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Bi-Covering Rough Sets

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Abstract. Rough set theory is a useful tool for knowledge discovery and data mining. Covering-based rough sets are important generalizations of the classical rough sets. Recently, the concept of the neighborhood has been applied to define different types of covering rough sets. In this paper, based on the notion of bi-neighborhood, four types of bi-neighborhoods related bi-covering rough sets were defined with their properties being discussed. We first show some basic properties of the introduced bi-neighborhoods. We then explore the relationships between the considered bi-covering rough sets and investigate the properties of them. Also, we show that new notions may be viewed as a generalization of the previous studies covering rough sets. Finally, figures are presented to show that the collection of all lower and upper approximations (bi-neighborhoods of all elements in the universe) introduced in this paper construct a lattice in terms of the inclusion relation $\subseteq$.

1. Introduction

Rough set theory was initially developed by Pawlak [8] as a new mathematical methodology to deal with the vagueness and uncertainty in information systems.

There are two directions for generalizing rough set theory, on one hand, Yao [16] do a generalization for rough set theory by replacing the equivalence relation by an arbitrary binary relation. On the other hand, Zakowski [18] proposed the primary idea of covering-based rough set approximation operators, Pomykala [9] and Willim Zhu [20] are replacing the partition arising from the equivalence relation to cover the universe. To enlarge the application scope of rough set theory, researchers have continued the study of covering rough sets [1–6, 12–14, 17, 21, 22, 24] as a natural extension of classical rough sets. In [10, 11, 18] the neighborhood of arbitrary element in a universe with covering was defined. Based on the concept of neighborhood, more types of covering rough sets were defined and their properties were also investigated [19, 22]. Different types of covering rough sets are often important to define, since different definitions can suit different applications and to pursue better lower and upper approximations of a vague set. This motivates the research of extending covering rough set models.

In this paper, we introduce four notions of bi-neighborhoods. We study the properties of the introduced bi-neighborhoods. Based on the introduced bi-neighborhoods, we define new types of covering rough sets and study their basic properties. We compare the accuracy of approximations of given rough sets

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concerning different pairs of the introduced approximations and investigate the relationship among different bi-neighborhood related bi-covering rough sets defined in this paper and some references. The rest of this paper is organized as follows. Section 2 reviews the main results from existing studies about Pawlak's rough set theory, neighborhood and covering. The major contributions of this paper are covered in Sections 3 and 4. We begin in Subsection 3.1 by defining a new type of bi-covering based rough sets from the concepts of bi-neighborhood. Subsection 3.2, to assist us to understand this concept of bi-neighborhood, we give an alternate representation of the upper approximation. This representation is also useful for us to prove the properties of this type of rough sets. Section 4 presents detailed properties of lower and upper approximations for this new type of rough sets. Also, we study the independence between the lower and the upper approximation operations of this new type of rough sets. The conclusion is in Section 5.

2. Preliminaries

In this section, we present concepts such as rough sets, neighborhood and covering.

2.1. Fundamentals of Pawlak's Rough Sets

Let U be a finite set called the universe, and R be an equivalence relation on U. Denote by U/R the family of all equivalence classes induced by R. U/R gives a partition of U. For any $A \subseteq U$, the lower and upper approximations of A are defined as below:

$\underline{R}(A) = \cup\{Y_i \in U/R : Y_i \subseteq A\}$, $\underline{R}(A)$ is the lower approximation of A.

$\overline{R}(A) = \cup\{Y_i \in U/R : Y_i \cap A \neq \phi\}$, $\overline{R}(A)$ is the upper approximation of A.

It follows that $\underline{R}(A) \subseteq A \subseteq \overline{R}(A)$.

According to Pawlak's definition, A is called a rough set if $\underline{R}(A) \neq \overline{R}(A)$.

Let $A^C = U - A$, we have the following basic properties of Pawlak's rough sets:

L_1	$\underline{R}(A) \subseteq A$	H_1	$A \subseteq \overline{R}(A)$
L_2	$\underline{R}(\phi) = \phi$	H_2	$\overline{R}(\phi) = \phi$
L_3	$\underline{R}(U) = U$	H_3	$\overline{R}(U) = U$
L_4	$\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$	H_4	$\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$
L_5	$A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$	H_5	$A \subseteq B \Rightarrow \overline{R}(A) \subseteq \overline{R}(B)$
L_6	$\underline{R}(A) \cup \underline{R}(B) \subseteq \underline{R}(A \cup B)$	H_6	$\overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$
L_7	$\underline{R}(A^C) = [\overline{R}(A)]^C$	H_7	$\overline{R}(A^C) = [\underline{R}(A)]^C$
L_8	$\underline{R}(\underline{R}(A)) = \underline{R}(A)$	H_8	$\overline{R}(\overline{R}(A)) = \overline{R}(A)$
L_9	$\underline{R}([\underline{R}(A)]^C) = [\underline{R}(A)]^C$	H_9	$\overline{R}([\overline{R}(A)]^C) = [\overline{R}(A)]^C$
L_{10}	$\forall K \in U/R \Rightarrow \underline{R}(K) = K$	H_{10}	$\forall K \in U/R \Rightarrow \overline{R}(K) = K$.

In [15, 16, 23] binary relation-based rough sets are different from covering-based rough sets, thus we introduce the neighborhood concept into covering-based rough sets.

2.2. Neighborhood and Covering

Suppose R is an arbitrary binary relation on U, the pair (U, R) is called an approximation space. Concerning R, we can define the $R-$left and $R-$right neighborhoods of an element x in U as follows:

$l_R(x) = \{y : y \in U, yRx\}$ and $r_R(x) = \{y : y \in U, xRy\}$, respectively.

Yao [15, 16] defined the operators $\underline{R}, \overline{R}$ from $P(U)$ to $P(U)$ by

$\underline{R}(X) = \{x : r_R(x) \subseteq X\}$ and $\overline{R}X = \{x : r_R(x) \cap X \neq \phi\}$,

$\underline{R}(X)$ is called a lower approximation of X and $\overline{R}(X)$ an upper approximation of X. Note that the definitions of the lower and upper approximations is not unique. For example, we can use the $R-$left neighborhood to define the lower and upper approximations. There are generalized rough sets induced by an arbitrary binary relation in [6, 7, 16].

Definition 2.1. ([2]) Let U be a domain of discourse, $\mathbb{C} = \{K : K \in \mathbf{K}\}$, a family of nonempty subsets of U. If $U = \cup\{K : K \in \mathbf{K}\}$, then $\mathbb{C}$ is called a covering of U. Let $U \neq \phi$ be a finite set and R be any binary relation on U. Set the following:

right cover (briefly, r-cover): $\mathbb{C}_r = \{xR : \forall x \in U \text{ and } U = \underset{x\in U}{\cup} xR\}$,

left cover (breifly, l-cover): $\mathbb{C}_l = \{Rx : \forall x \in U \text{ and } U = \underset{x\in U}{\cup} Rx\}$,

$(U, R, \mathbb{C}_n), n \in \{r, l\}$ is a generalized covering approximation space (for short, $\mathcal{G}_n$-CAS).

Definition 2.2. ([2]) Let $(U, R, \mathbb{C}_n), n \in \{r, l\}$ be a $\mathcal{G}_n$-CAS. Set the neighborhoods $N_j(x), j \in \{r, l, u, i\}$ as the following:

(i) $r-$neighborhood: $N_r(x) = \cap\{K \in \mathbb{C}_r : x \in K\}$;

(ii) $l-$neighborhood: $N_l(x) = \cap\{K \in \mathbb{C}_l : x \in K\}$;

(iii) $u-$neighborhood: $N_u(x) = N_r(x) \cup N_l(x)$;

(iv) $i-$neighborhood: $N_i(x) = N_r(x) \cap N_l(x)$.

3. Bi-Neighborhood and New Types of Bi-Covering Approximation Space

In this section, we present bi-neighborhood and bi-covering concepts as a generalization of the neighborhood and covering resp.

3.1. Bi-Neighborhood

Here we give a generalization of the different types of neighborhoods introduced in [2]. Also, we introduce bi-neighborhood concepts into bi-covering concepts.

Definition 3.1. Let $U \neq \phi$, R_1, R_2 be any binary relations on a finite set U. Set $\mathbb{C}_{r_i} = \{xR_i \forall x \in U \text{ and } U = \underset{x\in U}{\cup} xR_i\}, i \in \{1, 2\}$ and $\mathbb{C}_{l_i} = \{R_i x : \forall x \in U \text{ and } U = \underset{x\in U}{\cup} R_i x\}, i \in \{1, 2\}$. We define the following:

right bicover (breifly, r-bicover): $\mathbb{C}_{r_{12}} = \underset{i\in\{1,2\}}{\cup} \{xR_i : \forall x \in U \text{ and } U = \underset{x\in U}{\cup} xR_i\}$.

left bicover (breifly, l-bicover): $\mathbb{C}_{l_{12}} = \underset{i\in\{1,2\}}{\cup} \{R_i x : \forall x \in U \text{ and } U = \underset{x\in U}{\cup} R_i x\}$.

$(U, R_1, R_2, \mathbb{C}_{l_{12}}, \mathbb{C}_{r_{12}})$ is a generalized bi-covering approximation space (for short, $\mathcal{G}_n$-BCAS) and written as $(U, R_1, R_2, \mathbb{C}_n), n \in \{r_{12}, l_{12}\}$.

Proposition 3.2. *If R_i (resp. R_i^{-1}) is a serial relation on U for $i \in \{1, 2\}$, then $U = \underset{x\in U}{\cup} R_i x, i \in \{1, 2\}$ (resp. $U = \underset{x\in U}{\cup} xR_i, i \in \{1, 2\}$).*

Proof. Let R_i be a serial relation on U, $i \in \{1, 2\}$. It is clear that $\underset{x\in U}{\cup} R_i x \subseteq U, i \in \{1, 2\}$. Conversely, let $y \in U$. There exists at least $z \in U$ such that $(y, z) \in R_i$ and $i \in \{1, 2\}$. Thus $y \in R_i z$ and this implies that $y \in \underset{x\in U}{\cup} R_i x, i \in \{1, 2\}$. Hence $U = \underset{x\in U}{\cup} R_i x, i \in \{1, 2\}$. Similarly, way we can prove the case between parentheses. □

Definition 3.3. Let $(U, R_1, R_2, \mathbb{C}_n), n \in \{r_{12}, l_{12}\}$ be a $\mathcal{G}_n$-BCAS. Set the neighborhoods $N_j(x), j \in \lambda$ as the following:

(i) $r_{12}-$neighborhood: $N_{r_{12}}(x) = \cap\{K \in \mathbb{C}_{r_{12}} : x \in K\}$.

(ii) $l_{12}-$neighborhood: $N_{l_{12}}(x) = \cap\{K \in \mathbb{C}_{l_{12}} : x \in K\}$.

(iii) $r_{12} \vee l_{12}-$neighborhood: $N_{r_{12}\vee l_{12}}(x) = N_{r_{12}}(x) \cup N_{l_{12}}(x)$.

(iv) $r_{12} \wedge l_{12}-$neighborhood: $N_{r_{12}\wedge l_{12}}(x) = N_{r_{12}}(x) \cap N_{l_{12}}(x)$.

In this paper, we use $\lambda = \{r_{12}, l_{12}, r_{12} \vee l_{12}, r_{12} \wedge l_{12}\}$.

In the following, we present some basic properties of the introduced neighborhoods.

Proposition 3.4. *Let $(U, R_1, R_2, \mathbb{C}_n)$, be a $\mathcal{G}_n$-BCAS. For all $j \in \lambda$, $x \in N_j(x)$, $\forall x \in U$.*

Proof. Follows directly from definitions. □

Proposition 3.5. *Let $(U, R_1, R_2, \mathbb{C}_n)$, be a $\mathcal{G}_n$-BCAS. For all $j \in \{r_{12}, l_{12}, r_{12} \wedge l_{12}\}$ if $x \in N_j(y)$ then $N_j(x) \subseteq N_j(y)$.*

Proof. When $j = r_{12}$, let $x \in N_{r_{12}}(y) = \cap\{K \in \mathbb{C}_{r_{12}}, y \in K\}$. Then x contained in any after set containing y. Thus $x \in yR_i$, and thus yR_ix. Let $z \in N_{r_{12}}(x) = \cap\{K \in \mathbb{C}_{r_{12}}, x \in K\}, z \in xR_i, i = 1, 2$. Then $z \in \cap\{K \in \mathbb{C}_{r_{12}}, y \in K\}, z \in N_{r_{12}}(y)$. Hence $N_{r_{12}}(x) \subseteq N_{r_{12}}(y)$. In a similar way we can prove the other cases. □

The previous proposition was not satisfied in the case of $j = r_{12} \vee l_{12}$. As in the following example we have, $x \in N_{r_{12}\vee l_{12}}(y)$ but $N_{r_{12}\vee l_{12}}(x) \nsubseteq N_{r_{12}\vee l_{12}}(y)$

Example 3.6. Let $U = \{a, b, c, d\}$.
When $R_1 = \{(a, a), (a, b), (a, d), (b, a), (b, d), (c, a), (c, d), (c, c), (d, a), (d, d)\}$, we have:
$aR_1 = \{a, b, d\}, bR_1 = \{a, d\}, cR_1 = \{a, c, d\}, dR_1 = \{a, d\}$,
$R_1a = U, R_1b = \{a\}, R_1c = \{c\}, R_1d = U$,
$N_{r_1}(a) = \{a, d\}, N_{r_1}(b) = \{a, b, d\}, N_{r_1}(c) = \{a, c, d\}, N_{r_1}(d) = \{a, d\}$,
$N_{l_1}(a) = \{a\}, N_{l_1}(b) = U, N_{l_1}(c) = \{c\}, N_{l_1}(d) = U$.
When $R_2 = \{(a, a), (a, b), (a, d), (b, a), (b, d), (c, a), (c, d), (c, c), (d, a)\}$, we have:
$aR_2 = \{a, b, d\}, bR_2 = \{a, d\}, cR_2 = \{a, c, d\}, dR_2 = \{a\}$,
$R_2a = U, R_2b = \{a\}, R_2c = \{c\}, R_2d = \{a, b, c\}$,
$N_{r_2}(a) = \{a\}, N_{r_2}(b) = \{a, b, d\}, N_{r_2}(c) = \{a, c, d\}, N_{r_2}(d) = \{a, d\}$,
$N_{l_2}(a) = \{a\}, N_{l_2}(b) = \{a, b, c\}, N_{l_2}(c) = \{c\}, N_{l_2}(d) = U$.
Now, we have
$N_{r_{12}}(a) = \{a\}, N_{r_{12}}(b) = \{a, b, d\}, N_{r_{12}}(c) = \{a, c, d\}, N_{r_{12}}(d) = \{a, d\}$,
$N_{l_{12}}(a) = \{a\}, N_{l_{12}}(b) = \{a, b, c\}, N_{l_{12}}(c) = \{c\}, N_{l_{12}}(d) = U$,
$N_{r_{12}\vee l_{12}}(c) = \{a, c, d\}, d \in N_{r_{12}\vee l_{12}}(c), N_{r_{12}\vee l_{12}}(d) = U \nsubseteq N_{r_{12}\vee l_{12}}(c)$.

The following remark shows that Yao's, Abd Elmonsef's and Pawlak's approximations may be viewed as special cases of our study.

Remark 3.7. In $(U, R_1, R_2, \mathbb{C}_n)$, if R_1, R_2 are equivalence relations on U, then all of $\mathbb{C}_n, n \in \{r_{12}, l_{12}\}$ reduced to a partition on U and thus equivalence classes of R_1, R_2. Also, all $j - nbd$ of x, $N_j(x)$ $\forall j \in \lambda$ are identical to equivalence classes containing x, i.e. $N_j(x) = [x]_R$ $\forall j \in \lambda$. From another view, if $R_1 = R_2$ which are equivalence relations, then the $\mathcal{G}_n$-BCAS becomes Pawlak's approximation space. Finally, if $R_1 = R_2$ which are any binary relations, then the $\mathcal{G}_n$-BCAS becomes Yao's approximation space and $\mathbb{C}_{r_{12}}$ and $\mathbb{C}_{l_{12}}$ became as Abd Elmonsef introduced.

Proposition 3.8. *Let $(U, R_1, R_2, \mathbb{C}_n)$, be a $\mathcal{G}_n$-BCAS. For all $j \in \lambda$, $\{N_j(x) : x \in U\}$ generates a cover of U.*

Proof. Since $x \epsilon N_j(x)$ $\forall j \in \lambda$, then $U = \bigcup_{x\in U} N_j(x)$. Thus $\{N_j(x) : x \in U\}$ represents a cover of U $\forall j \in \lambda$. □

Proposition 3.9. *Let $(U, R_1, R_2, \mathbb{C}_n)$, be a $\mathcal{G}_n$-BCAS.*
(i) $N_{r_{12}\wedge l_{12}}(x) \subseteq N_{r_{12}}(x) \subseteq N_{r_{12}\vee l_{12}}(x)$.
(ii) $N_{r_{12}\wedge l_{12}}(x) \subseteq N_{l_{12}}(x) \subseteq N_{r_{12}\vee l_{12}}(x)$.

Proof. From Definition 3.3, we have
(i) $N_{r_{12}\wedge l_{12}}(x) = N_{r_{12}}(x) \cap N_{l_{12}}(x) \subseteq N_{r_{12}}(x) \subseteq N_{r_{12}}(x) \cup N_{l_{12}}(x) = N_{r_{12}\vee l_{12}}(x)$.
(ii) $N_{r_{12}\wedge l_{12}}(x) = N_{r_{12}}(x) \cap N_{l_{12}}(x) \subseteq N_{l_{12}}(x) \subseteq N_{r_{12}}(x) \cup N_{l_{12}}(x) = N_{r_{12}\vee l_{12}}(x)$. □

Remark 3.10. From Proposition 3.9, we find that, the family of all introduced bi-neighborhoods of

$$x,\ \{N_{r_{12}}(x), N_{l_{12}}(x), N_{r_{12}\wedge l_{12}}(x), N_{r_{12}\vee l_{12}}(x)\},\ \forall x \epsilon U$$

equipped with the binary relation of inclusion $\subseteq$, constructs a lattice, see Figure 1.

Also, $\{U, \phi, N_{r_{12}}(x), N_{l_{12}}(x), N_{r_{12}\wedge l_{12}}(x), N_{r_{12}\vee l_{12}}(x)\}$ forms a topology on U for all $x \epsilon U$.

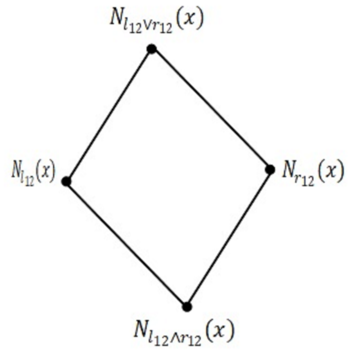

Figure 1:

3.2. New Rough Sets Approximations Depending on New Concepts

Based on $N_j(x)$, $j \in \lambda$ we now give definitions to the lower and upper approximations of new types of bi-covering rough sets.

Definition 3.11. Let $(U, R_1, R_2, \mathbb{C}_n)$ be a $\mathcal{G}_n$-BCAS. For all $j \in \lambda, A \subseteq U$ the $j-$lower approximation $\underline{R}_j(A)$ and $j-$upper approximation $\overline{R}_j(A)$ of A can be defined as follows:
(i) $\underline{R}_j(A) = \{x \in A : N_j(x) \subseteq A\}$.
(ii) $\overline{R}_j(A) = \{x \in U : N_j(x) \cap A \neq \phi\}$.

It is urgent to study the relation between the introduced approximations and each other as in the following proposition.

Theorem 3.12. *Let $(U, R_1, R_2, \mathbb{C}_n)$ be a $\mathcal{G}_n$-BCAS, $A \subseteq U$. Then:*
(i) $\underline{R}_{r_{12}\vee l_{12}}(A) \subseteq \underline{R}_{r_{12}}(A) \subseteq \underline{R}_{r_{12}\wedge l_{12}}(A)$ (resp. $\underline{R}_{r_{12}\vee l_{12}}(A) \subseteq \underline{R}_{l_{12}}(A) \subseteq \underline{R}_{r_{12}\wedge l_{12}}(A)$).
(ii) $\overline{R}_{r_{12}\wedge l_{12}}(A) \subseteq \overline{R}_{r_{12}}(A) \subseteq \overline{R}_{r_{12}\vee l_{12}}(A)$ (resp. $\overline{R}_{r_{12}\wedge l_{12}}(A) \subseteq \overline{R}_{l_{12}}(A) \subseteq \overline{R}_{r_{12}\vee l_{12}}(A)$).

Proof. (i) Let $x \in \underline{R}_{r_{12}\vee l_{12}}(A)$. Then $x \in A$ and $N_{r_{12}\vee l_{12}}(x) = N_{r_{12}}(x) \cup N_{l_{12}}(x) \subseteq A$. Thus $x \in A$ and $N_{r_{12}}(x) \subseteq A$ (resp. $N_{l_{12}}(x) \subseteq A$). And thus $x \in \underline{R}_{r_{12}}(A)$ (resp. $x \in \underline{R}_{l_{12}}(A)$). Therefore $\underline{R}_{r_{12}\vee l_{12}}(A) \subseteq \underline{R}_{r_{12}}(A)$ (resp. $\underline{R}_{r_{12}\vee l_{12}}(A) \subseteq \underline{R}_{l_{12}}(A)$).

Also, if $x \in \underline{R}_{r_{12}}(A)$(resp. $x \in \underline{R}_{l_{12}}(A)$). Then $N_{r_{12}}(x) \subseteq A$ (resp. $N_{l_{12}}(x) \subseteq A$). Therefore $x \in A$ and $N_{r_{12}\wedge l_{12}}(x) = N_{r_{12}}(x) \cap N_{l_{12}}(x) \subseteq A$. Thus $x \in A$ and $N_{r_{12}\wedge l_{12}}(x) \subseteq A$. And thus $x \in \underline{R}_{r_{12}\wedge l_{12}}(A)$. Hence $\underline{R}_{r_{12}}(A) \subseteq \underline{R}_{r_{12}\wedge l_{12}}(A)$ (resp. $\underline{R}_{l_{12}}(A) \subseteq \underline{R}_{r_{12}\wedge l_{12}}(A)$). So $\underline{R}_{r_{12}\vee l_{12}}(A) \subseteq \underline{R}_{r_{12}}(A) \subseteq \underline{R}_{r_{12}\wedge l_{12}}(A)$ (resp. $\underline{R}_{r_{12}\vee l_{12}}(A) \subseteq \underline{R}_{l_{12}}(A) \subseteq \underline{R}_{r_{12}\wedge l_{12}}(A)$).

(ii) Let $x \in \overline{R}_{r_{12}\wedge l_{12}}(A)$. Then $N_{r_{12}\wedge l_{12}}(x) \cap A \neq \phi$. Therefore $(N_{r_{12}}(x) \cap N_{l_{12}}(x)) \cap A \neq \phi$. Thus $(N_{r_{12}}(x) \cap A) \cap (N_{l_{12}}(x) \cap A) \neq \phi$. Hence $N_{r_{12}}(x) \cap A \neq \phi$ (resp. $N_{l_{12}}(x) \cap A \neq \phi$). And hence $x \in \overline{R}_{r_{12}}(A)$(resp. $x \in \overline{R}_{l_{12}}(A)$). Let $x \in \overline{R}_{r_{12}}(A)$ (resp. $x \in \overline{R}_{l_{12}}(A)$). Then $N_{r_{12}}(x) \cap A \neq \phi$ (resp. $N_{l_{12}}(x) \cap A \neq \phi$). Therefore $(N_{r_{12}}(x) \cap A) \cup (N_{l_{12}}(x) \cap A) \neq \phi$. Thus $(N_{r_{12}}(x) \cup N_{l_{12}}(x)) \cap A \neq \phi$. Hence $x \in \overline{R}_{r_{12}\vee l_{12}}(A)$. So $\overline{R}_{r_{12}\wedge l_{12}}(A) \subseteq \overline{R}_{r_{12}}(A) \subseteq \overline{R}_{r_{12}\vee l_{12}}(A)$ (resp. $\overline{R}_{r_{12}\wedge l_{12}}(A) \subseteq \overline{R}_{l_{12}}(A) \subseteq \overline{R}_{r_{12}\vee l_{12}}(A)$). □

Definition 3.13. Let $(U, R_1, R_2, \mathbb{C}_n)$ be a $\mathcal{G}_n$-BCAS. For all $j \in \lambda, A \subseteq U$, A is called a $j-$exact set if $\underline{R}_j(A) = \overline{R}_j(A)$. Otherwise A is called $j-$rough set.

It is easy to see the introduced lower approximation is a generalization of those in the other types of covering rough sets [9, 23], but the upper approximation is different.

Definition 3.14. Let $(U, R_1, R_2, \mathbb{C}_n)$ be a $\mathcal{G}_n$-BCAS. For all $j \in \lambda, A \subseteq U$, the $j-$boundary (resp. $j-$positive, $j-$negative) regions of A are defined respectively as follows $\mathrm{Bn}_j(A) = \overline{R}_j(A) - \underline{R}_j(A)$ (resp. $\mathrm{POS}_j(A) = \underline{R}_j(A)$, $\mathrm{NEG}_j(A) = U - \overline{R}_j(A)$).

Theorem 3.15. *Let* $(U, R_1, R_2, \mathbb{C}_n), n \in \{r_{12}, l_{12}\}$ *be a* $\mathcal{G}_n$*-BCAS,* $A \subseteq U$*. Then* $Bn_{r_{12}\wedge l_{12}}(A) \subseteq Bn_{r_{12}}(A) \subseteq Bn_{r_{12}\vee l_{12}}(A)$ *(resp.* $Bn_{r_{12}\wedge l_{12}}(A) \subseteq Bn_{l_{12}}(A) \subseteq Bn_{r_{12}\vee l_{12}}(A)$*).*

Proof. Obvious from Proposition 3.9 and Definitions 3.11, 3.14 and Theorem 3.12. □

Definition 3.16. Let $(U, R_1, R_2, \mathbb{C}_n)$ be a $\mathcal{G}_n$-BCAS. For all $j \in \lambda, \phi \neq A \subseteq U$, the $j-$accuracy of the approximation of A defined as $\alpha_j(A) = \frac{|\underline{R}_j(A)|}{|\overline{R}_j(A)|}$.

Theorem 3.17. *Let* $(U, R_1, R_2, \mathbb{C}_n), n \in \{r_{12}, l_{12}\}$ *be a* $\mathcal{G}_n$*-BCAS,* $A \subseteq U$*. Then* $\alpha_{r_{12}\wedge l_{12}}(A) \geqslant \alpha_{r_{12}}(A) \geqslant \alpha_{r_{12}\vee l_{12}}(A)$ *(resp.* $\alpha_{r_{12}\wedge l_{12}}(A) \geqslant \alpha_{l_{12}}(A) \geqslant \alpha_{r_{12}\vee l_{12}}(A)$*).*

Proof. Obvious from Proposition 3.9 and Definitions 3.14, 3.16 and Theorem 3.12. □

Remark 3.18. From Proposition 3.9 and Definition 3.11, we find that for any $A \subseteq U$, the set of all these lower and upper approximations given by the introduced approximations equipped with the binary relation of inclusion $\subseteq$, construct a lattice. From this, we can see that among the introduced pairs of lower and upper approximations, the pair $(\underline{R}_{r_{12}\wedge l_{12}}(A), \overline{R}_{r_{12}\wedge l_{12}}(A))$ is the best to describe A.

As we draw the Figure 2, where the lower element is a subset of the upper element when they are linked by one line.

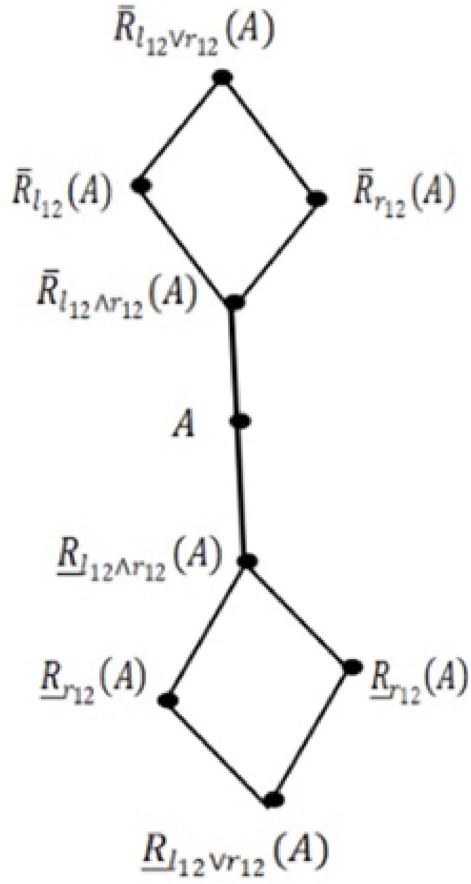

Figure 2:

Proposition 3.19. *Let $(U, R_1, R_2, \mathbb{C}_n), n \in \{r_{12}, l_{12}\}$ be a $\mathcal{G}_n$-BCAS, $A \subseteq U$. Then A is $r_{12} \vee l_{12}-$ exact $\Rightarrow$ A is $l_{12}-$exact$\Rightarrow$ A is $r_{12} \wedge l_{12}-$exact (resp. A is $r_{12} \vee l_{12}-$ exact $\Rightarrow$ A is $r_{12}-$exact$\Rightarrow$ A is $r_{12} \wedge l_{12}-$exact).*

Proof. Let $A \subseteq U$, A be $r_{12} \vee l_{12}-$ exact. Then $Bn_{r_{12}\vee l_{12}}(A) = \phi$. Since $Bn_{l_{12}}(A) \subseteq Bn_{r_{12}\vee l_{12}}(A)$, then $Bn_{l_{12}}(A) = \phi$. Therefore A be $l_{12}-$ exact. Thus

$$A \text{ is } r_{12} \vee l_{12} - \text{exact} \Rightarrow A \text{ is } l_{12} - \text{exact}. \tag{i}$$

Let $A \subseteq U$, A be $l_{12}-$ exact. Then $Bn_{l_{12}}(A) = \phi$. Since $Bn_{r_{12}\wedge l_{12}}(A) \subseteq Bn_{l_{12}}(A)$, then $Bn_{r_{12}\wedge l_{12}}(A) = \phi$. Therefore A be $r_{12} \wedge l_{12}-$ exact. Thus

$$A \text{ is } l_{12} - \text{exact} \Rightarrow A \text{ is } r_{12} \wedge l_{12} - \text{exact}. \tag{ii}$$

From (i), (ii) we have, A is $r_{12} \vee l_{12}-$ exact $\Rightarrow$ A is $l_{12}-$exact$\Rightarrow$ A is $r_{12} \wedge l_{12}-$exact. In the same way, we can prove the case inside parentheses. □

4. Properties of Different Pairs of Lower and Upper Approximations

It would be interesting to study the properties of new approximations.

Proposition 4.1. *Let $(U, R_1, R_2, \mathbb{C}_n)$ be a $\mathcal{G}_n$-BCAS, $A, B \subseteq U$. Then:*
(i) $\underline{R}_j(A) \subseteq A \subseteq \overline{R}_j(A)$.
(ii) $\underline{R}_j(U) = U = \overline{R}_j(U)$.
(iii) $\underline{R}_j(\phi) = \phi = \overline{R}_j(\phi)$.
(iv) If $A \subseteq B$ then $\underline{R}_j(A) \subseteq \underline{R}_j(B)$ and $\overline{R}_j(A) \subseteq \overline{R}_j(B)$.
(v) $\underline{R}_j(A) = [\overline{R}_j(A^C)]^C$.
(vi) $\overline{R}_j(A) = [\underline{R}_j(A^C)]^C$.
(vii) $\underline{R}_j(A \cap B) = \underline{R}_j(A) \cap \underline{R}_j(B)$ and $\overline{R}_j(A \cup B) = \overline{R}_j(A) \cup \overline{R}_j(B)$.
(viii) $\underline{R}_j(A) \cup \underline{R}_j(B) \subseteq \underline{R}_j(A \cup B)$ and $\overline{R}_j(A \cap B) \subseteq \overline{R}_j(A) \cap \overline{R}_j(B)$.

Proof. The proof of (i), (ii) and (iii) obvious from Definition 3.11. when $j = r_{12}$,

(iv) Let $x \in \underline{R}_{r_{12}}(A)$. Then $N_{r_{12}}(x) \subseteq A \subseteq B$. Therefore $x \in \underline{R}_{12}(B)$. Thus $\underline{R}_{12}(A) \subseteq \underline{R}_{12}(B)$. Let $x \in \overline{R}_{r_{12}}(A)$. Then $N_{r_{12}}(x) \cap A \neq \phi$. Therefore $N_{r_{12}}(x) \cap B \neq \phi$. Thus $x \in \overline{R}_{r_{12}}(B)$. Hence $\overline{R}_{r_{12}}(A) \subseteq \overline{R}_{r_{12}}(B)$.

(v) Let $x \in [\overline{R}_{r_{12}}(A^C)]^C$. Then $x \notin \overline{R}_{r_{12}}(A^C)$. Therefore $\exists N_{r_{12}}(x)$ s.t. $N_{r_{12}}(x) \cap A^C = \phi$. Thus $x \in N_{r_{12}}(x) \subseteq A$, and thus $x \in \underline{R}_{r_{12}}(A)$. Hence $[\overline{R}_{r_{12}}(A^C)]^C \subseteq \underline{R}_{r_{12}}(A)$. Let $x \in \underline{R}_{r_{12}}(A)$. Then $N_{r_{12}}(x) \subseteq A$. Thus $N_{r_{12}}(x) \cap A^C = \phi$ and thus $x \notin \overline{R}_j(A^C)$. Therefore $x \in [\overline{R}_j(A^C)]^C$. Hence $\underline{R}_{r_{12}}(A) \subseteq [\overline{R}_{r_{12}}(A^C)]^C$.

(vi) Let $x \in [\underline{R}_{r_{12}}(A^C)]^C$. Then $x \notin \underline{R}_{r_{12}}(A^C)$. Therefore $\nexists N_{r_{12}}(x)$ s.t. $N_{r_{12}}(x) \subseteq A^C$. Thus $\nexists N_{r_{12}}(x)$ s.t. $N_{r_{12}}(x) \cap A = \phi$, and thus $x \in \overline{R}_{r_{12}}(A)$. Hence $[\underline{R}_{r_{12}}(A^C)]^C \subseteq \overline{R}_{r_{12}}(A)$. Let $x \in \underline{R}_{r_{12}}(A)$. Then $N_{r_{12}}(x) \subseteq A$. Thus $N_{r_{12}}(x) \cap A^C = \phi$ and thus $x \notin \overline{R}_{r_{12}}(A^C)$. Therefore $x \in [\overline{R}_{r_{12}}(A^C)]^C$. Hence $\underline{R}_{r_{12}}(A) \subseteq [\overline{R}_{r_{12}}(A^C)]^C$. By reversing the previous steps we obtain that $[\overline{R}_{r_{12}}(A^C)]^C \subseteq \underline{R}_{r_{12}}(A)$.

(vii) Let $x \in \underline{R}_{r_{12}}(A) \cap \underline{R}_{r_{12}}(B)$. Then $x \in \underline{R}_{r_{12}}(A)$ and $x \in \underline{R}_{r_{12}}(B)$. Therefore $x \in N_{r_{12}}(x) \subseteq A$ and $x \in N_{r_{12}}(x) \subseteq B$. Thus $x \in N_{r_{12}}(x) \subseteq A \cap B$, and thus $x \in \underline{R}_{r_{12}}(A \cap B)$. Hence $\underline{R}_{r_{12}}(A) \cap \underline{R}_{r_{12}}(B) \subseteq \underline{R}_{r_{12}}(A \cap B)$.

Let $\underline{R}_{r_{12}}(A \cap B)$. Then $x \in N_{r_{12}}(x) \subseteq A \cap B$. Thus $x \in N_{r_{12}}(x) \subseteq A$ and $x \in N_{r_{12}}(x) \subseteq B$. Therefore $x \in \underline{R}_{r_{12}}(A)$ and $x \in \underline{R}_{r_{12}}(A)$. Hence $x \in \underline{R}_{r_{12}}(A) \cap \underline{R}_{r_{12}}(A)$. So $\underline{R}_{r_{12}}(A \cap B) \subseteq \underline{R}_{r_{12}}(A) \cap \underline{R}_{r_{12}}(A)$.

Now, we have $\underline{R}_{r_{12}}(A \cap B) = \underline{R}_{r_{12}}(A) \cap \underline{R}_{r_{12}}(A)$. In a similar way we can proof that $\overline{R}_{r_{12}}(A \cup B) = \overline{R}_{r_{12}}(A) \cup \overline{R}_{r_{12}}(B)$. □

An example to show that properties L_8 and H_8 in [8] are not satisfied in case of new concepts.

Example 4.2. In Example 3.6, taking $A = \{a, c, d\}$, we have $N_{r_{12\vee l_{12}}}(a) = \{a\}$,$N_{r_{12\vee l_{12}}}(b) = U$, $N_{r_{12\vee l_{12}}}(c) = \{a, c, d\}$, $N_{r_{12\vee l_{12}}}(d) = U$, $\underline{R}_{r_{12\vee l_{12}}}(A) = \{a, c\}$, $\underline{R}_{r_{12\vee l_{12}}}(\underline{R}_{r_{12\vee l_{12}}}(A)) = \{a\}$, $\underline{R}_{r_{12\vee l_{12}}}(\underline{R}_{r_{12\vee l_{12}}}(A)) = \{a\} \neq \{a, c\} = \underline{R}_{r_{12\vee l_{12}}}(A)$.

One can find another example to show that $\overline{R}_{r_{12}}(\overline{R}_{r_{12}}(A)) \neq \overline{R}_{r_{12}}(A)$.

When $A = \{d\}$, we have $\overline{R}_{r_{12\vee l_{12}}}(A) = \{b, c, d\}, [\overline{R}_{r_{12\vee l_{12}}}(A)]^C = \{a\}, \overline{R}_{r_{12\vee l_{12}}}([\overline{R}_{r_{12\vee l_{12}}}(A)]^C) = \overline{R}_{r_{12\vee l_{12}}}(\{a\}) = U$. Hence $\overline{R}_{r_{12\vee l_{12}}}([\overline{R}_{r_{12\vee l_{12}}}(A)]^C) \neq [\overline{R}_{r_{12\vee l_{12}}}(A)]^C$.

The property (viii) in Proposition 4.1, the inclusion relation can not be replaced by the equal relation as shown in the following.

Example 4.3. In Example 3.6, Let $A = \{a, c\}, B = \{d\}$. Then $\underline{R}_{r_{12\vee l_{12}}}(A) = \{a\}, \underline{R}_{r_{12\vee l_{12}}}(B) = \phi, \underline{R}_{r_{12\vee l_{12}}}(A \cup B) = \underline{R}_{r_{12\vee l_{12}}}(\{a, c, d\}) = \{a, c\}$. Therefore $\underline{R}_{r_{12\vee l_{12}}}(A \cup B) \not\subseteq \underline{R}_{r_{12\vee l_{12}}}(A) \cup \underline{R}_{r_{12\vee l_{12}}}(B)$.

Also, when $A = \{a, c\}, B = \{d\}$, $\overline{R}_{r_{12\vee l_{12}}}(A) = \{a, b, c, d\}, \overline{R}_{r_{12\vee l_{12}}}(B) = \{b, c, d\}$, $\overline{R}_{r_{12\vee l_{12}}}(A) \cap \overline{R}_{r_{12\vee l_{12}}}(B) = \{b, c, d\}$. $A \cap B = \phi, \overline{R}_{r_{12\vee l_{12}}}(A \cap B) = \phi$, hence $\overline{R}_{r_{12\vee l_{12}}}(A) \cap \overline{R}_{r_{12\vee l_{12}}}(B) \not\subseteq \overline{R}_{r_{12\vee l_{12}}}(A \cap B)$.

An example to show that property L_9 in [8] is not satisfied in case of new concepts.

Example 4.4. In Example 3.6. When $A = \{a, c\}$, we have

$$\underline{R}_{r_{12\vee l_{12}}}(A) = \{a\}, [\underline{R}_{r_{12\vee l_{12}}}(A)]^C = \{b, c, d\}, \underline{R}_{r_{12\vee l_{12}}}([\underline{R}_{r_{12\vee l_{12}}}(A)]^C) = \underline{R}_{r_{12\vee l_{12}}}(\{b, c, d\}) = \phi.$$

Hence $\underline{R}_{r_{12\vee l_{12}}}([\underline{R}_{r_{12\vee l_{12}}}(A)]^C) \neq [\underline{R}_{r_{12\vee l_{12}}}(A)]^C$.

In the following table, we collect properties and results obtained in this section. We call all the operators from $P(U)$ to $P(U)$ such as $\underline{R}_{r_{12}}, \underline{R}_{l_{12}}, \underline{R}_{r_{12}\vee l_{12}}$ and $\underline{R}_{r_{12}\wedge l_{12}}$ the lower approximation operators on U, and all the operators such as $\overline{R}_{r_{12}}, \overline{R}_{l_{12}}, \overline{R}_{r_{12}\vee l_{12}}$ and $\overline{R}_{r_{12}\wedge l_{12}}$ the upper approximation operators on U. In the table below we list codes to show whether these approximations satisfy the properties (L1)–(H10). In the table, figure ″1″ denotes ″yes″, ″0″ denotes ″not″ and there is an example and ″*″ denotes ″not″ and we can similarly add an example.

	$\underline{R}_{r_{12}}$	$\underline{R}_{l_{12}}$	$\underline{R}_{r_{12}\vee l_{12}}$	$\underline{R}_{r_{12}\wedge l_{12}}$		$\overline{R}_{r_{12}}$	$\overline{R}_{l_{12}}$	$\overline{R}_{r_{12}\vee l_{12}}$	$\overline{R}_{r_{12}\wedge l_{12}}$
L_1	1	1	1	1	H_1	1	1	1	1
L_2	1	1	1	1	H_2	1	1	1	1
L_3	1	1	1	1	H_3	1	1	1	1
L_4	1	1	1	1	H_4	1	1	1	1
L_5	1	1	1	1	H_5	1	1	1	1
L_6	1	1	1	1	H_6	1	1	1	1
L_7	1	1	1	1	H_7	1	1	1	1
L_8	*	*	0	*	H_8	0	*	*	*
L_9	*	*	0	*	H_9	0	*	*	*

5. Conclusion

In this paper, based on the notions of bi-neighborhoods, we defined some types of bi-covering rough sets. The relationship between the considered types of bi-covering rough sets was given. We show that Yao's, Abd Elmosef's and Pawlak's approximations may be viewed as special cases of our study. So the suggested models help in decreasing the boundary region of any vague concept in an information system. Therefore these notions are more precise than other one that have been identified by other authors. Consequently, in decision making, our methods are very interesting. Finally, we find that the family of all introduced bi-neighborhoods of x, $\{N_{r_{12}}(x), N_{l_{12}}(x), N_{r_{12}\wedge l_{12}}(x), N_{r_{12}\vee l_{12}}(x)\}$ $\forall$ $x \epsilon U$, equipped with the binary relation of inclusion $\subseteq$ constructs a lattice. Also, the lower and upper approximations given by the introduced approximations equipped with the binary relation of inclusion $\subseteq$, constructs a lattice. From this, we can see that among the introduced pairs of lower and upper approximations, the pair $(\underline{R}_{r_{12}\wedge l_{12}}(A), \overline{R}_{r_{12}\wedge l_{12}}(A))$ is the best to describe A.

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