independent of $\Delta$. Consequently, we see that

$$E\sup_{-\tau\leq t\leq T} |z(t\wedge\theta_{\Delta,R}) - z_\Delta(t\wedge\theta_{\Delta,R})|^q \leq S(l)\Delta^{\frac{ql-1}{2l}}.$$

Using (72) and (73), we have

$$E\sup_{-\tau\leq t\leq T} |x(t\wedge\theta_{\Delta,R}) - x_\Delta(t\wedge\theta_{\Delta,R})|^q \leq S(l)\Delta^{\frac{ql-1}{2l}},$$

which implies

$$E|x(T\wedge\theta_{\Delta,R}) - x_\Delta(T\wedge\theta_{\Delta,R})|^q \leq S(l)\Delta^{\frac{ql-1}{2l}}.$$

Finally, we can conclude that

$$E\left(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R}>T\}}\right) = E\left(|e_\Delta(T\wedge\theta_{\Delta,R})|^q I_{\{\theta_{\Delta,R}>T\}}\right) \leq E|x(T\wedge\theta_{\Delta,R}) - x_\Delta(T\wedge\theta_{\Delta,R})|^q \leq S(l)\Delta^{\frac{q}{2}(1-\frac{1}{l})}.$$

For example, if we fix $l = 2$, we can choose small enough $\Delta$, such that (70) holds, which completes the proof. □

In order to illustrate the previous theoretical results, we provide the following example.

**Example 3.8.** *Consider the following one-dimensional neutral stochastic differential equation with time-dependent delay*

$$d\left[x(t) + \frac{1}{27}\sin x(t-\delta(t))\right] = f(x(t), x(t-\delta(t)))dt + g(x(t), x(t-\delta(t)))dB(t), \quad t \geq 0, \tag{74}$$

*with the initial condition $\xi(\theta) = 1$, $\theta \in [-\tau, 0]$, $\tau = 2$, where the delay function is defined as $\delta(t) = 1 - \frac{1}{4}\sin t$, $t \geq 0$ and*

$$f(x,y) = a_1 + a_2|y|^{\frac{6}{5}} - a_3x^5, \qquad g(x,y) = a_4|x|^{\frac{5}{2}} + a_5y, \quad x, y \in R,$$

*and $a_1, \cdots, a_5 \in R$ with $a_3 > 0$. Clearly, the coefficients $f$ and $g$ are locally Lipschitz continuous, namely, they satisfy Assumption $\mathcal{H}_1$, while $u(x) = -1/27\sin x$ satisfies Assumption $\mathcal{H}_2$ for $k = 1/27$. Since $\delta'(t) \leq \frac{1}{4} = \bar{\delta}$, Assumption $\mathcal{H}_3$ holds. Moreover, for any $p \geq 2$ and any $a \in (0, 1]$, we have*

$$\begin{aligned}(x - au(y))f(x,y) + \frac{p-1}{2}|g(x,y)|^2 \leq |a_1||x| + |a_2||x||y|^{\frac{6}{5}} - a_3x^6 + \frac{a|a_1|}{27} + \frac{a|a_2|}{27}|y|^{\frac{6}{5}} \\ + \frac{aa_3}{27}|x|^5 + (p-1)a_4^2|x|^5 + (p-1)a_5^2y^2.\end{aligned} \tag{75}$$

*Now, by the Young inequality (51) we obtain*

$$|x||y|^{\frac{6}{5}} \leq \frac{2}{5}|x|^{\frac{5}{2}} + \frac{3}{5}y^2, \qquad a|a_2||y|^{\frac{6}{5}} \leq \frac{2}{5}a|a_2| + \frac{3}{5}a|a_2|y^2. \tag{76}$$

*Substituting (76) into (75) we get*

$$\begin{aligned}(x - au(y))f(x,y) + \frac{p-1}{2}|g(x,y)|^2 &\leq \frac{a}{27}\left[|a_1| + \frac{2}{5}|a_2|\right] + \left[\frac{3}{5}|a_2|\left(1 + \frac{a}{27}\right) + (p-1)a_5^2\right]y^2 \\ &\quad + |a_1||x| + \left[\frac{aa_3}{27} + (p-1)a_4^2\right]|x|^5 + \frac{2}{5}|a_2||x|^{\frac{5}{2}} - a_3x^6 \\ &\leq K(1 + x^2 + ay^2),\end{aligned}$$

*where*

$$K = \frac{a}{27}\left[|a_1| + \frac{2}{5}|a_2|\right] \vee \frac{1}{a}\left[\frac{3}{5}|a_2|\left(1 + \frac{a}{27}\right) + (p-1)a_5^2\right] \vee \tilde{a}$$

*and*

$$\tilde{a} = \sup_{u \geq 0}\left\{|a_1|u + \left[\frac{aa_3}{27} + (p-1)a_4^2\right]u^5 + \frac{2}{5}|a_2|u^{\frac{5}{2}} - a_3u^6\right\} < \infty.$$

*Thus, Assumption $\mathcal{H}_4'$ holds for any $p \geq 2$ and any $a \in (0, 1]$. Particularly, for $a = 1$ we find that Assumption $\mathcal{H}_4$ is fulfilled as well. The initial condition $\xi(\theta) = 1$, $\theta \in [-\tau, 0]$ satisfies Assumption $\mathcal{H}_5$. Noting that*

$$|\delta(t) - \delta(s)| \leq \frac{1}{4}|t - s|, \quad t, s \geq 0,$$

*we find that $\mathcal{H}_6$ holds with $\eta = 1/4$. Also, since $k(3 + \lfloor\eta\rfloor)^2 = 1/3 < 1$, the condition (30) holds. To apply Theorem 3.7, we still need to design functions $\mu$ and $h$ satisfying (7), (8) and (9). Note that, for every $r \geq 1$, the inequality (7) is satisfied with $\mu(r) = \bar{a}r^5$, where $\bar{a} = (|a_1| + |a_2| + a_3) \vee (|a_4| + |a_5|)$ and $\mu^{-1}(r) = \bar{a}^{-\frac{1}{5}}r^{\frac{1}{5}}$ for $r \geq 1$. For $\epsilon_0 \in (0, 1)$ we can choose $\bar{\epsilon} \in [\epsilon_0, 1)$ and define $h(\Delta) = \Delta^{-(1-\bar{\epsilon})}$ for $\Delta \in (0, \Delta^*]$. Letting $\Delta^* \in (0, 1)$ be sufficiently small, we can make (8) and (9) hold.*

*So, by Theorem 3.7, we can conclude that the truncated Euler–Maruyama solutions will converge to the exact solution $x(t)$ of Eq. (74) in the sense (64) for every $q \in [2, p)$.*

## 4. Appendix: The Euler–Maruyama method under the global Lipschitz condition

For the purpose of the following consideration we impose the following hypothesis.
$\mathcal{H}_1'$ : (The global Lipschitz condition) There exists a constant $K_1 > 0$ such that, for all $x, y, \bar{x}, \bar{y} \in R^d$,

$$|f(x,y) - f(\bar{x},\bar{y})|^2 \vee |g(x,y) - g(\bar{x},\bar{y})|^2 \leq K_1(|x-y|^2 + |\bar{x}-\bar{y}|^2).$$

In this section we introduce the following existence and uniqueness theorem. The proof of this result can be found in [6].

**Theorem 4.1.** *Let Assumption $\mathcal{H}_2$ be satisfied and let $f$ and $g$ satisfy the local Lipschitz condition, that is, there exists a constant $R_n > 0$ such that, for all $x, y, \bar{x}, \bar{y} \in R^d$, with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq n$,*

$$|f(x,y) - f(\bar{x},\bar{y})|^2 \vee |g(x,y) - g(\bar{x},\bar{y})|^2 \leq R_n(|x-\bar{x}|^2 + |y-\bar{y}|^2). \tag{77}$$

*Assume also that both $f$ and $g$ satisfy the linear growth condition, i.e. there exists a constant $K > 0$ such that, for all $x, y \in R^d$,*

$$|f(x,y)|^2 \vee |g(x,y)|^2 \leq K(1 + |x|^2 + |y|^2). \tag{78}$$

*Then, for any initial condition $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^d)$, there exists a unique global solution $x = \{x(t), t \geq -\tau\}$ of Eq. (1).*

Now, we see that if Assumption $\mathcal{H}_1'$ holds, then for any $x, y \in R^d$, the condition (78) is fulfilled with $K = 2(K_1 \vee |f(0,0)|^2 \vee |g(0,0)|^2)$. Clearly, Assumptions, $\mathcal{H}_1'$ and $\mathcal{H}_2$ guarantee the existence and uniqueness of solution to Eq. (1).

In the sequel we will establish the convergence of the Euler–Maruyama approximate solution $y$ corresponding to Eq. (1) in the $L^p$-sense on the finite time interval $[0, T]$.

Primarily, we define the discrete-time Euler–Maruyama approximate solution $Y$ of Eq. (1) on the equidistant partition $t_k = k\Delta$, for $k \in \{-(M+1), -M, \ldots, -1, 0, 1, \ldots, N\}$ of the interval $[0, T]$. In order for this solution to be well defined, set

$$\delta(-\Delta) = \delta(0), \qquad Y(-(M+1)\Delta) = \xi(-M\Delta). \tag{79}$$

Then, define

$$Y(t_k) = \xi(t_k), \qquad k = -M, -(M-1), \ldots, 0, \tag{80}$$

$$\begin{aligned} Y(t_{k+1}) = {} & Y(t_k) + u(Y(t_k - I_\Delta[\delta(t_k)]\Delta)) - u(Y(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta)) \\ & + f(Y(t_k), Y(t_k - I_\Delta[\delta(t_k)]\Delta))\Delta + g(Y(t_k), Y(t_k - I_\Delta[\delta(t_k)]\Delta))\Delta B_k, \quad k \in \{0, 1, \ldots, N\}. \end{aligned} \tag{81}$$

Let us introduce the step-processes

$$z_1(t) = \sum_{k=0}^{N-1} Y(t_k) I_{[t_k, t_{k+1})}(t), \quad z_2(t) = \sum_{k=-1}^{N-1} Y(t_k - I_\Delta[\delta(t_k)]\Delta) I_{[t_k, t_{k+1})}(t), \ \ t \in [0, T] \tag{82}$$

and define the linear combination of $Y(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta)$ and $Y(t_k - I_\Delta[\delta(t_k)]\Delta)$ as

$$\bar{Z}_k(t) = z_2(t_{k-1}) + \frac{t - t_k}{\Delta}(z_2(t_k) - z_2(t_{k-1})), \quad t \in [t_k, t_{k+1}), k = 0, 1, 2, \ldots, N-1. \tag{83}$$

For convenience, denote that

$$\bar{Z}_k(t) = \left(1 - \frac{t - t_k}{\Delta}\right) z_2(t_{k-1}) + \frac{t - t_k}{\Delta} z_2(t_k), \tag{84}$$

$$z_3(t) = \sum_{k=0}^{N-1} \bar{Z}_k(t) I_{[t_k, t_{k+1})}(t). \tag{85}$$

Then, we define the continuous-time Euler–Maruyama approximate solution $\{y(t), t \geq -\tau\}$ such that $y(t) = \xi(t)$, $t \in [-\tau, 0]$, while, for $t \in [0, T]$, we have that

$$y(t) = \xi(0) + u(z_3(t)) - u(Y(-\Delta - I_\Delta[\delta(-\Delta)]\Delta)) + \int_0^t f(z_1(s), z_2(s))ds + \int_0^t g(z_1(s), z_2(s))dB(s). \tag{86}$$

Clearly, for $t \in [t_k, t_{k+1})$, Eq. (86) can be written as

$$y(t) = Y(t_k) + u(\bar{Z}_k(t)) - u(Y(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta)) + \int_{t_k}^t f(z_1(s), z_2(s))ds + \int_{t_k}^t g(z_1(s), z_2(s))dB(s). \tag{87}$$

It is useful to observe that $y(t_k) = Y(t_k)$ for every $k \geq -M$, namely the discrete and continuous-time Euler–Maruyama solutions coincide at the grid points $t_k$. Now, bearing in mind that, for $t \in [0, T]$, there exist a unique integer $k \geq 0$ such that $t \in [t_k, t_{k+1})$, we see from (82), (84) and (85) that, for $p > 1$,

$$|z_3(t)|^p \leq \sup_{-\tau \leq s \leq t} |y(s)|^p. \tag{88}$$

Now, we are in position to establish the $p$-th moment boundedness of the Euler–Maruyama solution.

**Lemma 4.2.** *If Assumptions $\mathcal{H}_1'$ and $\mathcal{H}_2$ hold, together with $E\sup_{-\tau \leq s \leq 0} |\xi(s)|^p < \infty$, then, for any $p \geq 2$,*

$$E \sup_{-\tau \leq t \leq T} |y(t)|^p \leq H, \qquad T \geq 0, \tag{89}$$

*where $H$ is a positive constant independent of $\Delta$.*

*Proof.* Let $\epsilon > 0$ and $t \in [0, T]$ be arbitrary. For any integer $R \geq R_0$, define the stopping time

$$\rho_R = \inf\{t \geq 0 : |y(t)| \vee |z_1(t)| \vee z_3(t)| \geq R\}, \quad \inf \emptyset = 0, \tag{90}$$

where $R_0$ satisfies (58). Then, the sequence of stopping times $\{\rho_R, R \geq R_0\}$ is increasing and $\lim_{R\to+\infty} \rho_R = +\infty$ a.s. On the basis of the elementary inequality (33) and assumption (4), we have that

$$\begin{aligned} |y(t \wedge \rho_R)|^p &= |y(t \wedge \rho_R) - u(z_3(t \wedge \rho_R)) + u(z_3(t \wedge \rho_R))|^p \\ &\leq \left[1 + \epsilon^{\frac{1}{p-1}}\right]^{p-1} \left( |y(t \wedge \rho_R) - u(z_3(t \wedge \rho_R))|^p + k^p \frac{|z_3(t \wedge \rho_R)|^p}{\epsilon} \right). \end{aligned}$$

Letting $\epsilon = \left[\frac{k}{1-k}\right]^{p-1}$ and using (88) we obtain

$$|y(t \wedge \rho_R)|^p \leq \frac{1}{(1-k)^{p-1}} |y(t \wedge \rho_R) - u(z_3(t \wedge \rho_R))|^p + k \sup_{-\tau \leq s \leq t} |y(s \wedge \rho_R)|^p.$$

Therefore, we have that

$$E \sup_{-\tau \leq s \leq t} |y(s \wedge \rho_R)|^p \leq \frac{1}{1-k} E \sup_{-\tau \leq s \leq 0} |\xi(s)|^p + \frac{1}{(1-k)^p} E \sup_{0 \leq s \leq t} |y(s \wedge \rho_R) - u(z_3(s \wedge \rho_R))|^p. \tag{91}$$

Applying the Hölder inequality on (86), for $t \in [0, T]$, we observe that

$$\begin{aligned} &E \sup_{0 \leq s \leq t} |y(s \wedge \rho_R) - u(z_3(s \wedge \rho_R))|^p \\ &\qquad \leq 3^{p-1} E|\xi(0) - u(Y(-\Delta - I_\Delta[\delta(-\Delta)]\Delta))|^p + 3^{p-1} T^{p-1} \int_0^t E|f(z_1(s \wedge \rho_R), z_2(s \wedge \rho_R))|^p ds \\ &\qquad +3^{p-1} E \sup_{0 \leq s \leq t} \left| \int_0^{s \wedge \rho_R} g(z_1(r), z_2(r))dB(r) \right|^p. \end{aligned} \tag{92}$$

Using the assumption (4) and bearing in mind (79) and (80), we get

$$E|\xi(0) - u(Y(-\Delta - I_\Delta[\delta(-\Delta)]\Delta))|^p \leq E(|\xi(0)| + k|Y(-\Delta - I_\Delta[\delta(-\Delta)]\Delta)|)^p \leq (1+k)^p E \sup_{-\tau \leq s \leq 0} |\xi(s)|^p. \tag{93}$$

On the other hand, from Assumption $\mathcal{H}_1'$, that is, from (78), we have

$$E|f(z_1(s),z_2(s))|^p \vee E|g(z_1(s),z_2(s))|^p \le K_2(1+E|z_1(s)|^p+E|z_2(s)|^p), \quad s\in[0,t\wedge\rho_R],$$

where $K_2 = 3^{\frac{p}{2}-1}K^{\frac{p}{2}}$. Now, by the Burkholder–Davis–Gandy inequality we find that

$$E\sup_{0\le s\le t}\left|\int_0^{s\wedge\rho_R} g(z_1(r),z_2(r))dB(r)\right|^p \le c_pT^{\frac{p}{2}-1}K_2\int_0^t(1+E|z_1(s\wedge\rho_R)|^p+E|z_2(s\wedge\rho_R)|^p)ds, \tag{94}$$

where $c_p$ is a universal constant which depends only on $p$. Therefore, substituting (93) and (94) into (92) and using the definition (82) of the step-processes $z_1$ and $z_2$, we have

$$\begin{aligned} &E\sup_{0\le s\le t}|y(s\wedge\rho_R)-u(z_3(s\wedge\rho_R))|^p \\ &\le 3^{p-1}(1+k)^pE\sup_{-\tau\le s\le 0}|\xi(s)|^p+3^{p-1}T^{\frac{p}{2}-1}(T^{\frac{p}{2}}+c_p)K_2\int_0^t(1+E|z_1(s\wedge\rho_R)|^p+E|z_2(s\wedge\rho_R)|^p)ds \\ &\le S_1+S_2\int_0^t E\sup_{-\tau\le r\le s}|y(r\wedge\rho_R)|^pds, \end{aligned} \tag{95}$$

where

$$S_1 = 3^{p-1}\left[(1+k)^pE\sup_{-\tau\le s\le 0}|\xi(s)|^p+T^{\frac{p}{2}}(T^{\frac{p}{2}}+c_p)K_2\right],\ S_2=3^{p-1}2T^{\frac{p}{2}-1}(T^{\frac{p}{2}}+c_p)K_2.$$

Now, substituting (95) into (91) we obtain

$$E\sup_{-\tau\le s\le t}|y(s\wedge\rho_R)|^p \le S_1'+S_2'\int_0^t E\sup_{-\tau\le r\le s}|y(r\wedge\rho_R)|^pds,$$

where

$$S_1' = \frac{1}{1-k}E\sup_{-\tau\le s\le 0}|\xi(s)|^p+\frac{S_1}{(1-k)^p},\quad S_2'=\frac{S_2}{(1-k)^p}.$$

Applying the Gronwall–Bellman lemma, we find that

$$E\sup_{-\tau\le s\le t}|y(s\wedge\rho_R)|^p \le S_1'e^{S_2'T}\equiv H.$$

Letting $R\to+\infty$, we conclude from the previous inequality that (89) holds. □

**Lemma 4.3.** *If Assumptions $\mathcal{H}_1'$, $\mathcal{H}_2$, $\mathcal{H}_3$, $\mathcal{H}_5$ and $\mathcal{H}_6$ hold, together with the assumptions (30) and*

$$E\sup_{-\tau\le s\le 0}|\xi(s)|^p<\infty,$$

*then, for any integer $l>1$ and $p\ge 2$,*

$$E\sup_{-M\le k\le N}|Y(t_k)-Y(t_{k-1})|^p \le \bar{H}(l)\Delta^{\frac{pl-1}{2l}}, \tag{96}$$

*where $\bar{H}(l)$ is a positive constant, which is dependent on $l$, but independent of $\Delta$.*

*Proof.* On the basis of (79), (80) and Assumption $\mathcal{H}_5$, we observe that

$$E\sup_{-M\le k\le 0}|Y(t_k)-Y(t_{k-1})|^p = E\sup_{-M+1\le k\le 0}|\xi(t_k)-\xi(t_{k-1})|^p \le C_\xi\Delta^{\frac{p}{2}}.$$

When $k\in\{1,2,\dots,N\}$, from (81), we have

$$\begin{aligned} Y(t_k)-Y(t_{k-1}) &= u(Y(t_{k-1}-I_\Delta[\delta(t_{k-1})]\Delta))-u(Y(t_{k-2}-I_\Delta[\delta(t_{k-2})]\Delta)) \\ &\quad + f(Y(t_{k-1}),Y(t_{k-1}-I_\Delta[\delta(t_{k-1})]\Delta))\Delta+g(Y(t_{k-1}),Y(t_{k-1}-I_\Delta[\delta(t_{k-1})]\Delta))\Delta B_{k-1}. \end{aligned}$$

Applying the elementary inequality (33) and Assumption $\mathcal{H}_2$ we get

$$\begin{aligned} E\sup_{1\le k\le N}|Y(t_k)-Y(t_{k-1})|^p \le& \left[1+\epsilon^{\frac{1}{p-1}}\right]^{p-1}\frac{k^p}{\epsilon}E\sup_{1\le k\le N}|Y(t_{k-1}-I_\Delta[\delta(t_{k-1})]\Delta)-Y(t_{k-2}-I_\Delta[\delta(t_{k-2})]\Delta)|^p\\ &+2^{p-1}\left[1+\epsilon^{\frac{1}{p-1}}\right]^{p-1}E\sup_{1\le k\le N}|f(Y(t_{k-1}),Y(t_{k-1}-I_\Delta[\delta(t_{k-1})]\Delta))|^p\Delta^p\\ &+2^{p-1}\left[1+\epsilon^{\frac{1}{p-1}}\right]^{p-1}E\sup_{1\le k\le N}|g(Y(t_{k-1}),Y(t_{k-1}-I_\Delta[\delta(t_{k-1})]\Delta))\Delta B_{k-1}|^p. \end{aligned} \tag{97}$$

In the sequel, we will deal with these three terms separately.

Applying the same arguments which are used for the estimate (37), we get

$$E\sup_{1\le k\le N+1}|Y(t_{k-1}-I_\Delta[\delta(t_{k-1})]\Delta)-Y(t_{k-2}-I_\Delta[\delta(t_{k-2})]\Delta)|^p\le(3+\lfloor\eta\rfloor)^pE\sup_{-M\le k\le N}|Y(t_k)-Y(t_{k-1})|^p. \tag{98}$$

On the basis of Assumption $\mathcal{H}_1'$, that is (78), and Lemma 4.2, we see that

$$\begin{aligned} E\sup_{1\le k\le N}|f(Y(t_{k-1}),Y(t_{k-1}-I_\Delta[\delta(t_{k-1})]\Delta))|^p &\le 3^{\frac{p}{2}-1}K^{\frac{p}{2}}E\sup_{1\le k\le N}\left[1+|Y(t_{k-1})|^p+|Y(t_{k-1}-I_\Delta[\delta(t_{k-1})]\Delta)|^p\right]\\ &\le 3^{\frac{p}{2}-1}K^{\frac{p}{2}}\Big(1+2E\sup_{-\tau\le t\le T}|y(t)|^p\Big)\\ &\le 3^{\frac{p}{2}-1}K^{\frac{p}{2}}(1+2H). \end{aligned} \tag{99}$$

By the Hölder inequality, (78) and Lemma 4.2, for any integer $l>1$,

$$\begin{aligned} &E\sup_{1\le k\le N}|g(Y(t_{k-1}),Y(t_{k-1}-I_\Delta[\delta(t_{k-1})]\Delta))\Delta B_{k-1}|^p\\ &\le E\left(\sup_{1\le k\le N}|g(Y(t_{k-1}),Y(t_{k-1}-I_\Delta[\delta(t_{k-1})]\Delta))|^p\sup_{1\le k\le N}|\Delta B_{k-1}|^p\right)\\ &\le\left[E\left(\sup_{1\le k\le N}|g(Y(t_{k-1}),Y(t_{k-1}-I_\Delta[\delta(t_{k-1})]\Delta))|^{\frac{2pl}{2l-1}}\right)\right]^{\frac{2l-1}{2l}}\left[E\left(\sup_{0\le k\le N-1}|\Delta B_k|^{2pl}\right)\right]^{\frac{1}{2l}}\\ &\le(3K)^{\frac{p}{2}}(1+2H)^{\frac{2l-1}{2l}}\left[\sum_{k=0}^{N-1}E|\Delta B_k|^{2pl}\right]^{\frac{1}{2l}}. \end{aligned} \tag{100}$$

Substituting (40) into (100), we obtain

$$E\sup_{1\le k\le N}|g(Y(t_{k-1}),Y(t_{k-1}-I_\Delta[\delta(t_{k-1})]\Delta))\Delta B_{k-1}|^p\le(3K)^{\frac{p}{2}}(1+2H)^{\frac{2l-1}{2l}}\left[\sum_{k=0}^{N-1}m^{pl}(2pl-1)!!\Delta^{pl}\right]^{\frac{1}{2l}}\le\bar{M}(l)\Delta^{\frac{pl-1}{2l}}, \tag{101}$$

where $\bar{M}(l)=(3K)^{\frac{p}{2}}(1+2H)^{\frac{2l-1}{2l}}m^{\frac{p}{2}}(T(2pl-1)!!)^{\frac{1}{2l}}$. Now, substituting (98), (99) and (101) into (97), we get

$$\begin{aligned} E\sup_{1\le k\le N}|Y(t_k)-Y(t_{k-1})|^p\le&\left[1+\epsilon^{\frac{1}{p-1}}\right]^{p-1}\frac{k^p}{\epsilon}(3+\lfloor\eta\rfloor)^pE\sup_{-M\le k\le N}|Y(t_k)-Y(t_{k-1})|^p\\ &+\left[1+\epsilon^{\frac{1}{p-1}}\right]^{p-1}2^{p-1}\left(3^{\frac{p}{2}-1}K^{\frac{p}{2}}(1+2H)\Delta^p+\bar{M}(l)\Delta^{\frac{pl-1}{2l}}\right). \end{aligned}$$

Choosing $\epsilon = \left[\frac{\sqrt{k}}{1-\sqrt{k}}\right]^{p-1}$, recalling that $\Delta \in (0,1)$ and applying Assumption $\mathcal{H}_5$, we have

$$E\sup_{-M\le k\le N}|Y(t_k)-Y(t_{k-1})|^p \le C_\xi \Delta^{\frac{pl-1}{2l}} + k^{\frac{p}{2}+\frac{1}{2}}(3+\lfloor\eta\rfloor)^p E\sup_{-M\le k\le N}|Y(t_k)-Y(t_{k-1})|^p + \frac{2^{p-1}}{(1-\sqrt{k})^{p-1}}\left(3^{\frac{p}{2}-1}K^{\frac{p}{2}}(1+2H)+\bar{M}(l)\right)\Delta^{\frac{pl-1}{2l}}.$$

Therefore, bearing in mind (30), we obtain (96), where

$$\bar{H}(l) = \frac{C_\xi}{1-\sqrt{k}} + \frac{2^{p-1}}{(1-\sqrt{k})^p}\left(3^{\frac{p}{2}-1}K^{\frac{p}{2}}(1+2H)+\bar{M}(l)\right),$$

which completes the proof. □

**Lemma 4.4.** *Let the conditions of Lemmas 4.2 and 4.3 hold. Then, for any integer $l>1$ and $p\ge 2$, we have that*

$$E\sup_{0\le t\le T}|y(t-\delta(t))-z_3(t)|^p \le D(l)\Delta^{\frac{pl-1}{2l}}, \tag{102}$$

*where $D(l)$ is a positive constant, which is dependent on $l$, but independent of $\Delta$.*

*Proof.* Fix any $t\in(0,T]$. Let $k\in\{0,1,\dots,N-1\}$ and let $k_t\in\{-M,-M+1,\dots,N-1\}$, such that $t\in[t_k,t_{k+1})$ and $t-\delta(t)\in[t_{k_t},t_{k_t+1})$.

Observe that, for $t_{k-1}-I_\Delta[\delta(t_{k-1})]\Delta \le t_{k_t}\le t-\delta(t)$, on the basis of $\mathcal{H}_6$, we have that

$$|t_{k_t}-t_{k-1}+I_\Delta[\delta(t_{k-1})]\Delta| \le |t-\delta(t)-t_{k-1}+I_\Delta[\delta(t_{k-1})]\Delta| \le (4+\lfloor 2\eta\rfloor)\Delta. \tag{103}$$

On the other hand, if $t_{k_t}\le t-\delta(t)<t_{k-1}-I_\Delta[\delta(t_{k-1})]\Delta$, in a view of $\mathcal{H}_6$, we get

$$|t_{k_t}-t_{k-1}+I_\Delta[\delta(t_{k-1})]\Delta| \le (5+\lfloor 2\eta\rfloor)\Delta. \tag{104}$$

From (103) and (104) we obtain

$$|k_t-(k-1-I_\Delta[\delta(t_{k-1})])| \le 5+\lfloor 2\eta\rfloor. \tag{105}$$

By the definition (85) of the step-process $z_3$, we see that

$$E\sup_{0\le t\le T}|y(t-\delta(t))-z_3(t)|^p \le 2^{p-1}E\sup_{0\le t\le T}|y(t-\delta(t))-Y(t_{k_t})|^p + 2^{p-1}E\sup_{0\le t\le T}|Y(t_{k_t})-\bar{Z}_k(t)|^p. \tag{106}$$

In order to estimate $E\sup_{0\le t\le T}|y(t-\delta(t))-Y(t_{k_t})|^p$, we discuss the following two cases.

*Case 1:* If $k_t<0$ then, on the basis of Assumption $\mathcal{H}_5$, we obtain

$$E\sup_{0\le t\le T,k_t<0}|y(t-\delta(t))-Y(t_{k_t})|^p \le E\sup_{0\le s,t\le T,|s-t|\le\Delta}|\xi(t)-\xi(s)|^p \le C_\xi\Delta^{\frac{p}{2}}. \tag{107}$$

*Case 2:* If $k_t\ge 0$, then, from (87), we have that

$$\begin{aligned} &E\sup_{0\le t\le T,k_t\ge 0}|y(t-\delta(t))-Y(t_{k_t})|^p \\ &\le 3^{p-1}E\sup_{0\le t\le T,k_t\ge 0}|u(\bar{Z}_{k_t}(t-\delta(t))-u(Y(t_{k_t-1}-I_\Delta[\delta(t_{k_t-1})]\Delta))|^p \\ &\quad + 3^{p-1}E\sup_{0\le t\le T,k_t\ge 0}\left|\int_{t_{k_t}}^{t-\delta(t)} f(z_1(s),z_2(s))ds\right|^p + 3^{p-1}E\sup_{0\le t\le T,k_t\ge 0}\left|\int_{t_{k_t}}^{t-\delta(t)} g(z_1(s),z_2(s))dB(s)\right|^p. \end{aligned} \tag{108}$$

Thus, using Assumption $\mathcal{H}_2$ and definitions (82) and (83), we get

$$\begin{aligned}
&E \sup_{0\le t\le T, k_t\ge 0} |u(\bar{Z}_{k_t}(t-\delta(t)) - u(Y(t_{k_t-1} - I_\Delta[\delta(t_{k_t-1})]\Delta))|^p \\
&\quad\le k^p E \sup_{0\le t\le T, k_t\ge 0} \left|\frac{t-\delta(t)-t_{k_t}}{\Delta}(z_2(t_{k_t}) - z_2(t_{k_t-1}))\right|^p \\
&\quad\le k^p E \sup_{0\le t\le T, k_t\ge 0} \left|Y(t_{k_t} - I_\Delta[\delta(t_{k_t})]\Delta) - Y(t_{k_t-1} - I_\Delta[\delta(t_{k_t-1})]\Delta)\right|^p.
\end{aligned}$$

By the estimate (98) and Lemma 4.3, we conclude that

$$\begin{aligned}
E \sup_{0\le t\le T, k_t\ge 0} |u(\bar{Z}_{k_t}(t-\delta(t)) - u(Y(t_{k_t-1} - I_\Delta[\delta(t_{k_t-1})]\Delta))|^p &\le k^p(3+\lfloor\eta\rfloor)^p E \sup_{-M\le k\le N} |Y(t_k) - Y(t_{k-1})|^p \\
&\le k^p(3+\lfloor\eta\rfloor)^p \bar{H}(l)\Delta^{\frac{pl-1}{2l}}. \qquad (109)
\end{aligned}$$

Applying the Hölder inequality, (78) and Lemma 4.2 we see that

$$\begin{aligned}
E \sup_{0\le t\le T, k_t\ge 0} \left|\int_{t_{k_t}}^{t-\delta(t)} f(z_1(s), z_2(s))ds\right|^p &\le 3^{\frac{p}{2}-1}K^{\frac{p}{2}}\Delta^{p-1} E \sup_{0\le t\le T, k_t\ge 0} \int_{t_{k_t}}^{t-\delta(t)} (1+|z_1(s)|^p + |z_2(s)|^p)ds \\
&\le 3^{\frac{p}{2}-1}K^{\frac{p}{2}}\Delta^{p-1} \int_0^T \left(1 + 2E \sup_{-\tau\le t\le T} |y(t)|^p\right)ds \\
&\le 3^{\frac{p}{2}-1}K^{\frac{p}{2}} T(1+2H)\Delta^{p-1}. \qquad (110)
\end{aligned}$$

On the other hand, for any integer $l > 1$, applying the Holder inequality we find that

$$\begin{aligned}
&E \sup_{0\le t\le T, k_t\ge 0} \left|\int_{t_{k_t}}^{t-\delta(t)} g(z_1(s), z_2(s))dB(s)\right|^p \\
&\quad\le \left(E \sup_{0\le t-\delta(t)\le T, k_t\ge 0} \left|g(Y(t_{k_t}), Y(t_{k_t} - I_\Delta[\delta(t_{k_t})]\Delta))\right|^{\frac{2pl}{2l-1}}\right)^{\frac{2l-1}{2l}} \left(E \sup_{0\le t-\delta(t)\le T, k_t\ge 0} |B(t-\delta(t)) - B(k_t)|^{2pl}\right)^{\frac{1}{2l}}. \qquad (111)
\end{aligned}$$

From the Doob martingale inequality, we have

$$\begin{aligned}
E \sup_{0\le t-\delta(t)\le T, k_t\ge 0} |B(t-\delta(t)) - B(k_t)|^{2pl} &\le E \sup_{0\le k_t\le N-1} \sup_{k_t\Delta\le t-\delta(t)\le (k_t+1)\Delta} |B(t-\delta(t)) - B(k_t)|^{2pl} \\
&\le \sum_{k_t=0}^{N-1} E \sup_{k_t\Delta\le t-\delta(t)\le (k_t+1)\Delta} |B(t-\delta(t)) - B(k_t)|^{2pl} \\
&\le \left(\frac{2pl}{2pl-1}\right)^{2pl} \sum_{k_t=0}^{N-1} E|B(t_{k_t+1}) - B(t_{k_t})|^{2pl}. \qquad (112)
\end{aligned}$$

Substituting (112) into (111) and using the same procedure as in obtaining (101), we get

$$\begin{aligned}
&E \sup_{0\le t\le T, k_t\ge 0} \left|\int_{t_{k_t}}^{t-\delta(t)} g(z_1(s), z_2(s))dB(s)\right|^p \\
&\quad\le \left(\frac{2pl}{2pl-1}\right)^p \left(E \sup_{0\le t-\delta(t)\le T, k_t\ge 0} \left|g(Y(t_{k_t}), Y(t_{k_t} - I_\Delta[\delta(t_{k_t})]\Delta))\right|^{\frac{2pl}{2l-1}}\right)^{\frac{2l-1}{2l}} \left(\sum_{k_t=0}^{N-1} E|B(t_{k_t+1}) - B(t_{k_t})|^{2pl}\right)^{\frac{1}{2l}} \\
&\quad\le \left(\frac{2pl}{2pl-1}\right)^p \bar{M}(l)\Delta^{\frac{pl-1}{2l}}. \qquad (113)
\end{aligned}$$

Moreover, substituting (109), (110) and (113) into (108) and recalling that $\Delta \in (0,1)$, we obtain

$$E \sup_{0\le t\le T, k_t\ge 0} |y(t-\delta(t)) - Y(t_{k_t})|^p \le R_1(l)\Delta^{\frac{pl-1}{2l}}, \tag{114}$$

where

$$R_1(l) = 3^{p-1}\left(k^p(3+\lfloor\eta\rfloor)^p\bar{H}(l) + 3^{\frac{p}{2}-1}K^{\frac{p}{2}}T(1+2H) + \left(\frac{2pl}{2pl-1}\right)^p \bar{M}(l)\right).$$

Consequently, from (107) and (114), we find that

$$E \sup_{0\le t\le T} |y(t-\delta(t)) - Y(t_{k_t})|^p \le M(l)\Delta^{\frac{pl-1}{2l}},$$

where $M(l) = C_\xi \vee R_1(l)$. Therefore, (106) becomes

$$E \sup_{0\le t\le T} |y(t-\delta(t)) - z_3(t)|^p \le 2^{p-1}M(l)\Delta^{\frac{pl-1}{2l}} + 2^{p-1}E \sup_{0\le t\le T} |Y(t_{k_t}) - \bar{Z}_k(t)|^p. \tag{115}$$

In order to estimate the expression $E\sup_{0\le t\le T}|Y(t_{k_t}) - \bar{Z}_k(t)|^p$ from (115), we will use the definition (83). Thus, we have that

$$Y(t_{k_t}) - \bar{Z}_k(t) = Y(t_{k_t}) - z_2(t_{k-1}) - \frac{t-t_k}{\Delta}(z_2(t_k) - z_2(t_{k-1})), \quad t \in [t_k, t_{k+1}).$$

On the basis of (82), (98), (105) and Lemma 4.3, we obtain

$$\begin{aligned} &E \sup_{0\le t\le T} |Y(t_{k_t}) - \bar{Z}_k(t)|^p \\ &\le 2^{p-1}E \sup_{0\le t\le T} |Y(t_{k_t}) - Y(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta)|^p + 2^{p-1}E \sup_{1\le k\le N} |Y(t_k - I_\Delta[\delta(t_k)]\Delta) - Y(t_{k-1} - I_\Delta[\delta(t_{k-1})]\Delta)|^p \\ &\le 2^{p-1}\left((5+\lfloor 2\eta\rfloor)^p + (3+\lfloor\eta\rfloor)^p\right) E \sup_{-M\le k\le N} |Y(t_k) - Y(t_{k-1})|^p \\ &\le 2^{p-1}\left((5+\lfloor 2\eta\rfloor)^p + (3+\lfloor\eta\rfloor)^p\right)\bar{H}(l)\Delta^{\frac{pl-1}{2l}}. \end{aligned} \tag{116}$$

Finally, on the basis of (115) and (116), together with (106), we get (102), where

$$D(l) = 2^{p-1}M(l) + 4^{p-1}\left((5+\lfloor 2\eta\rfloor)^p + (3+\lfloor\eta\rfloor)^p\right)\bar{H}(l),$$

which completes the proof. □

**Theorem 4.5.** *Let the conditions of Lemmas 4.2, 4.3 and 4.4 hold. Then, for any integer $l > 1$ and $p > 0$,*

$$E \sup_{-\tau\le t\le T} |x(t) - y(t)|^p \le S(l)\Delta^{\frac{pl-1}{2l}},$$

*where $S(l)$ is a positive constant, which is dependent on $l$, but independent of $\Delta$.*

*Proof.* First, we consider the case when $p \ge 2$. Let $\tau_R$ and $\rho_R$ be the stopping times defined by (59) and (90), respectively. Now, for any integer $R \ge R_0$, define the stopping time

$$\theta_R = \tau_R \wedge \rho_R,$$

where $R_0$ satisfies (58). Then the sequence of stopping times $\{\theta_R, R \geq R_0\}$ is increasing and $\lim_{R\to+\infty} \theta_R = +\infty$ a.s. For any $t \in [0,T]$ and $\epsilon \in (0,1)$, from the elementary inequality (33), we have

$$\begin{aligned}
&|x(t\wedge\theta_R) - y(t\wedge\theta_R)|^p \\
&\leq \left[1+\epsilon^{\frac{1}{p-1}}\right]^{p-1}\frac{1}{\epsilon}|u(x(t\wedge\theta_R-\delta(t\wedge\theta_R)) - u(z_3(t\wedge\theta_R)) - u(\xi(-\delta(0))) + u(Y(-\Delta-[\delta(-\Delta)/\Delta]\Delta))|^p \\
&+ \left[1+\epsilon^{\frac{1}{p-1}}\right]^{p-1} 2^{p-1}\left|\int_0^{t\wedge\theta_R} f(x(s),x(s-\delta(s))) - f(z_1(s),z_2(s))ds\right|^p \\
&+ \left[1+\epsilon^{\frac{1}{p-1}}\right]^{p-1} 2^{p-1}\left|\int_0^{t\wedge\theta_R} g(x(s),x(s-\delta(s))) - g(z_1(s),z_2(s))dB(s)\right|^p. \qquad (117)
\end{aligned}$$

On the basis of Assumption $\mathcal{H}_2$, we obtain

$$\begin{aligned}
&|u(x(t\wedge\theta_R-\delta(t\wedge\theta_R)) - u(z_3(t\wedge\theta_R)) - u(\xi(-\delta(0))) + u(Y(-\Delta-[\delta(-\Delta)/\Delta]\Delta))|^p \\
&\quad\leq \left[1+\epsilon^{\frac{1}{p-1}}\right]^{p-1}\frac{1}{\epsilon}|u(x(t\wedge\theta_R-\delta(t\wedge\theta_R))) - u(z_3(t\wedge\theta_R))|^p \\
&\quad+ \left[1+\epsilon^{\frac{1}{p-1}}\right]^{p-1}|u(Y(-\Delta-I_\Delta[\delta(-\Delta)]\Delta)) - u(\xi(-\delta(0)))|^p \\
&\quad\leq \left[1+\epsilon^{\frac{1}{p-1}}\right]^{2p-2}\frac{1}{\epsilon^2}k^p|x(t\wedge\theta_R-\delta(t\wedge\theta_R)) - y(t\wedge\theta_R-\delta(t\wedge\theta_R))|^p \\
&\quad+ \left[1+\epsilon^{\frac{1}{p-1}}\right]^{2p-2}\frac{1}{\epsilon}k^p|y(t\wedge\theta_R-\delta(t\wedge\theta_R)) - z_3(t\wedge\theta_R)|^p \\
&\quad+ \left[1+\epsilon^{\frac{1}{p-1}}\right]^{p-1}k^p|Y(-\Delta-I_\Delta[\delta(-\Delta)]\Delta)) - \xi(-\delta(0))|^p.
\end{aligned}$$

Note that $\xi(-\delta(0)) = y(-\delta(0))$ by the definition of the continuous-time Euler–Maruyama solution $y$ and also that $Y(-\Delta - I_\Delta[\delta(-\Delta)]\Delta)) = z_3(0)$. So,

$$\begin{aligned}
&\sup_{0\leq s\leq t}|u(x(s\wedge\theta_R-\delta(s\wedge\theta_R)) - u(z_3(s\wedge\theta_R)) - u(\xi(-\delta(0))) + u(Y(-\Delta-I_\Delta[\delta(-\Delta)]\Delta))|^p \\
&\quad\leq \left[1+\epsilon^{\frac{1}{p-1}}\right]^{2p-2}\frac{1}{\epsilon^2}k^p\sup_{-\tau\leq s\leq t}|x(s\wedge\theta_R) - y(s\wedge\theta_R)|^p \\
&\quad+ \left[1+\epsilon^{\frac{1}{p-1}}\right]^{p-1}k^p\left(\left[1+\epsilon^{\frac{1}{p-1}}\right]^{p-1}\frac{1}{\epsilon}+1\right)\sup_{0\leq s\leq t}|y(s\wedge\theta_R-\delta(s\wedge\theta_R)) - z_3(s\wedge\theta_R)|^p.
\end{aligned}$$

Since the solutions $x$ and $y$ satisfy the same initial condition, on the basis of (117) we find that

$$\begin{aligned}
&E\sup_{-\tau\leq s\leq t}|x(s\wedge\theta_R) - y(s\wedge\theta_R)|^p \\
&\leq \left[1+\epsilon^{\frac{1}{p-1}}\right]^{3p-3}\frac{1}{\epsilon^3}k^pE\sup_{-\tau\leq s\leq t}|x(s\wedge\theta_R) - y(s\wedge\theta_R)|^p \\
&+ \left[1+\epsilon^{\frac{1}{p-1}}\right]^{2p-2}\frac{k^p}{\epsilon}\left(\left[1+\epsilon^{\frac{1}{p-1}}\right]^{p-1}\frac{1}{\epsilon}+1\right)E\sup_{0\leq s\leq t}|y(s\wedge\theta_R-\delta(s\wedge\theta_R)) - z_3(s\wedge\theta_R)|^p \\
&+ \left[1+\epsilon^{\frac{1}{p-1}}\right]^{p-1}2^{p-1}\left[T^{p-1}\int_0^t E\left|f(x(s\wedge\theta_R),x(s\wedge\theta_R-\delta(s\wedge\theta_R))) - f(z_1(s\wedge\theta_R),z_2(s\wedge\theta_R))\right|^p ds\right. \\
&\left.+\ c_pT^{\frac{p}{2}-1}\int_0^t E|g(x(s\wedge\theta_R),x(s\wedge\theta_R-\delta(s\wedge\theta_R))) - g(z_1(s\wedge\theta_R),z_2(s\wedge\theta_R))|^p ds\right], \qquad (118)
\end{aligned}$$

where $c_p$ is a universal constant which depends only on $p$. Using Lemma 4.4, choosing $\epsilon = \left(\frac{\sqrt[3]{k}}{1-\sqrt[3]{k}}\right)^{p-1}$ and applying Assumption $\mathcal{H}_1'$, the expression (118) becomes

$$
\begin{aligned}
&E \sup_{-\tau\le s\le t} |x(s\wedge\theta_R) - y(s\wedge\theta_R)|^p \\
&\quad\le \frac{(\sqrt[3]{k})^{p+2}\left[1+(\sqrt[3]{k})^{p-1}\right]D(l)}{(1-k)(1-\sqrt[3]{k})^{p-1}}\Delta^{\frac{pl-1}{2l}} + \frac{2^{\frac{3p}{2}-2}T^{\frac{p}{2}-1}(T^{\frac{p}{2}}+c_p)K_1^{\frac{p}{2}}}{(1-k)(1-\sqrt[3]{k})^{p-1}} \\
&\quad\times \int_0^t (E|x(s\wedge\theta_R) - z_1(s\wedge\theta_R)|^p + E|x(s\wedge\theta_R - \delta(s\wedge\theta_R)) - z_2(s\wedge\theta_R)|^p)ds. \qquad (119)
\end{aligned}
$$

For $u\in[0,t\wedge\theta_R]$, let $k_u$ be an integer such that $u\in[k_u\Delta,(k_u+1)\Delta\wedge\theta_R)$. Recalling the definition (82) of the step-process $z_1$ and using the procedure which gave the estimate (114), we obtain, for $s\in[0,t\wedge\theta_R]$,

$$
\begin{aligned}
E|x(s)-z_1(s)|^p &\le 2^{p-1}E\sup_{-\tau\le u\le s}|x(u)-y(u)|^p + 2^{p-1}E\sup_{0\le u\le s,k_u\ge 0}|y(u)-Y(k_u\Delta)|^p \\
&\le 2^{p-1}E\sup_{-\tau\le u\le s}|x(u)-y(u)|^p + 2^{p-1}R_1(l)\Delta^{\frac{pl-1}{2l}}. \qquad (120)
\end{aligned}
$$

Using the definitions (83) and (85) we have that

$$
z_3(s)-z_2(s) = z_2(t_{k-1}) + \frac{s-t_k}{\Delta}(z_2(t_k)-z_2(t_{k-1})) - z_2(t_k), \qquad (121)
$$

whenever $s\in[t_k,t_{k+1})$, $k=0,1,\dots,N-1$. This fact, together with (82), (98) and Lemma 4.3, implies that

$$
E|z_3(s)-z_2(s)|^p \le E\sup_{1\le k\le N+1}|Y(t_{k-1}-I_\Delta[\delta(t_{k-1})]\Delta) - Y(t_{k-2}-I_\Delta[\delta(t_{k-2})]\Delta)|^p \le (3+\lfloor\eta\rfloor)^p\bar{H}(l)\Delta^{\frac{pl-1}{2l}}. \qquad (122)
$$

On the other hand, using Lemma 4.4 and the estimate (122), we get

$$
\begin{aligned}
E|x(s-\delta(s))-z_2(s)|^p &\le 3^{p-1}E|x(s-\delta(s))-y(s-\delta(s))|^p + 3^{p-1}E|y(s-\delta(s))-z_3(s)|^p + 3^{p-1}E|z_3(s)-z_2(s)|^p \\
&\le 3^{p-1}E|x(s-\delta(s))-y(s-\delta(s))|^p + 3^{p-1}(D(l)+(3+\lfloor\eta\rfloor)^p\bar{H}(l))\Delta^{\frac{pl-1}{2l}}. \qquad (123)
\end{aligned}
$$

Now, substituting (120) and (123) into (119), we obtain

$$
\begin{aligned}
&E\sup_{-\tau\le s\le t}|x(s\wedge\theta_R)-y(s\wedge\theta_R)|^p \\
&\quad\le \frac{(\sqrt[3]{k})^{p+2}\left[1+(\sqrt[3]{k})^{p-1}\right]D(l)}{(1-k)(1-\sqrt[3]{k})^{p-1}}\Delta^{\frac{pl-1}{2l}} \\
&\quad+ \frac{2^{\frac{3p}{2}-2}T^{\frac{p}{2}-1}(T^{\frac{p}{2}}+c_p)K_1^{\frac{p}{2}}(2^{p-1}+3^{p-1})}{(1-k)(1-\sqrt[3]{k})^{p-1}}\int_0^t E\sup_{-\tau\le u\le s}|x(u\wedge\theta_R)-y(u\wedge\theta_R)|^p ds \\
&\quad+ \frac{2^{\frac{3p}{2}-2}T^{\frac{p}{2}}(T^{\frac{p}{2}}+c_p)K_1^{\frac{p}{2}}\left(2^{p-1}R_1(l)+3^{p-1}(D(l)+(3+\lfloor\eta\rfloor)^p\bar{H}(l))\right)}{(1-k)(1-\sqrt[3]{k})^{p-1}}\Delta^{\frac{pl-1}{2l}}.
\end{aligned}
$$

The application of the Gronwall–Bellman lemma yields

$$
E\sup_{-\tau\le s\le t}|x(s\wedge\theta_R)-y(s\wedge\theta_R)|^p \le S_0(l)\Delta^{\frac{pl-1}{2l}},
$$

where

$$
S_0(l) = \tilde{S}(l)e^{T\bar{S}}, \quad \bar{S} = \frac{2^{\frac{3p}{2}-2}T^{\frac{p}{2}-1}(T^{\frac{p}{2}}+c_p)K_1^{\frac{p}{2}}(2^{p-1}+3^{p-1})}{(1-k)(1-\sqrt[3]{k})^{p-1}}
$$

$$
\tilde{S}(l) = \frac{(\sqrt[3]{k})^{p+2}\left[1+(\sqrt[3]{k})^{p-1}\right]D(l)}{(1-k)(1-\sqrt[3]{k})^{p-1}} + \frac{2^{\frac{3p}{2}-2}T^{\frac{p}{2}}(T^{\frac{p}{2}}+c_p)K_1^{\frac{p}{2}}\left(2^{p-1}R_1(l)+3^{p-1}(D(l)+(3+\lfloor\eta\rfloor)^p\bar{H}(l))\right)}{(1-k)(1-\sqrt[3]{k})^{p-1}}.
$$

Consequently, we have that

$$E \sup_{-\tau \le s \le T} |x(s \wedge \theta_R) - y(s \wedge \theta_R)|^p \le S_0(l) \Delta^{\frac{pl-1}{2l}}.$$

Letting $R \to +\infty$ we conclude from the previous inequality that

$$E \sup_{-\tau \le s \le T} |x(s) - y(s)|^p \le S_0(l) \Delta^{\frac{pl-1}{2l}}.$$

Now, for $p \in (0,2)$, on the basis of the Holder inequality and previous part of the proof, we obtain

$$E \sup_{-\tau \le t \le T} |x(t) - y(t)|^p \le \left( E \sup_{-\tau \le t \le T} |x(t) - y(t)|^2 \right)^{\frac{p}{2}} \le S_0^{\frac{p}{2}}(l) \Delta^{\frac{2l-1}{2l} \frac{p}{2}} < S_0^{\frac{p}{2}}(l) \Delta^{\frac{pl-1}{2l}}.$$

Thus, the proof is complete with $S(l) = S_0(l) \vee S_0^{\frac{p}{2}}(l)$. □

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## References

[1] Q. Guo, X. Mao, R. Yue, The truncated Euler–Maruyama method for stochastic differential delay equations, Numerical Algorithms (2018) 78:599-624.
[2] L. Hu, X. Li, X. Mao, Convergence rate and stability of the truncated Euler–Maruyama method for stochastic differential equations, J. Comput. Appl. Math. 337 (2018) 274–289.
[3] N. Jacob, Y. Wang, C. Yuan, Stochastic differential delay equations with jumps, under nonlinear growth condition, Stochastics: An International Journal of Probability and Stochastics Processes 81 (2009) 571–588.
[4] P. E. Kloeden, E. Platen, Numerical Solutions of Stochastic Differential Equations, Springer, Berlin, 1992.
[5] Q. Luo, X. Mao, Y. Shen, New criteria on exponential stability of neutral stochastic differential delay equations, Systems Control Lett. 55 (2006) 826–834.
[6] X. Mao, Stochastic Differential Equations and Applications, Horvood, Chichester, UK, 1997.
[7] X. Mao, Asymptotic properties of neutral stochastic differential delay equations, Stochastics and Stochastics Reports 68 (2000) 273–295.
[8] X. Mao, Numerical solutions of stochastic differential delay equations under local Lipschitz condition, J. Comput. Appl. Math. 151 (2003) 215–227.
[9] X. Mao, M.J. Rassias, Khasminskii-type theorems for stochastic differential delay equations, Stochastic Analysis and Applications (2005) 1045–1069.
[10] X. Mao, Numerical solutions of stochastic differential delay equations under the generalized Khasminskii-type conditions, Appl. Math. Comput. 217 (2011) 5512–5524.
[11] X. Mao, The truncated Euler–Maruyama method for stochastic differential equations, J. Comput. Appl. Math. 290 (2015) 370–384.
[12] M. Milošević, Highly nonlinear neutral stochastic differential equations with time-dependent delay and the Euler–Maruyama method, Math. Comput. Modelling 54 (2011) 2235–2251.
[13] L. Tan, C. Yuan, Convergence rates of truncated EM scheme for NSDDEs, arXiv:1801.05952v1.
[14] F. Wu, X. Mao, Numerical solutions of neutral stochastic functional differential equations, SIAM J. NUMER. ANAL. 46 (2008) 1821–1841.
[15] M. Xue, S. Zhou, S. Hu, Stability of nonlinear neutral stochastic functional differential equations, Journal of Applied Mathematics (2010), doi: 10.1155/2010/425762.
[16] Z. Yu, Almost surely asymptotic stability of exact and numerical solutions for neutral stochastic pantograph equations, Abstract and Applied Analysis, 2011, doi:10.1155/2011/14079.