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Inscription on Statistical Convergence of Order α

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Abstract. In this article we recall a remarkable result stated as *"For a fixed α, $0 < \alpha \leq 1$, the set of all bounded statistically convergent sequences of order α is a closed linear subspace of m (m is the set of all bounded real sequences endowed with the sup norm)"* by Bhunia et al. (Acta Math. Hungar. 130 (1-2) (2012), 153–161) and to develop the objective of this perception we demonstrate that the set of all bounded statistically convergent sequences of order α may not form a closed subspace in other sequence spaces. Also we determine two different sequence spaces in which the set of all statistically convergent sequences of order α (irrespective of boundedness) forms a closed set.

1. Introduction

The phenomenon of statistical convergence had been appeared as a generalization of usual convergence in the middle of twentieth century. Circumstantially this is to mention that in the same year 1951, Fast [4] and Steinhaus [12] intimated this prime notion but in independent ways. In this description we now proceed to the definition of statistical convergence. Meanwhile we delineate asymptotic density.

Let $\mathbb{N}$ denote the set of all natural numbers and $A \subseteq \mathbb{N}$. The expression $A(m, n)$ (where $m, n \in \mathbb{N}$) denotes the cardinality of the set $A \cap [m, n]$. The upper and lower asymptotic (or natural) densities of the set A are respectively defined as

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{A(1,n)}{n} \text{ and } \underline{d}(A) = \liminf_{n\to\infty} \frac{A(1,n)}{n}.$$

If $\overline{d}(A) = \underline{d}(A)$ is satisfied, we say that the asymptotic density of A exists and is denoted by $d(A)$. Specifically $d(A) = \lim_{n\to\infty} \frac{A(1,n)}{n}$.

Now we illustrate the definition of statistical convergence: A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to a real number c if for any $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - c| \geq \varepsilon\}$ has the asymptotic density zero. We symbolize this as $x_n \xrightarrow{St} c$ or $St - \lim x = c$. We shall also use the notation m_0 to denote the set of all statistically convergent sequences with real entries.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be statistically bounded [13] if there exists a positive real number M such that the asymptotic density of the set $\{n \in \mathbb{N} : |x_n| \geq M\}$ is zero. For more references please go through [2, 3, 5–7, 9, 11].

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In such an incredible manner Šalát [8] amplified this subject and introduced many indispensable results. We quote below a conventional one.

Theorem 1.1. *[8] The set of all bounded statistically convergent sequences of real numbers form a closed linear subspace of the norm linear space m (the set of all bounded real sequences endowed with the sup norm).*

On the other hand, an interesting concept regarding to the statistical convergence was materialized by Bhunia et al. [1] and termed as statistical convergence of order α. The depiction is as follows:
Let α be any real number such that $0 < \alpha \leq 1$, then upper and lower α-asymptotic densities of the set $A(\subset \mathbb{N})$ are respectively defined as

$$\overline{d^{\alpha}}(A) = \limsup_{n\to\infty} \frac{A(1,n)}{n^{\alpha}} \text{ and } \underline{d^{\alpha}}(A) = \liminf_{n\to\infty} \frac{A(1,n)}{n^{\alpha}}.$$

If $\overline{d^{\alpha}}(A) = \underline{d^{\alpha}}(A)$, then we say that the α-asymptotic density (or, α-natural density) of A exists and it is denoted by $d^{\alpha}(A)$. Clearly $d^{\alpha}(A) = \lim_{n\to\infty} \frac{A(1,n)}{n^{\alpha}}$.

A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent of order α to c if for every $\varepsilon > 0$, $d^{\alpha}(\{n \in \mathbb{N} : |x_n - c| \geq \varepsilon\}) = 0$. We express this as $St^{\alpha} - \lim x = c$ or $x_n \xrightarrow{St^{\alpha}} c$. The set of all statistically convergent sequences of order α $(0 < \alpha \leq 1)$ is denoted by m_0^{α}.

Succeedingly in the year 2010, Bhunia et al. [1] did an extension of Šalát's Theorem 1.1 for statistical convergence of order α but keeping the sequence space m (where m is the set of all bounded real sequences endowed with the sup norm) unaltered. This result has an relevant role in our study and so we state it.

Theorem 1.2. *[1] For a fixed α, $0 < \alpha \leq 1$, the set $m_0^{\alpha} \cap m$ is a closed linear subspace of m (where m is the set of all bounded real sequences endowed with the sup norm).*

In due course of time, from this aforementioned discussion we could scrutinize the objective "closeness" of the set of all bounded statistically convergent sequences of order α (for any α in $0 < \alpha \leq 1$) had been explained only over the space m (m is the set of all bounded real sequences endowed with the sup norm). Our aspiration is to develop the characteristic of the above mentioned set apart from the sequence space m and to do so we take two foremost sequence spaces such as Hilbert-Cube space and Frechet sequence space in which Theorem 1.2 does not hold. Conjointly we explore two sequence spaces in which the set of all statistically convergent sequences of order α regardless of boundedness been closed.

2. Main Results

In this section our first context of observation is Hilbert-Cube Space.

2.1. Hilbert-Cube Space

Let H_∞ be the set of all real sequences $\{x_n\}_{n\in\mathbb{N}}$ such that $0 \leq x_n \leq 1$ for all $n \in \mathbb{N}$. Let the distance function ρ be defined by

$$\rho(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} |x_k - y_k|$$

where $x = \{x_k\}_{k\in\mathbb{N}}$ and $y = \{y_k\}_{k\in\mathbb{N}}$ are elements in H_∞. This distance function ρ forms a metric on H_∞. The space H_∞ is called the Hilbert-Cube Space.

While reformulating the Theorem 1.2 based over the Hilbert-Cube Space, the objective of closeness violates. Our following first example asserts this.

Example 2.1. *The space $m_0^{\alpha} \cap H_\infty$ is not a closed subspace of H_∞.*

Proof. In our line of proof we assume a sequence $\{x^{(n)}\}_{n\in\mathbb{N}}$ in the form $x^{(n)} = \{x_k^{(n)}\}_{k\in\mathbb{N}}$ belongs to $m_0^\alpha \cap H_\infty$ for $n \in \mathbb{N}$ and that $\{x^{(n)}\}_{n\in\mathbb{N}}$ converges to x in H_∞ but x does not belong to $m_0^\alpha \cap H_\infty$.

Let $x^{(n)} = \{x_k^{(n)}\}_{k\in\mathbb{N}}$ where

$$x_k^{(n)} = \begin{cases} 1, \ k \in \{1^2, 2^2, 3^2, ...\} \cup \{3, 5, ..., 2n-1\}, \\ 0, \ \text{otherwise}. \end{cases}$$

Clearly each $x^{(n)} = \{x_k^{(n)}\}_{k\in\mathbb{N}}$ ($n = 1, 2, 3, ...$) is statistically convergent of order α to 0 (where $\frac{1}{2} < \alpha \leq 1$). Setting $x = \{x_k\}_{k\in\mathbb{N}}$ where

$$x_k = \begin{cases} 1, \ k \in \{1^2, 2^2, 3^2, ...\} \cup \{2k-1 : k \in \mathbb{N}\}, \\ 0, \ \text{otherwise}. \end{cases}$$

Now,

$$\lim_{n\to\infty} \rho(x^{(n)}, x) = \lim_{n\to\infty} \left(\sum_{k=1}^{\infty} \frac{1}{2^k} |x_k^{(n)} - x_k| \right) = \lim_{n\to\infty} \left(\frac{1}{2^{2n+1}} + \frac{1}{2^{2n+3}} + \frac{1}{2^{2n+5}} + ... \right)$$

$$= \lim_{n\to\infty} \left[\frac{1}{2^{2n+1}} (1 + \frac{1}{2^2} + \frac{1}{2^4} + ...) \right] = \lim_{n\to\infty} \frac{1}{2^{2n+1}} . \frac{4}{3} = 0.$$

This implies $\lim_{n\to\infty} x^{(n)} = x$ in H_∞.

For $0 < \varepsilon < 1$ we consider two sets

$$A(\varepsilon) = \{k \in \mathbb{N} : |x_k - 0| \geq \varepsilon\} = \{1^2, 2^2, 3^2, ...\} \cup \{2k-1 : k \in \mathbb{N}\},$$

$$B(\varepsilon) = \{k \in \mathbb{N} : |x_k - 1| \geq \varepsilon\} = \mathbb{N} \setminus (\{1^2, 2^2, 3^2, ...\} \cup \{2k-1 : k \in \mathbb{N}\}).$$

Therefore we get

$$d^\alpha(A(\varepsilon)) = d^\alpha(B(\varepsilon)) = \begin{cases} \infty, \ \text{if } \alpha \in (\frac{1}{2}, 1), \\ \frac{1}{2}, \text{if } \alpha = 1. \end{cases}$$

As a conclusion we observe that the sequence $x = \{x_k\}_{k\in\mathbb{N}}$ does not belong to $m_0^\alpha \cap H_\infty$. Hence our assertion follows. □

2.2. *Fréchet Sequence Space*

Let F be the set of all real sequences. Also let the distance function σ be defined by

$$\sigma(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|x_k - y_k|}{1 + |x_k - y_k|}$$

where $x = \{x_k\}_{k\in\mathbb{N}}$ and $y = \{y_k\}_{k\in\mathbb{N}}$ are in F. This distance function σ forms a metric over F and hence F is called the Fréchet sequence space.

Our next aim is to discuss the closeness of the space $m_0^\alpha \cap F$.

Example 2.2. *The space $m_0^\alpha \cap F$ is not a closed subspace of F. In particular, the space m_0^α is not a closed subspace of F.*

Proof. Let us consider $x^{(n)} = \{x_k^{(n)}\}_{k\in\mathbb{N}} \in m_0^\alpha$ for $n = 1, 2, 3, ...$ and $\{x^{(n)}\}_{n\in\mathbb{N}}$ be convergent to x where $x = \{x_k\}_{k\in\mathbb{N}}$ is in (F, σ). Our target is to show that $x \notin m_0^\alpha$.

Let $\frac{1}{2} < \alpha \leq 1$. Define the sequences $\{x_k^{(n)}\}_{k\in\mathbb{N}}$ for $n \in \mathbb{N}$ and $x = \{x_k\}_{k\in\mathbb{N}}$ respectively as follows:

$$x_k^{(n)} = \begin{cases} 1, & k \in \{1^2, 2^2, 3^2, ...\} \cup \{3, 5, ..., 2n-1\}, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and } x_k = \begin{cases} 1, & k \in \{1^2, 2^2, 3^2, ...\} \cup \{2k-1 : k \in \mathbb{N}\}, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly each $x^{(n)} = \{x_k^{(n)}\}_{k\in\mathbb{N}}$ $(n = 1, 2, 3, ...)$ is statistically convergent of order α to 0.

Now

$$\lim_{n\to\infty} \sigma(x^{(n)}, x) = \lim_{n\to\infty} \left(\sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|x_k^{(n)} - x_k|}{1 + |x_k^{(n)} - x_k|} \right) = \lim_{n\to\infty} \left(\sum_{\substack{k=2n+1 \\ k\in\{2m-1:m\in\mathbb{N}\}\setminus\{m^2:m\in\mathbb{N}\}}}^{\infty} \frac{1}{2^k} \times \frac{1}{2} \right) \leq \lim_{n\to\infty} \frac{1}{2^{2n+1}} . \frac{4}{3} = 0.$$

This implies $x^{(n)} \to x$ in F. But for each $0 < \varepsilon < 1$ and $x = \{x_k\}_{k\in\mathbb{N}}$,

$$A(\varepsilon) = \{k \in \mathbb{N} : |x_k - 0| \geq \varepsilon\} = \{1^2, 2^2, 3^2, ...\} \cup \{2k-1 : k \in \mathbb{N}\}.$$

Hence the sequence $x = \{x_k\}_{k\in\mathbb{N}}$ is not statistically convergent of order α (where $\frac{1}{2} < \alpha \leq 1$). So $x \notin m_0^\alpha$. □

Remark 2.3. *From the above Example 2.2 we can also draw a conclusion that the set of all bounded statistically convergent sequences of order α is not a closed subspace in the Fréchet sequence space.*

2.3. *Another Sequence Spaces*

In this segment we introduce two effectual sequence spaces so that not only the set of all bounded statistically convergent sequences of order α form a closed set but also the set of all statistically convergent sequences of order α form a closed set.

Suppose (X, ϱ) be any complete metric space. We introduce the set m_ϱ^α (where $0 < \alpha \leq 1$) as follows:

$$m_\varrho^\alpha = \{x = \{x_n\}_{n\in\mathbb{N}} : \ x_n \in X \text{ for all } n \in \mathbb{N} \text{ and } St^\alpha - \lim x \in X\}.$$

This set m_ϱ^α plays an important role on the following two theorems.

Our first approach to a metric on the sequence space X^ω (where $\omega = \{1, 2, 3, ...\}$) as

$$\tau(x, y) = \sup_{k\in\mathbb{N}} \left(\frac{\varrho(x_k, y_k)}{1 + \varrho(x_k, y_k)} \right)$$

where $x = \{x_k\}_{k\in\mathbb{N}}$ and $y = \{y_k\}_{k\in\mathbb{N}}$ belong to X^ω. This distance function τ forms a metric on X^ω.

Theorem 2.4. *The set m_ϱ^α forms a closed subspace of (X^ω, τ) for a fixed α ranges in $0 < \alpha \leq 1$.*

Proof. We yield to expound that in the sequence space X^ω, if any sequence $\{x^{(n)}\}_{n\in\mathbb{N}}$ in m_ϱ^α be such that $\{x^{(n)}\}_{n\in\mathbb{N}}$ converges to x in X^ω then x belongs to m_ϱ^α.

Suppose $x^{(n)} = \{x_k^{(n)}\}_{k\in\mathbb{N}} \xrightarrow{St^\alpha} \xi_n$ where $n \in \mathbb{N}$. This implies $\{x_k^{(n)}\}_{k\in\mathbb{N}} \xrightarrow{St} \xi_n$ where $n \in \mathbb{N}$.

We prove the theorem in two steps:

Step (i): To show $\{\xi_n\}_{n\in\mathbb{N}}$ is a convergent sequence in X.

Step (ii): Finally $x = \{x_k\}_{k\in\mathbb{N}} \xrightarrow{St^\alpha} \xi$, where $\lim_{n\to\infty} \xi_n = \xi$.

We begin with step (i):

Since $\tau(x^{(n)}, x) \to 0$ as $n \to \infty$, then for each $0 < \varepsilon < 1$ there exists a natural number n_0 such that

$$\tau(x^{(n)}, x^{(j)}) < \frac{\varepsilon}{4} \text{ for all } n, j \geq n_0$$

$$\Rightarrow \sup_{k\in\mathbb{N}}\{\frac{\varrho(x_k^{(n)}, x_k^{(j)})}{1 + \varrho(x_k^{(n)}, x_k^{(j)})}\} < \frac{\varepsilon}{4} \text{ for all } n, j \geq n_0$$

$$\Rightarrow \frac{\varrho(x_k^{(n)}, x_k^{(j)})}{1 + \varrho(x_k^{(n)}, x_k^{(j)})} < \frac{\varepsilon}{4} \text{ for all } n, j \geq n_0 \text{ and } k \in \mathbb{N}$$

$$\Rightarrow \varrho(x_k^{(n)}, x_k^{(j)}) < \frac{\varepsilon}{4-\varepsilon} < \frac{\varepsilon}{3} \text{ for all } n, j \geq n_0 \text{ and } k \in \mathbb{N}.$$

Since $\{x_k^{(n)}\}_{k\in\mathbb{N}} \xrightarrow{St} \xi_n$ then there exists $A_n(\subseteq \mathbb{N})$ such that $d(A_n) = 1$ and $\lim_{\substack{k\to\infty\\ k\in A_n}} x_k^{(n)} = \xi_n$

$$\Rightarrow \varrho(x_k^{(n)}, \xi_n) < \frac{\varepsilon}{3} \text{ for all } k \in A_n \text{ and } k \geq l_0 \text{ (where } l_0 \in \mathbb{N} \text{ depends on } \varepsilon).$$

In a similar manner as $\{x_k^{(j)}\}_{k\in\mathbb{N}} \xrightarrow{St} \xi_j$ so there exists $A_j(\subseteq \mathbb{N})$ such that $d(A_j) = 1$ and $\lim_{\substack{k\to\infty\\ k\in A_j}} x_k^{(j)} = \xi_j$

$$\Rightarrow \varrho(x_k^{(j)}, \xi_j) < \frac{\varepsilon}{3} \text{ for all } k \in A_j \text{ and } k \geq q_0 \text{ (where } q_0 \in \mathbb{N} \text{ depends on } \varepsilon).$$

Now choose a natural number k such that $k \in A_n \cap A_j$ and $k \geq \max\{l_0, q_0\}$. Thus we derive that

$$\varrho(\xi_n, \xi_j) \leq \varrho(x_k^{(n)}, \xi_n) + \varrho(x_k^{(j)}, \xi_j) + \varrho(x_k^{(n)}, x_k^{(j)}) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ for all } n, j \geq n_0.$$

Hence $\{\xi_n\}_{n\in\mathbb{N}}$ forms a Cauchy sequence in the complete metric space (X, ϱ). Consequently $\lim_{n\to\infty} \xi_n = \xi \in X$.

Now we proceed to step (ii):

Since $\tau(x^{(n)}, x) \to 0$ and $\varrho(\xi_n, \xi) \to 0$ as $n \to \infty$, therefore we get

$$\tau(x^{(p)}, x) < \frac{\varepsilon}{4} \text{ for all } p \geq a_0 \text{ and } \varrho(\xi_p, \xi) < \frac{\varepsilon}{3} \text{ for all } p \geq b_0,$$

$$\Rightarrow \tau(x^{(p)}, x) < \frac{\varepsilon}{4} \text{ and } \varrho(\xi_p, \xi) < \frac{\varepsilon}{3} \text{ for all } p \geq \max\{a_0, b_0\}.$$

If we choose a fixed p such that $p \geq \max\{a_0, b_0\}$ then

$$\varrho(x_k^{(p)}, x_k) < \frac{\varepsilon}{3} \text{ and } \varrho(\xi_p, \xi) < \frac{\varepsilon}{3} \text{ for all } k \in \mathbb{N}.$$

Applying above inequalities we get $\varrho(x_k, \xi) \leq \varrho(x_k^{(p)}, \xi_p) + \varrho(\xi_p, \xi) + \varrho(x_k^{(p)}, x_k) \leq \varrho(x_k^{(p)}, \xi_p) + \frac{2\varepsilon}{3}$ for all $k \in \mathbb{N}$ which procures $\{k \in \mathbb{N} : \varrho(x_k, \xi) \geq \varepsilon\} \subseteq \{k \in \mathbb{N} : \varrho(x_k^{(p)}, \xi_p) + \frac{2\varepsilon}{3} \geq \varepsilon\} = \{k \in \mathbb{N} : \varrho(x_k^{(p)}, \xi_p) \geq \frac{\varepsilon}{3}\}$.

Accordingly $d^\alpha(\{k \in \mathbb{N} : \varrho(x_k^{(p)}, \xi_p) \geq \frac{\varepsilon}{3}\}) = 0$ and as a subsequence $d^\alpha(\{k \in \mathbb{N} : \varrho(x_k, \xi) \geq \varepsilon\}) = 0$. This shows that $x = \{x_k\}_{k\in\mathbb{N}} \xrightarrow{St^\alpha} \xi$. Hence we reach to the desired conclusion. □

Finally, let us define another metric on the sequence space X^ω as

$$D(x,y) = \sup_{k\in\mathbb{N}}\{\bar{\varrho}(x_k, y_k)\},$$

where $x = \{x_k\}_{k\in\mathbb{N}}$, $y = \{y_k\}_{k\in\mathbb{N}}$ are elements of X^ω and $\bar{\varrho}$ is the standard bounded metric on X (i.e., $\bar{\varrho} = \min\{\varrho, 1\}$). This distance function D forms a metric on X^ω and is called uniform metric on X^ω

Theorem 2.5. *For a fixed α $(0 < \alpha \leq 1)$ the set m_ϱ^α is a closed subspace of X^ω (X^ω-endowed with uniform metric).*

Proof. Let $x^{(n)} = \{x_k^{(n)}\}_{k\in\mathbb{N}} \in m_\varrho^\alpha$ $(n = 1,2,3,...)$ and $\lim_{n\to\infty} x^{(n)} = x$ where $x = \{x_k\}_{k\in\mathbb{N}} \in X^\omega$, i.e., $\lim_{n\to\infty} D(x^{(n)}, x) = 0$. We shall show that $x \in m_\varrho^\alpha$. By our assumption, each $x^{(n)}$ is statistically convergent of order α in X and let $St^\alpha - \lim x^{(n)} = \xi_n$ $(n = 1,2,3,...)$. We require two steps to prove the theorem.

Step (I) : $\{\xi_n\}_{n\in\mathbb{N}}$ is a convergent sequence in X.

Step (II) : $St^\alpha - \lim x = \lim \xi_n$.

(I) : Let $0 < \delta < 1$. Since $\lim_{n\to\infty} x^{(n)} = x$ so $\{x^{(n)}\}_{n\in\mathbb{N}}$ is Cauchy in X^ω. Then there exists $k_0 \in \mathbb{N}$ such that for $u, v \in \mathbb{N}$ and $u, v > k_0$ we have

$$\begin{aligned} & D(x^{(u)}, x^{(v)}) < \delta/3 \\ \Rightarrow\ & \sup_{k\in\mathbb{N}}\{\bar{\varrho}(x_k^{(u)}, x_k^{(v)})\} < \delta/3 < 1 \\ \Rightarrow\ & \bar{\varrho}(x_k^{(u)}, x_k^{(v)}) < \delta/3 < 1 \text{ for all } k \in \mathbb{N}, \\ \Rightarrow\ & \varrho(x_k^{(u)}, x_k^{(v)}) < \delta/3 \text{ for all } k \in \mathbb{N}. \end{aligned}$$

We know that, if $\{x_k^{(n)}\}_{k\in\mathbb{N}} \xrightarrow{St^\alpha} \xi_n$ then $\{x_k^{(n)}\}_{k\in\mathbb{N}} \xrightarrow{St} \xi_n$ where $n \in \mathbb{N}$.
Since $\{x_k^{(u)}\}_{k\in\mathbb{N}} \xrightarrow{St} \xi_u$ then there exists $A_u (\subseteq \mathbb{N})$ such that $d(A_u) = 1$ and $\lim_{\substack{k\to\infty \\ k\in A_u}} x_k^{(u)} = \xi_u$

$$\Rightarrow\ \varrho(x_k^{(u)}, \xi_u) < \frac{\delta}{3} \text{ for all } k \in A_u \text{ and } k \geq k_1 \text{ (where } k_1 \in \mathbb{N} \text{ depends on } \delta).$$

Similarly, $\{x_k^{(v)}\}_{k\in\mathbb{N}} \xrightarrow{St} \xi_v$ then there exists $A_v (\subseteq \mathbb{N})$ such that $d(A_v) = 1$ and $\lim_{\substack{k\to\infty \\ k\in A_v}} x_k^{(v)} = \xi_v$

$$\Rightarrow\ \varrho(x_k^{(v)}, \xi_v) < \frac{\delta}{3} \text{ for all } k \in A_v \text{ and } k \geq k_2 \text{ (where } k_2 \in \mathbb{N} \text{ depends on } \delta).$$

Since $d(A_u \cap A_v) = 1$ so there exists a natural number k such that $k \in A_u \cap A_v$ and $k \geq \max\{k_1, k_2\}$ which together imply

$$\varrho(\xi_u, \xi_v) \leq \varrho(x_k^{(u)}, \xi_u) + \varrho(x_k^{(v)}, \xi_v) + \varrho(x_k^{(u)}, x_k^{(v)}) < \delta \text{ for all } u, v \geq k_0.$$

Hence $\{\xi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in a complete metric space (X, ϱ). So conveniently $\lim_{n\to\infty} \xi_n = \xi \in X$.

(II): Let $0 < \varepsilon < 1$ be given. Choose $v_0 \in \mathbb{N}$ such that for $v \in \mathbb{N}$ and $v > v_0$ we have simultaneously $\varrho(\xi_v, \xi) < \varepsilon/3$ and $D(x^{(v)}, x) < \varepsilon/3$. Also we have $\varrho(x_k^{(v)}, x_k) < \varepsilon/3$ for all $k \in \mathbb{N}$. Now for each $k \in \mathbb{N}$ we have

$$\bar{\varrho}(x_k, \xi) \leq \bar{\varrho}(x_k, x_k^{(v)}) + \bar{\varrho}(x_k^{(v)}, \xi_v) + \bar{\varrho}(\xi_v, \xi) < \frac{\varepsilon}{3} + \bar{\varrho}(x_k^{(v)}, \xi_v) + \frac{\varepsilon}{3}. \qquad (1)$$

Let $A(\varepsilon) = \{k \in \mathbb{N} : \varrho(x_k, \xi) \geq \varepsilon\}$ and $A_v(\varepsilon/3) = \{k \in \mathbb{N} : \varrho(x_k^{(v)}, \xi_v) \geq \varepsilon/3\}$. Then $(A(\varepsilon))^c = \{k \in \mathbb{N} : \varrho(x_k, \xi) < \varepsilon\} = \{k \in \mathbb{N} : \bar{\varrho}(x_k, \xi) < \varepsilon\}$ and $(A_v(\varepsilon/3))^c = \{k \in \mathbb{N} : \varrho(x_k^v, \xi_v) < \varepsilon/3\} = \{k \in \mathbb{N} : \bar{\varrho}(x_k^{(v)}, \xi_v) < \varepsilon/3\}$, where c

stands for complement.

From (1) now it easily follows that $(A_v(\varepsilon/3))^c \subset (A(\varepsilon))^c \Rightarrow A(\varepsilon) \subset A_v(\varepsilon/3)$. Since $d^\alpha(A_v(\epsilon)) = 0$ so $d^\alpha(A(\epsilon)) = 0$. Hence $St^\alpha - \lim x = \xi$ and this proves the result. □

Remark 2.6. *In particular if we consider $X = \mathbb{R}$ and ϱ is the usual metric on $\mathbb{R}$ then from Theorem 2.4 and Theorem 2.5 we get the set of all statistically convergent sequences of order α forms a closed subspace w.r.t the metrics τ and D.*

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