Filomat 35:7 (2021), 2403–2424
https://doi.org/10.2298/FIL2107403Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

(Weighted Pseudo) Almost Automorphic Solutions in Distribution for Fractional Stochastic Differential Equations Driven by Levy Noise

Min Yang[a]

[a]*College of Mathematics, Taiyuan University of Technology, Taiyuan 030024, PR China*

Abstract. In this paper, by using contraction principle, fractional calculus and stochastic analysis, we study the existence and uniqueness of (weighted pseudo) almost automorphic solutions in distribution for fractional stochastic differential equations driven by Lévy noise. An example is presented to illustrate the application of the abstract results.

1. Introduction

In the literature, several generalized functions were presented to study almost periodic functions, such as almost automorphic functions, asymptotically almost periodic functions, asymptotically almost automorphic functions, pseudo-almost periodic functions, pseudo-almost automorphic functions and weighted pseudo-almost automorphic functions. The properties of these functions have been extensively studied, see the monographs of Corduneanu [12], N'Guérékata [24], the works [2,3, 5-14, 16-20, 23, 24, 27, 30,33] and references therein.

The almost automorphic functions play a significant role in characterizing recurrence, randomness, and complexity of dynamical systems which have been seen as an important generalization of the classical almost periodic functions. Since they have been introduced by Bochner [5], they have been considerably investigated and undergone some interesting, natural and powerful generalizations.

Besides, noise or stochastic perturbation is unavoidable and omnipresent in nature as well as in man-made systems. Therefore, it is of great significance to consider stochastic effects into the investigation of fractional differential systems. Most of the current studies on almost automorphic solutions for stochastic differential equations are concerned with equations perturbed by Brownian motion. We refer the reader to [9,25-27] for more details.

However, many real models involve jump perturbations, or more general Lévy noise. Wang and Liu [29] first introduced the concept of Poisson square-mean almost periodicity and studied the existence, uniqueness and stability of square-mean almost period solutions for stochastic evolution equations driven by Lévy noise. Liu and Sun [22] introduced the concepts of Poisson square-mean almost automorphy and almost automorphy in distribution, established the existence of solutions which are almost automorphic in distribution for some semilinear stochastic differential equations with infinite dimensional Lévy noise.

2010 *Mathematics Subject Classification*. Primary 34A08, 34K14; Secondary 60H05, 60G51
Keywords. Fractional stochastic differential equations; (Weighted pseudo) almost automorphic solutions in distribution; Sectorial operator; Lévy noise.
Received: 15 July 2020; Revised: 01 October 2020; Accepted: 17 December 2020
Communicated by Miljana jovanović
Email address: yangm58@mail2.sysu.edu.cn (Min Yang)

Li [21] established the existence and uniqueness of weighted pseudo almost automorphic solutions for the following nonautonomous stochastic partial differential equations driven by Lévy noise.

$$\begin{aligned} dY(t) &= A(t)Y(t)dt + f(t, Y(t))dt + g(t, Y(t))dW(t) \\ &+ \int_{|x|_V<1} F(t, Y(t-), x)\tilde{N}(dt, dx) + \int_{|x|_V\geq 1} G(t, Y(t-), x)N(dt, dx), \end{aligned} \tag{1.1}$$

where $A(t)$ satisfies the Acquistapace-Terreni condition [1]. $f : \mathbb{R} \times \mathcal{L}^2(P, H) \to \mathcal{L}^2(P, H), g : \mathbb{R} \times \mathcal{L}^2(P, H) \to L(V, \mathcal{L}^2(P, H)), F : \mathbb{R} \times \mathcal{L}^2(P, H) \times V \to \mathcal{L}^2(P, H), G : \mathbb{R} \times \mathcal{L}^2(P, H) \times V \to \mathcal{L}^2(P, H)$ are stochastic processes, W, $\tilde{N}$ and N are the Lévy-Itô decomposition components of the two-sided Lévy process.

Besides, fractional calculus has gained much attention due to their extensive applications in the fields such as physics, fluid mechanics viscoelasticity, heat conduction in materials with memory, chemistry and engineering. In recent years, notable contributions have been made in theory and applications of fractional differential equations, one can refer to [34,35] and the references therein.

However, up to now, most of the studies on (weighted pseudo) almost automorphic solutions for stochastic differential equations are concerned with integer-order differential equations perturbed by Brownian motion, (weighted pseudo) almost automorphic solutions for fractional stochastic differential equations driven by Lévy noise have still rarely been treated in the literature. Motivated by these facts, in this paper, by using sectorial operator, we mainly study the existence and uniqueness of (weighted pseudo) almost automorphic solutions in distribution to the system

$$\begin{cases} dY(t) = \displaystyle\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} AY(s)dsdt + f(t, Y_t)dt + g(t, Y_t)d\omega(t) \\ + \displaystyle\int_{|u|_U<1} F(t, Y(t-), u)\tilde{N}(dt, du) + \int_{|u|_U\geq 1} G(t, Y(t-), u)N(dt, du), \\ Y_0 = \varphi \in \mathcal{B}, \end{cases} \tag{1.2}$$

where $1 < \alpha < 2$, $A : D(A) \subseteq H \to H$ is a linear densely defined operator of sectorial type on a complex Banach space H, for $\theta \in (-\infty, 0]$, the history $Y_t : (-\infty, 0] \to \mathcal{B}$ defined by $Y_t(\theta) = Y(t+\theta)$ belongs to the phase space $\mathcal{B}$ defined axiomatically in Section 2; f, g, F, G are functions satisfying some additional conditions to be specified later. The convolution integral in (1.2) is understood in the Riemann–Liouville sense.

We organize this paper as follows. In the next section, we recall several basic definitions and useful lemmas. In Section 3, by using sectorial operator, we obtain very general results on the existence and uniqueness of (weighted pseudo) almost automorphic solutions in distribution for the semilinear problem (1.2) under Lipschitz hypothesis on the nonlinearity. Finally, in Section 4, we apply our theory to an example which enables us to a better understanding of the work and hence attract the attention of researchers who are entering the subject.

2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be the complete probability space and $(H, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space. The notation $\mathcal{L}^2(P, H)$ stands for the space of all $H-$valued random variables x such that $E\|x\|^2 = \int_\Omega \|x(\omega)\|^2 dP < \infty$. For $x \in \mathcal{L}^2(P, H)$, let $\|x\|_2 = (\int_\Omega \|x(\omega)\|^2 dP)^{\frac{1}{2}}$. It is clear that $\mathcal{L}^2(P, H)$ is a Hilbert space equipped with the norm $\|\cdot\|_2$. Suppose that $\{\hat{q}(t) : t \in J\}$ be the Poisson point process, taking its values in a measurable space $(V, B(V))$ with a σ finite intensity measure $\nu(dx)$. Let $N(ds, dx)$ be the Poisson counting measure, which is induced by $\hat{q}(\cdot)$ and the compensating martingale measure denoted by $\hat{N}(ds, dx) = N(ds, dx) - \nu(dx)ds$. We assume that the filtration is generated by the Poisson point process $\hat{q}(t)$ and is augmented; that is, $\mathcal{F}_t = \sigma\{N((0, s], s \leq t, A \in B(Z)\} \vee \mathcal{N}, t \in J$, where N is the class of $P-$null sets.

A closed linear operator A is said to be sectorial of type μ if there exist $0 < \theta < \frac{\pi}{2}, M > 0$ and $\mu \in \mathbb{R}$ such that the spectrum of A is contained in the sector $\mu + S_\theta = \{\mu + \lambda : \lambda \in \mathbb{C} \setminus \{0\}, |arg(-\lambda)| < \theta\}$, and $\|\lambda I - A\|^{-1} \leq \dfrac{M}{|\lambda - \mu|}$ for all $\lambda \notin \mu + S_\theta$. (see [15])

In order to give an operator theoretical approach to system (1.2) we recall the following definition.

Definition 2.1 *([15,31].) Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X. We call A the generator of a solution operator if there exist $\mu \in \mathbb{R}$ and a strongly continuous function $T_\alpha : \mathbb{R}^+ \to \mathcal{B}(X)$ such that $\{\lambda^\alpha : Re\lambda > \mu\} \subseteq \rho(A)$ and $\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{-\lambda t}T_\alpha(t)xdt$ for all $Re\lambda > \mu,\ x \in X$. In this case, $T_\alpha(t)$ is called the solution operator generated by A which satisfies $T_\alpha(0) = I$. We observe that the power function λ^α is uniquely defined as $\lambda^\alpha = |\lambda^\alpha| e^{\alpha i \arg(\lambda)}$ with $-\pi < \arg(\lambda) < \pi$.*

We note that if A is a sectorial operator of type μ with $0 < \theta \le \pi(1 - \frac{\alpha}{2})$, then A is the generator of a solution operator given by $T_\alpha(t) := \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}d\lambda, t > 0$, where γ is a suitable path lying outside the sector $\mu + \Sigma_\theta$ (cf. [13]). In 2007, Cuesta [13, Theorem 1] proved that if A is a sectorial operator of type $\mu < 0$, for $M > 0$ and $0 < \theta \le \pi(1 - \frac{\alpha}{2})$, then there exists $C > 0$ such that

$$\|T_\alpha(t)\| \le \frac{CM}{1 + |\mu| t^\alpha},\ \ t \ge 0. \tag{2.1}$$

In this paper, the phase space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will denote a seminormed linear space of $\mathcal{F}_0$-measurable functions mapping from $(-\infty, 0]$ into H which satisfies the following axioms:

(1) If $x : (-\infty, b] \to H,\ b \ge 0$, is such that $x\mid_{[0,b]} \in \mathcal{B}$, then, for every $t \in [0, b]$, the following conditions hold:

(a) $x_t \in \mathcal{B}$;

(b) $\|x(t)\| \le \mathcal{N}_1\|x_t\|_{\mathcal{B}}$;

(c) $\|x_t\|_{\mathcal{B}} \le \mathcal{N}_2(t)\sup\{\|x(s)\| : 0 \le s \le t\} + \mathcal{N}_3(t)\|x_\eta\|_{\mathcal{B}}$;

where $\mathcal{N}_1$ is a constant, $\mathcal{N}_2, \mathcal{N}_3 : [0,\infty) \to [1,\infty)$, $\mathcal{N}_2$ is continuous, $\mathcal{N}_3$ is locally bounded, $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ are independent of $x(\cdot)$.

(2) For the function $x(\cdot)$ in (1) the function $t \to x_t$ is continuous from $[0, b]$ into $\mathcal{B}$.

(3) The space $\mathcal{B}$ is complete.

Let $J_1 := (-\infty, b]$ and $C(J_1, \mathcal{L}^2(P, H))$ be the Banach space of all continuous maps from J_1 into $\mathcal{L}^2(P, H)$ satisfying the condition $\sup_{t\in J_1} E\|x_t\|^2 < \infty$. Let C be the closed subspace of all continuous process x that belongs to the space $C(J_1, \mathcal{L}^2(P, H))$ consisting of $\mathcal{F}_t-$adapted measurable processes such that $\mathcal{F}_0-$adapted processes $\phi \in \mathcal{B}$. Let $\|\cdot\|_C$ be a seminorm in C defined by

$$E\|x\|_C^2 = E(\sup_{t\in J_1}\|x_t\|_{\mathcal{B}}^2),$$

where

$$E\|x\|_{\mathcal{B}}^2 \le \tilde{\mathcal{N}}_2 E\|\phi\|_{\mathcal{B}}^2 + \tilde{\mathcal{N}}_3 \sup\{E\|x(s)\|^2 : 0 \le s \le b\},$$

$\tilde{\mathcal{N}}_2 = \sup_{t\in J_1}\{\mathcal{N}_2(t)\}$ and $\tilde{\mathcal{N}}_3 = \sup_{t\in J_1}\{\mathcal{N}_3(t)\}$. It is easy to verify that C, equipped with the norm topology as defined earlier, is a Banach space.

Remark 2.1 *In the rest of this paper, we always suppose that A is a sectorial of type $\mu < 0$ with angle θ satisfying $0 < \theta \le \pi(1 - \frac{\pi}{2})$, M and C are the constants introduced above.*

We now recall some notations and properties related to weighted pseudo almost automorphic functions.

Let $(H, \|\cdot\|)$ and $(V, |\cdot|)$ be real separable Hilbert spaces. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with given filtration $(\mathcal{F}_t)_{t\ge0}$ satisfying the usual conditions, i.e., the filtration is right continuous and $\mathcal{F}_0$ contains all $P-$null sets. Denoted by $L(V, H)$ the family of all linear bounded operators from V into H. $\mathcal{L}^2(P, H)$ stands for the space of all $H-$valued random variable Y such that $E\|Y\|^2 = \int_\Omega \|Y\|^2 dP < \infty$, which is a Banach space equipped with the norm $\|Y\|_2 = (E\|Y\|^2)^{\frac{1}{2}}$. We consider a Lévy process with values in V.

2.1. Lévy process.

Assume that $L = (L(t), t \ge 0)$ is a $V-$valued Lévy process, then L is stochastically continuous, has independent and stationary increments, and satisfies $L(0) = 0$ almost surely. We define the process of jumps of L by $\Delta L(t) = L(t) - L(t-)$ for each $t \geq 0$. Since every Lévy process is càdlàg, then there exists $\Omega_0 \in F$ with $P(\Omega_0) = 1$ such that $t \to L(t)(\omega)$ is càdlàg for all $\omega \in \Omega_0$. Define the counting Poisson random measure N on $V \setminus 0$ by

$$N(t,E)(\omega) = \sharp\{0 \leq s \leq t : \Delta L(s)(\omega) \in E\} = \sum_{0\leq s\leq t} \chi_E(\Delta L(s)(\omega)) < \infty, t \geq 0,$$

where $\omega \in \Omega_0, E \in \mathcal{B}(V \setminus \{0\})$ with $0 \notin \bar{E}$, the closure of E in V. $\sharp$ means the counting, $\mathcal{B}(V \setminus \{0\}$ is the Borel σ-field on $V \setminus \{0\}$ and χ_E is the characteristic function. If $\omega \in \Omega_0^c$, then $N(t,E)(\omega) = 0$. Note that for each $\omega \in \Omega_0, t \geq 0$, the set function $E \to N(t,E)(\omega)$ is a counting measure and $E(N(t,E)) = \int N(t,E)(\omega)dP(\omega)$ is a Borel measure. We write $\nu(\cdot) = E(N(1,\cdot))$ and call it the intensity measure associated with L. The compensated Poisson random measure $\tilde{N}$ is defined by

$$\tilde{N}(t,dx) = N(t,dx) - t\nu(dx).$$

Proposition 2.1 *([4, 21, 22].) If L is a V -valued Lévy process, then there exist $a \in V$, a $V-$valued Wiener process W with convariance operator Q, and an independent Poisson random measure on $\mathbb{R}^+ \times (V \setminus \{0\})$ such that for each $t \geq 0$,*

$$L(t) = at + W(t) + \int_{\|x\|_V<1} x\tilde{N}(t,dx) + \int_{\|x\|_V\geq 1} xN(t,dx), \tag{2.2}$$

where the Poisson random measure N has the intensity measure ν which satisfies

$$\int (|x|_V^2 \bigwedge 1)\nu(dx) < \infty.$$

Let $L_1(t)$ and $L_2(t), t \geq 0$, be two independent, identically distributed Lévy processes. Set

$$L(t) := \begin{cases} L_1(t), & t \geq 0, \\ -L_2((-t)-), & t < 0. \end{cases} \tag{2.3}$$

Then L is a two-sided Lévy process defined on the filtered probability space $(\Omega, F, P, (\mathcal{F}_t)_{t\in\mathbb{R}})$. We assume that Q is a positive, self-adjoint and trace class operator on V.

Remark 2.2 *It follows from (2.2) that $\int_{|x|_V\geq 1} \nu(dx) < \infty$. For convenience, we denote*

$$c := \int_{|x|_V\geq 1} \nu(dx).$$

2.2. Almost automorphic process and weighted pseudo almost automorphic process.

In this subsection, we present some preliminaries which are used throughout this paper.

Definition 2.2 *([26].) A stochastic process $Y : \mathbb{R} \to \mathcal{L}^2(P,H)$ is said to be $\mathcal{L}^2-$bounded if there exists a constant $M > 0$ such that $E\|Y(t)\|^2 = \int_\Omega \|Y(t)\|^2 dP \leq M$.*

Definition 2.3 *([26].) A stochastic process $Y : \mathbb{R} \to \mathcal{L}^2(P,H)$ is said to be $\mathcal{L}^2$-continuous if for any $s \in \mathbb{R}$,*

$$\lim_{t\to s} E\|Y(t) - Y(s)\|^2 = 0.$$

Denote by $SBC(\mathbb{R}, \mathcal{L}^2(P,H))$ the collection of all the $\mathcal{L}^2-$bounded and $\mathcal{L}^2-$ continuous processes.

Remark 2.3 *([21,26].) $SBC(\mathbb{R}, \mathcal{L}^2(P,H))$ is a Banach space equipped with the norm $\|Y\|_\infty = \sup_{t\in\mathbb{R}}(E\|Y(t)\|^2)^{\frac{1}{2}}$.*

Let $\mathcal{U}$ be the set of all functions which are positive and locally integrable over $\mathbb{R}$. For given $r > 0$ and $\rho \in \mathcal{U}$, define $m(r,\rho) = \int_{-r}^{r} \rho(t)dt$ and $\mathcal{U}_\infty = \{\rho \in \mathcal{U} | \lim_{r\to\infty} m(r,\rho) = +\infty\}$.

Definition 2.4 *([27].) For $\rho \in \mathcal{U}_\infty$, define a class of stochastic processes*

$$SBC_0(\mathbb{R}, \rho) = \{Y \in SBC(\mathbb{R}, \mathcal{L}^2(P,H)) | \lim_{r\to+\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} E\|Y(t)\|^2 \rho(t)dt = 0\}.$$

Remark 2.4 *([21].) $SBC_0(\mathbb{R}, \rho)$ is a linear closed subspace of $SBC(\mathbb{R}, \mathcal{L}^2(P, H))$.*
Remark 2.5 *([21].) $SBC_0(\mathbb{R}, \rho)$ equipped with the norm $\|Y\|_\infty$ is a Banach space.*
Definition 2.5 *([21].) An $\mathcal{L}^2$-continuous stochastic process $Y : \mathbb{R} \to \mathcal{L}^2(P, H)$ is said to be square-mean almost automorphic if every sequence of real numbers $\{s_n'\}$ has a subsequence $\{s_n\}$ such that for some stochastic process $\tilde{Y} : \mathbb{R} \to \mathcal{L}^2(P, H)$, $\lim_{n\to\infty} E\|Y(t + s_n) - \tilde{Y}(t)\|^2 = 0$ and $\lim_{n\to\infty} E\|Y(t - s_n) - \tilde{Y}(t)\|^2 = 0$ hold for each $t \in \mathbb{R}$.*

The collection of all square-mean almost automorphic process $Y : \mathbb{R} \to \mathcal{L}^2(P, H)$ is denoted by $SAA(\mathbb{R}, \mathcal{L}^2(P, H))$. It is a Banach space with the norm $\|Y\|_\infty$.
Remark 2.6 *([29].) If $Y \in SAA(\mathbb{R}, \mathcal{L}^2(P, H))$, then Y is bounded, that is, $\|Y\|_\infty < \infty$. Similarly, any square-mean almost automorphic function $g : \mathbb{R} \to L(V, \mathcal{L}^2(P, H))$ is bounded, i.e., $\sup_{s\in\mathbb{R}} \|g(s)\|_{L(V,\mathcal{L}^2(P,H))} < \infty$.*
Proposition 2.11 ([29].) Let $f : \mathbb{R}\times\mathcal{L}^2(P, H) \to \mathcal{L}^2(P, H), (t, Y) \mapsto f(t, Y)$ be square-mean almost automorphic in $t \in \mathbb{R}$ for each $Y \in \mathcal{L}^2(P, H)$, and assume that f satisfies the Lipschitz condition in the following sense:

$$E\|f(t, Y) - f(t, Z)\|^2 \leq L\|Y - Z\|^2$$

for all $Y, Z \in \mathcal{L}^2(P, H)$, and for each $t \in \mathbb{R}$, where $L > 0$ is independent of t. Then for any almost automorphic process $Y : \mathbb{R} \to \mathcal{L}^2(P, H)$, the stochastic process $F : \mathbb{R} \to \mathcal{L}^2(P, H)$ given by $F(t) := f(t, Y(t))$ is square-mean almost automorphic.
Definition 2.6 *([26].) An $\mathcal{L}^2-$continuous stochastic process $f : \mathbb{R} \to \mathcal{L}^2(P, H)$, is said to be square-mean weighted pseudo almost automorphic with respect to ρ if it can be decomposed as $f = g + \varphi$, where $g \in SAA(\mathbb{R}, \mathcal{L}^2(P, H))$ and $\varphi \in SBC(\mathbb{R}, \rho)$.*

The collection of all square-mean weighted pseudo almost automorphic processes with respect to ρ is denoted by $SWPAA(\mathbb{R}, \rho)$.
Definition 2.7 *([33].)A set D is said to be translation invariant if for any $f(t) \in D$, $f(t + \tau) \in D$ for any $\tau \in \mathbb{R}$.*

Denote

$$\mathcal{U}^{inv} = \{\rho \in \mathcal{U}_\infty | SBC_0(R, \rho) \text{ is translation invariant}\}.$$

Lemma 2.1 *([26].) For $\rho \in \mathcal{U}^{inv}$, $SWPAA(\mathbb{R}, \rho)$ equipped with the norm $\|Y\|_\infty$ is a Banach space. Denote*

$$SAA(\mathbb{R} \times \mathcal{L}^2(P, H), \mathcal{L}^2(P, H)) = \{g(t, Y) \in SAA(\mathbb{R}, \mathcal{L}^2(P, H)) | \text{ for any } Y \in \mathcal{L}^2(P, H)\}$$

and

$$SBC_0(\mathbb{R} \times \mathcal{L}^2(P, H), \rho) = \{\varphi(t, Y) \in SBC_0(\mathbb{R}, \rho) | \text{ for any } Y \in \mathcal{L}^2(P, H)\}.$$

Definition 2.8 *([26].) An $\mathcal{L}^2$-continuous stochastic process $f : \mathbb{R} \times \mathcal{L}^2(P, H) \to \mathcal{L}^2(P, H)$ is said to be square-mean weighted pseudo almost automorphic with respect to $\rho \in \mathcal{U}_\infty$ for any $Y \in \mathcal{L}^2(P, H)$ in t if it can be decomposed as $f = g + \varphi$, where $g \in SAA(\mathbb{R} \times \mathcal{L}^2(P, H), \mathcal{L}^2(P, H))$ and $\varphi \in SBC_0(\mathbb{R} \times \mathcal{L}^2(P, H), \rho)$. We denote all such stochastic process by $SWPAA(\mathbb{R} \times \mathcal{L}^2(P, H), \rho)$.*
Lemma 2.2 *([26].) Suppose $\rho \in \mathcal{U}^{inv}$, $f(t, x) \in SWPAA(\mathbb{R} \times \mathcal{L}^2(P, H), \rho)$, and there exists a constant $L > 0$ such that for any $x, y \in \mathcal{L}^2(P, H)$,*

$$E\|f(t, x) - f(t, y)\|^2 \leq LE\|x - y\|^2.$$

Then, for any $x \in SWPAA(\mathbb{R}, \rho)$, we have $f(t, x) \in SWPAA(\mathbb{R}, \rho)$.
Definition 2.9 *([19].) A continuous stochastic process $f : \mathbb{R}\times\mathcal{L}^2(P, H), \rho \to L(V, \mathcal{L}^2(P, H))$ is said to be square-mean almost automorphic in $t \in \mathbb{R}$ for all $Y \in \mathcal{L}^2(P, H)$ if for every sequence of real numbers $\{s_n'\}$, there exists a subsequence $\{s_n\}$ such that for some function $\tilde{f} : \mathbb{R} \times \mathcal{L}^2(P, H) \to L(V, \mathcal{L}^2(P, H))$*

$$\lim_{n\to\infty} E\|f(t + s_n, Y) - \tilde{f}(t, Y)\|^2_{L(V,\mathcal{L}^2(P,H))} = 0,$$

and

$$\lim_{n\to\infty} E\|\tilde{f}(t - s_n, Y) - f(t, Y)\|^2_{L(V,\mathcal{L}^2(P,H))} = 0$$

for all $Y \in \mathcal{L}^2(P, H)$ *and each* $t \in \mathbb{R}$.

The collection of all square-mean almost automorphic stochastic process $f : \mathbb{R}\times\mathcal{L}^2(P, H) \to L(V, \mathcal{L}^2(P, H))$ is denoted by $SAA(\mathbb{R} \times \mathcal{L}^2(P, H), L(V, \mathcal{L}^2(P, H)))$.

Definition 2.10 *([24].) A stochastic process* $J(t, x) : \mathbb{R} \times V \to \mathcal{L}^2(P, H)$ *is said to be Poisson stochastically bounded if there exists a constant* $M > 0$ *such that*

$$\int_V E\|J(t, x)\|^2 \nu(dx) \le M,$$

for all $t \in \mathbb{R}$.

We denote the collection of all Poisson stochastically bounded processes by $PSB(\mathbb{R} \times V, \mathcal{L}^2(P, H))$.

Definition 2.11 *([21].) A stochastic process* $J(t, x) : \mathbb{R}\times V \to \mathcal{L}^2(P, H)$ *is said to be Poisson stochastically continuous if*

$$\lim_{t\to s} \int_V E\|J(t, x) - J(s, x)\|^2 \nu(dx) = 0.$$

The collection of all Poisson stochastically bounded and continuous processes is denoted by $PSBC(\mathbb{R} \times V, \mathcal{L}^2(P, H))$.

Definition 2.12 *([21].) Let* $F : \mathbb{R}\times\mathcal{L}^2(P, H)\times V \to \mathcal{L}^2(P, H), (t, Y, x) \mapsto F(t, Y, x)$, *F is said to be Poisson stochastically continuous, if*

$$\int_V E\|F(t, Y, x) - F(t', Y', x)\|^2 \nu(dx) \to 0 \ \text{as} \ (t, Y) \to (t', Y').$$

By $PSC(\mathbb{R}\times\mathcal{L}^2(P, H)\times V \to \mathcal{L}^2(P, H))$, resp. $PSBC(\mathbb{R}\times\mathcal{L}^2(P, H)\times V \to \mathcal{L}^2(P, H))$, we denote the collection of all Poisson stochastically continuous processes, resp. Poisson stochastically bounded and continuous processes.

Definition 2.13 *([21].) A stochastic process* $J(t, x) \in PSBC_0(\mathbb{R} \times V, \mathcal{L}^2(P, H))$, *provided that* $J(t, x) \in PSBC(\mathbb{R} \times V, \mathcal{L}^2(P, H))$ *and* $\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \int_V E\|J(t, x)\|^2 \nu(dx)dt = 0$.

Definition 2.14 *([21].) For* $\rho \in \mathcal{U}_\infty$, *define a class of stochastic processes* $PSBC_0(\mathbb{R} \times V, \rho) = \{J(t, x) \in PSBC(\mathbb{R} \times V, \mathcal{L}^2(P, H)) | \lim_{r\to+\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \int_V E\|J(t, x)\|^2 \rho(t)\nu(dx)dt = 0\}$.

Denote

$$\mathcal{U}_p^{inv} = \{\rho \in \mathcal{U}_\infty | PSBC_0(\mathbb{R} \times V, \rho) \ \text{ is translation invariant }\}.$$

Remark 2.7 *([21].) If* $\rho \equiv 1$, *it is obvious that* $PSBC_0(\mathbb{R} \times V, \rho)$ *reduces to* $PSBC_0(\mathbb{R} \times V, \mathcal{L}^2(P, H))$.

Definition 2.15 *([29].) A stochastic process* $F : \mathbb{R}\times\mathcal{L}^2(P, H)\times V \to \mathcal{L}^2(P, H), (t, Y, x) \to F(t, Y, x)$ *is said to be Poisson square-mean almost automorphic in* $t \in \mathbb{R}$ *for each* $Y \in \mathcal{L}^2(P, H)$ *if F is Poisson stochastically continuous and for every sequence of real numbers* s_n', *there exists a subsequence* s_n *such that for some function* $\tilde{F} : \mathbb{R}\times\mathcal{L}^2(P, H)\times V \to \mathcal{L}^2(P, H)$ *with* $\int_V E\|\tilde{F}(t, Y, x)\|^2 \nu(dx) < \infty$ *is Poisson stochastically continuous such that*

$$\lim_{n\to\infty} \int_V E\|F(t + s_n, Y, x) - \tilde{F}(t, Y, x)\|^2 \nu(dx) = 0,$$

and

$$\lim_{n\to\infty} \int_V E\|\tilde{F}(t - s_n, Y, x) - F(t, Y, x)\|^2 \nu(dx) = 0,$$

for all $Y \in \mathcal{L}^2(P, H)$ *and each* $t \in \mathbb{R}$.

We denoted by $PSAA(\mathbb{R} \times \mathcal{L}^2(P, H) \times V \to \mathcal{L}^2(P, H))$ the collection of all Poisson square-mean almost automorphic stochastic processes $F : \mathbb{R} \times \mathcal{L}^2(P, H) \times V \to \mathcal{L}^2(P, H)$.

In the following lemma, we give some properties of Poisson square-mean almost automorphic stochastic processes.

Lemma 2.3 *([29].) If $F, F_1, F_2 : \mathbb{R} \times \mathcal{L}^2(P,H) \times V \to \mathcal{L}^2(P,H)$ are all Poisson square-mean almost automorphic stochastic processes in t for each $Y \in \mathcal{L}^2(P,H)$, then*

(1) $F_1 + F_2$ is Poisson square-mean almost automorphic.

(2) λF is Poisson square-mean almost automorphic for every scalar λ.

(3) For every $Y \in \mathcal{L}^2(P,H)$, there exists a constant $M > 0$ such that

$$\sup_{t\in\mathbb{R}} \int_V E\|F(t,Y,x)\|^2 \nu(dx) < M.$$

Lemma 2.4 *([29].) Let $F : \mathbb{R} \times \mathcal{L}^2(P,H) \times V \to \mathcal{L}^2(P,H), (t,Y,x) \to F(t,Y,x)$ be Poisson square-mean almost automorphic in $t \in \mathbb{R}$ for each $Y \in \mathcal{L}^2(P,H)$, and assume that F satisfies the Lipschitz condition in the following sense:*

$$\int_V E\|F(t,Y,x) - F(t,Z,x)\|^2 \nu(dx) < LE\|Y - Z\|^2,$$

for all $Y, Z \in \mathcal{L}^2(P,H)$ and each $t \in \mathbb{R}$, where $L > 0$ is independent of t. Then for any square-mean almost automorphic process $Y : \mathbb{R} \to \mathcal{L}^2(P,H)$, the stochastic process $J : \mathbb{R} \times V \to \mathcal{L}^2(P,H)$ given by $J(t,x) := F(t,Y(t),x)$ is Poisson square-mean almost automorphic.

Definition 2.16 *([29].) A stochastic process $J : \mathbb{R} \times V \to \mathcal{L}^2(P,H)$ is said to be Poisson square-mean pseudo almost automorphic in $t \in \mathbb{R}$ for each $Y \in \mathcal{L}^2(P,H)$ if J is Poisson stochastically continuous and it can be decomposed as $J = g + \phi$, where $g \in PSAA(\mathbb{R} \times V, \mathcal{L}^2(P,H))$ and $\phi \in PSBC_0(\mathbb{R} \times V, \mathcal{L}^2(P,H))$.*

The collection of all such stochastic processes is denoted by $PSPAA(\mathbb{R} \times V, \mathcal{L}^2(P,H))$.

Definition 2.17 *([21].) A stochastic process $J : \mathbb{R} \times V \to \mathcal{L}^2(P,H)$ is said to be Poisson square-mean weighted pseudo almost automorphic about $\rho \in U_\infty$ in $t \in \mathbb{R}$ if F is Poisson stochastically continuous and it can be decomposed as $F = g + \phi$, where $g \in PSAA(\mathbb{R} \times V, \mathcal{L}^2(P,H))$ and $\phi \in PSBC_0(\mathbb{R} \times V, \rho)$.*

The collection of all such stochastic processes is denoted by $PSWPAA(\mathbb{R} \times V, \rho)$.

Set

$$PSAA(\mathbb{R} \times \mathcal{L}^2(P,H) \times V, \rho) = \{F(t,Y,x) \in PSAA(\mathbb{R} \times V, \rho) | \text{ for any } Y \in \mathcal{L}^2(P,H)\},$$

and

$$PSBC_0(\mathbb{R} \times \mathcal{L}^2(P,H) \times V, \rho) = \{F(t,Y,x) \in PSBC_0(\mathbb{R} \times V, \rho) | \text{ for any } Y \in \mathcal{L}^2(P,H)\}.$$

Lemma 2.5 *([21].) Assume $J \in PSBC(\mathbb{R} \times V, \mathcal{L}^2(P,H))$. Then $J \in PSBC_0(\mathbb{R} \times V, \rho)$, where $\rho \in U_\infty$ if and only if for any $\varepsilon > 0$,*

$$\lim_{r\to+\infty} \frac{1}{m(r,\rho)} \int_{M_{r,\varepsilon(\varphi)}} \rho(t)dt = 0,$$

where

$$M_{r,\varepsilon(\varphi)} = \{t \in [-r,r] | \int_V \|E(J,x)\|^2 \nu(dx) \geq \varepsilon\}.$$

Theorem 2.1 *([21].) If $\rho \in U_\infty$, $F = g + \varphi \in PSWPAA(\mathbb{R} \times \mathcal{L}^2(P,H) \times V, \rho)$ with $g \in PSAA(\mathbb{R} \times \mathcal{L}^2(P,H) \times V, \mathcal{L}^2(P,H))$ and $\varphi \in PSBC_0(\mathbb{R} \times \mathcal{L}^2(P,H) \times V, \rho)$. Assume that F and g are Lipschitzian in Y uniformly in $t \in \mathbb{R}$, that is for all $Y, Z \in \mathcal{L}^2(P,H)$ and $t \in \mathbb{R}$,*

$$\int_V E\|F(t,Y,x) - F(t,Z,x)\|^2 \nu(dx) \leq LE\|Y - Z\|^2,$$

$$\int_V E\|g(t,Y,x) - g(t,Z,x)\|^2 \nu(dx) \leq LE\|Y - Z\|^2,$$

for some constant $L > 0$ independent of t. Then for any $Y \in SWPAA(\mathbb{R}, \rho)$, the stochastic process $J : \mathbb{R} \times V \to \mathcal{L}^2(P,H)$ given by $J(t,x) := F(t,Y(t),x)$ is Poisson square-mean weighted pseudo almost automorphic.

The collection of all square-mean bi-almost automorphic processes is denoted by $SBAA(\mathbb{R}\times\mathbb{R},\mathcal{L}^2(P,H))$. Define by $\mathcal{P}(H)$ the space of all Borel probability measures on H with the β metric (see [21], [22, 9.3]): $\beta(\mu,\eta) := \sup\{|\int f d\mu - \int f d\eta| : \|f\|_{BL} \le 1\}, \mu,\eta,\in \mathcal{P}(H)$, where f are Lipschitz continuous real-valued functions on H with

$$\|f\|_{BL} = \|f\|_L + \|f\|_H, \|f\|_L = \sup_{x\neq y}\frac{|f(x)-f(y)|}{\|x-y\|}, \|f\|_H = \sup_{x\in H}|f(x)|.$$

Definition 2.18 *([21,22].) An H-valued stochastic process $Y(t)$ is said to be almost automprphic in distribution if its law $\mu(t)$ is a $\mathcal{P}(H)-$valued almost automorphic mapping, i.e. for every sequence of real number s_n', there exist a subsequence s_n and a $\mathcal{P}(H)-$valued mapping $\tilde{\mu}(t)$ such that $\lim_{n\to\infty}\beta(\mu(t+s_n),\tilde{\mu}(t)) = 0$ and $\beta(\mu(t+s_n),\tilde{\mu}(t)) = 0$ hold for each $t\in\mathbb{R}$.*

Definition 2.19 *([21,22].) An $H-$valued stochastic process $Y(t)$ is said to be weighted pseudo almost automorphic in distribution with respect to $\rho\in\mathcal{U}_\infty$, provided that it can be decomposed as $Y = \phi+\psi$, where ϕ is almost automorphic in distribution and $\psi\in SBC_0(\mathbb{R},\rho)$.*

Lemma 2.6 *([31].) Let $g\in SAA(J,\mathcal{L}^2(P,H))$. If Γ is the function defined by $\Gamma g(t) = \int_0^t T_\alpha(t-s)g(s)ds$, for every $t\in J$, then $\Gamma g(t)\in SAA(J,\mathcal{L}^2(P,H))$.*

Lemma 2.7 *([26].) Let $u\in WPAA(\mathbb{R},\rho)$ and assume that $\mathcal{B}$ is a uniform fading memory space. Then the function $t\to u_t$ belongs to $WPAA(\mathbb{R},\rho)$.*

If $Y(t)$ is $\mathcal{L}^2-$bounded, it is easy to see that the stochastic process

$$\begin{aligned}Y(t) &= T_\alpha(t)\varphi(0) + \int_0^t T_\alpha(t-s)f(s,Y_s)ds + \int_0^t T_\alpha(t-s)g(s,Y_s)d\omega(s)\\ &+\int_0^t\int_{|x|_V<1}T_\alpha(t-s)F(s,Y(s-),x)\tilde{N}(ds,dx) + \int_0^t\int_{|x|_V\ge 1}T_\alpha(t-s)G(s,Y(s-),x)N(ds,dx)\end{aligned} \tag{2.4}$$

satisfies the equation (1.2), thus it is a mild solution of (1.2).

Let

$$\begin{aligned}\hat{Y}(t) &= \int_{-\infty}^t T_\alpha(t-s)f(s,Y_s)ds + \int_{-\infty}^t T_\alpha(t-s)g(s,Y_s)d\omega(s)\\ &+\int_{-\infty}^t\int_{|x|_V<1}T_\alpha(t-s)F(s,Y(s-),x)\tilde{N}(ds,dx) + \int_{-\infty}^t\int_{|x|_V\ge 1}T_\alpha(t-s)G(s,Y(s-),x)N(ds,dx).\end{aligned} \tag{2.5}$$

Now one can easily get

$$\begin{aligned}Y(t)-\hat{Y}(t) &= T_\alpha(t)\varphi(0) - \int_t^\infty T_\alpha(t-s)f(s,Y_s)ds - \int_t^\infty T_\alpha(t-s)g(s,Y_s)d\omega(s)\\ &-\int_t^\infty\int_{|x|_V<1}T_\alpha(t-s)F(s,Y(s-),x)N(ds,dx) - \int_t^\infty\int_{|x|_V\ge 1}T_\alpha(t-s)G(s,Y(s-),x)\tilde{N}(ds,dx).\end{aligned} \tag{2.6}$$

Thus we have $Y(t)-\hat{Y}(t)\to 0$ as $t\to\infty$.

3. Main results

This section is mainly concerned with the existence and uniqueness of (weighted pseudo) almost automorphic solutions in distribution.

Definition 3.1. An $\mathcal{F}_t$-progressively measurable process $\{\hat{Y}(t) : t\in(-\infty,b] = J_1\}$ is called a mild solution to (1.2) if $Y_0 = \varphi\in\mathcal{B}$, $\int_{-\infty}^b E\|Y(s)\|^2 ds < \infty$, a.s., and for each $t\in J_1$

$$\begin{aligned}\hat{Y}(t) &= \int_{-\infty}^t T_\alpha(t-s)f(s,Y_s)ds + \int_{-\infty}^t T_\alpha(t-s)g(s,Y_s)d\omega(s)\\ &+\int_{-\infty}^t\int_{|x|_V<1}T_\alpha(t-s)F(s,Y(s-),x)\tilde{N}(ds,dx) + \int_{-\infty}^t\int_{|x|_V\ge 1}T_\alpha(t-s)G(s,Y(s-),x)N(ds,dx).\end{aligned} \tag{3.1}$$

Before stating and proving the main results, we introduce the following hypothesis.

(H_1) The solution operator $T_\alpha(s)x \in SBAA(\mathbb{R}, \mathcal{L}^2(P,H))$ uniformly for all x in any bounded subset of $\mathcal{L}^2(P,H)$.

Our first theorem is for the existence and uniqueness of almost automorphic solutions in distribution.

Theorem 3.1. *Assume that (H_1) holds, and $f \in SAA(\mathbb{R}\times\mathcal{B}, \mathcal{L}^2(P,H)), g \in SAA(\mathbb{R}\times\mathcal{B}, L(V, \mathcal{L}^2(P,H))), F, G \in PSAA(\mathbb{R}\times\mathcal{L}^2(P,H)\times V, \mathcal{L}^2(P,H))$. Suppose that f, g, F and G satisfy the Lipschitz conditions in Y uniformly for t, that is, there exists a constant $L > 0$ such that for all $Y, Z \in \mathcal{L}^2(P,H)$ and $t \in \mathbb{R}$,*

$$E\|f(t,Y) - f(t,Z)\|^2 \le LE\|Y - Z\|^2, \tag{3.2}$$

$$E\|(g(t,Y) - g(t,Z))Q^{\frac{1}{2}}\|^2 \le LE\|Y - Z\|^2, \tag{3.3}$$

$$\int_{|x|_V<1} E\|F(t,Y,x) - F(t,Z,x)\|^2\nu(dx) \le LE\|Y - Z\|^2, \tag{3.4}$$

$$\int_{|x|_V\ge1} E\|G(t,Y,x) - G(t,Z,x)\|^2\nu(dx) \le LE\|Y - Z\|^2. \tag{3.5}$$

Then equation (1.2) has a unique $\mathcal{L}^2$-bounded almost automorphic solution in distribution.

Proof. We now shall prove the process (3.1) is the unique almost automorphic solution in distribution for (1.2) with three steps.

Step 1. $\mathcal{L}^2-$bounded solution is $\mathcal{L}^2-$continuous.

Assume $Y(t)$ is an $\mathcal{L}^2-$bounded solution of (1.2), then it satisfies (3.1), by the Cauchy-Schwarz inequality, Itô isometry and properties of the integral for the Poisson random measure, we have for $t \ge t_0$,

$$\begin{aligned}
E\|Y(t) - Y(t_0)\|^2 &\le 4E\|\int_t^{t_0} T_\alpha(t-s)f(s,Y_s)ds\|^2 \\
&+ 4E\|\int_t^{t_0} T_\alpha(t-s)g(s,Y_s)dW(s)\|^2 \\
&+ 4E\|\int_t^{t_0}\int_{|x|_V<1} T_\alpha(t-s)F(s,Y(s-),x)\tilde{N}(ds,dx)\|^2 \\
&+ 4E\|\int_t^{t_0}\int_{|x|_V\ge1} T_\alpha(t-s)G(s,Y(s-),x)\tilde{N}(ds,dx) \\
&+ \int_t^{t_0}\int_{|x|_V\ge1} T_\alpha(t-s)G(s,Y(s-),x)\nu(dx)ds\|^2 \\
&\le 4E\|\int_t^{t_0} T_\alpha(t-s)f(s,Y_s)ds\|^2 \\
&+ 4E\|\int_t^{t_0} T_\alpha(t-s)g(s,Y_s)dW(s)\|^2 \\
&+ 4E\|\int_t^{t_0}\int_{|x|_V<1} T_\alpha(t-s)F(s,Y(s-),x)\tilde{N}(ds,dx)\|^2 \\
&+ 4\Big(2E\|\int_t^{t_0}\int_{|x|_V\ge1} T_\alpha(t-s)G(s,Y(s-),x)\tilde{N}(ds,dx)\|^2 \\
&+ 2E\|\int_t^{t_0}\int_{|x|_V\ge1} T_\alpha(t-s)G(s,Y(s-),x)\nu(dx)ds\|^2\Big)
\end{aligned}$$

$$\le 4M^2\int_t^{t_0} E\|f(s,Y_s)\|^2ds + 4M^2\|\int_t^{t_0} E\|g(s,Y_s)\|^2_{L(V,\mathcal{L}^2(P,H))}ds$$
$$+4M^2E\|\int_t^{t_0}\int_{|x|_V<1} E\|F(s,Y(s-),x)\|^2\nu(dx)ds$$
$$+4M^2\Big(2\int_t^{t_0}\int_{|x|_V\ge 1} E\|G(s,Y(s-),x)\|^2\nu(dx)ds$$
$$+2\int_t^{t_0}\int_{|x|_V\ge 1} E\|G(s,Y(s-),x)\|^2\nu(dx)ds\Big).$$

Since $T_\alpha(\cdot)$ is integrable on $[0,+\infty)$, then $E\|Y(t)-Y(t_0)\|^2\to 0$ as $t\to t_0$. By a similar arguments as in Step 1 in the proof of [22, Theorem 3.2], it follows that

$$\sup_{s\in\mathbb{R}} E\|f(s,Y_s)\|^2 < M, \tag{3.6}$$
$$\sup_{s\in\mathbb{R}} E\|g(s,Y_s)\|^2_{L(V,\mathcal{L}^2(P,H))} < M, \tag{3.7}$$
$$\sup_{s\in\mathbb{R}} \int_{|x|_V<1} E\|F(s,Y,x)\|^2\nu(dx) < M, \tag{3.8}$$
$$\sup_{s\in\mathbb{R}} \int_{|x|_V\ge 1} E\|G(s,Y,x)\|^2\nu(dx) < M. \tag{3.9}$$

Thus, $E\|Y(t)-Y(t_0)\|^2\to 0$ as $t\to t_{0^+}$. Similarly, we have $E\|Y(t)-Y(t_0)\|^2\to 0$ as $t\to t_{0^-}$. Therefore, $Y(t)$ is $\mathcal{L}^2-$continuous.

Step 2. Existence and uniqueness of $\mathcal{L}^2-$bounded solutions.

Let Φ be the operator defined on $SBC(\mathbb{R},\mathcal{L}^2(P,H))$ by

$$\begin{aligned}(\Phi Y)(t) &= \int_{-\infty}^t T_\alpha(t-s)f(s,Y_s)ds + \int_{-\infty}^t T_\alpha(t-s)g(s,Y_s)d\omega(s)\\ &+\int_{-\infty}^t\int_{|x|_V<1} T_\alpha(t-s)F(s,Y(s-),x)\tilde{N}(ds,dx)\\ &+\int_{-\infty}^t\int_{|x|_V\ge 1} T_\alpha(t-s)G(s,Y(s-),x)N(ds,dx).\end{aligned} \tag{3.10}$$

If $Y(t)$ is $\mathcal{L}^2$-bounded, from (H1) and (3.6)-(3.9), we can derive that $(\Phi Y)(t)$ is $\mathcal{L}^2$- bounded.

$$E\|(\Phi Y)(t)-(\Phi Y)(t_0)\|^2 \le 4E\|\int_t^{t_0} T_\alpha(t-s)f(s,Y_s)ds\|^2$$
$$+4E\|\int_t^{t_0} T_\alpha(t-s)g(s,Y_s)d\omega(s)\|^2$$
$$+4E\|\int_t^{t_0}\int_{|x|_V<1} T_\alpha(t-s)F(s,Y(s-),x)\tilde{N}(ds,dx)\|^2$$
$$+4E\|\int_t^{t_0}\int_{|x|_V\ge 1} T_\alpha(t-s)G(s,Y(s-),x)N(ds,dx)\|^2$$
$$\le 4M^2\int_t^{t_0} E\|f(s,Y_s)\|^2ds + 4M^2E\|\int_t^{t_0} E\|g(s,Y_s)\|^2_{L(V,\mathcal{L}^2(P,H))}d\omega(s)$$
$$+4M^2E\|\int_t^{t_0}\int_{|x|_V<1} E\|F(s,Y(s-),x)\|^2\nu(dx)ds$$
$$+4M^2(2\int_t^{t_0}\int_{|x|_V<1} E\|G(s,Y(s-),x)\|^2\nu(dx)ds$$
$$+2\int_t^{t_0}\int_{|x|_V\ge 1} E\|G(s,Y(s-),x)\|^2\nu(dx)ds\|^2).$$

From the proof of Step 1, $(\Phi Y)(t)$ is an $\mathcal{L}^2-$continuous process as long as $Y(t)$ is an $\mathcal{L}^2-$bounded process. Thus Φ maps $SBC(\mathbb{R}, \mathcal{L}^2(P, H))$ into itself. We next show that Φ is a contraction operator. Similar to the Step 2 in the proof of [22, Theorem 3.2], we can obtain the operator Φ is a contraction. Therefore, Φ has a unique fixed point $Y^* \in SBC(\mathbb{R}, \mathcal{L}^2(P, H))$ which is the unique $\mathcal{L}^2-$bounded mild solution of (1.2).

Step 3. Almost automorphy of the $\mathcal{L}^2-$bounded solution.

Let $\{s_n\}$ be an arbitrary sequence of real numbers. Since $f \in SAA(\mathbb{R}, \mathcal{L}^2(P, H)), g \in SAA(\mathbb{R}, L(V, \mathcal{L}^2(P, H)))$, $F, G \in PSAA(\mathbb{R} \times V, \mathcal{L}^2(P, H))$, there exists a subsequence $\{s_n^{'}\}$ of $\{s_n\}$ and some functions $\tilde{f}, \tilde{g}, \tilde{F}, \tilde{G}$ such that

$$\lim_{n\to\infty} E\|f(t + s_n, Y) - \tilde{f}(t, Y)\|^2 = 0,$$

$$\lim_{n\to\infty} E\|\tilde{f}(t - s_n, Y) - f(t, Y)\|^2 = 0,$$

$$\lim_{n\to\infty} E\|g(t + s_n, Y, x) - \tilde{g}(t, Y, x)\|^2_{L(V, \mathcal{L}^2(P,H))} = 0,$$

$$\lim_{n\to\infty} E\|\tilde{g}(t - s_n, Y, x) - g(t, Y, x)\|^2_{L(V, \mathcal{L}^2(P,H))} = 0,$$

$$\lim_{n\to\infty} \int_{|x|_V<1} E\|F(t + s_n, Y, x) - \tilde{F}(t, Y, x)\|^2 \nu(dx) = 0,$$

$$\lim_{n\to\infty} \int_{|x|_V<1} E\|\tilde{F}(t - s_n, Y, x) - F(t, Y, x)\|^2 \nu(dx) = 0,$$

and

$$\lim_{n\to\infty} \int_{|x|_V\geq 1} E\|G(t + s_n, Y, x) - \tilde{G}(t, Y, x)\|^2 \nu(dx) = 0,$$

$$\lim_{n\to\infty} \int_{|x|_V\geq 1} E\|\tilde{G}(t - s_n, Y, x) - G(t, Y, x)\|^2 \nu(dx) = 0,$$

for each $t \in \mathbb{R}$, and $Y \in \mathcal{L}^2(P, H)$. By (H1), there exists a solution operator $T_\alpha^{'}(t - s)$ and a bounded subset B of $\mathcal{L}^2(P, H)$ such that for each $y \in B$,

$$\lim_{n\to\infty} E\|T_\alpha(t - s + s_n)y - T_\alpha^{'}(t - s)y\|^2 = 0,$$

and

$$\lim_{n\to\infty} E\|T_\alpha(t - s + s_n)y - T_\alpha^{'}(t - s)y\|^2 = 0.$$

From the estimation of solution operator $T_\alpha(\cdot)$, we get

$$E\|T_\alpha(t + s + s_n)y\|^2 \leq \frac{(CM)^2}{1 + |\mu|(t + s + s_n)^\alpha} E\|y\|^2 \tag{3.11}$$

for all $t > s$ and $y \in B$. Let $\tilde{Y}(t)$ satisfy the integral equation

$$\begin{aligned}\tilde{Y}(t) = &\int_{-\infty}^t T_\alpha^{'}(t - s)\tilde{f}(s, Y_s)ds + \int_{-\infty}^t T_\alpha^{'}(t - s)\tilde{g}(s, Y_s)d\omega(s) \\ &+ \int_{-\infty}^t \int_{|x|_V<1} T_\alpha^{'}(t - s)\tilde{F}(s, Y(s-), x)\tilde{N}(ds, dx) \\ &+ \int_{-\infty}^t \int_{|x|_V\geq 1} T_\alpha^{'}(t - s)\tilde{G}(s, Y(s-), x)N(ds, dx).\end{aligned} \tag{3.12}$$

According to (3.11) and Steps 1-2, it follows that $Y(t)$ is unique and $\mathcal{L}^2-$bounded. Let $\omega_n(\sigma) := \omega(\sigma + s_n) - \omega(s_n), N_n(\sigma, x) := N(\sigma + s_n, x) - N(s_n, x)$ and $\tilde{N}_n(\sigma, x) := \tilde{N}(\sigma + s_n, x) - \tilde{N}(s_n, x)$ for each $\sigma \in \mathbb{R}$. It is obvious that ω_n is a Q-Wiener process with the same law as ω, N_n is also a Poisson random measure and has the same law as N. In addition, $\tilde{N}_n$ is the compensated Poisson measure of N_n and has the same law as $\tilde{N}$. Let $\sigma = s - s_n$, we have

$$\begin{aligned}Y(t+s_n) = &\int_{-\infty}^{t} T_\alpha(t-\sigma)f(\sigma+s_n, Y_{\sigma+s_n})d\sigma\\&+\int_{-\infty}^{t} T_\alpha(t-\sigma)g(\sigma+s_n, Y_{\sigma+s_n})d\omega_n(\sigma)\\&+\int_{-\infty}^{t}\int_{|x|_V<1} T_\alpha(t-\sigma)F(\sigma+s_n, Y((\sigma+s_n)-), x)\tilde{N}_n(d\sigma, dx)\\&+\int_{-\infty}^{t}\int_{|x|_V\geq 1} T_\alpha(t-\sigma)G(\sigma+s_n, Y((\sigma+s_n)-), x)N_n(d\sigma, dx).\end{aligned}\tag{3.13}$$

We now consider the process

$$\begin{aligned}Y^{(n)}(t) = &\int_{-\infty}^{t} T_\alpha(t-\sigma)f(\sigma+s_n, Y^{(n)}_\sigma)d\sigma\\&+\int_{-\infty}^{t} T_\alpha(t-\sigma)g(\sigma+s_n, Y^{(n)}_\sigma)d\omega(\sigma)\\&+\int_{-\infty}^{t}\int_{|x|_V<1} T_\alpha(t-\sigma)F(\sigma+s_n, Y^{(n)}(\sigma-), x)\tilde{N}(d\sigma, dx)\\&+\int_{-\infty}^{t}\int_{|x|_V\geq 1} T_\alpha(t-\sigma)G(\sigma+s_n, Y^{(n)}(\sigma-), x)N(d\sigma, dx).\end{aligned}\tag{3.14}$$

It is obvious that $Y(t+s_n)$ has the same distribution as $Y^{(n)}(t)$ and $Y^{(n)}(t)$ is unique and $\mathcal{L}^2$−bounded. Meanwhile, we can get

$$\begin{aligned}&E\|Y^{(n)}(t)-\tilde{Y}(t)\|^2\\&\leq 4E\|\int_{-\infty}^{t} T_\alpha(t-\sigma)f(\sigma+s_n, Y^{(n)}_\sigma)d\sigma - \int_{-\infty}^{t} T'_\alpha(t-\sigma-s_n)\tilde{f}(\sigma, \tilde{Y}_\sigma)d\sigma\|^2\\&+4E\|\int_{-\infty}^{t} T_\alpha(t-\sigma)g(\sigma+s_n, Y^{(n)}_\sigma)d\omega(\sigma)\\&-\int_{-\infty}^{t} T'_\alpha(t-\sigma-s_n)\tilde{g}(\sigma, \tilde{Y}_\sigma)d\omega(\sigma)\|^2\\&+4E\|\int_{-\infty}^{t}\int_{|x|_V<1} T_\alpha(t-\sigma)F(\sigma+s_n, Y^{(n)}(\sigma-), x)\tilde{N}(d\sigma, dx)\\&-\int_{-\infty}^{t}\int_{|x|_V<1} T'_\alpha(t-\sigma-s_n)\tilde{F}(\sigma, \tilde{Y}(\sigma-), x)\tilde{N}(d\sigma, dx)\|^2\\&+4E\|\int_{-\infty}^{t}\int_{|x|_V\geq 1} T_\alpha(t-\sigma)G(\sigma+s_n, Y^{(n)}(\sigma-), x)\tilde{N}(d\sigma, dx)\\&-\int_{-\infty}^{t}\int_{|x|_V\geq 1} T'_\alpha(t-\sigma-s_n)\tilde{G}(\sigma, \tilde{Y}(\sigma-), x)\tilde{N}(d\sigma, dx)\|^2\\&\leq 4E\|\int_{0}^{\infty} T_\alpha(\sigma)f(t-\sigma+s_n, Y^{(n)}_{t-\sigma})d\sigma - \int_{0}^{\infty} T'_\alpha(\sigma-s_n)\tilde{f}(t-\sigma, \tilde{Y}_{t-\sigma})d\sigma\|^2\\&+4E\|\int_{0}^{\infty} T_\alpha(\sigma)g(t-\sigma+s_n, Y^{(n)}_{t-\sigma})d\omega(\sigma) - \int_{0}^{\infty} T'_\alpha(\sigma-s_n)\tilde{g}(t-\sigma, \tilde{Y}_{t-\sigma})d\omega(\sigma)\|^2\\&+4E\|\int_{0}^{\infty}\int_{|x|_V<1} T_\alpha(\sigma)F(t-\sigma+s_n, Y^{(n)}((t-\sigma)-), x)\tilde{N}(d\sigma, dx)\\&-\int_{0}^{\infty}\int_{|x|_V<1} T'_\alpha(\sigma-s_n)\tilde{F}(t-\sigma, \tilde{Y}((t-\sigma)-), x)\tilde{N}(d\sigma, dx)\|^2\end{aligned}$$

$$
\begin{aligned}
&+4E\|\int_0^{\infty}\int_{|x|_V\geq 1} T_\alpha(\sigma)G(t-\sigma+s_n, Y^{(n)}((t-\sigma)-),x)\tilde{N}(d\sigma,dx)\\
&-\int_0^{\infty}\int_{|x|_V\geq 1} T_\alpha'(\sigma-s_n)\tilde{G}(t-\sigma,\tilde{Y}((t-\sigma)-),x)\tilde{N}(d\sigma,dx)\|^2 := I_1+I_2+I_3+I_4.
\end{aligned}
\tag{3.15}
$$

Since $T_\alpha(\cdot)$ is integrable on $[0,+\infty)$, then there exists constant $N>0$ such that $\int_0^\infty \|T_\alpha(\sigma)\|d\sigma < N$. We now evaluate each term respectively,

$$
\begin{aligned}
I_1 &= 4E\|\int_0^{\infty} T_\alpha(\sigma)f(t-\sigma+s_n, Y_{t-\sigma}^{(n)})d\sigma - \int_0^{\infty} T_\alpha'(\sigma-s_n)\tilde{f}(t-\sigma,\tilde{Y}_{t-\sigma})d\sigma\|^2\\
&\leq 12E\|\int_0^{\infty} T_\alpha(\sigma)[f(t-\sigma+s_n, Y_{t-\sigma}^{(n)})-f(t-\sigma+s_n,\tilde{Y}_{t-\sigma})]d\sigma\|^2\\
&\quad + 12E\|\int_0^{\infty} T_\alpha(\sigma)[f(t-\sigma+s_n,\tilde{Y}_{t-\sigma})-\tilde{f}(t-\sigma,\tilde{Y}_{t-\sigma})]d\sigma\|^2\\
&\quad + 12E\|\int_0^{\infty} [T_\alpha(\sigma)-T_\alpha'(\sigma-s_n)]\tilde{f}(t-\sigma,\tilde{Y}_{t-\sigma})d\sigma\|^2\\
&\leq 12\int_0^{\infty}\|T_\alpha(\sigma)\|d\sigma\int_0^{\infty}\|T_\alpha(\sigma)\|d\sigma L E\|Y^{(n)}(t-\sigma))-\tilde{Y}(t-\sigma))\|^2\\
&\quad + 12\int_0^{\infty}\|T_\alpha(\sigma)\|d\sigma\int_0^{\infty}\|T_\alpha(\sigma)\|E\|f(t-\sigma+s_n,\tilde{Y}_{t-\sigma})-\tilde{f}(t-\sigma,\tilde{Y}_{t-\sigma})\|^2 d\sigma\\
&\quad + 12\int_0^{\infty}\|T_\alpha(\sigma)-T_\alpha'(\sigma-s_n)\|^2 E\|\tilde{f}(t-\sigma,\tilde{Y}_{t-\sigma})\|^2 d\sigma\\
&\leq 12N^2 L E\|Y^{(n)}(t-\sigma))-\tilde{Y}(t-\sigma))\|^2+\chi_1^n,
\end{aligned}
\tag{3.16}
$$

where

$$
\begin{aligned}
\chi_1^n &= 12\int_0^{\infty}\|T_\alpha(\sigma)\|d\sigma\int_0^{\infty}\|T_\alpha(\sigma)\|E\|f(t-\sigma+s_n,\tilde{Y}_{t-\sigma})-\tilde{f}(t-\sigma,\tilde{Y}_{t-\sigma})\|^2 d\sigma\\
&\quad + 12\int_0^{\infty}\|T_\alpha(\sigma)-T_\alpha'(\sigma-s_n)\|^2 E\|\tilde{f}(t-\sigma,\tilde{Y}_{t-\sigma})\|^2 d\sigma.
\end{aligned}
$$

From Remark 2.6 and that f is square-mean almost automorphic in t. $Y(\cdot)$ is bounded in $\mathcal{L}^2(P,H)$, we have $\sup_{\sigma\in\mathbb{R}}\|f(\sigma+s_n,\tilde{Y}(\sigma))\|_2<\infty$, so $\sup_{\sigma\in\mathbb{R}}\|\tilde{f}(\sigma,\tilde{Y}(\sigma))\|_2<\infty$. According to inequality (3.11) and Lebesgue dominated convergence theorem, it follows that

$$
\lim_{n\to\infty}\int_0^{\infty}\|T_\alpha(\sigma)\|E\|f(t-\sigma+s_n,\tilde{Y}_{t-\sigma})-\tilde{f}(t-\sigma,\tilde{Y}_{t-\sigma})\|^2 d\sigma = 0,
$$

$$
\lim_{n\to\infty}\int_0^{\infty}\|T_\alpha(\sigma)-T_\alpha'(\sigma-s_n)\|^2 E\|\tilde{f}(t-\sigma,\tilde{Y}_{t-\sigma})\|^2 d\sigma = 0.
$$

Hence

$$
\lim_{n\to\infty}\chi_1^n = 0. \tag{3.17}
$$

For I_2, by Itô isometry, we have

$$
\begin{aligned}
I_2 &= 4E\|\int_0^{\infty} T_\alpha(\sigma)g(t-\sigma+s_n, Y_{t-\sigma}^{(n)})d\omega(\sigma) - \int_0^{\infty} T_\alpha'(\sigma-s_n)\tilde{g}(t-\sigma,\tilde{Y}_{t-\sigma})d\omega(\sigma)\|^2\\
&\leq 12E\|\int_0^{\infty} T_\alpha(\sigma)[g(t-\sigma+s_n, Y_{t-\sigma}^{(n)})-g(t-\sigma+s_n,\tilde{Y}_{t-\sigma})]d\omega(\sigma)\|^2\\
&\quad + 12E\|\int_0^{\infty} T_\alpha(\sigma)[g(t-\sigma+s_n,\tilde{Y}_{t-\sigma})-\tilde{g}(t-\sigma,\tilde{Y}_{t-\sigma})]d\omega(\sigma)\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 12E\|\int_0^\infty [T_\alpha(\sigma) - T'_\alpha(\sigma - s_n)]\tilde{g}(t-\sigma, \tilde{Y}_{t-\sigma})d\omega(\sigma)\|^2 \\
&\leq 12\int_0^\infty \|T_\alpha(\sigma)\|d\sigma \int_0^\infty \|T_\alpha(\sigma)\|d\sigma Q^{\frac{1}{2}} LE\|Y^{(n)}(t-\sigma)) - \tilde{Y}(t-\sigma))\|^2 \\
&+ 12\int_0^\infty \|T_\alpha(\sigma)\|d\sigma \int_0^\infty \|T_\alpha(\sigma)\|E\|[g(t-\sigma+s_n, \tilde{Y}_{t-\sigma}) - \tilde{g}(t-\sigma, \tilde{Y}_{t-\sigma})]Q^{\frac{1}{2}}\|^2_{L(V,\mathcal{L}^2(P,H))}d\sigma \\
&+ 12\int_0^\infty \|T_\alpha(\sigma) - T'_\alpha(\sigma - s_n)\|^2 E\|[\tilde{g}(t-\sigma, \tilde{Y}_{t-\sigma})]Q^{\frac{1}{2}}\|^2_{L(V,\mathcal{L}^2(P,H))}d\sigma \\
&\leq 12N^2 Q^{\frac{1}{2}} LE\|Y^{(n)}(t-\sigma)) - \tilde{Y}(t-\sigma))\|^2 + \chi_2^n,
\end{aligned} \tag{3.18}
$$

where

$$
\begin{aligned}
\chi_2^n = {}& 12\int_0^\infty \|T_\alpha(\sigma)\|d\sigma \int_0^\infty \|T_\alpha(\sigma)\|E\|[g(t-\sigma+s_n, \tilde{Y}_{t-\sigma}) - \tilde{g}(t-\sigma, \tilde{Y}_{t-\sigma})]Q^{\frac{1}{2}}\|^2_{L(V,\mathcal{L}^2(P,H))}d\sigma \\
&+ 12\int_0^\infty \|T_\alpha(\sigma) - T'_\alpha(\sigma - s_n)\|^2 E\|\tilde{g}(t-\sigma, \tilde{Y}_{t-\sigma})Q^{\frac{1}{2}}\|^2_{L(V,\mathcal{L}^2(P,H))}d\sigma.
\end{aligned}
$$

Similarly, we can derive

$$
\lim_{n\to\infty} \chi_2^n = 0. \tag{3.19}
$$

For I_3, Applying the properties of integral for Poisson random measure, we can obtain

$$
\begin{aligned}
I_3 &\leq 12E\|\int_0^\infty \int_{|x|_V<1} T_\alpha(\sigma)[F(t-\sigma+s_n, Y^{(n)}((t-\sigma)-), x) - F(t-\sigma+s_n, \tilde{Y}((t-\sigma)-), x)]\tilde{N}(d\sigma, dx)\|^2 \\
&+ 12E\|\int_0^\infty \int_{|x|_V<1} T_\alpha(\sigma)[F(t-\sigma+s_n, \tilde{Y}((t-\sigma)-), x) - \tilde{F}(t-\sigma, \tilde{Y}((t-\sigma)-), x)]\tilde{N}(d\sigma, dx)\|^2 \\
&+ 12E\|\int_0^\infty \int_{|x|_V<1} [T_\alpha(\sigma) - T'_\alpha(\sigma - s_n)]\tilde{F}(\sigma, \tilde{Y}(\sigma-), x)\tilde{N}(d\sigma, dx)\|^2 \\
&\leq 12\int_0^\infty \int_{|x|_V<1} \|T_\alpha(\sigma)\|^2 E\|F(t-\sigma+s_n, Y^{(n)}((t-\sigma)-), x) - F(t-\sigma+s_n, \tilde{Y}((t-\sigma)-), x)\|^2 \tilde{N}(d\sigma, dx) \\
&+ 12\int_0^\infty \int_{|x|_V<1} \|T_\alpha(\sigma)\|^2 E\|F(t-\sigma+s_n, \tilde{Y}((t-\sigma)-), x) - \tilde{F}(\sigma, \tilde{Y}((t-\sigma)-), x)\|^2 \tilde{N}(d\sigma, dx) \\
&+ 12\int_0^\infty \int_{|x|_V<1} \|T_\alpha(\sigma) - T'_\alpha(\sigma - s_n)\|^2 E\|\tilde{F}(\sigma, \tilde{Y}((t-\sigma)-), x)\|^2 \tilde{N}(d\sigma, dx) \\
&\leq 12\int_0^\infty \|T_\alpha(\sigma)\|d\sigma \times \int_0^\infty \int_{|x|_V<1} \|T_\alpha(\sigma)\|E\|F(t-\sigma+s_n, Y^{(n)}((t-\sigma)-), x) \\
&- F(t-\sigma+s_n, \tilde{Y}((t-\sigma)-), x)\|^2 \nu(dx)d\sigma \\
&+ 12\int_0^\infty \|T_\alpha(\sigma)\|d\sigma \int_0^\infty \int_{|x|_V<1} \|T_\alpha(\sigma)\|E\|F(t-\sigma+s_n, Y^{(n)}((t-\sigma)-), x) \\
&- \tilde{F}(t-\sigma, \tilde{Y}(t-\sigma), x)\|^2 \nu(dx)d\sigma \\
&+ 12\int_0^\infty \int_{|x|_V<1} \|T_\alpha(\sigma) - T'_\alpha(\sigma - s_n)\|^2 E\|\tilde{F}(t-\sigma, \tilde{Y}((t-\sigma)-), x)\|^2 \nu(dx)d\sigma \\
&\leq 12N\int_0^\infty T_\alpha(\sigma)d\sigma LE\|Y^{(n)}(t-\sigma) - \tilde{Y}(t-\sigma)\|^2 \\
&+ 12\int_0^\infty \int_{|x|_V<1} \|T_\alpha(\sigma)\|^2 E\|F(t-\sigma+s_n, Y^{(n)}((t-\sigma)-), x) - \tilde{F}(t-\sigma, \tilde{Y}((t-\sigma)-), x)\|^2 \nu(dx)d\sigma \\
&+ 12\int_0^\infty \int_{|x|_V<1} \|T_\alpha(\sigma) - T'_\alpha(\sigma - s_n)\|^2 E\|\tilde{F}(t-\sigma, \tilde{Y}((t-\sigma)-), x)\|^2 \nu(dx)d\sigma \\
&\leq 12N^2 LE\|Y^{(n)}(t-\sigma)) - \tilde{Y}(t-\sigma))\|^2 + \chi_3^n,
\end{aligned} \tag{3.20}
$$

where

$$\chi_3^n = 12\int_0^\infty\int_{|x|_V<1}\|T_\alpha(\sigma)\|^2E\|F(t-\sigma+s_n,Y^{(n)}((t-\sigma)-),x)-\tilde{F}(t-\sigma,\tilde{Y}((t-\sigma)-),x)\|^2\nu(dx)d\sigma$$
$$+12\int_0^\infty\int_{|x|_V<1}\|T_\alpha(\sigma)-T_\alpha^{'}(\sigma-s_n)\|^2E\|\tilde{F}(t-\sigma,\tilde{Y}((t-\sigma)-),x)\|^2\nu(dx)d\sigma.$$

From (3.4), we have

$$\sup_{t-\sigma\in\mathbb{R}}\int_{|x|_V<1}E\|F(t-\sigma+s_n,Y^{(n)}((t-\sigma)-),x)-F(t-\sigma+s_n,\tilde{Y}(0-),x)\|^2\nu(dx)\le LE\|Y^{(n)}(t-\sigma)-\tilde{Y}(0)\|^2<\infty.$$

By (3) of Lemma 2.3,

$$\sup_{t-\sigma\in\mathbb{R}}\int_{|x|_V<1}E\|\tilde{F}(t-\sigma+s_n,\tilde{Y}(0),x)\|^2\nu(dx)<M\tilde{Y}(0),$$

where $M\tilde{Y}(0)$ is a constant. Thus

$$\sup_{t-\sigma\in\mathbb{R}}\int_{|x|_V<1}E\|\tilde{F}(t-\sigma,\tilde{Y}((t-\sigma)-),x)\|^2\nu(dx)<\infty.$$

From assumption (H_1) and (3.8), we can derive the formula

$$\int_0^\infty\int_{|x|_V<1}\|T_\alpha(\sigma)-T_\alpha^{'}(\sigma-s_n)\|^2E\|\tilde{F}(t-\sigma,\tilde{Y}((t-\sigma)-),x)\|^2\nu(dx)d\sigma$$

tends to zero as n tends to infinity. Hence

$$\lim_{n\to\infty}\chi_3^n=0. \tag{3.21}$$

Besides,

$$I_4\le 12E\|\int_0^\infty\int_{|x|_V\ge1}T_\alpha(\sigma)[G(t-\sigma+s_n,Y_n((t-\sigma)-),x)-G(t-\sigma+s_n,\tilde{Y}((t-\sigma)-),x)]N(d\sigma,dx)\|^2$$
$$+12E\|\int_0^\infty\int_{|x|_V\ge1}T_\alpha(\sigma)[G(t-\sigma+s_n,\tilde{Y}((t-\sigma)-),x)-\tilde{G}(t-\sigma,\tilde{Y}((t-\sigma)-),x)]N(d\sigma,dx)\|^2$$
$$+12E\|\int_0^\infty\int_{|x|_V\ge1}[T_\alpha(\sigma)-T_\alpha^{'}(\sigma-s_n)]\tilde{G}(\sigma,\tilde{Y}((t-\sigma)-),x)N(d\sigma,dx)\|^2$$
$$\le 24E\|\int_0^\infty\int_{|x|_V\ge1}T_\alpha(\sigma)[G(t-\sigma+s_n,Y^{(n)}((t-\sigma)-),x)-G(t-\sigma+s_n,\tilde{Y}((t-\sigma)-),x)]\tilde{N}(d\sigma,dx)\|^2$$
$$+24E\|\int_0^\infty\int_{|x|_V\ge1}T_\alpha(\sigma)[G(t-\sigma+s_n,Y^{(n)}((t-\sigma)-),x)-G(t-\sigma+s_n,\tilde{Y}((t-\sigma)-),x)]\nu(dx)d\sigma\|^2$$
$$+24E\|\int_0^\infty\int_{|x|_V\ge1}T_\alpha(\sigma)[G(t-\sigma+s_n,\tilde{Y}((t-\sigma)-),x)-\tilde{G}(t-\sigma,\tilde{Y}((t-\sigma)-),x)]\tilde{N}(d\sigma,dx)\|^2$$

$$
\begin{aligned}
&+24E\|\int_0^\infty\int_{|x|_V\geq 1}T_\alpha(\sigma)[G(t-\sigma+s_n,\tilde{Y}((t-\sigma)-),x)-\tilde{G}(t-\sigma,\tilde{Y}((t-\sigma)-),x)]\nu(dx)d\sigma\|^2\\
&+24E\|\int_0^\infty\int_{|x|_V\geq 1}[T_\alpha(\sigma)-T_\alpha^{'}(\sigma-s_n)]\tilde{G}(t-\sigma,\tilde{Y}((t-\sigma)-),x)\tilde{N}(d\sigma,ds)\|^2\\
&+24E\|\int_0^\infty\int_{|x|_V\geq 1}[T_\alpha(\sigma)-T_\alpha^{'}(\sigma-s_n)]\tilde{G}(\sigma,\tilde{Y}((t-\sigma)-),x)\nu(dx)d\sigma\|^2\\
&\leq 24\int_0^\infty\|T_\alpha(\sigma)\|d\sigma\int_0^\infty\int_{|x|_V\geq 1}\|T_\alpha(\sigma)\|E\|G(t-\sigma+s_n,Y^{(n)}((t-\sigma)-),x)\\
&-G(t-\sigma+s_n,\tilde{Y}((t-\sigma)-),x)\|^2\tilde{N}(d\sigma,dx)\\
&+24\int_0^\infty\|T_\alpha(\sigma)\|d\sigma\int_0^\infty\int_{|x|_V\geq 1}\|T_\alpha(\sigma)\|E\|G(t-\sigma+s_n,Y^{(n)}((t-\sigma)-),x)\\
&-G(t-\sigma+s_n,\tilde{Y}((t-\sigma)-),x)\|^2\nu(dx)d\sigma \qquad (3.22)\\
&+24\int_0^\infty\|T_\alpha(\sigma)\|d\sigma\int_0^\infty\int_{|x|_V\geq 1}\|T_\alpha(\sigma)\|E\|G(t-\sigma+s_n,\tilde{Y}((t-\sigma)-),x)\\
&-\tilde{G}(t-\sigma,\tilde{Y}((t-\sigma)-),x)\|^2\tilde{N}(d\sigma,dx)\\
&+24\int_0^\infty\|T_\alpha(\sigma)\|d\sigma\int_0^\infty\int_{|x|_V\geq 1}\|T_\alpha(\sigma)\|E\|G(t-\sigma+s_n,\tilde{Y}((t-\sigma)-),x)\\
&-\tilde{G}(t-\sigma,\tilde{Y}((t-\sigma)-),x)\|^2\nu(dx)d\sigma\\
&+24\int_0^\infty\int_{|x|_V\geq 1}\|T_\alpha(\sigma)-T_\alpha^{'}(\sigma-s_n)\|^2E\|\tilde{G}(t-\sigma,\tilde{Y}((t-\sigma)-),x)\|^2\tilde{N}(d\sigma,dx)\\
&+24\int_0^\infty\int_{|x|_V\geq 1}\|T_\alpha(\sigma)-T_\alpha^{'}(\sigma-s_n)\|^2E\|\tilde{G}(t-\sigma,\tilde{Y}((t-\sigma)-),x)\|^2\nu(dx)d\sigma\\
&\leq 48N^2LE\|Y^{(n)}(t-\sigma)-\tilde{Y}(t-\sigma)\|^2+\chi_4^n,
\end{aligned}
$$

where

$$
\begin{aligned}
\chi_4^n=\ &24\int_0^\infty\|T_\alpha(\sigma)\|d\sigma\int_0^\infty\int_{|x|_V\geq 1}\|T_\alpha(\sigma)\|E\|G(t-\sigma+s_n,\tilde{Y}((t-\sigma)-),x)\\
&-\tilde{G}(t-\sigma,\tilde{Y}((t-\sigma)-),x)\|^2\tilde{N}(d\sigma,dx)\\
&+24\int_0^\infty\|T_\alpha(\sigma)\|d\sigma\int_0^\infty\int_{|x|_V\geq 1}\|T_\alpha(\sigma)\|E\|G(t-\sigma+s_n,\tilde{Y}((t-\sigma)-),x)\\
&-\tilde{G}(t-\sigma,\tilde{Y}((t-\sigma)-),x)\|^2\nu(dx)d\sigma\\
&+24\int_0^\infty\int_{|x|_V\geq 1}\|T_\alpha(\sigma)-T_\alpha^{'}(\sigma-s_n)\|^2E\|\tilde{G}(t-\sigma,\tilde{Y}((t-\sigma)-),x)\|^2\tilde{N}(d\sigma,dx)\\
&+24\int_0^\infty\int_{|x|_V\geq 1}\|T_\alpha(\sigma)-T_\alpha^{'}(\sigma-s_n)\|^2E\|\tilde{G}(t-\sigma,\tilde{Y}((t-\sigma)-),x)\|^2\nu(dx)d\sigma.
\end{aligned}
$$

By similar arguments as above, we can get

$$\lim_{n\to\infty}\chi_4^n=0. \tag{3.23}$$

From (3.17)-(3.23), it follows that

$$E\|Y^{(n)}(t)-\tilde{Y}(t)\|^2\leq\sum_{i=1}^4\chi_1^n+(72+12Q^{\frac{1}{2}})N^2LE\|Y^{(n)}(t-\sigma))-\tilde{Y}(t-\sigma))\|^2.$$

We oberave that $Y(t+s_n)$ has the same distribution as $Y_n(t)$, from [30, Remark 2.12], we have $Y(t+s_n)\to\tilde{Y}(t)$ in distribution as $n\to\infty$. Similarly, we have that $\tilde{Y}(t-s_n)\to Y(t)$ in distribution as $n\to\infty$ for each $t\in\mathbb{R}$. This makes the proof complete. □

The second theorem is for the existence and uniqueness of weighted pseudo almost automorphic solutions in distribution to system (1.2).

Theorem 3.2. *Assume that* (H_1) *holds, and* $f\in SWPAA(\mathbb{R}\times\mathcal{B},\mathcal{L}^2(P,H))$, $g\in SWPAA(\mathbb{R}\times\mathcal{B},L(V,\mathcal{L}^2(P,H)))$, $F,G\in PSWPAA(\mathbb{R}\times\mathcal{L}^2(P,H)\times V,\mathcal{L}^2(P,H))$. *Suppose that* $f=m+l, g=p+q$ *with* $m\in SAA(\mathbb{R}\times\mathcal{B},\mathcal{L}^2(P,H))$,

$l, q \in SBC_0(\mathbb{R}\times\mathcal{B}, \mathcal{L}^2(P,H), \rho), p \in SAA(\mathbb{R}\times\mathcal{B}, L(V, \mathcal{L}^2(P,H))), F = h+\varphi, G = \alpha+\beta$ with $h, \alpha \in PSAA(\mathbb{R}\times \mathcal{L}^2(P,H)\times V, \mathcal{L}^2(P,H))$ and $\varphi, \beta \in PSBC_0(\mathbb{R}\times\mathcal{L}^2(P,H)\times V, \rho)$, where $\rho \in \mathcal{U}^{inv}\cap\mathcal{U}_p^{inv}$ and $t\in\mathbb{R}$. In addition, suppose that f, m, g, p, h, α, F and G satisfy the Lipschitz conditions in Y uniformly for t, that is, for all $Y, Z \in \mathcal{L}^2(P,H)$ and $t \in \mathbb{R}$, there exists a constant $L > 0$ independent of t, such that for all $Y, Z \in \mathcal{L}^2(P,H)$ and $t\in\mathbb{R}$,

$$E\|f(t,Y)-f(t,Z)\|^2 \le LE\|Y-Z\|^2,$$

$$E\|m(t,Y)-m(t,Z)\|^2 \le LE\|Y-Z\|^2,$$

$$E\|(g(t,Y)-g(t,Z))Q^{\frac{1}{2}}\|^2_{L(V,\mathcal{L}^2(P,H))} \le LE\|Y-Z\|^2,$$

$$E\|(p(t,Y)-p(t,Z))Q^{\frac{1}{2}}\|^2_{L(V,\mathcal{L}^2(P,H))} \le LE\|Y-Z\|^2,$$

$$\int_{|x|_V<1} E\|F(t,Y,x)-F(t,Z,x)\|^2\nu(dx) \le LE\|Y-Z\|^2,$$

$$\int_{|x|_V<1} E\|h(t,Y,x)-h(t,Z,x)\|^2\nu(dx) \le LE\|Y-Z\|^2,$$

$$\int_{|x|_V\ge1} E\|G(t,Y,x)-G(t,Z,x)\|^2\nu(dx) \le LE\|Y-Z\|^2,$$

$$\int_{|x|_V\ge1} E\|\alpha(t,Y,x)-\alpha(t,Z,x)\|^2\nu(dx) \le LE\|Y-Z\|^2.$$

Then (1.2) has a unique $\mathcal{L}^2$−bounded mild solution, which is weighted pseudo almost automorphic in distribution.

Proof. One can easily see that if $Y(t)$ is $\mathcal{L}^2$−bounded, then $Y(t)$ is a mild solution of (1.2). From the assumptions, we have $f(t,Y_t) = m(t,Y_t)+l(t,Y_t), g(t,Y_t) = p(t,Y_t)+q(t,Y_t), F(t,Y(t-),x) = h(t,Y(t-),x)+\varphi(t,Y(t-),x), G(t,Y(t-),x) = \alpha(t,Y(t-),x)+\beta(t,Y(t-),x)$. Then we have

$$\begin{aligned}
Y(t) &= \int_{-\infty}^t T_\alpha(t-s)f(s,Y_s)ds + \int_{-\infty}^t T_\alpha(t-s)g(s,Y_s)d\omega(s)\\
&+ \int_{-\infty}^t\int_{|x|_V<1} T_\alpha(t-s)F(s,Y(s-),x)\tilde N(ds,dx)\\
&+ \int_{-\infty}^t\int_{|x|_V\ge1} T_\alpha(t-s)G(s,Y(s-),x)N(ds,dx)\\
&= [\int_{-\infty}^t T_\alpha(t-s)m(s,Y_s)ds + \int_{-\infty}^t T_\alpha(t-s)p(s,Y_s)d\omega(s)\\
&+ \int_{-\infty}^t\int_{|x|_V<1} T_\alpha(t-s)h(s,Y(s-),x)\tilde N(ds,dx)\\
&+ \int_{-\infty}^t\int_{|x|_V\ge1} T_\alpha(t-s)\alpha(s,Y(s-),x)N(ds,dx)]\\
&+[\int_{-\infty}^t T_\alpha(t-s)l(s,Y_s)ds + \int_{-\infty}^t T_\alpha(t-s)q(s,Y_s)d\omega(s)\\
&+ \int_{-\infty}^t\int_{|x|_V<1} T_\alpha(t-s)\varphi(s,Y(s-),x)\tilde N(ds,dx)\\
&+ \int_{-\infty}^t\int_{|x|_V\ge1} T_\alpha(t-s)\beta(s,Y(s-),x)N(ds,dx)]\\
&:= Y_1(t)+Y_2(t).
\end{aligned} \tag{3.24}$$

From the proof of Theorem 3.1, we have that Y_1 is the unique almost automorphic solution in distribution. We now in a position to prove that $Y_2 \in SBC_0(\mathbb{R},\rho)$. $Y_2(t)$ is $\mathcal{L}^2$−continuous and $\mathcal{L}^2$−bounded would be showed first. Set $\Lambda(t) = \int_{-\infty}^t T_\alpha(t-s)m(s,Y_s)ds + \int_{-\infty}^t T_\alpha(t-s)q(s,Y_s)d\omega(s)$. By similar arguments as [13, Theorem

4.1], it follows that $\Lambda(t)$ is $\mathcal{L}^2-$continuous and $\mathcal{L}^2-$bounded. Since $\varphi, \beta \in PSBC_0(\mathbb{R} \times \mathcal{L}^2(P, H) \times V, \rho)$, then φ, β are both Poisson stochastically bounded and Poisson stochastically continuous. Similarly, we can show that $\int_{-\infty}^{t}\int_{|x|_V<1} T_\alpha(t-s)\varphi(s, Y(s-), x)\tilde{N}(ds, dx) + \int_{-\infty}^{t}\int_{|x|_V\geq 1} T_\alpha(t-s)\beta(s, Y(s-), x)N(ds, dx)$ is $\mathcal{L}^2-$continuous. According to the properties of the integral for the Poisson random measure, we have

$$\begin{aligned}
&E\|\int_{-\infty}^{t}\int_{|x|_V<1} T_\alpha(t-s)\varphi(s, Y(s-), x)\tilde{N}(ds, dx)\\
&+\int_{-\infty}^{t}\int_{|x|_V\geq 1} T_\alpha(t-s)\beta(s, Y(s-), x)N(ds, dx)\|^2\\
&\leq 2E\|\int_{-\infty}^{t}\int_{|x|_V<1} T_\alpha(t-s)\varphi(s, Y(s-), x)\tilde{N}(ds, dx)\|^2\\
&+4E\|\int_{-\infty}^{t}\int_{|x|_V\geq 1} T_\alpha(t-s)\beta(s, Y(s-), x)\tilde{N}(ds, dx)\|^2\\
&+4E\|\int_{-\infty}^{t}\int_{|x|_V\geq 1} T_\alpha(t-s)\beta(s, Y(s-), x)\nu(dx)ds\|^2\\
&\leq 2\int_{-\infty}^{t}\int_{|x|_V<1} \|T_\alpha(t-s)\|^2 E\|\varphi(s, Y(s-), x)\|^2\tilde{N}(ds, dx)\\
&+4\int_{-\infty}^{t}\int_{|x|_V\geq 1} \|T_\alpha(t-s)\|^2 E\|\beta(s, Y(s-), x)\|^2\tilde{N}(ds, dx)\\
&+4\int_{-\infty}^{t}\int_{|x|_V\geq 1} \|T_\alpha(t-s)\|^2 E\|\beta(s, Y(s-), x)\|^2\nu(dx)ds.
\end{aligned} \tag{3.25}$$

Since $T_\alpha(\cdot)$ is integral on $[0, \infty)$ and $\varphi, \beta \in PSBC_0(\mathbb{R} \times \mathcal{L}^2(P, H) \times V, \rho)$, we have

$$\begin{aligned}
&E\|\int_{-\infty}^{t}\int_{|x|_V<1} T_\alpha(t-s)\varphi(s, Y(s-), x)\nu(dx)ds\\
&\leq \int_{0}^{\infty}\int_{|x|_V<1} \|T_\alpha(s)\|^2 E\|\varphi(t-s, Y((t-s)-), x)\|^2\nu(dx)ds\\
&\leq \int_{0}^{\infty}\frac{CM}{1+|\mu|s^\alpha}ds\int_{0}^{\infty}\frac{CM}{1+|\mu|s^\alpha}ds\int_{|x|_V<1} E\|\varphi(t-s, Y((t-s)-), x)\|^2\nu(dx)\\
&\leq (CM)^2\int_{|x|_V<1} E\|\varphi(t-s, Y((t-s)-), x)\|^2\nu(dx) < \infty.
\end{aligned} \tag{3.26}$$

Similarly,

$$\begin{aligned}
&E\|\int_{-\infty}^{t}\int_{|x|_V\geq 1} T_\alpha(t-s)\beta(s, Y(s-), x)\nu(dx)ds\|^2\\
&\leq \int_{0}^{\infty}\frac{CM}{1+|\mu|s^\alpha}ds\int_{|x|_V\geq 1}\nu(dx)\int_{0}^{\infty}\frac{CM}{1+|\mu|s^\alpha}ds\int_{|x|_V\geq 1} E\|\beta(t-s, Y((t-s)-), x)\|^2\nu(dx)ds\\
&\leq (CM)^2\int_{|x|_V\geq 1} E\|\beta(t-s, Y((t-s)-), x)\|^2\nu(dx) < \infty.
\end{aligned} \tag{3.27}$$

By (3.25)-(3.27), we have

$$\begin{aligned}
&E\|\int_{-\infty}^{t}\int_{|x|_V<1} T_\alpha(t-s)\varphi(s, Y(s-), x)\nu(dx)ds\\
&+\int_{-\infty}^{t}\int_{|x|_V\geq 1} T_\alpha(t-s)\beta(s, Y(s-), x)\nu(dx)ds\|^2 < \infty.
\end{aligned}$$

Then $Y_2(t)$ is $\mathcal{L}^2-$bounded. We next show that $\lim_{r\to+\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r} E\|Y_2(t)\|^2\rho(t)dt < \infty$. From the definition of

$Y_2(t)$, we can derive

$$\begin{aligned}
&\frac{1}{m(r,\rho)}\int_{-r}^{r}E\|Y_2(t)\|^2\rho(t)dt\\
&\leq \frac{1}{m(r,\rho)}\int_{-r}^{r}E\Big\|\int_{-\infty}^{t}T_\alpha(t-s)l(s,Y_s)ds+\int_{-\infty}^{t}T_\alpha(t-s)q(s,Y_s)d\omega(s)\\
&+\int_{-\infty}^{t}\int_{|x|_V<1}T_\alpha(t-s)\varphi(s,Y(s-),x)\nu(dx)ds\\
&+\int_{-\infty}^{t}\int_{|x|_V\geq 1}T_\alpha(t-s)\beta(s,Y(s-),x)\nu(dx)ds\Big\|^2\rho(t)dt \qquad (3.28)\\
&\leq \frac{2}{m(r,\rho)}\int_{-r}^{r}E\Big\|\int_{-\infty}^{t}T_\alpha(t-s)l(s,Y_s)ds+\int_{-\infty}^{t}T_\alpha(t-s)q(s,Y_s)d\omega(s)\Big\|^2\rho(t)dt\\
&+\frac{2}{m(r,\rho)}\int_{-r}^{r}E\Big\|\int_{-\infty}^{t}\int_{|x|_V<1}T_\alpha(t-s)\varphi(s,Y(s-),x)\nu(dx)ds\\
&+\int_{-\infty}^{t}\int_{|x|_V\geq 1}T_\alpha(t-s)\beta(s,Y(s-),x)\nu(dx)ds\Big\|^2\rho(t)dt.
\end{aligned}$$

By a similar way as the proof of [28, Theorem 4.1], we can derive

$$\lim_{r\to+\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}E\Big\|\int_{-\infty}^{t}T_\alpha(t-s)l(s,Y_s)ds+\int_{-\infty}^{t}T_\alpha(t-s)q(s,Y_s)d\omega(s)\Big\|^2\rho(t)dt=0.$$

In addition,

$$\begin{aligned}
&\frac{1}{m(r,\rho)}\int_{-r}^{r}E\|\int_{-\infty}^{t}\int_{|x|_V<1}T_\alpha(t-s)\varphi(s,Y(s-),x)\tilde{N}(ds,dx)\\
&+\int_{-\infty}^{t}\int_{|x|_V\geq 1}T_\alpha(t-s)\beta(s,Y(s-),x)N(ds,dx)\|^2\rho(t)dt\\
&\leq \frac{2}{m(r,\rho)}\int_{-r}^{r}E\|\int_{-\infty}^{t}\int_{|x|_V<1}T_\alpha(t-s)\varphi(s,Y(s-),x)\tilde{N}(ds,dx)\|^2\rho(t)dt\\
&+\frac{2}{m(r,\rho)}\int_{-r}^{r}E\|\int_{-\infty}^{t}\int_{|x|_V<1}T_\alpha(t-s)\beta(s,Y(s-),x)N(ds,dx)\|^2\rho(t)dt\\
&\leq \frac{2}{m(r,\rho)}\int_{-r}^{r}E\|\int_{-\infty}^{t}\int_{|x|_V<1}T_\alpha(t-s)\varphi(s,Y(s-),x)\tilde{N}(ds,dx)\|^2\rho(t)dt\\
&+\frac{4}{m(r,\rho)}\int_{-r}^{r}E\|\int_{-\infty}^{t}\int_{|x|_V<1}T_\alpha(t-s)\beta(s,Y(s-),x)\tilde{N}(ds,dx)\|^2\rho(t)dt\\
&+\frac{4}{m(r,\rho)}\int_{-r}^{r}E\|\int_{-\infty}^{t}\int_{|x|_V<1}T_\alpha(t-s)\beta(s,Y(s-),x)\nu(dx)ds\|^2\rho(t)dt,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{m(r,\rho)}\int_{-r}^{r}E\|\int_{-\infty}^{t}\int_{|x|_V<1}T_\alpha(t-s)\varphi(s,Y(s-),x)\tilde{N}(ds,dx)\|^2\rho(t)dt\\
&\leq \frac{1}{m(r,\rho)}\int_{-r}^{r}\rho(t)dtE\|\int_{0}^{\infty}\int_{|x|_V<1}T_\alpha(s)\varphi(t-s,Y((t-s)-),x)\nu(dx)ds\|^2\\
&\leq \frac{1}{m(r,\rho)}\int_{-r}^{r}\rho(t)dtE\|\int_{0}^{\infty}\int_{|x|_V<1}\frac{CM}{1+\mu s^\alpha}\varphi(t-s,Y((t-s)-),x)\nu(dx)ds\|^2\\
&\leq \frac{(CM)^2}{m(r,\rho)}\int_{-r}^{r}\rho(t)dt\int_{0}^{\infty}\int_{|x|_V<1}\frac{1}{1+\mu s^\alpha}E\|\varphi(t-s,Y((t-s)-),x)\|^2\nu(dx)ds.
\end{aligned}$$

According to Fubini theorem, we derive that

$$\begin{aligned}&\frac{(CM)^2}{m(r,\rho)}\int_{-r}^{r}\rho(t)dt\int_0^{\infty}\int_{|x|_V<1}\frac{1}{1+\mu s^{\alpha}}E\|\varphi(t-s,Y((t-s)-),x)\|^2\nu(dx)ds\\&\leq\frac{CM}{m(r,\rho)}\int_0^{\infty}\frac{1}{1+|\mu|s^{\alpha}}ds\int_{-r}^{r}\int_{|x|_V<1}E\|\varphi(t-s,Y((t-s)-),x)\|^2\nu(dx)\rho(t)dt.\end{aligned}$$

Since $\rho\in\mathcal{U}^{inv},\varphi\in PSBC_0(\mathbb{R}\times\mathcal{L}^2(P,H)\times V,\rho)$, it follows that

$$\lim_{r\to+\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}E\|\int_0^{\infty}\int_{|x|_V<1}T_{\alpha}(s)\varphi(t-s,Y((t-s)-),x)\nu(dx)ds\|^2\rho(t)dt\to 0. \tag{3.29}$$

Similarly,

$$\lim_{r\to+\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}E\|\int_0^{\infty}\int_{|x|_V\geq 1}T_{\alpha}(s)\beta(t-s,Y((t-s)-),x)\nu(dx)ds\|^2\rho(t)dt\to 0. \tag{3.30}$$

Combing (3.28), (3.29) with (3.30), we have

$$\lim_{r\to+\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}E\|Y_2(t)\|^2\rho(t)dt=0.$$

Therefore $Y_2(t)\in SBC_0(\mathbb{R},\rho)$. The proof is complete. □

4. Applications

In what follows, we use the previous theory to verify the existence and uniqueness of almost automorphic solutions in distribution for a class of fractional partial integro-differential equations. We are concerned with the following system

$$\begin{cases}\frac{\partial}{\partial t}v(t,\vartheta)=J_t^{\alpha-1}(\frac{\partial^2}{\partial\vartheta}-c)v(t,\vartheta)\\ +h(t)\int_{-\infty}^{0}k(s)v(t+s,\vartheta)ds+a(t)\int_{-\infty}^{0}b(s)v(t-s,\vartheta)d\omega(s)+\bar{h}(t,u,z)\frac{\partial Z}{\partial t},\ \ t\geq 0,\ c>0,\ \vartheta\in[0,\pi],\\ v(\theta,\vartheta)=\varphi(\theta,\vartheta),\quad \theta\in(-\infty,0],\end{cases} \tag{4.1}$$

where $1<\alpha<2$, Let $H=\mathcal{L}^2([0,\pi],\mathbb{R},\|\cdot\|_{\mathcal{L}^2})$, $h(t),a(t)\in SAA(\mathbb{R},\mathcal{L}^2(P,H))$, $k,b\in C_b([-\infty,0],\mathbb{R})$ and satisfy some particular conditions specified later. Define the operator A on H by $Av=(\frac{\partial^2}{\partial\vartheta}-c)v$. $D(A)=\{v\in H=L^2([0,\pi],\mathbb{R}):v''\in L^2[0,\pi],\ v(0)=v(\pi)=0\}$. It is well known that A is sectorial of type $\omega=-c<0$. Let $u(t)(\vartheta)=v(t,\vartheta)$ for $t\in[0,\infty)$, $f(t,u_t)(\vartheta)=h(t)\int_{-\infty}^{0}k(s)v(t+s,\vartheta)ds$, $g(t,u_t)(\vartheta)=a(t)\int_{-\infty}^{0}b(s)v(t+s,\vartheta)d\omega(s)$, $\bar{h}(t,u,z)dZ:=\int_{\|z\|_U<1}H(t,X,z)\tilde{N}(dt,dz)+\int_{\|z\|_U\geq 1}H(t,X,z)N(dt,dz)$, with $Z(t,x)=\int_{\|z\|_U<1}z\tilde{N}(dt,dz)+\int_{\|z\|_U\geq 1}zN(dt,dz)$, $H(t,X,z)=\bar{h}(t,u,z)z$. Here we may assume for simplicity that the Lévy pure jump process Z on $\mathcal{L}^2(0,\pi)$ is decomposed as above by the Lévy-Itô decomposition theorem and $J_t^{\alpha-1}u(t)=\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}u(s)ds$. Let $\{\hat{q}(t):t\in J\}$ be the Poisson point process taking values in the space $K=[0,+\infty)$ with a $\sigma-$finite intensity measure $(\Omega,\mathcal{F}_t,P)$ on the complete probability space. Let $N(ds,d\eta)$ be the Poisson counting measure which is induced by $\hat{q}(\cdot)$ and the compensating martingale measure is denoted by $\tilde{N}(ds,dz)=N(ds,dz)-\nu(dz)ds$, then (4.1) can be rewritten as an abstract system of the form (1.2). Furthermore, suppose that the assumptions of Theorem 3.1 hold, then from Theorem 3.1, system (4.1) admits an almost automorphic solution in distribution.

5. Conclusion

In this paper, we have considered a class of fractional integro-differential equations driven by Lévy noise. Since $T_\alpha(\cdot)$ generated by sectorial operator is integral in t on $[0,\infty)$, it enables us to use sectorial operator to establish the existence and uniqueness of (weighted pseudo) almost automorphic solutions in distribution for system (1.2). In particular, the fixed point technique, fractional calculus and stochastic analysis are used for achieving the sufficient conditions to ensure the existence and uniqueness of (weighted pseudo) almost automorphic solutions in distribution for fractional integro-differential equations driven by Lévy noise with an illustrative example. Our future work will be focused on investigating the Stepanov-like almost automorphic solutions for fractional stochastic differential equations with Poison jumps or Lévy noise.

Acknowledgments

This work was supported by National Natural Science Foundation of China (Grant No. 12001393) and Natural Science Foundation of Shanxi (201901D211103).

References

[1] P. Acquistapace, B. Terreni, A unified approach to abstract linear parabolic equations, Rendiconti Del Seminario Matematico Della Universita Di Padova 78(1987) 47-107.
[2] R.P. Agarwal, B. de Andrade, C. Cuevas, Weighted pseudo-almost periodic solutions of a class of semilinear fractional differential equations, Nonlinear Analysis: Real World Applications 11(2010) 3532–3554.
[3] F. Andrade, C. Cuevas, C. Silva, H. Soto, Asymptotic periodicity for hyperbolic evolution equations and applications, Applied Mathematics and Computation 269(2015) 169–195.
[4] D. Applebaum, Lévy Processes and Stochastic Calculus 2nd edn, Cambridge University Press, Cambridge, 2009.
[5] S. Bochner, A new approach to almost automorphicity, Proceedings of the National Academy of Sciences of the United States of America 48(1962) 2039–2043.
[6] A. Caicedo, C. Cuevas, G.M. Mophou, G.M. ŃGuérékata, Asymptotic behavior of solutions of some semilinear functional differential and integrodifferential equations with infinite delay in Banach spaces, Journal of the Franklin Institute 349(2012) 1–24.
[7] J. Cao, Q. Yang, Z. Huang, Existence and exponential stability of almost automorphic mild solutions for stochastic functional differential equations, Stochastics 83 (2011) 259-275.
[8] Y.K. Chang, T.W. Feng, Properties on measure pseudo almost automorphic functions and applications to fractional differential equations in Banach spaces, Electronic Journal of Differential Equations 2018(2018), No. 47, pp. 1-14.
[9] Y.K. Chang, Z.H. Zhao, G.M. ŃGuérékata, A new composition theorem for square-mean almost automorphic functions and applications to stochastic differential equations, Nonlinear Analysis: Theory, Methods & Applications 74(6)(2011) 2210-2219.
[10] X.X. Chen, X.Y. Hu, Weighted pseudo almost periodic solutions of neutral functional differential equations, Nonlinear Analysis: Real World Applications 12(2011) 601–610.
[11] Z. Chen, W. Lin, Square-mean almost automorphic process and its application to stochastic evolution equations, Journal of Functional Analysis 261(2011) 69–89.
[12] C. Corduneanu, Almost Periodic Functions, New York: Wiley, 1968; Reprinted, New York: Chelsea, 1989.
[13] E. Cuesta, Asymptotic behavior of the solutions of fractional integro-differential equations and some time discretizations, Dynamics of Continuous Discrete and Impulsive Systems (2007) 277–285.
[14] C. Cuevas, E. Hernández, Pseudo-almost periodic solutions for abstract partial functional differential equations, Applied Mathematics Letters 22(2009) 534–538.
[15] C. Cuevas, H.R. Henríquez, H. Soto, Asymptotically periodic solutions of fractional differential equations, Applied Mathematics and Computation 236(2014) 524–545.
[16] T. Diagana, Almost automorphic type and almost periodic type functions in abstract spaces, New York: Springer, 2013.
[17] J.P.C. dos Santos, C. Cuevas, Asymptotically almost automorphic solutions of abstract fractional integro-differential neutral equations, Applied Mathematics Letters 23(9)(2011) 960-965.
[18] J.P.C. dos Santos, S.M. Guzzo, M.N. Rabelo, Asymptotically almost periodic solutions for abstract partial neutral integro-differential equation, Advance in Difference Equations (2010). Art. ID 310951, 26 pp.
[19] M.M. Fu, Z.X. Liu, Square-mean almost automorphic solutions for some stochastic differential equations, Proceedings of the American Mathematical Society 138(2010) 3689-3701.
[20] D.S. Ji, C.Y. Zhang, Translation invariance of weighted pseudo almost periodic functions and related problems, Journal of Mathematical Analysis and Applications (391)2012 350–362.
[21] K.X. Li, Weighted pseudo almost automorphic solutions for nonautonomous SPDEs driven by Lévy noise, Journal of Mathematical Analysis and Applications 427(2015) 686-721.
[22] Z.X. Liu, K. Sun, Almost automorphic solutions for stochastic differential equations driven by Lévy noise, Journal of Functional Analysis 266(2014) 1115-1149.

[23] G.M. Mophou, Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations, Applied Mathematics and Computation 217(2011) 7579–7587.
[24] G.M. N'Guérékata, Topics in Almost Automorphy, New York: Springer, 2005.
[25] C. Rajivganthi, P. Muthukumar, Almost automorphic solutions for fractional stochastic differential equations and its optimal control, Optimal Control Applications & Methods (2015). DOI: 10.1002/oca.2186.
[26] P. Revathi, R. Sakthivel, Y. Ren, S.M. Anthoni, Existence of almost automorphic mild solutions to non-autonomous neutral stochastic differential equations, Applied Mathematics and Computation 230 (2014) 639–649.
[27] R. Sakthivel, P. Revathi, S.M. Anthoni, Existence of pseudo almost automorphic mild solutions to stochastic fractional differential equations, Nonlinear Analysis: Theory, Methods & Applications 75(2012) 3339–3347.
[28] K. Sun, Y. Wang, Almost automorphic solutions for stochastic differential equations driven by Lévy noise with exponential dichotomy, Stochastic Analysis and Applications (2016) 1–26.
[29] Y. Wang, Z.X. Liu, Almost periodic solutions for stochastic differential equations with Lévy noise, Nonlinearity 25(2012) 2803-2821.
[30] Z.N. Xia, Weighted pseudo periodic solutions of neutral functional differential equations, Electronic Journal of Differential Equations, 191(2014) 1–17.
[31] M. Yang, Q.R. Wang, Pseudo asymptotically periodic solutions for fractional integro-differential neutral equations, SCIENCE CHINA Mathematics, 62(9)(2019) 1705–1718.
[32] R. Zhang, Y.K. Chang, G.M. ŃGuérékata, New composition theorems of Stepanov-like almost automorphic functions and applications to nonautonomous evolution equations, Nonlinear Analysis: Real World Applications 13 (2012) 2866–2879.
[33] Z.M. Zheng, H.S. Ding, On completeness of the space of weighted pseudo almost automorphic functions, Journal of Functional Analysis 268 (10) (2015) 3211–3218.
[34] Y. Zhou, Basic Theory of Fractional Differential Equations, New Jersey: World Scientific, 2014.
[35] Y. Zhou, F. Jiao, Existence of mild solutions for fractional neutral evolution equations, Computers & Mathematics with Applications 59(3)(2010) 1063-1077.