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On Topological Conjugacy of Some Chaotic Dynamical Systems on the Sierpinski Gasket

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Abstract. The dynamical systems on the classical fractals can naturally be obtained with the help of their iterated function systems. In the recent years, different ways have been developed to define dynamical systems on the self similar sets. In this paper, we give composition functions by using expanding and folding mappings which generate the classical Sierpinski Gasket via the escape time algorithm. These functions also indicate dynamical systems on this fractal. We express the dynamical systems by using the code representations of the points. Then, we investigate whether these dynamical systems are topologically conjugate (equivalent) or not. Finally, we show that the dynamical systems are chaotic in the sense of Devaney and then we also compute and compare the periodic points.

1. Introduction

Fractals are one of the popular research subjects and have a great importance because of the different applications (see [9, 11, 13, 15, 18, 19]). Moreover, there is a strong relationship between chaos, fractals and dynamical systems. Thus, there also have been many studies involving these subjects in common (see [1, 3, 4, 8, 10, 17, 22, 24]). For instance, Barnsley defined a dynamical system on the right Sierpinski triangle in [4] through the related iterated function systems and by using this manner, many dynamical systems can be constructed on the different fractals. There are also different ways to define any dynamical systems on the fractals considering their structures. For example, the Cantor set, Cantor dust, the classical Sierpinski gasket (S), the right Sierpinski gasket, the Koch curve, the Vicsek fractal and the Sierpinski carpet are obtained by using the composition functions defined in [2] via escape time algorithm and these composition functions also state dynamical systems on the related fractal sets. Moreover, the dynamical systems can be expressed by using the code representations of the points on these fractals. In order to show that whether a dynamical system is chaotic or not, we need an intrinsic metric defined on the code set of the fractal set. The intrinsic metrics on the classical fractals such as the Sierpinski gasket, Sierpinski-like gasket, Sierpinski tetrahedron, Sierpinski gasket SG(3) and Vicsek fractal have been recently defined in [3, 12, 20, 21, 23, 25] by using the code representations of their points. The geometrical and topological properties of these fractals can be expressed more clearly (see [12, 23, 26]) thanks to these intrinsic metrics. Moreover, it is more understandable to show that the dynamical systems defined on the Sierpinski gasket

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and the Sierpinski tetrahedron are chaotic by using the intrinsic metric (see [3, 24]). Note that, the chaotic dynamical system $\{S; F\}$ given in [24] is constructed by using expanding and folding mappings (see Figure 1) and also the composition function F is defined as follows:

$$F = f_4 \circ f_3 \circ f_2 \circ f_1 \tag{1}$$

where $f_1, f_2, f_3, f_4 : \mathbb{R}^2 \to \mathbb{R}^2$ such that,

$$\begin{aligned}
f_1(x,y) &= (2x, 2y)\,, \\
f_2(x,y) &= \left(x, \tfrac{\sqrt{3}}{2} - \left|y - \tfrac{\sqrt{3}}{2}\right|\right), \\
f_3(x,y) &= \left(-\tfrac{\sqrt{3}}{2}\left|\tfrac{\sqrt{3}}{2}(x-1) - \tfrac{y}{2}\right| + \tfrac{(x-1)+y\sqrt{3}}{4} + 1, \tfrac{\sqrt{3}}{2}\left(\tfrac{x-1}{2} + \tfrac{y\sqrt{3}}{2}\right) + \tfrac{1}{2}\left|\tfrac{\sqrt{3}(x-1)-y}{2}\right|\right), \\
f_4(x,y) &= \left(-\tfrac{\sqrt{3}}{2}\left|\tfrac{\sqrt{3}}{2}(x-1) + \tfrac{y}{2}\right| + \tfrac{(x-1)-y\sqrt{3}}{4} + 1, \tfrac{\sqrt{3}}{2}\left(\tfrac{y\sqrt{3}}{2} - \tfrac{(x-1)}{2}\right) - \tfrac{1}{2}\left|\tfrac{\sqrt{3}(x-1)+y}{2}\right|\right).
\end{aligned}$$

This function is expressed on the code set of the Sierpinski gasket in the following proposition (for details see [24]). Thus, it is easily shown that $\{S; F\}$ is chaotic dynamical system by using Proposition 1.2.

Definition 1.1. Let X be a point on S which is denoted by $x_1x_2x_3\ldots x_i\ldots$ where $x_i \in \{0,1,2\}$, $i \in \mathbb{N}$. The sequence $x_1x_2x_3\ldots x_i\ldots$ is called a code representation of the point $X \in S$ (for details see [24]).

Proposition 1.2. *If $x_1x_2x_3\ldots$ is the code representation of an arbitrary point X of S, then the function $F : S \to S$ defined in (1) is expressed by $F(X) = Y$ such that the code representation of Y is $y_1y_2y_3\ldots$, where $y_i \equiv x_{i+1} + x_1 \pmod 3$ for $x_i, y_i \in \{0,1,2\}$ and $i = 1,2,3,\ldots$.*

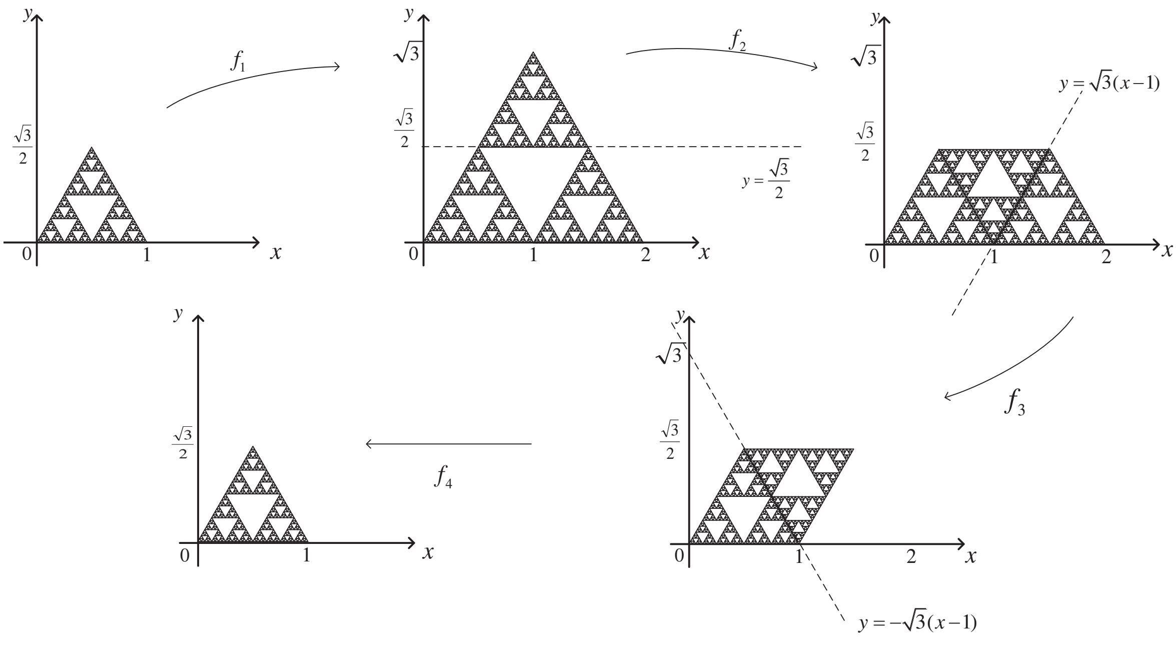

Figure 1: The functions f_1, f_2, f_3, f_4 on S.

In the present paper, we first define new functions G and T, by using different folding mappings in (3) and (8) respectively. Note that, the functions G and T generate the classical Sierpinski gasket via escape time algorithm, although F do not give. We express the functions G and T on the code set of the Sierpinski gasket in Proposition 3.1 and Proposition 4.1. Moreover, we show that $\{S; G\}$ is chaotic in Theorem 3.2 and we conclude that $\{S; T\}$ is also chaotic dynamical system in Corollary 4.5. To this end, we use Proposition 4.4 which states that these dynamical system are topologically conjugate. On the other hand, we determine that $\{S; G\}$ and $\{S; T\}$ are not topologically conjugate with $\{S; F\}$ in Corollary 3.3.

2. Preliminaries

We now recall the definition of chaotic dynamical systems, equivalent (in other words topologically conjugate) dynamical systems and the intrinsic metric formula on the Sierpinski gasket by using the code representation of the points.

Definition 2.1. Let (X, d) be a metric space. A transformation $f : X \to X$ is a dynamical system and represented by $\{X; f\}$ (see [4]).

Definition 2.2. If a dynamical system $\{X; f\}$ satisfies the following three conditions, then it is chaotic in the sense of Devaney:

Sensitivity dependence on the initial condition: there exists $\epsilon > 0$ such that, for any $x \in X$ and any ball $B(x, \delta)$ with radius $\delta > 0$, there is $y \in B(x, \delta)$ and an integer $n \geq 0$ satisfying $d(f^n(x), f^n(y)) > \epsilon$.

Topologically transitive: for any open subsets U and V of the metric space (X, d), there exists an integer n such that $U \cap f^n(V) \neq \emptyset$.

Density of periodic points: there exist periodic points of f which is sufficiently close to any point of X (see [8]).

Definition 2.3. Two dynamical systems $\{X_1; f_1\}$ and $\{X_2; f_2\}$ are said to be equivalent or topologically conjugate, if there is a homeomorphism $\theta : X_1 \to X_2$ such that $f_2 = \theta \circ f_1 \circ \theta^{-1}$ (or that means $\forall x \in X_1$, $\theta(f_1(x)) = f_2(\theta(x))$). In this case, f_1 and f_2 are called conjugate maps and θ is called a conjugacy (see [4]).

In order to compare the dynamical systems, we also use the following proposition.

Proposition 2.4. ([14]) *If $f : X \to X$ and $g : Y \to Y$ are conjugate maps via a conjugacy $h : X \to Y$ such that $h \circ f = g \circ h$, then*

- *$h \circ f^n = g^n \circ h$ for $n = 1, 2, 3, \ldots$.*
- *If x^* is a point of period n for f, then $h(x^*)$ is a point of period n for g. That means $\{x_1, x_2, \ldots, x_n\}$ is a cycle of period n for f if and only if $\{h(x_1), \ldots, h(x_n)\}$ is a cycle of period n for g.*
- *f is transitive if and only if g is transitive.*
- *f has a dense set of periodic points if and only if g has a dense set of periodic points.*

To define intrinsic metrics on a fractal set is a substantial matter. As seen from the literature, the intrinsic metrics can be defined on the same sets with different ways (see [6, 7, 16, 25]). In this study, we use the following intrinsic metric defined in [25] by using the code representations of the point on the classical Sierpinski Gasket.

Theorem 2.5. *Let $a_1a_2\ldots a_{k-1}a_ka_{k+1}\ldots$ and $b_1b_2\ldots b_{k-1}b_kb_{k+1}\ldots$ be two representations respectively of the points A and B on the equilateral Sierpinski Gasket such that $a_i = b_i$ for $i = 1, 2, \ldots, k-1$ and $a_k \neq b_k$. The distance $d(A, B)$ between A and B is determined by the following formula:*

$$d(A,B) = \min\left\{\sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i}, \frac{1}{2^k} + \sum_{i=k+1}^{\infty} \frac{\gamma_i + \delta_i}{2^i}\right\}, \tag{2}$$

where

$$\alpha_i = \begin{cases} 0, & a_i = b_k \\ 1, & a_i \neq b_k \end{cases}, \qquad \beta_i = \begin{cases} 0, & b_i = a_k \\ 1, & b_i \neq a_k \end{cases},$$

$$\gamma_i = \begin{cases} 0, & a_i \neq a_k \text{ and } a_i \neq b_k \\ 1, & \text{otherwise} \end{cases}, \qquad \delta_i = \begin{cases} 0, & b_i \neq b_k \text{ and } b_i \neq a_k \\ 1, & \text{otherwise} \end{cases}.$$

3. The Construction of a Chaotic Dynamical System on S which is not Equivalent to $\{S; F\}$

In this section, we firstly define a composition function G by using an expanding and two folding mappings. Then, we express this function on the code set of Sierpinski Gasket and we show that this dynamical system is chaotic in the sense of Devaney. Finally, we compare the dynamical systems $\{S; G\}$ and $\{S; F\}$ by means of the topological conjugacy.

Suppose that the composition function G is defined as

$$G = g_3 \circ g_2 \circ g_1 \tag{3}$$

where $g_i : \mathbb{R}^2 \to \mathbb{R}^2$ $(i = 1, 2, 3)$ such that

$$\begin{aligned} g_1(x,y) &= (2x, 2y) \\ g_2(x,y) &= \left(-\frac{1}{2}\left|\frac{x-2+y\sqrt{3}}{2}\right| - \frac{\sqrt{3}}{2}\left(\frac{y-\sqrt{3}(x-2)}{2}\right) + 2, \frac{y-\sqrt{3}(x-2)}{4} - \frac{\sqrt{3}}{2}\left|\frac{x-2+y\sqrt{3}}{2}\right| \right) \\ g_3(x,y) &= (1 - |x-1|, y). \end{aligned}$$

The function g_1 is an expanding mapping. The function g_2 is a folding mapping with respect to the line $y = -\frac{\sqrt{3}}{3}x + \frac{2\sqrt{3}}{3}$ that moves the points from the upper hand side of this line to the lower hand side. The function g_3 is also a folding mapping that moves the points from the right hand side of the line $x = 1$ to the left hand side. Obviously $\{S; G\}$ is a dynamical system. Since it is difficult to show that whether this dynamical system is chaotic or not by using the equation (3), we express this function on the code set of the Sierpinski Gasket with following proposition.

Proposition 3.1. *Let $x_1x_2x_3\ldots$ and $y_1y_2y_3\ldots$ be the code representations of the points X and Y respectively for $x_i, y_i \in \{0, 1, 2\}$ and $i = 1, 2, 3, \ldots$ where the function $G : S \to S$ defined in (3) is expressed by $G(X) = Y$ such that*

if $x_1 = 0$, then

$$G(x_1x_2x_3\ldots) = x_2x_3x_4\ldots$$

if $x_1 = 1$, then there are two cases:

Case 1:

$$G(111\ldots10x_{k+1}x_{k+2}x_{k+3}\ldots) = y_1y_2y_3\ldots y_ky_{k+1}\ldots$$

$$y_i = \begin{cases} 0, & x_{i+1} = 1 \\ 1, & x_{i+1} = 0 \quad (i \geq 1), \\ 2, & x_{i+1} = 2 \end{cases}$$

Case 2:

$$G(111\ldots12x_{k+1}x_{k+2}x_{k+3}\ldots) = y_1y_2y_3\ldots y_ky_{k+1}\ldots$$

$$y_i = \begin{cases} 0, & x_{i+1} = 1 \\ 1, & x_{i+1} = 2 \quad (i \geq 1), \\ 2, & x_{i+1} = 0 \end{cases}$$

(Note that, due to the above rules, $G(\overline{1}) = \overline{0}$ is obtained).

if $x_1 = 2$, then

$$G(x_1x_2x_3\ldots) = y_1y_2y_3y_4\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 2 \\ 1, & x_{i+1} = 1 \quad (i \geq 1). \\ 2, & x_{i+1} = 0 \end{cases}$$

Proof. We must show that the function G is well defined on the code set of the Sierpinski Gasket. According to the cases of x_1, the four different rules are valid (see Figure 2, Figure 3, Figure 4 and Figure 5). If X has unique code representation, then $G(X)$ has also. Suppose that X has two different code representations such as $x_1x_2x_3\ldots x_n\alpha\beta\beta\beta\ldots$ and $x_1x_2x_3\ldots x_n\beta\alpha\alpha\alpha\ldots$. There are two different code representations of some points such as $0\overline{1}, 1\overline{0}, 0\overline{2}, 2\overline{0}, 1\overline{2}, 2\overline{1}, 00\overline{1}, 01\overline{0}, 01\overline{2}, 02\overline{1}, 00\overline{2}, 02\overline{0}, 11\overline{0}, 10\overline{1}, 12\overline{0}, 10\overline{2}, 11\overline{2}, 12\overline{1}, 20\overline{2}, 22\overline{0}, 21\overline{2}, 22\overline{1}, 20\overline{1}, 21\overline{0}$.

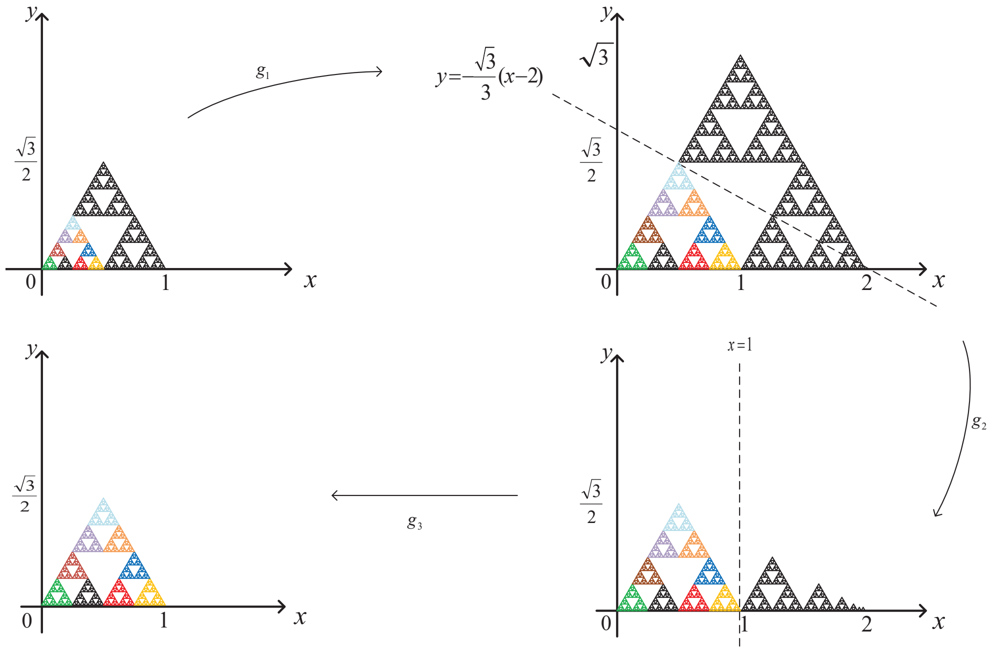

Figure 2: The images of the code sets S_0, S_{0x_2} and $S_{0x_2x_3}$ under G

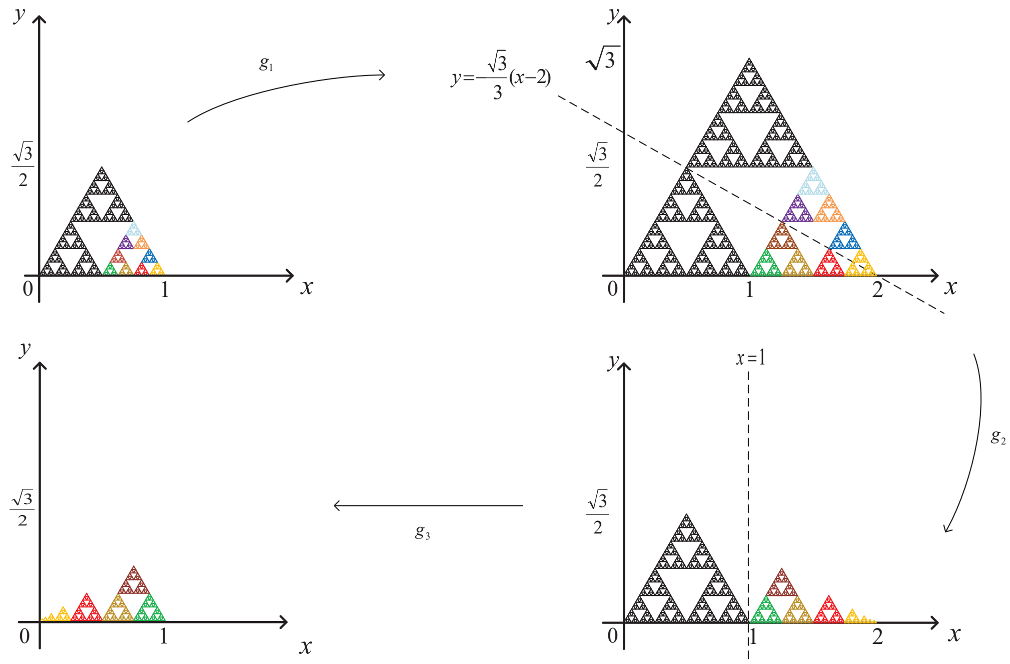

Figure 3: The images of the code sets S_{10}, S_{110}, S_{1110} under G

The images of these points are

$$\begin{array}{llllll} G(0\overline{1}) = & \overline{1}, & G(0\overline{2}) = & \overline{2}, & G(1\overline{2}) = & \overline{1}, \\ G(1\overline{0}) = & \overline{1}, & G(2\overline{0}) = & \overline{2}, & G(2\overline{1}) = & \overline{1}, \end{array}$$

$$\begin{array}{llllll} G(01\overline{0}) = & 1\overline{0}, & G(02\overline{0}) = & 2\overline{0}, & G(01\overline{2}) = & 1\overline{2}, \\ G(00\overline{1}) = & 0\overline{1}, & G(00\overline{2}) = & 0\overline{2}, & G(02\overline{1}) = & 2\overline{1}, \\ G(11\overline{0}) = & 0\overline{1}, & G(11\overline{2}) = & 0\overline{1}, & G(12\overline{0}) = & 1\overline{2}, \\ G(10\overline{1}) = & 1\overline{0}, & G(12\overline{1}) = & 1\overline{0}, & G(10\overline{2}) = & 1\overline{2}, \\ G(20\overline{2}) = & 2\overline{0}, & G(21\overline{2}) = & 1\overline{0}, & G(20\overline{1}) = & 2\overline{1} \\ G(22\overline{0}) = & 0\overline{2}, & G(22\overline{1}) = & 0\overline{1}, & G(21\overline{0}) = & 1\overline{2} \end{array}.$$

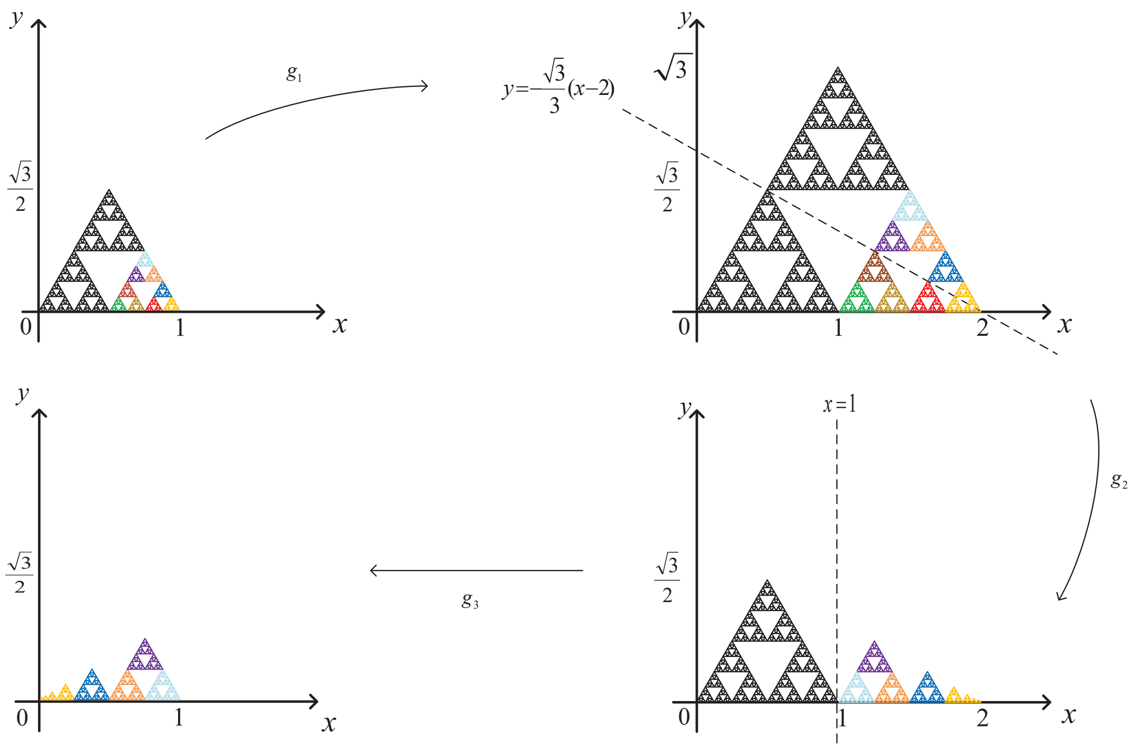

Figure 4: The images of the code sets S_{12}, S_{112}, S_{1112} under G

According to the above results, we can clearly see that the images of different code representations of any points indicate the same points on the Sierpinski Gasket. Since there are all possible cases for the function G, it is enough to investigate the images of the above points. By this means, we can determine the images of any other points. In the general form, if $\sigma = x_1x_2x_3\ldots x_n$, where $x_n \in \{0, 1, 2\}$ then $\sigma\alpha\overline{\beta}$ and $\sigma\beta\overline{\alpha}$ are the different code representations of same points and thus the images of these pairs of points indicate the same addresses independently of σ. □

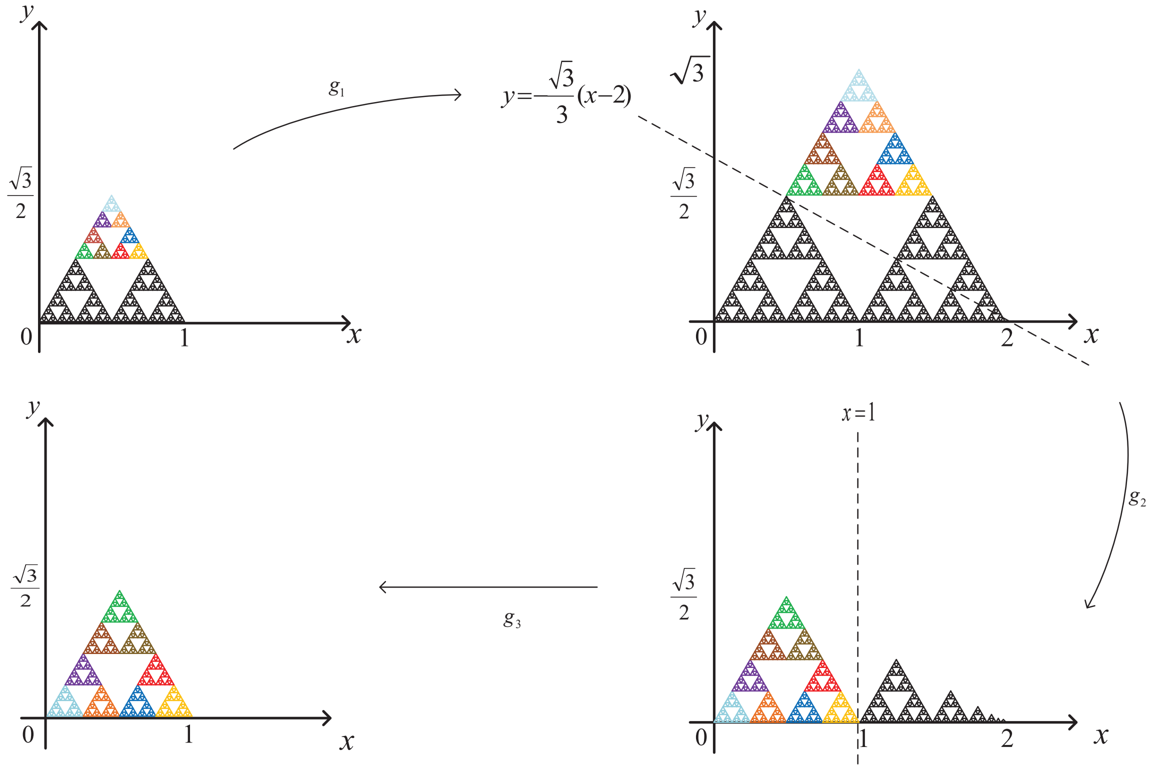

Figure 5: The images of the code sets S_2, S_{2x_2} ve $S_{2x_2x_3}$ under G

By using the intrinsic metric defined in [25], we show that the dynamical system $\{S; G\}$ is chaotic in the sense of Devaney.

Theorem 3.2. *The dynamical system $\{S; G\}$ is chaotic in the sense of Devaney.*

Proof. Initially, we prove that G is sensitive dependence to initial conditions. Let $A \in S$ be an arbitrary point whose code representation is $a_1a_2 \ldots a_{k-1}a_ka_{k+1} \ldots$ where $a_k \in \{0, 1, 2\}$ for $k \in \mathbb{N}$. For any δ, there exists $k \in \mathbb{N}$ which satisfies $\frac{1}{2^{k-2}} < \delta$. If a point $B \in S$ with the code representation $a_1a_2 \ldots a_{k-1}b_kb_{k+1} \ldots$ is chosen, where $b_i \neq a_k$ for $i = k, k+1, k+2, \ldots$, then one can easily show that

$$d(A, B) < \delta$$

from the intrinsic metric given in (2). Note that the images of the different sth ($s \geq k$) terms of A and B under the G^{k-1} are different. Therefore, the images of points A and B under G^{k-1} are

$$G^{k-1}(A) = C$$

$$G^{k-1}(B) = D,$$

where $d_k \neq c_k$ and $d_{k+i} \neq c_k$ $(i = 1, 2, 3, \ldots)$, the code representations of C and D are $c_kc_{k+1}c_{k+2} \ldots$ and $d_kd_{k+1}d_{k+2} \ldots$ respectively and we get the desired result as follows:

$$d(G^{k-1}(A), G^{k-1}(B)) \geq \frac{1}{2}.$$

In order to show that G is topologically transitive, consider nonempty open sets U and V in S. For any $A \in U$ with the code representation $a_1 a_2 \ldots a_k \ldots$, there is a natural number k such that $B(A, \frac{1}{2^{k-1}}) \subset U$. Easy calculations show that

$$U' = \{a_1 a_2 \ldots a_k x_{k+1} x_{k+2} x_{k+3} \ldots \mid a_1, \ldots, a_k \text{ are fixed}\} \subseteq B(A, \frac{1}{2^{k-1}})$$

where $x_i \in \{0, 1, 2\}$ are arbitrary for $i = k+1, k+2, k+3, \ldots$. Since y_j are determined according to the arbitrary x_j where $j = k+1, k+2, k+3, \ldots$, the image of U' under G^k is obtained as

$$G^k(U') = \{y_{k+1} y_{k+2} y_{k+3} \ldots \mid y_i \in \{0, 1, 2\}, i = k+1, k+2, \ldots\}.$$

Hence, we get $G^k(U') = S$ and $G^k(U) = S$. That is, there exists $k \in \mathbb{N}$ which satisfies

$$G^k(U) \cap V \neq \emptyset.$$

For an arbitrary element $A \in S$ and open subset $U \subseteq S$, there exists $k \in \mathbb{N}$ such that $B(A, \frac{1}{2^{k-1}}) \subset U$. Let us define

$$U' = \{a_1 a_2 \ldots a_k x_{k+1} x_{k+2} \ldots \mid a_i \text{ are fixed and } x_j \text{ are arbirtrary }\}$$

for every $i = 1, 2, \ldots, k$ and $j = k+1, k+2, \ldots$. We know that $U' \subset B\left(A, \frac{1}{2^{k-1}}\right)$, and thus, by using the definition of G, we obtain y_j corresponding to arbitrary x_j for $j = k+1, k+2, k+3, \ldots$ such that

$$G^k(\{a_1 a_2 \ldots a_k x_{k+1} x_{k+2} \ldots\}) = \{y_{k+1} y_{k+2} y_{k+3} \ldots\}.$$

Consequently, we have $a_i = y_{k+i}$ and $x_{k+i} = y_{2k+i}$ for $i = 1, 2, 3, \ldots, k$. This shows that there are $k-$ periodic points in any neighbourhood of A. This completes the proof. □

3.1. The Computation of the Periodic Points of G

We now compute the periodic points of G. Let the code representation of a point A on S be $a_1 a_2 a_3 a_4 \ldots$. We must solve

$$G(a_1 a_2 a_3 a_4 \ldots) = a_1 a_2 a_3 a_4 \ldots$$

to determine all fixed points. For the different values of a_1, the different rules of G are valid.

• If $a_1 = 0$, then the previous equations gives:

$$G(0 a_2 a_3 \ldots a_k \ldots) = a_2 a_3 a_4 \ldots a_{k+1} \ldots = a_1 a_2 a_3 \ldots a_{k+1} \ldots . \tag{4}$$

By using the equation (4), we find $a_2 = a_3 = a_4 = \cdots = a_{k+1} = \cdots = 0$. Thus one of the fixed points is $\overline{0} = 000 \ldots$.

• If $a_1 = 1$, then there are two cases:

$$G(1 a_2 a_3 \ldots a_k \ldots) = b_1 b_2 b_3 \ldots b_k \ldots = 1 a_2 a_3 \ldots a_k \ldots \tag{5}$$

and we therefore get $b_1 = a_1 = 1, b_2 = a_2,\ b_3 = a_3$ and so on.

If we consider Case 1 given in Proposition 3.1, we get $a_2 = 0$ and $b_2 = a_2 = 0$, since $b_1 = 1$. It follows that $a_3 = 1$, $b_3 = a_3 = 1$ and $a_4 = 0$. So, by solving the equation (5), we find

$$a_i = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 0, & \text{if } i \text{ is even.} \end{cases}$$

So, another fixed point of G is $\overline{10} = 101010 \ldots$.

If we consider Case 2 given in Proposition 3.1, then we obtain $a_2 = 2$ since $b_1 = 1$. So, we have $b_2 = a_2 = 2$ and then $a_3 = 0$, $b_3 = a_3 = 0$ and $a_4 = 1$. So, by solving (5) we compute for $k = 0, 1, 2, \ldots,$

$$a_i = \begin{cases} 1, & \text{if } i = 3k+1 \\ 2, & \text{if } i = 3k+2 \\ 0, & \text{if } i = 3k+3. \end{cases}$$

It means that $\overline{120} = 120120\ldots$ is also a fixed point of G.

• If $a_1 = 2$, then

$$G(2a_2a_3\ldots a_k\ldots) = b_1b_2b_3\ldots b_k\ldots = 2a_2a_3\ldots a_k\ldots \tag{6}$$

and from the rule of G, we get $b_1 = 2, b_2 = a_2,\ b_3 = a_3\ldots.$ Because of $b_1 = 2$, it follows that $a_2 = 0$, $b_2 = a_2 = 0$, $b_3 = a_3 = 2$ and $a_4 = 0$. If we solve the equation (6), then we find

$$a_i = \begin{cases} 2, & \text{if } i \text{ is odd} \\ 0, & \text{if } i \text{ is even} \end{cases}$$

and the fourth fixed point is $\overline{20} = 2020\ldots.$ Hence, we observe that G has a number of fixed points different from F defined in (1).

Any periodic points can be calculated in a similar fashion. For example, the 2− periodic points of G are computed as

$$\bullet\overline{0220} = 02200220\ldots,\quad \bullet\overline{0110} = 01100110\ldots$$

$$\bullet\overline{011220} = 011220011220\ldots,\quad \bullet\overline{112200} = 112200112200\ldots$$

$$\bullet\overline{1100} = 11001100\ldots,\quad \bullet\overline{1020} = 10201020\ldots$$

$$\bullet\overline{1210} = 12101210\ldots,\quad \bullet\overline{2200} = 22002200\ldots.$$

In order to compute any 2−periodic points of G, we must solve the following equation

$$G^2(a_1a_2a_3a_4\ldots) = a_1a_2a_3a_4\ldots.$$

For example, if $a_1 = 0$, then

$$G^2(0a_2a_3\ldots a_k\ldots) = G(a_2a_3a_4\ldots a_{k+1}\ldots) = 0a_2a_3\ldots a_k\ldots.$$

Depending on a_2, the valid rule is determined. If $a_2 = 2$, then we solve the equation

$$G^2(02a_3\ldots a_k\ldots) \quad = \quad G(2a_3a_4\ldots a_{k+1}\ldots) = 02a_3\ldots a_k\ldots. \tag{7}$$

Hence, we calculate $a_3 = 2$, $a_4 = 0$, $a_5 = 0$, $a_6 = 2$. By the equation (7), we obtain one of the 2− periodic point $\overline{0220}$ and others can be obtained in the same way.

Corollary 3.3. *The dynamical systems $\{S; F\}$ and $\{S; G\}$ are not topologically conjugate or equivalent.*

Proof. These dynamical systems have different number of fixed points. Thus, according to Proposition 2.4, they are not topologically conjugate or equivalent. □

4. The Construction of a Chaotic Dynamical System on the Sierpinski Gasket which is Equivalent to $\{S; G\}$

By using expanding and folding maps, we define a new function T different from F and G on the Sierpinski gasket. Let $t_i : \mathbb{R}^2 \to \mathbb{R}^2$, $(i = 1, 2, 3, 4)$

$$\begin{aligned} t_1(x, y) &= (2x, 2y) \\ t_2(x, y) &= \left(\frac{1}{2}\left|\frac{x-y\sqrt{3}}{2}\right| + \frac{\sqrt{3}}{2}\left(\frac{y+x\sqrt{3}}{2}\right), \frac{y+x\sqrt{3}}{4} - \frac{\sqrt{3}}{2}\left|\frac{x-y\sqrt{3}}{2}\right|\right) \\ t_3(x, y) &= (1 + |x-1|, y) \\ t_4(x, y) &= (x-1, y). \end{aligned}$$

Thus, the composition function T on the Sierpinski Gasket is defined such that

$$T = t_4 \circ t_3 \circ t_2 \circ t_1. \tag{8}$$

The mapping t_1 doubles the Sierpinski gasket, while t_2 and t_3 are folding mappings that take the points from the left hand sides of the lines $y = \frac{\sqrt{3}}{3}x$ and $x = 1$ to the right hand sides respectively. Also, t_4 is a translation mapping. This shows that T states a dynamical system on the Sierpinski gasket and $\{S; T\}$ is expressed on the code set of the Sierpinski gasket in the following proposition.

Proposition 4.1. *Suppose that $x_1x_2x_3\ldots$ and $y_1y_2y_3\ldots$ are the code representations of the points X and Y respectively for $x_i, y_i \in \{0, 1, 2\}$ and $i = 1, 2, 3, \ldots$. Then the function $T : S \to S$ defined in (8) is expressed by $T(X) = Y$ as follows:*
If $x_1 = 0$, then there are two cases:
Case 1:

$$T(000\ldots01x_{k+1}x_{k+2}x_{k+3}\ldots) = y_1y_2\ldots y_ky_{k+1}\ldots$$

$$y_i = \begin{cases} 0, & x_{i+1} = 1 \\ 1, & x_{i+1} = 0 \quad (i \geq 1). \\ 2, & x_{i+1} = 2 \end{cases}$$

Case 2:

$$T(000\ldots02x_{k+1}x_{k+2}x_{k+3}\ldots) = y_1y_2\ldots y_ky_{k+1}\ldots$$

$$y_i = \begin{cases} 0, & x_{i+1} = 2 \\ 1, & x_{i+1} = 0 \quad (i \geq 1). \\ 2, & x_{i+1} = 1 \end{cases}$$

(Note that, due to the above rules, $T(\overline{0}) = \overline{1}$ is obtained).
If $x_1 = 1$, then

$$T(x_1x_2x_3\ldots) = x_2x_3x_4\ldots.$$

If $x_1 = 2$, then

$$T(x_1x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 1, & x_{i+1} = 2 \\ 2, & x_{i+1} = 1 \quad (i \geq 1). \\ 0, & x_{i+1} = 0 \end{cases}$$

Proof. Depending on the cases of x_1, the four different rules must be applied (see Figure 6, Figure 7, Figure 8, Figure 9). If X has unique code representation, then the result is obvious. If X has two different code representations such as $x_1x_2x_3\ldots x_n\alpha\beta\beta\beta\ldots$ and $x_1x_2x_3\ldots x_n\beta\alpha\alpha\alpha\ldots$, it is enough to check the images of $0\overline{1}, 1\overline{0}, 0\overline{2}, 2\overline{0}, 1\overline{2}, 2\overline{1}, 00\overline{1}, 01\overline{0}, 01\overline{2}, 02\overline{1}, 00\overline{2}, 02\overline{0}$ $11\overline{0}, 10\overline{1}, 12\overline{0}, 10\overline{2}, 11\overline{2}, 12\overline{1}, 20\overline{2}, 22\overline{0}, 21\overline{2}, 22\overline{1}, 20\overline{1}, 21\overline{0}$.

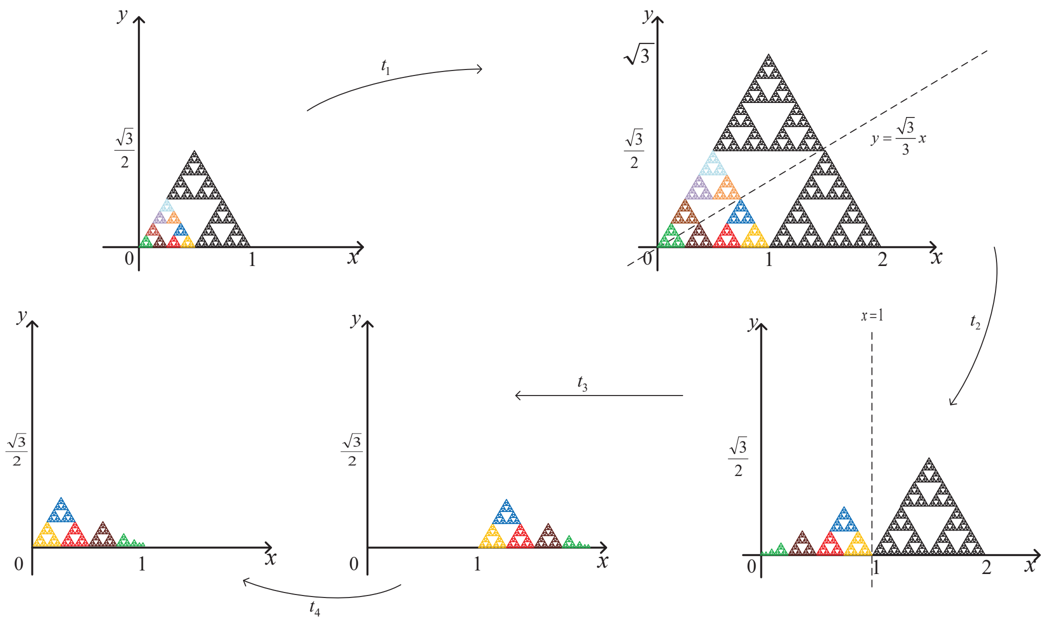

Figure 6: The images of the code sets S_{01}, S_{001}, S_{0001} under T

The images of these points under T according to the rules given in Proposition 4.1 are obtained as follows:

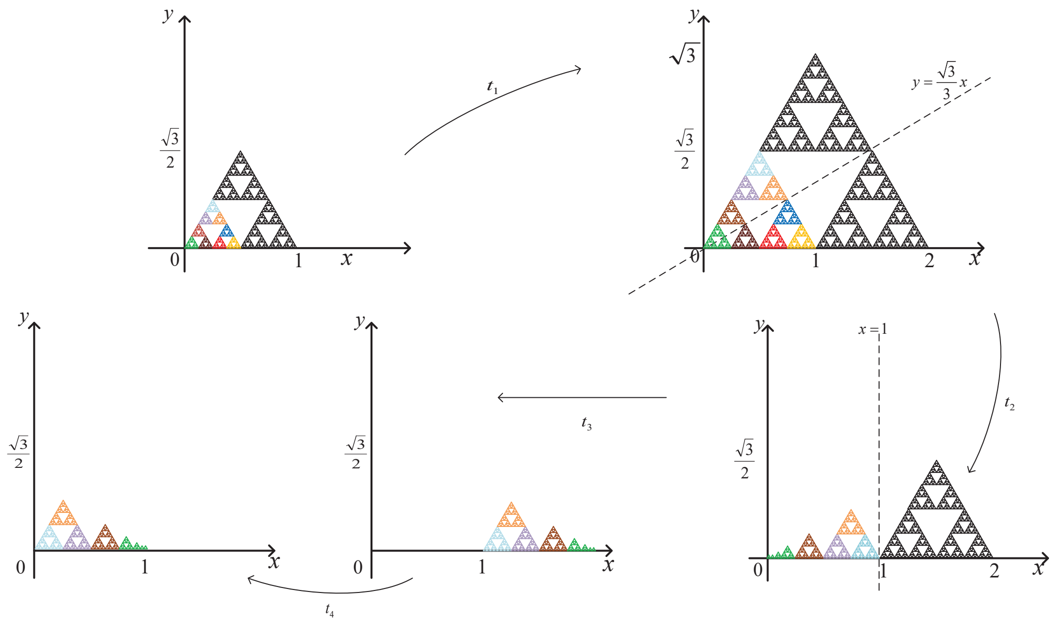

Figure 7: The images of the code sets S_{02}, S_{002}, S_{0002} under T

$$\begin{array}{llllll}
T(0\overline{1}) = & \overline{0}, & T(0\overline{2}) = & \overline{0}, & T(1\overline{2}) = & \overline{2}, \\
T(1\overline{0}) = & \overline{0}, & T(2\overline{0}) = & \overline{0}, & T(2\overline{1}) = & \overline{2}, \\
\\
T(00\overline{1}) = & 1\overline{0}, & T(00\overline{2}) = & 1\overline{0}, & T(02\overline{1}) = & 0\overline{2}, \\
T(01\overline{0}) = & 0\overline{1}, & T(02\overline{0}) = & 0\overline{1}, & T(01\overline{2}) = & 0\overline{2}, \\
\\
T(11\overline{0}) = & 1\overline{0}, & T(11\overline{2}) = & 1\overline{2}, & T(12\overline{0}) = & 2\overline{0}, \\
T(10\overline{1}) = & 0\overline{1}, & T(12\overline{1}) = & 2\overline{1}, & T(10\overline{2}) = & 0\overline{2}, \\
\\
T(20\overline{2}) = & 0\overline{1}, & T(21\overline{2}) = & 2\overline{1}, & T(20\overline{1}) = & 0\overline{2} \\
T(22\overline{0}) = & 1\overline{0}, & T(22\overline{1}) = & 1\overline{2}, & T(21\overline{0}) = & 2\overline{0}
\end{array}.$$

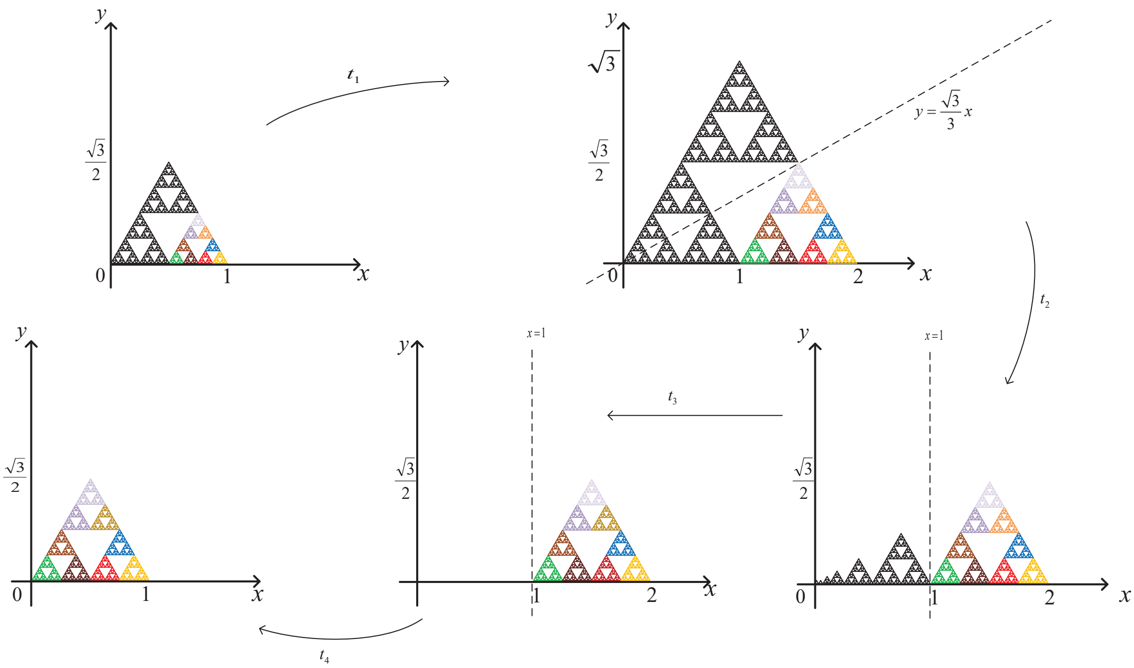

Figure 8: The images of the code sets S_1, S_{1x_2} and $S_{1x_2x_3}$ under T

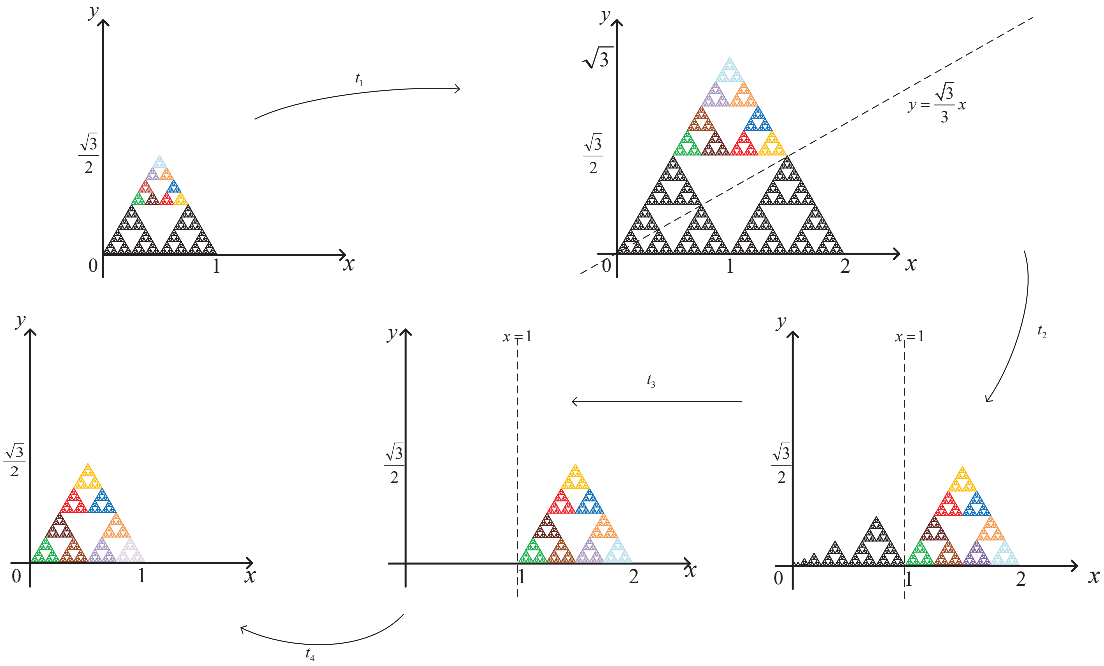

Figure 9: The images of the code sets S_2, S_{2x_2} and $S_{2x_2x_3}$ under T

In the general form, for $\sigma = x_1x_2x_3\dots x_n$, $\sigma\alpha\overline{\beta}$ and $\sigma\beta\overline{\alpha}$ are the different code representations of the same points and the images of these pairs of points indicate the same addresses independently of σ. □

In the following, we give a conjugacy H between T and G:

Lemma 4.2. *Let the code representations of the points X, $X' \in S$ be $x_1x_2x_3\dots$ and $x'_1x'_2x'_3\dots$ respectively where $x_i, x'_i \in \{0,1,2\}$ for $i \in \mathbb{N}$. If the function $H : S \to S$ is defined by*

$$H(X) = X',\ x'_i = \begin{cases} 1, & x_i = 0 \\ 0, & x_i = 1 \\ 2, & x_i = 2 \end{cases}, \tag{9}$$

then we get $H(G(X)) = T(H(X))$ for $X \in S$.

Proof. It is clear from the definitions of the functions F and G. □

Lemma 4.3. *For all $X, Y \in S$, $d(H(X), H(Y)) = d(X, Y)$.*

Proof. Suppose that the code representations of X and Y are $x_1x_2x_3\dots$ and $y_1y_2y_3\dots$ respectively where $x_i, y_i \in \{0,1,2\}$ and $x_i = y_i$ for $i = 1,2,3,\dots k-1$ and $x_k \neq y_k$. Due to the definition of H, the images of the points are $H(X) = x'_1x'_2x'_3\dots x'_{k-1}x'_kx'_{k+1}\dots$ and $H(Y) = y'_1y'_2y'_3\dots y'_{k-1}y'_ky'_{k+1}\dots$ where $x'_i = y'_i$ for $i = 1,2,3,\dots k-1$ and $x'_k \neq y'_k$. Moreover,

$$x_{k+i} = y_k \iff x'_{k+i} = y'_k \text{ and } x_{k+i} \neq y_k \iff x'_{k+i} \neq y'_k$$

$$y_{k+i} = x_k \iff y'_{k+i} = x'_k \text{ and } y_{k+i} \neq x_k \iff y'_{k+i} \neq x'_k.$$

Therefore, considering the metric d, we obtain the desired result $d(H(X), H(Y)) = d(X, Y)$. □

Since H, which is given in (9), is also surjective, we have the following proposition.

Proposition 4.4. *The function H defined in (9) is a homeomorphism.*

Corollary 4.5. $\{S;T\}$ *is a chaotic dynamical system.*

Proof. Since $\{S;G\}$ is chaotic and S is compact and T is continuous, we conclude from Proposition 2.4 and [5] that $\{S;T\}$ is a chaotic dynamical system. □

The computation of periodic points of $\{S;T\}$ by using the homeomorphism H

From Proposition 4.4, we conclude that $\{S;T\}$ and $\{S;G\}$ are equivalent dynamical systems. Hence $\{S;T\}$ and $\{S;G\}$ have the same number of periodic points. By the help of H defined in (9), the periodic points of $\{S;T\}$ can be easily found, while the periodic points of $\{S;G\}$ are known.

Since the fixed points of G are

$$\bullet\overline{0} = 000\ldots,\ \bullet\overline{20} = 2020\ldots,\ \bullet\overline{10} = 1010\ldots,\ \bullet\overline{120} = 120120\ldots,$$

the fixed points of T are computed as

$$\bullet H(\overline{0}) = \overline{1},\ \bullet H(\overline{20}) = \overline{21},\ \bullet H(\overline{10}) = \overline{01},\ \bullet H(\overline{120}) = \overline{021}.$$

Similarly, 2−periodic points of T are

$$\bullet H(\overline{0220}) = \overline{1221},\quad \bullet H(\overline{0110}) = \overline{1001},\quad \bullet H(\overline{011220}) = \overline{100221}$$

$$\bullet H(\overline{112200}) = \overline{002211},\quad \bullet H(\overline{1100}) = \overline{0011},\quad \bullet H(\overline{1020}) = \overline{0121}$$

$$\bullet H(\overline{1210}) = \overline{0201},\quad \bullet H(\overline{2200}) = \overline{2211}.$$

5. Conclusion

In the present paper, we give chaotic dynamical systems $\{S;G\}$ and $\{S;T\}$ on the Sierpinski Gasket and we compare the dynamical systems $\{S;F\}$ given in [24], $\{S;G\}$ and $\{S;T\}$. With a similar way, different dynamical systems can be constructed on many fractals and can be compared whether they are topologically conjugate or not.

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