Filomat 35:7 (2021), 2349–2360
https://doi.org/10.2298/FIL2107349H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Identities Related to Generalized Derivations and Jordan $(*, \star)$-Derivations

Amin Hosseini[a]

[a] *Kashmar Higher Education Institute, Kashmar, Iran*

Abstract. The main purpose of this research is to characterize generalized (left) derivations and Jordan $(*, \star)$-derivations on Banach algebras and rings using some functional identities. Let $\mathcal{A}$ be a unital semiprime Banach algebra and let $F, G : \mathcal{A} \to \mathcal{A}$ be linear mappings satisfying $F(x) = -x^2G(x^{-1})$ for all $x \in Inv(\mathcal{A})$, where $Inv(\mathcal{A})$ denotes the set of all invertible elements of $\mathcal{A}$. Then both F and G are generalized derivations on $\mathcal{A}$. Another result in this regard is as follows. Let $\mathcal{A}$ be a unital semiprime algebra and let $n > 1$ be an integer. Let $f, g : \mathcal{A} \to \mathcal{A}$ be linear mappings satisfying $f(a^n) = na^{n-1}g(a) = ng(a)a^{n-1}$ for all $a \in \mathcal{A}$. If $g(\mathbf{e}) \in Z(\mathcal{A})$, then f and g are generalized derivations associated with the same derivation on $\mathcal{A}$. In particular, if $\mathcal{A}$ is a unital semisimple Banach algebra, then both f and g are continuous linear mappings. Moreover, we define a $(*, \star)$-ring and a Jordan $(*, \star)$-derivation. A characterization of Jordan $(*, \star)$-derivations is presented as follows. Let $\mathcal{R}$ be an $n!$-torsion free $(*, \star)$-ring, let $n > 1$ be an integer and let $d : \mathcal{R} \to \mathcal{R}$ be an additive mapping satisfying $d(a^n) = \sum_{j=1}^{n} a^{\star\, n-j} d(a) a^{*\, j-1}$ for all $a \in \mathcal{R}$. Then d is a Jordan $(*, \star)$-derivation on $\mathcal{R}$. Some other functional identities are also investigated.

1. Introduction and preliminaries

The notion of generalized derivation was defined by Brešar [9]. Let $\mathcal{A}$ be an algebra. A linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called a generalized derivation if there exists a derivation $d : \mathcal{A} \to \mathcal{A}$ such that $f(ab) = f(a)b + ad(b)$ for all $a, b \in \mathcal{R}$. We say f is inner if there exist $c, b \in \mathcal{A}$ such that $f(a) = ca - ab$ for all $a \in \mathcal{A}$. For results concerning generalized derivations we refer the reader to [1, 9, 15, 18, 20, 21]. Brešar and Vukman introduced the concept of a left derivation as follows. An additive mapping $d : \mathcal{R} \to \mathcal{X}$, where $\mathcal{R}$ is a ring and $\mathcal{X}$ is a left $\mathcal{R}$ module is called a left derivation if $d(rs) = rd(s) + sd(r)$ for all $r, s \in \mathcal{R}$. An additive mapping $d : \mathcal{R} \to \mathcal{X}$ is called a Jordan left derivation if $d(r^2) = 2rd(r)$ for all $r \in \mathcal{R}$. Inspired by the definition of a generalized derivation, Ashraf and Ali [2] introduced the notion of a generalized left derivation as follows: an additive mapping $G : \mathcal{R} \to \mathcal{R}$ is called a generalized left derivation (resp. generalized Jordan left derivation) if there exists a left derivation $\delta : \mathcal{R} \to \mathcal{R}$ (resp. Jordan left derivation $\delta : \mathcal{R} \to \mathcal{R}$) such that $G(xy) = xG(y) + y\delta(x)$ (resp. $G(x^2) = xG(x) + x\delta(x)$) holds for all $x, y \in \mathcal{R}$. In the same paper they gave an example of a generalized Jordan left derivation which is not a generalized left derivation (see [2, Example 1.1]). The main aim of this article is to characterize generalized (left) derivations and Jordan $(*, \star)$-derivations using functional identities. ”A functional identity (FI) can be informally described as

2010 *Mathematics Subject Classification*. Primary 47B47; Secondary 39B05
Keywords. generalized derivation; generalized left derivation; Jordan $(*, \star)$-derivation; Banach algebra; $(*, \star)$-ring
Received: 12 June 2020; Accepted: 28 June 2020
Communicated by Dragan S. Djordjević
Email address: a.hosseini@kashmar.ac.ir, hussi.kashm@gmail.com (Amin Hosseini)

an identical relation involving (arbitrary) elements in a ring together with ("unknown") functions; more precisely, elements are multiplied by values of functions. The goal of the general *FI theory* is to determine the form of these functions, or, when this is not possible, to determine the structure of the ring admitting the *FI* in question. This theory has turned out to be a powerful tool for solving a variety of problems in different areas". For more material about *FI theory*, see, e.g. [5]. Let us give a brief background of this study. We know that every derivation is a Jordan derivation, but the converse is in general not true (see [14, Example 2.8]). So, it is a natural and classical question in which algebras (and rings) a Jordan derivation is necessarily a derivation. In an important and interesting article, Herstein [16] answered this question. Indeed, he achieved a fundamental result which began characterizing the derivations on rings. He proved that any Jordan derivation on a prime ring of characteristic different from two is a derivation. A brief proof of Herstein's result can be found in [8] and in 1975, Cusack [12] generalized Herstein's result to 2-torsion free semiprime rings (see also [7] for an alternative proof). Recently, the author along with Ajda Fošner [17] have investigated the same problem for (σ, τ)-derivations from a C^*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-module $\mathcal{M}$. It should be mentioned that Herstein theorem has been fairly generalized by Beidar et al [4]. Vukman [10, 24, 25] used some interesting functional identities to characterize derivations and left derivations. Now, we come to the part where we talk about the achievements of this article. Let $\mathcal{A}$ be a unital semiprime Banach algebra and let $F, G : \mathcal{A} \to \mathcal{A}$ be linear mappings satisfying $F(x) = xG(x^{-1})$ for all $x \in Inv(\mathcal{A})$, where $Inv(\mathcal{A})$ denotes the set of all invertible elements of $\mathcal{A}$. Then $F = 0 = G$. Vukman [25, Theorem 5] proved a theorem stating if $\mathcal{A}$ is a unital semisimple Banach algebra and $F, G : \mathcal{A} \to \mathcal{A}$ are linear mappings satisfying $F(x) = -x^2G(x^{-1})$ for all $x \in Inv(\mathcal{A})$, then $F = G$ and further, $F(a) = aF(\mathbf{e})$ for all $a \in \mathcal{A}$. We extend that theorem as follows. Let $\mathcal{A}$ be a unital semiprime Banach algebra and let $F, G : \mathcal{A} \to \mathcal{A}$ be linear mappings satisfying $F(x) = -x^2G(x^{-1})$ for all $x \in Inv(\mathcal{A})$. Then both F and G are generalized (left) derivations on $\mathcal{A}$. In addition, some consequences of this result are presented. Another objective of this study is to characterize Jordan $(*, \star)$-derivations. Let $\mathcal{R}$ be a ring. An additive mapping $x \mapsto x^*$ of $\mathcal{R}$ into itself is called an involution in case $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in \mathcal{R}$. A ring $\mathcal{R}$ equipped with an involution '$*$' is called a $*$-ring. Let $\mathcal{R}$ be a $*$-ring. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a $*$-derivation (resp. a Jordan $*$-derivation) if $d(xy) = d(x)y^* + xd(y)$ (resp. $d(x^2) = d(x)x^* + xd(x)$) holds for all $x, y \in \mathcal{R}$. For instance, see [3, 11, 19, 22] and references therein for details on this topic. Inspiring by the definition of $*$-ring, $*$-derivation and Jordan $*$-derivation, we introduce $(*, \star)$-ring, $(*, \star)$-derivation and Jordan $(*, \star)$-derivation and then a characterization of Jordan $(*, \star)$-derivations is presented. In order to characterize the Jordan $(*, \star)$-derivations on the $(*, \star)$-rings, we prove the following result. Let $\mathcal{R}$ be an $n!$-torsion free $(*, \star)$-ring, $n > 1$ be an integer and let $d : \mathcal{R} \to \mathcal{R}$ be an additive mapping satisfying

$$d(a^n) = \sum_{j=1}^{n} a^{\star^{n-j}} d(a) a^{*^{j-1}}$$

for all $a \in \mathcal{R}$. Then d is a Jordan $(*, \star)$-derivation on $\mathcal{R}$.

2. Results and proofs

Throughout this section, without further mention, $\mathbf{e}$ stands for the unit element of any unital algebra or unital ring. Recall that an algebra (or ring) $\mathcal{A}$ is called prime if for $a, b \in \mathcal{A}$, $a\mathcal{A}b = \{0\}$ implies $a = 0$ or $b = 0$, and is semiprime if for $a \in \mathcal{A}$, $a\mathcal{A}a = \{0\}$ implies $a = 0$. A ring $\mathcal{R}$ is said to be n-torsion free, where $n > 1$ is an integer, if for any $x \in \mathcal{R}$, $nx = 0$ implies that $x = 0$. The center of an algebra (or ring) $\mathcal{A}$ is

$$Z(\mathcal{A}) = \{c \in \mathcal{A} \mid ac = ca \ for \ all \ a \in \mathcal{A}\}.$$

Here, we establish our first theorem.

Theorem 2.1. *Let $\mathcal{A}$ be a unital semiprime Banach algebra and let $F, G : \mathcal{A} \to \mathcal{A}$ be linear mappings satisfying*

$$F(x) = xG(x^{-1}) \tag{1}$$

for all $x \in Inv(\mathcal{A})$. Then $F = 0 = G$.

Proof. Let b be an arbitrary element of $\mathcal{A}$ and let $n > 1$ be a real number such that $\|\frac{b}{n-1}\| < 1$. It is evident that $\|\frac{b}{n}\| < 1$, too. Letting $a = n\mathbf{e} + b$, we have $\frac{a}{n} = \mathbf{e} + \frac{b}{n} = \mathbf{e} - \frac{-b}{n}$. Since $\|\frac{-b}{n}\| < 1$, it follows from [13, Theorem 1.4.2] that $\mathbf{e} - \frac{-b}{n}$ is invertible and consequently, a is invertible. Similarly, we can show that $\mathbf{e} - a$ is also an invertible element of $\mathcal{A}$. In the following, we use the well-known Hua identity $a^2 = a - \left(a^{-1} + (\mathbf{e} - a)^{-1}\right)^{-1}$. Applying (1), we have the following expressions:

$$\begin{aligned}
F(a^2) &= F\Big(a - \left(a^{-1} + (\mathbf{e} - a)^{-1}\right)^{-1}\Big)\\
&= F(a) - F\Big(\left(a^{-1} + (\mathbf{e} - a)^{-1}\right)^{-1}\Big)\\
&= F(a) - \left(a^{-1} + (\mathbf{e} - a)^{-1}\right)^{-1} G(a^{-1} + (\mathbf{e} - a)^{-1})\\
&= F(a) - \left(a^{-1} + (\mathbf{e} - a)^{-1}\right)^{-1} G(a^{-1}) - \left(a^{-1} + (\mathbf{e} - a)^{-1}\right)^{-1} G((\mathbf{e} - a)^{-1})\\
&= F(a) - \left(a^{-1} + (\mathbf{e} - a)^{-1}\right)^{-1} a^{-1}F(a) - \left(a^{-1} + (\mathbf{e} - a)^{-1}\right)^{-1} (\mathbf{e} - a)^{-1}F(\mathbf{e} - a)\\
&= F(a) - \left(\mathbf{e} + a(\mathbf{e} - a)^{-1}\right)^{-1} F(a) - \left(\mathbf{e} + a^{-1} - \mathbf{e}\right)^{-1} F(\mathbf{e} - a)\\
&= F(a) - \left(\mathbf{e} + a(\mathbf{e} - a)^{-1}\right)^{-1} F(a) - aF(\mathbf{e} - a)\\
&= F(a) - (\mathbf{e} - a)F(a) - aF(\mathbf{e} - a)\\
&= 2aF(a) - aF(\mathbf{e}),
\end{aligned}$$

which means that

$$F(a^2) = 2aF(a) - aF(\mathbf{e}) \tag{2}$$

Therefore, we have $F(n^2\mathbf{e} + 2nb + b^2) = 2(n\mathbf{e} + b)F(n\mathbf{e} + b) - (n\mathbf{e} + b)F(\mathbf{e})$ and so,

$$F(b^2) - 2bF(b) + bF(\mathbf{e}) = n^2F(\mathbf{e}) + 2nbF(\mathbf{e}) - nF(\mathbf{e}). \tag{3}$$

Putting $b = 0$ in (3), we have $(n^2 - n)F(\mathbf{e}) = 0$ which implies that $F(\mathbf{e}) = 0$. Considering this fact along with (2), we get that $F(a^2) = 2aF(a)$ and since $a = n\mathbf{e} + b$, we deduce that $F(b^2) = 2bF(b)$ for any $b \in \mathcal{A}$. Thus, F is a Jordan left derivation and since $\mathcal{A}$ is a semiprime algebra, it follows from [25, Theorem 2] that F is a derivation mapping $\mathcal{A}$ into $Z(\mathcal{A})$. Defining $H = F - G$, we have $H(x) = F(x) - G(x) = F(x) - xF(x^{-1})$ for any $x \in Inv(\mathcal{A})$. Clearly, $H(\mathbf{e}) = 0$ and also for any invertible element $x \in \mathcal{A}$, we have the following statements:

$$xH(x^{-1}) = xF(x^{-1}) - F(x) = -H(x) = K(x).$$

According to the first part of the proof, $K(= -H)$ is a derivation mapping $\mathcal{A}$ into $Z(\mathcal{A})$. Hence, $K(x) = -x^2K(x^{-1})$ for any $x \in Inv(\mathcal{A})$. This equation along with $K(x) = xF(x^{-1}) - F(x)$ $(x \in Inv(\mathcal{A}))$ imply that

$$(\mathbf{e} - x)xF(x^{-1}) = (\mathbf{e} - x)F(x) \tag{4}$$

for any $x \in Inv(\mathcal{A})$. Replacing x by a in (4), we get that

$$(\mathbf{e} - a)aF(a^{-1}) = (\mathbf{e} - a)F(a).$$

Multiplying the above relation from the left by $(\mathbf{e} - a)^{-1}$, we get that

$$F(a) = aF(a^{-1}) \tag{5}$$

Comparing (1) and (5), we infer that $F(a^{-1}) = G(a^{-1})$. We have $a^{-1}G(a) = G(a^{-1})$ and consequently, $G(a) = aG(a^{-1}) = F(a)$. Since $a = n\mathbf{e} + b$ and $G(\mathbf{e}) = F(\mathbf{e}) = 0$, $F(b) = G(b)$ for any $b \in \mathcal{A}$ which means that $F = G$. This fact along with (1) imply that

$$F(x) = xF(x^{-1}) \tag{6}$$

for any $x \in Inv(\mathcal{A})$. Moreover, we know that F is a derivation mapping $\mathcal{A}$ into $Z(\mathcal{A})$. So, for any $x \in Inv(\mathcal{A})$, we have

$$F(x) = -x^2 F(x^{-1}) \tag{7}$$

Comparing (6) and (7), we see that $F(x^{-1}) = -xF(x^{-1})$. Replacing x by $-x$ in the previous equation, we obtain that $(\mathbf{e} - x)F(x^{-1}) = 0$ for any $x \in Inv(\mathcal{A})$. Replacing x by a in the former equation and using the assumption that $\mathbf{e} - a$ is an invertible element of $\mathcal{A}$, we get that $F(a^{-1}) = 0$ and it follows from (5) that $F(a) = 0$. Considering $a = n\mathbf{e} + b$ and using the fact that $F(\mathbf{e}) = 0$, we obtain that $0 = F(a) = F(n\mathbf{e} + b) = F(b)$ for any $b \in \mathcal{A}$, as desired. □

There is a consequence of the above theorem as follows.

Corollary 2.2. *Let $\mathcal{A}$ be a unital semiprime Banach algebra and let $F : \mathcal{A} \to \mathcal{A}$ be a linear mapping satisfying*

$$F(x) = x[x^{-1}, y_0] \tag{8}$$

for any $x \in Inv(\mathcal{A})$ and some $y_0 \in \mathcal{A}$. Then $y_0 \in Z(\mathcal{A})$.

Proof. Defining $G : \mathcal{A} \to \mathcal{A}$ by $G(a) = [a, y_0]$, we see that $F(x) = xG(x^{-1})$ for any $x \in Inv(\mathcal{A})$. It follows from Theorem 2.1 that $F = G = 0$. Therefore, $[a, y_0] = 0$ for all $a \in \mathcal{A}$ which means that $y_0 \in Z(\mathcal{A})$. □

[25, Theorem 5] asserts that if $\mathcal{A}$ is a semisimple Banach algebra and $F, G : \mathcal{A} \to \mathcal{A}$ are linear mappings satisfying $F(x) = -x^2 G(x^{-1})$ for all $x \in Inv(\mathcal{A})$, then $F = G$ and further, $F(a) = aF(\mathbf{e})$ for all $a \in \mathcal{A}$. We extend that theorem as follows.

Theorem 2.3. *Let $\mathcal{A}$ be a unital semiprime Banach algebra and let $F, G : \mathcal{A} \to \mathcal{A}$ be linear mappings satisfying*

$$F(x) = -x^2 G(x^{-1}) \tag{9}$$

for all $x \in Inv(\mathcal{A})$. Then both F and G are generalized (left) derivations on $\mathcal{A}$.

Proof. Let a be an arbitrary element of $\mathcal{A}$. Similar to what was mentioned in the proof of Theorem 2.1, we can choose a positive integer n such that both $b = n\mathbf{e}+a$ and $\mathbf{e}-b$ are invertible elements of $\mathcal{A}$. Now, suppose that $F(\mathbf{e}) = 0$. We have $F(\mathbf{e}) = 0$ if and only if $G(\mathbf{e}) = 0$. Note that $\mathbf{e} = (\mathbf{e} - b)(\mathbf{e} - b)^{-1} = (\mathbf{e} - b)^{-1} - b(\mathbf{e} - b)^{-1}$ and so, $b(\mathbf{e} - b)^{-1} = (\mathbf{e} - b)^{-1} - \mathbf{e}$ and also we have, $(\mathbf{e} - b)^{-1} b = (\mathbf{e} - b)^{-1} - \mathbf{e}$. Hence, $b(\mathbf{e} - b)^{-1} = (\mathbf{e} - b)^{-1} b$. It follows from (9) that $G(x^{-1}) = -x^{-2} F(x)$ for all $x \in Inv(\mathcal{A})$. Considering $c = (\mathbf{e} - b)^{-1} b$, we have

$$\begin{aligned}
F(b) &= -b^2 G(b^{-1}) = -b^2 G(b^{-1} - \mathbf{e}) \\
&= -b^2 G(b^{-1}(\mathbf{e} - b)) \\
&= -b^2 G(((\mathbf{e} - b)^{-1} b)^{-1}) \\
&= -b^2 G(c^{-1}) \\
&= -b^2(-c^{-2} F(c)) \\
&= b^2((\mathbf{e} - b)^{-1} b)^{-2} F(c) \\
&= b^2\Big(((\mathbf{e} - b)^{-1} b)^{-1}\Big)^2 F((\mathbf{e} - b)^{-1} b) \\
&= (\mathbf{e} - b)^2 F((\mathbf{e} - b)^{-1} - \mathbf{e}) \\
&= (\mathbf{e} - b)^2 F((\mathbf{e} - b)^{-1}),
\end{aligned}$$

which means that

$$F(b) = (\mathbf{e} - b)^2 F((\mathbf{e} - b)^{-1}) \tag{10}$$

Replacing b by $(\mathbf{e}-b)$ in (10) and using the assumption that $F(\mathbf{e})=0$, we obtain that

$$F(b) = -b^2F(b^{-1}) \tag{11}$$

Applying (9) and (11), we have $G(b) = -b^2F(b^{-1}) = F(b)$ and since $b = n\mathbf{e} + a$, $G(a) = F(a)$. Since a is an arbitrary element of $\mathcal{A}$, $F = G$. This fact along with (9) imply that $F(x) = -x^2F(x^{-1})$ $(x \in Inv(\mathcal{A}))$ and it follows from the proof of [25, Theorem 5] that F is a Jordan left derivation on $\mathcal{A}$. Since $\mathcal{A}$ is a semiprime algebra, [25, Theorem 2] implies that F is a derivation mapping $\mathcal{A}$ into $Z(\mathcal{A})$. Now, we define a linear mapping $H : \mathcal{A} \to \mathcal{A}$ by $H(a) = G(a) - aG(\mathbf{e})$. Evidently, $H(\mathbf{e}) = 0$ and also $-x^2H(x^{-1}) = -x^2G(x^{-1}) + xG(\mathbf{e})$. Consequently, we have

$$F(x) + xG(\mathbf{e}) = -x^2H(x^{-1}) \tag{12}$$

for any $x \in Inv(\mathcal{A})$. Defining $K : \mathcal{A} \to \mathcal{A}$ by $K(a) = F(a)+aG(\mathbf{e})$, we have $K(x) = -x^2H(x^{-1})$ for any $x \in Inv(\mathcal{A})$. Obviously, K and H are linear mappings and $K(\mathbf{e}) = H(\mathbf{e}) = 0$. According to the first part of the proof, we infer that $K(a) = H(a)$ for all $a \in \mathcal{A}$, which means that $F(a) + aG(\mathbf{e}) = G(a) - aG(\mathbf{e})$ for all $a \in \mathcal{A}$. Hence,

$$G - F = R_{2G(\mathbf{e})} \tag{13}$$

Since $F(x) = -x^2G(x^{-1})$ and $G(x) = -x^2F(x^{-1})$, one can easily get that $\mathfrak{L}(x) = -x^2\mathfrak{L}(x^{-1})$ for all $x \in Inv(\mathcal{A})$, where $\mathfrak{L} = F + G$. According to the proof of [25, Theorem 5], $\mathfrak{L}$ is a Jordan left derivation and it follows from [25, Theorem 2] that $\mathfrak{L}$ is a derivation mapping $\mathcal{A}$ into $Z(\mathcal{A})$. Since $G - F = R_{2G(\mathbf{e})}$ and $F + G = \mathfrak{L}$, we deduce that

$$G = R_{G(\mathbf{e})} + \frac{1}{2}\mathfrak{L}, \tag{14}$$

which means that G is a generalized derivation on $\mathcal{A}$. Moreover, we have

$$F = \mathfrak{L} - G = \mathfrak{L} - R_{G(\mathbf{e})} - \frac{1}{2}\mathfrak{L} = \frac{1}{2}\mathfrak{L} - R_{G(\mathbf{e})} = \frac{1}{2}\mathfrak{L} + R_{-G(\mathbf{e})} = \frac{1}{2}\mathfrak{L} + R_{F(\mathbf{e})}. \tag{15}$$

It means that F is a generalized derivation on $\mathcal{A}$. It is clear that both F and G are generalized (left) derivations on $\mathcal{A}$. □

In the following, there are some immediate consequences of the above theorem.

Corollary 2.4. *[25, Theorem 5] Let $\mathcal{A}$ be a unital semisimple Banach algebra and let $F, G : \mathcal{A} \to \mathcal{A}$ be linear mappings.*

(i) If F and G satisfy $F(x) = -x^2G(x^{-1})$ for all $x \in Inv(\mathcal{A})$, then $F = -G$ and furthermore $F(a) = aF(e)$ for all $a \in \mathcal{A}$.

(ii) If F and G satisfy $F(x) = x^2G(x^{-1})$ for all $x \in Inv(\mathcal{A})$, then $F = G$ and furthermore $F(a) = aF(e)$ for all $a \in \mathcal{A}$.

Proof. (i) It follows from Theorem 2.3 that $\mathfrak{L} = F + G$ and $F = \frac{1}{2}\mathfrak{L} + R_{F(\mathbf{e})}$, where $\mathfrak{L}$ is a left derivation on $\mathcal{A}$ mapping $\mathcal{A}$ into $Z(\mathcal{A})$. According to [25, Theorem 4], $\mathfrak{L} = 0$. So, $F = -G$ and further $F(a) = aF(\mathbf{e})$ for all $a \in \mathcal{A}$.
(ii) Indeed, we can write $F(x) = -x^2(-G)(x^{-1})$ for all $x \in Inv(\mathcal{A})$. Now, part (i) completes the proof. □

Corollary 2.5. *Let $\mathcal{A}$ be a unital semiprime Banach algebra and let $f, g_1, g_2 : \mathcal{A} \to \mathcal{A}$ be linear mappings satisfying $f(ab) = bg_1(a) + ag_2(b)$ for all $a, b \in \mathcal{A}$. Then f, g_1, g_2 are generalized derivations. In particular, if $\mathcal{A}$ is semisimple, then f, g_1, g_2 are continuous.*

Proof. Clearly, we have $f(\mathbf{e}) = a^{-1}g_1(a) + ag_2(a^{-1})$ for all invertible elements $a \in \mathcal{A}$. A routine calculation shows that $g_2(a) - af(\mathbf{e}) = -a^2 g_1(a^{-1})$ for any invertible element $a \in \mathcal{A}$. Defining a linear mapping $F_1 : \mathcal{A} \to \mathcal{A}$ by $F_1(b) = g_2(b) - bf(\mathbf{e})$ $(b \in \mathcal{A})$, we have

$$F_1(a) = -a^2 g_1(a^{-1}),$$

for all invertible elements $a \in \mathcal{A}$. It follows from Theorem 2.3 that there exists a left derivation $\mathfrak{L}_1 : \mathcal{A} \to \mathcal{A}$ such that $g_1 = R_{g_1(\mathbf{e})} + \frac{1}{2}\mathfrak{L}_1$. We know that $\mathfrak{L}_1$ is a derivation mapping $\mathcal{A}$ into $Z(\mathcal{A})$ and so g_1 is a generalized derivation. Reasoning like above, there exists a derivation $\mathfrak{L}_2 : \mathcal{A} \to \mathcal{A}$ mapping $\mathcal{A}$ into $Z(\mathcal{A})$ such that $g_2 = R_{g_2(\mathbf{e})} + \frac{1}{2}\mathfrak{L}_2$, and so g_2 is a generalized derivation. Our next task is to show that f is a generalized derivation. Since $f = g_1 + R_{g_2(\mathbf{e})}$ and $g_1 = R_{g_1(\mathbf{e})} + \frac{1}{2}\mathfrak{L}_1$, we deduce that $f = \frac{1}{2}\mathfrak{L}_1 + R_{f(\mathbf{e})}$. Similarly, we have

$$f = R_{g_1(\mathbf{e})} + g_2 = R_{g_1(\mathbf{e})} + R_{g_2(\mathbf{e})} + \frac{1}{2}\mathfrak{L}_2 = \frac{1}{2}\mathfrak{L}_2 + R_{f(\mathbf{e})}.$$

Therefore, f is a generalized derivation associated with both $\mathfrak{L}_1$ and $\mathfrak{L}_2$. If $\mathcal{A}$ is simisimple, then in view of [23, Theorem 2.3.2], the derivations $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are continuous and consequently, f, g_1 and g_2 are also continuous linear mappings. □

Corollary 2.6. *Let $\mathcal{A}$ be a unital semisimple Banach algebra and let $F, G : \mathcal{A} \to \mathcal{A}$ be linear mappings satisfying*

$$[F(x), y] = x^2[G(x^{-1}), y] \tag{16}$$

for any $x \in Inv(\mathcal{A})$ and $y \in \mathcal{A}$. Then $F(\mathcal{A}), G(\mathcal{A}) \subseteq Z(\mathcal{A})$ and furthermore

- *$F(a) = F(\boldsymbol{e})a + K(a)$,*
- *$G(a) = G(\boldsymbol{e})a + S(a)$,*

where K and S are linear mappings from $\mathcal{A}$ into $Z(\mathcal{A})$.

Proof. Let y_0 be an arbitrary element of $\mathcal{A}$. Defining the mappings $f, g : \mathcal{A} \to \mathcal{A}$ by $f(a) = [F(a), y_0]$ and $g(a) = [G(a), y_0]$, we see that $f(x) = x^2 g(x^{-1})$ for all $x \in Inv(\mathcal{A})$. It follows from Corollary 2.4 that $f = g$ and $f(a) = af(\mathbf{e})$ for all $a \in \mathcal{A}$. Indeed, we have

$$[F(a), y_0] = [G(a), y_0], \tag{17}$$

and

$$[F(a), y_0] = a[F(\mathbf{e}), y_0] \tag{18}$$

It follows from (17) that $[F(a) - G(a), y_0] = 0$ for all $a \in \mathcal{A}$ and since y_0 is an arbitrary element of $\mathcal{A}$, $F(a) - G(a) \in Z(\mathcal{A})$ for all $a \in \mathcal{A}$. Now, suppose that $F(\mathbf{e}) \in Z(\mathcal{A})$. It follows from (18) that $[F(a), y_0] = 0$ for all $a \in \mathcal{A}$ and since y_0 is an arbitrary element of $\mathcal{A}$, $F(\mathcal{A}) \subseteq Z(\mathcal{A})$. Suppose that $H : \mathcal{A} \to \mathcal{A}$ is an additive mapping satisfying

$$[H(x), y] = x^2[F(x^{-1}) - x^{-1}F(\mathbf{e}), y] \tag{19}$$

for any $x \in Inv(\mathcal{A})$ and $y \in \mathcal{A}$. Therefore, we have

$$x^2[H(x^{-1}), y] = [F(x) - xF(\mathbf{e}), y] \tag{20}$$

for all $x \in Inv(\mathcal{A})$. Defining a linear mapping $K : \mathcal{A} \to \mathcal{A}$ by $K(a) = F(a) - aF(\mathbf{e})$, we have

$$[K(x), y] = x^2[H(x^{-1}), y] \tag{21}$$

for any $x \in Inv(\mathcal{A})$ and $y \in \mathcal{A}$. Clearly, $K(\mathbf{e}) \in Z(\mathcal{A})$ and based on the first part of the proof, $K(\mathcal{A}) \subseteq Z(\mathcal{A})$. Thus, $F(a) = aF(\mathbf{e}) + K(a)$ for all $a \in \mathcal{A}$, where K is a mapping from $\mathcal{A}$ into $Z(\mathcal{A})$. According to (18), we have

$$[F(a), y] = a[F(\mathbf{e}), y] = [aF(\mathbf{e}), y] - [a, y]F(\mathbf{e})$$

Since $K(\mathcal{A}) \subseteq Z(\mathcal{A})$, we have

$$[y, a]F(\mathbf{e}) = [F(a) - aF(\mathbf{e}), y] = [K(a), y] = 0$$

for all $a, y \in \mathcal{A}$. We can deduce from [26, Lemma 1.3] that there exists an ideal $\mathcal{I}$ of $\mathcal{A}$ such that $F(\mathbf{e}) \in \mathcal{I} \subseteq Z(\mathcal{A})$. Using this fact along with (18), we obtain that $[F(a), y] = a[F(\mathbf{e}), y] = 0$ for all $a, y \in \mathcal{A}$ and so, $F(\mathcal{A}) \subseteq Z(\mathcal{A})$. Using the above argument, we get that $G(\mathcal{A}) \subseteq Z(\mathcal{A})$ and also there exists a linear mapping $S : \mathcal{A} \to Z(\mathcal{A})$ such that $G(a) = G(\mathbf{e})a + S(a)$ for all $a \in \mathcal{A}$. Moreover, note that there exists an ideal $\mathcal{J}$ of $\mathcal{A}$ such that $G(\mathbf{e}) \in \mathcal{J} \subseteq Z(\mathcal{A})$. Considering $F(\mathbf{e}) - G(\mathbf{e}) = \psi(\mathbf{e})$ and $\eta = L_{\psi(\mathbf{e})} + K - S : \mathcal{A} \to Z(\mathcal{A})$, we have

$$\begin{aligned} F(a) &= F(\mathbf{e})a + K(a) \\ &= (\psi(\mathbf{e}) + G(\mathbf{e}))a + K(a) \\ &= G(a) - S(a) + \psi(\mathbf{e})a + K(a) \\ &= G(a) + L_{\psi(\mathbf{e})}(a) + (K - S)(a) \\ &= G(a) + \eta(a). \end{aligned}$$

□

In the following theorem, we present another characterization of generalized derivations on semiprime algebras.

Theorem 2.7. *Let $\mathcal{A}$ be a unital semiprime algebra and $n > 1$ be an integer. Let $f, g : \mathcal{A} \to \mathcal{A}$ be linear mappings satisfying*

$$f(a^n) = na^{n-1}g(a) = ng(a)a^{n-1}$$

for all $a \in \mathcal{A}$. If $g(\mathbf{e}) \in Z(\mathcal{A})$, then f and g are generalized derivations associated with the same derivation on $\mathcal{A}$.

Proof. According to the aforementioned assumption, we have

$$f(a^n) = na^{n-1}g(a), \qquad a \in \mathcal{A}. \tag{22}$$

Let c be an arbitrary element of $Z(\mathcal{A})$. Replacing a by $a + c$ in (22), we have

$$f\Big(\sum_{k=0}^{n} \binom{n}{k} a^{n-k}c^k\Big) = n \sum_{k=0}^{n-1} \binom{n-1}{k} a^{n-k-1}c^k \,(g(a) + g(c))$$

Using equation (22) and collecting together terms of the above-mentioned expressions involving the same number of factors of c, it is obtained that

$$\sum_{k=1}^{n-1} \lambda_k(a, c) = 0, \qquad a \in \mathcal{A}, \tag{23}$$

where

$$\lambda_k(a, c) = \binom{n}{k} f\big(a^{n-k}c^k\big) - (n-k)\binom{n}{k} c^k a^{n-k-1} g(a) - k\binom{n}{k} c^{k-1} a^{n-k} g(c).$$

Replacing c by $c, 2c, ..., (n-1)c$ in equation (23), we obtain a system of $(n-1)$ homogeneous equations as follows:

$$\begin{cases} \sum_{i=1}^{n-1} \lambda_i(x, c) = 0, \\ \sum_{i=1}^{n-1} \lambda_i(x, 2c) = 0, \\ \quad \vdots \\ \sum_{i=1}^{n-1} \lambda_i(x, (n-1)c) = 0. \end{cases}$$

We see that the coefficient matrix of the above system is:

$$Y = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & 2^2 & 2^3 & \cdots & 2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ (n-1) & (n-1)^2 & (n-1)^3 & \cdots & (n-1)^{n-1} \end{bmatrix}$$

We know that the determinant of a square Vandermonde matrix is nonzero, and it implies that the above-mentioned system has only a trivial solution. In particular, $\lambda_{n-2}(a, c) = 0$. So

$$0 = \lambda_{n-2}(a, c) = \binom{n}{n-2} f(a^2c^{n-2}) - 2\binom{n}{n-2} c^{n-2}ag(a) - (n-2)\binom{n}{n-2} c^{n-3}a^2g(c).$$

Therefore, we have

$$f(a^2c^{n-2}) = 2c^{n-2}ag(a) + (n-2)c^{n-3}a^2g(c). \tag{24}$$

Putting $c = \mathbf{e}$ in (24) and defining the linear mapping $h : \mathcal{A} \to \mathcal{A}$ by $h(a) = 2g(a) + (n-2)g(\mathbf{e})a$ $(a \in \mathcal{A})$, we get that

$$f(a^2) = 2ag(a) + (n-2)a^2g(\mathbf{e}) = ah(a), \qquad a \in \mathcal{A}. \tag{25}$$

Reasoning like above, we obtain that

$$f(a^2) = 2g(a)a + (n-2)g(\mathbf{e})a^2 = h(a)a, \qquad a \in \mathcal{A}. \tag{26}$$

Replacing a by $a + b$ in (25) and (26), we get that

$$f(ab + ba) = ah(b) + bh(a) = h(a)b + h(b)a, \qquad a, b \in \mathcal{A}. \tag{27}$$

Hence, we have

$$2f(a \circ b) = h(a) \circ b + a \circ h(b),$$

for all $a, b \in \mathcal{A}$, where $a \circ b = ab + ba$. It follows from [6, Theorem 4.3] that f is a $\{\frac{h}{2}, \frac{h}{2}\}$-derivation, which means that

$$f(ab) = \frac{h}{2}(a)b + a\frac{h}{2}(b) = g(a)b + ag(b) + (n-2)g(\mathbf{e})ab, \qquad a, b \in \mathcal{A}. \tag{28}$$

So, we have

$$f(a) = g(a) + (n-1)g(\mathbf{e})a = (g + L_{(n-1)g(\mathbf{e})})(a), \qquad a \in \mathcal{A}. \tag{29}$$

Comparing (28) and (29), we get that

$$g(ab) = g(a)b + ag(b) - g(\mathbf{e})ab, \qquad a, b \in \mathcal{A}. \tag{30}$$

Considering $d = g - L_{g(\mathbf{e})}$ and using (30), we have the following expressions:

$$\begin{aligned} d(ab) &= g(ab) - g(\mathbf{e})ab \\ &= g(a)b + ag(b) - g(\mathbf{e})ab - g(\mathbf{e})ab \\ &= \big(g(a) - g(\mathbf{e})a\big)b + a\big(g(b) - g(\mathbf{e})b\big) \\ &= d(a)b + ad(b), \end{aligned}$$

which means that d is a derivation on $\mathcal{A}$. Therefore, g is a generalized derivation associated with the derivation d. It follows from (29) that

$$f = g + L_{(n-1)g(\mathbf{e})} = d + L_{g(\mathbf{e})} + L_{(n-1)g(\mathbf{e})} = d + L_{ng(\mathbf{e})},$$

which means that f is a generalized derivation associated with the derivation d on $\mathcal{A}$. This yields the desired result. □

Corollary 2.8. *Let $\mathcal{A}$ be a unital semisimple Banach algebra and let $n > 1$ be an integer. Let $f, g : \mathcal{A} \to \mathcal{A}$ be linear mappings satisfying*

$$f(a^n) = na^{n-1}g(a) = ng(a)a^{n-1}$$

for all $a \in \mathcal{A}$. If $g(e) \in Z(\mathcal{A})$, then both f and g are continuous.

Proof. According to the previous theorem, there exists a derivation d on $\mathcal{A}$ such that $f = d + L_{ng(\mathbf{e})}$ and $g = d + L_{g(\mathbf{e})}$. It follows from [23, Theorem 2.3.2] that derivation d is continuous and consequently, both f and g are continuous linear mappings on $\mathcal{A}$. □

In the following, we are going to present a functional identity to characterize Jordan $(*, \star)$-derivations. Let $\mathcal{R}$ be a ring. An additive mapping $x \mapsto x^*$ of $\mathcal{R}$ into itself is called an involution on $\mathcal{R}$ if it satisfies the following conditions:
(i) $(x^*)^* = x$,
(ii) $(xy)^* = y^*x^*$ for all $x, y \in \mathcal{R}$.

A ring $\mathcal{R}$ equipped with an involution '$*$' is called a $*$-ring. Let $\mathcal{R}$ be a $*$-ring. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a $*$-derivation (resp. a Jordan $*$-derivation) if $d(xy) = d(x)y^* + xd(y)$ (resp. $d(x^2) = d(x)x^* + xd(x)$) holds for all $x, y \in \mathcal{R}$.

Definition 2.9. *Let $*$ and $\star$ be involutions on $\mathcal{R}$. The ring $\mathcal{R}$ is called a $(*, \star)$-ring if it is both a $*$-ring and a $\star$-ring.*

Now, we provide an example of a $(*, \star)$-ring.

Example 2.10. *Let*

$$\mathcal{R} = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} : \alpha, \beta \in \mathbb{C} \right\}$$

Clearly, $\mathcal{R}$ is a ring. Define the mappings $*, \star : \mathcal{R} \to \mathcal{R}$ by

$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}^* = \begin{bmatrix} \overline{\alpha} & 0 \\ 0 & \overline{\beta} \end{bmatrix} \text{ and } \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}^\star = \begin{bmatrix} \overline{\beta} & 0 \\ 0 & \overline{\alpha} \end{bmatrix},$$

where $\overline{z}$ is the conjugate of z for all $z \in \mathbb{C}$. A simple calculation shows that $\mathcal{R}$ is a $(*, \star)$-ring.

Definition 2.11. *Let $\mathcal{R}$ be a $(*,\star)$-ring. An additive mapping $d:\mathcal{R}\to\mathcal{R}$ is called a $(*,\star)$-derivation (resp. a Jordan $(*,\star)$-derivation) on $\mathcal{R}$ if $d(xy)=d(x)y^*+x^\star d(y)$ (resp. $d(x^2)=d(x)x^*+x^\star d(x)$) holds for all $x,y\in\mathcal{R}$.*

Example 2.12. *Let $\mathcal{R}$ be a $(*,\star)$-ring and let c be an arbitrary fixed element of $\mathcal{R}$. Then an additive mapping $d:\mathcal{R}\to\mathcal{R}$ defined by $d(x)=x^\star c-cx^*$, is a Jordan $(*,\star)$-derivation on $\mathcal{R}$. It is clear that, if $\mathcal{R}$ is a commutative $(*,\star)$-ring, then the additive mapping d is a $(*,\star)$-derivation on $\mathcal{R}$.*

In the next theorem, we present a characterization of Jordan $(*,\star)$-derivations.

Theorem 2.13. *Let $\mathcal{R}$ be an $n!$-torsion free $(*,\star)$-ring, let $n>1$ be an integer and let $d:\mathcal{R}\to\mathcal{R}$ be an additive mapping satisfying*

$$d(a^n)=\sum_{j=1}^{n}a^{\star\, n-j}d(a)a^{*\, j-1} \tag{31}$$

for all $a\in\mathcal{R}$. Then d is a Jordan $(*,\star)$-derivation on $\mathcal{R}$.

Proof. Let c be an element of $Z(\mathcal{R})$ such that $c^*=c=c^\star$ and $d(c)=0$. Replacing a by $a+c$ in equation (31), we have

$$\begin{aligned}
d\Big(\sum_{i=0}^{n}\binom{n}{i}a^{n-i}c^i\Big) &= \sum_{j=1}^{n}\Big(\sum_{k_1=0}^{n-j}\binom{n-j}{k_1}a^{\star\, n-j-k_1}c^{k_1}d(a)\sum_{k_2=0}^{j-1}\binom{j-1}{k_2}a^{*\, j-1-k_2}c^{k_2}\Big)\\
&= \sum_{k_1=0}^{n-1}\binom{n-1}{k_1}a^{\star\, n-1-k_1}c^{k_1}d(a)+\\
&\sum_{k_1=0}^{n-2}\binom{n-2}{k_1}a^{\star\, n-2-k_1}c^{k_1}d(a)\sum_{k_2=0}^{1}\binom{1}{k_2}a^{*\, 1-k_2}c^{k_2}+\\
&\dots+\sum_{k_1=0}^{1}\binom{1}{k_1}a^{\star\, 1-k_1}c^{k_1}d(a)\sum_{k_2=0}^{n-2}\binom{n-2}{k_2}a^{*\, n-2-k_2}c^{k_2}+\\
&d(a)\sum_{k_2=0}^{n-1}\binom{n-1}{k_2}a^{*\, n-1-k_2}c^{k_2}.
\end{aligned}$$

Rearranging the above relations with respect to involving equal number of factors of c, we arrive at

$$\sum_{i=1}^{n-1}\gamma_i(a,a^*,a^\star,c)=0, \tag{32}$$

where,

$$\gamma_i(a,a^*,a^\star,c)=\binom{n}{i}d(a^{n-i}c^i)-\sum_{k=1}^{n-i}\binom{n}{i}a^{\star\, n-i-k}c^i d(a)a^{*\, k-1}. \tag{33}$$

Replacing c by c, $2c$, $3c$,..., $(n-1)c$ in equation (32), we obtain a system of $n-1$ homogeneous equations as follows:

$$\begin{cases} \sum_{i=1}^{n-1} \gamma_i(a, a^*, a^\star, c) = 0 \\ \sum_{i=1}^{n-1} \gamma_i(a, a^*, a^\star, 2c) = 0 \\ \sum_{i=1}^{n-1} \gamma_i(a, a^*, a^\star, 3c) = 0 \\ \vdots \\ \sum_{i=1}^{n-1} \gamma_i(a, a^*, a^\star, (n-1)c) = 0 \end{cases}$$

It is observed that the coefficient matrix of the above system is:

$$\mathcal{V} = \begin{bmatrix} \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \cdots & \binom{n}{n-1} \\ 2\binom{n}{1} & 2^2\binom{n}{2} & 2^3\binom{n}{3} & \cdots & 2^{n-1}\binom{n}{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (n-1)\binom{n}{1} & (n-1)^2\binom{n}{2} & (n-1)^3\binom{n}{3} & \cdots & (n-1)^{n-1}\binom{n}{n-1} \end{bmatrix}$$

In view of the determinant of a square Vandermonde matrix, one can easily get that

$$det\ \mathcal{V} = \left(\prod_{k=1}^{n-1}\binom{n}{k}\right)(n-1)! \prod_{1\leq i<j\leq n-1}(i-j).$$

It is evident that the determinant of $\mathcal{V}$ is nonzero, i.e $det\ \mathcal{V} \neq 0$, and it implies that the system has only a trivial solution. In particular, $\gamma_{n-2}(a, a^*, a^\star, c) = 0$. It means that

$$\binom{n}{n-2}d(a^2c^{n-2}) - \sum_{k=1}^{2}\binom{n}{n-2}a^{\star 2-k}c^{n-2}d(a)a^{*k-1} = 0. \tag{34}$$

Since $\mathcal{R}$ is an n!-torsion free ring, it follows from equation (34) that

$$d(a^2c^{n-2}) = a^\star c^{n-2}d(a) + c^{n-2}d(a)a^*. \tag{35}$$

Using the assumption that $\mathbf{e}^* = \mathbf{e} = \mathbf{e}^\star$ along with equation (31), we get that $d(\mathbf{e}) = 0$. Replacing c by $\mathbf{e}$ in equation (35), we obtain that $d(a^2) = d(a)a^* + a^\star d(a)$ and it means that d is a Jordan $(*, \star)$-derivation on $\mathcal{R}$. □

Acknowledgements The author thanks the referee for his/her careful reading of the manuscript and suggesting valuable comments that improved the quality of this work.

References

[1] E. Albas and N. Argas, Generalized derivations of prime rings, *Algebra Colloq.* **11** (2004), 399–410.
[2] M. Ashraf and S. Ali, On generalized Jordan left derivations in rings, *Bull. Korean Math. Soc.* **45** (2008), 253–261.
[3] M. Ashraf and M. A. Siddeeque, On ∗-n-derivations in rings with involution, *Georgian Math. J.*, **22** (1) (2015), 9—18.
[4] K. I. Beidar, M. Brešar, M. A. Chebotar and W.S. Martindale 3^{rd}, On Herstein's Lie map conjectures II. *J. Algebra*, **238** (2001), 239-–264.
[5] M. Brešar, M. A. Chebotar and W.S. Martindale 3^{rd}, Functional Identities, Birkhäuser Verlag, Basel-Boston-Berlin, 2007.
[6] M. Brešar, Jordan $\{g, h\}$-derivations on tensor products of algebras, Linear Multilinear Algebra. **64** 11 (2016), 2199–2207.

[7] M. Brešar, Jordan derivations on semiprime rings, *Proc. Amer. Math. Soc.* **140** 4 (1988), 1003–1006.
[8] M. Brešar and J. Vukman, Jordan derivations on prime rings, Bull. Austral. Math. Soc. **37** (1988), 321–322.
[9] M. Brešar, On the distance of the composition of two derivations to the generalized derivations, Glasg. Math. J. **33** (1991), 89—93.
[10] M. Brešar and J. Vukman, On Left Derivations and Related Mappings, Proc. Amer. Math. Soc. **110** (1990), 7-–16.
[11] M. Brešar and J. Vukman, On some additive mappings in rings with involution, *Aequ. Math* **38** (1989), 178–185.
[12] J. Cusack, Jordan derivations on rings, *Proc.Amer. Math. Soc.* **53** (1975), 1104–1110.
[13] G. Dales, et al., Introduction to Banach algebras, Operators and harmonic analysis, Cambridge University press, Cambridge, 2002.
[14] A. Hosseini and A. Fošner, The Image of Jordan Left Derivations on Algebras, Bol. Soc. Paran. Mat., (**38**) 6 (2020), 53—61.
[15] B. Hvala, Generalized derivations in prime rings, Comm. Algebra, **26** (1998), 1147–1166.
[16] I. N. Herstein, Jordan derivations of prime rings, *Proc.Amer. Math. Soc.* **8** (1957) 1104–1110.
[17] A. Hosseini and A. Fošner, Identities related to (σ, τ)-Lie derivations and (σ, τ)-derivations, Boll. Unione Mat. Ital. **11** (2018), 395–401.
[18] W. Jing and S. Lu, Generalized Jordan derivations on prime rings and standard operator algebras, Taiwanese J. Math., **7** (2003), 605–613.
[19] I. Kosi-Ulbl and J. Vukman, On certain identity related to Jordan $*$-derivations, *Glas. Mat.*, **50**(70) (2015), 363 -– 371.
[20] Y. Li and D. Benkovič. Jordan generalized derivations on triangular algebras. Linear Multilinear Algebra. **59** (2011), 841—849.
[21] C. Lanski. Generalized derivations and nth power maps in rings. Communications in Algebra. **35** (2007), 3660-–3672.
[22] T-K. Lee and Y. Zhou, Jordan $*$-derivations of prime rings, *J. Algebra Appl.*, **13** (4) (2014), 1350126 (9 pages).
[23] S. Sakai, C^*-algebras and W^*algebras, Springer-Verlag, New York, 1971.
[24] J. Vukman and I. Kosi-Ulbl, A note on derivations in semiprime rings. , *International Journal of Mathematics and Mathematical Sciences.* **20** (2005), 3347—3350.
[25] J. Vukman, On left Jordan derivations of rings and Banach algebras, *Aequ. Math*. **75** (2008), 260—266.
[26] B. Zalar, On centralizers of semiprime rings, *Comment. Math. Univ. Carolinae*. **32** (4) (1991), 609–614.