Filomat 35:7 (2021), 2303–2315
https://doi.org/10.2298/FIL2107303D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Certain Observations on Statistical Variations of Bornological Covers

Subhankar Das[a], Debraj Chandra[b]

[a]*Department of Mathematics, Jadavpur University, Kolkata-700032, West Bengal, India*
[b]*Department of Mathematics, University of Gour Banga, Malda-732103, West Bengal, India*

Abstract. We primarily make a general approach to the study of open covers and related selection principles using the idea of statistical convergence in metric space. In the process we are able to extend some results in (Caserta et al. 2012; Chandra et al. 2020) where bornological covers and related selection principles in metric spaces have been investigated using the idea of strong uniform convergence (Beer and Levi, 2009) on a bornology. We introduce the notion of statistical-$\gamma_{\mathfrak{B}^s}$-cover, statistically-strong-$\mathfrak{B}$-Hurewicz and statistically-strong-$\mathfrak{B}$-groupable cover and study some of its properties mainly related to the selection principles and corresponding games. Also some properties like statistically-strictly Frèchet Urysohn, statistically-Reznichenko property and countable fan tightness have also been investigated in $C(X)$ with respect to the topology of strong uniform convergence $\tau^s_{\mathfrak{B}}$.

1. Introduction

We start by recalling the definition of asymptotic density. If $\mathbb{N}$ is the set of natural numbers and $K \subseteq \mathbb{N}$ then $K(n)$ denotes the set $\{k \in K : k \leq n\}$ and $|K(n)|$ is the cardinality of $K(n)$. The asymptotic density of K is defined by $d(K) = \lim\limits_{n\to\infty} \frac{|K(n)|}{n}$, provided the limit exists. Though this notion has long been used in Number Theory, Ergodic Theory etc., one of its most interesting applications has been in Analysis where the notion of asymptotic density was used to define the idea of statistical convergence by Fast [15] (see also [21, 26–28]), generalizing the idea of usual convergence of real sequences. A sequence $\{x_n : n \in \mathbb{N}\}$ in a topological space is said to converge statistically (in short, s-converge) to x if for any neighbourhood U of x, $d(\{n \in \mathbb{N} : x_n \notin U\}) = 0$. In [12], the authors had studied selection principles, function spaces and hyperspaces using the notion of statistical convergence in topological and uniform spaces. For more details of the study of statistical convergence in topological and function spaces related to selection principles see also [7, 11, 13] and references therein.

Recall that a bornology $\mathfrak{B}$ on a metric space (X, ρ) is a family of subsets of X that is closed under taking finite unions, is hereditary and forms a cover of X (see [16]). A base $\mathfrak{B}_0$ for a bornology $\mathfrak{B}$ is a subfamily

2010 *Mathematics Subject Classification*. Primary 54D20 ; Secondary 54A20, 54C35, 54A25
Keywords. Bornology, selection principles, topology of strong uniform convergence, statistical convergence, open $\mathfrak{B}^s$-cover, $\gamma_{\mathfrak{B}^s}$-cover, s-$\gamma_{\mathfrak{B}^s}$-cover, s-$\mathfrak{B}^s$-Hurewicz property, s-Reznichenko property, s-strictly Frèchet Urysohn property
Received: 06 June 2020; Revised: 16 September 2020; Accepted: 23 September 2020
Communicated by Ljubiša D.R. Kočinac
The first author is thankful to University Grants Commission (UGC), New Delhi-110002, India for granting UGC-NET Junior Research Fellowship (1183/(CSIR-UGC NET DEC.2017)) during the tenure of which this work was done.
Email addresses: subhankarjumh70@gmail.com (Subhankar Das), debrajchandra1986@gmail.com (Debraj Chandra)

of $\mathfrak{B}$ that is cofinal in $\mathfrak{B}$ with respect to inclusion i.e. for $B \in \mathfrak{B}$ there is a $B_0 \in \mathfrak{B}_0$ such that $B \subseteq B_0$. A base is called closed (compact) if all of its members are closed (compact). The family of all finite subsets $\mathcal{F}$ of X forms a bornology which is the smallest bornology on X and the family of all non empty subsets of X is the largest bornology on X. There are other important bornologies such as the family of all non empty ρ-bounded subsets and the family $\mathcal{K}$ of non empty subsets of X with compact closure.

In [4], Beer and Levi had introduced the notion of strong uniform continuity on a bornology. Let (X, ρ) and (Y, ρ') be metric spaces. A mapping $f : X \to Y$ is strongly uniformly continuous on a subset B of X if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\rho(x_1, x_2) < \delta$ and $\{x_1, x_2\} \cap B \neq \emptyset$ imply $\rho'(f(x_1), f(x_2)) < \varepsilon$. Also for a bornology $\mathfrak{B}$ on X, f is called strongly uniformly continuous on $\mathfrak{B}$ if f is strongly uniformly continuous on B for each $B \in \mathfrak{B}$. They had also introduced a new topology on Y^X the set of all function from X into Y, called the topology of strong uniform convergence and studied various properties in function spaces. This study has been further continued in [5].

In [6], the authors had studied open covers and related selection principles in function spaces with respect to the topology of strong uniform convergence on a bornology. Very recently in [8], a further advancement has been made in this direction (see also [3]). Motivated by [12], in this paper, we introduce statistical analogue of certain types of open covers and investigate the behaviour of related selection principles using the idea of strong uniform convergence on a bornology. Our main objective is to study some results of [6, 8] in a more general setup using the idea of statistical convergence. In Section 3, we introduce statistical version of certain types of bornological open covers and observe the behaviour of related selection principles including the α_i-properties. We also introduce the notions of statistically-strong-$\mathfrak{B}$-Hurewicz property and statistically-strong-$\mathfrak{B}$-groupable cover and obtain some game theoretic results. In Section 4, we concentrate on the function space $C(X)(= C(X, \mathbb{R}))$ associated with the topology of strong uniform convergence on $\mathfrak{B}$ and deal with some properties like statistically strictly Frèchet Urysohn, statistically Reznichenko and countable fan tightness.

2. Preliminaries

We follow the notations and terminologies of [2, 14, 16, 22]. Throughout the paper (X, ρ) stands for an infinite metric space and $\mathbb{N}$ stands for the set of positive integers. We first write down two classical selection principles formulated in general form in [17, 24] (see also the survey papers [18, 25] for more details).

For two nonempty classes of sets $\mathcal{A}$ and $\mathcal{B}$, we define

$\mathrm{S}_1(\mathcal{A}, \mathcal{B})$: For each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$, there is a sequence $\{b_n : n \in \mathbb{N}\}$ such that $b_n \in A_n$ for each n and $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$.

$\mathrm{S}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$: For each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$, there is a sequence $\{B_n : n \in \mathbb{N}\}$ of finite (possibly empty) sets such that $B_n \subseteq A_n$ for each n and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

There are infinitely long games corresponding to these selection principles.

$\mathrm{G}_1(\mathcal{A}, \mathcal{B})$ denotes the game for two players, ONE and TWO, who play a round for each positive integer n. In the n-th round ONE chooses a set A_n from $\mathcal{A}$ and TWO responds by choosing an element $b_n \in A_n$. TWO wins the play $\{A_1, b_1, \dots, A_n, b_n, \dots\}$ if $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$. Otherwise ONE wins.

$\mathrm{G}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$ denotes the game where in the n-th round ONE chooses a set A_n from $\mathcal{A}$ and TWO responds by choosing a finite (possibly empty) set $B_n \subseteq A_n$. TWO wins the play $\{A_1, B_1, \dots, A_n, B_n, \dots\}$ if $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$. Otherwise ONE wins.

We also define
$\mathrm{U}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$: For each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$, there is a sequence $\{B_n : n \in \mathbb{N}\}$ of finite (possibly empty) sets such that $B_n \subseteq A_n$ for each n and either $\{\cup B_n : n \in \mathbb{N}\} \in \mathcal{B}$ or for some n, $\cup B_n = X$ (from [17, 24]).

$\mathrm{CDR_{Sub}}(\mathcal{A},\mathcal{B})$: For each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there is a sequence $\{B_n : n \in \mathbb{N}\}$ of pairwise disjoint elements of $\mathcal{B}$ such that for each n, $B_n \subseteq A_n$ [24].

The following selection principles are defined in [19]. The symbol $\alpha_i(\mathcal{A},\mathcal{B})$ for $i = 1, 2, 3, 4$ denotes that for each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$, there is a $B \in \mathcal{B}$ such that
$\alpha_1(\mathcal{A},\mathcal{B})$: for each $n \in \mathbb{N}$, the set $A_n \setminus B$ is finite.
$\alpha_2(\mathcal{A},\mathcal{B})$: for each $n \in \mathbb{N}$, the set $A_n \cap B$ is infinite.
$\alpha_3(\mathcal{A},\mathcal{B})$: for infinitely many $n \in \mathbb{N}$, the set $A_n \cap B$ is infinite.
$\alpha_4(\mathcal{A},\mathcal{B})$: for infinitely many $n \in \mathbb{N}$, the set $A_n \cap B$ is non empty.

Also the statistical version of the above α_i properties are introduced and studied in [12]. In particular, the symbol s-$\alpha_4(\mathcal{A},\mathcal{B})$ denotes that for each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$, there is a $B \in \mathcal{B}$ and a set $K \subseteq \mathbb{N}$ with $d(K) = 1$ such that for each $k \in K$ the set $A_k \cap B$ is non empty.

For $x \in X$, we denote $\Omega_x = \{A \subseteq X : x \in \overline{A} \setminus A\}$ [20]. Also X is said to have countable fan tightness at x [1] if X satisfies $\mathrm{S_{fin}}(\Omega_x, \Omega_x)$.

Let $\mathcal{O}$ denote the collection of all open covers of X. An open cover $\mathcal{U}$ of X is called a γ-cover [24] if $\mathcal{U}$ is infinite and each point of X belongs to all but finitely many members of $\mathcal{U}$. The collection of all γ-covers of X is denoted by Γ. An open cover $\mathcal{U}$ of X is a large cover [24] if it is infinite and each point of X belongs to infinitely many elements of $\mathcal{U}$. The collection of all large covers of X is denoted by Λ. Now we recall some terminologies in terms of the statistical convergence. A subset $\mathcal{V}$ of a cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X is called statistically dense [12] in $\mathcal{U}$ if the set of indices of elements from $\mathcal{V}$ has asymptotic density 1. A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is said to be a statistical γ-cover (in short, s-γ-cover) [12] if for each $x \in X$, $d(\{n \in \mathbb{N} : x \notin U_n\}) = 0$. The collection of all s-γ-covers is denoted by s-Γ. For $x \in X$, the symbol s-Σ_x denotes the set of all sequences s-converge to x [12]. A space X is said to be statistically strictly Fréchet-Urysohn (in short, s-SFU) [12] if $\mathrm{S}_1(\Omega_x, s\text{-}\Sigma_x)$ holds for each $x \in X$. A cover $\mathcal{U}$ is said to be s-groupable [12] of X if it can be represented as a countable union of finite pairwise disjoint subfamilies $\mathcal{V}_n$ of $\mathcal{U}$ such that for $x \in X$, $d(\{n \in \mathbb{N} : x \notin \cup\mathcal{V}_n\}) = 0$. The collection of all s-groupable open covers is denoted by s-$\mathcal{O}^{gp}$. X is said to have the s-Reznichenko property at $x \in X$ [12] if each countable set A in Ω_x can be represented as a countable union of finite and pairwise disjoint subsets of A such that for each neighbourhood W of x, $d(\{n \in \mathbb{N} : W \cap A_n = \emptyset\}) = 0$. The collection of all such countable sets is denoted by s-Ω_x^{gp}.

Next we recall some classes of bornological covers of X. Let $\mathfrak{B}$ be a bornology on metric space (X, ρ). For $B \in \mathfrak{B}$ and $\delta > 0$, let $B^\delta = \bigcup_{x\in B} S(x, \delta)$, where $S(x, \delta) = \{y \in X : \rho(x, y) < \delta\}$. It can be easily observed that $\overline{B^\delta} \subseteq B^{2\delta}$ for every $B \in \mathfrak{B}$ and $\delta > 0$. A cover $\mathcal{U}$ is said to be a strong-$\mathfrak{B}$-cover (in short, $\mathfrak{B}^s$-cover)[5] if for $B \in \mathfrak{B}$ there exist a $U \in \mathcal{U}$ and a $\delta > 0$ such that $B^\delta \subseteq U$. If the members of $\mathcal{U}$ are open then $\mathcal{U}$ is called an open $\mathfrak{B}^s$-cover. The collection of all open $\mathfrak{B}^s$-covers is denoted by $\mathcal{O}_{\mathfrak{B}^s}$. X is said to be $\mathfrak{B}^s$-Lindelöf [6] if each $\mathfrak{B}^s$-cover contains a countable $\mathfrak{B}^s$-subcover. An open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is said to be a $\gamma_{\mathfrak{B}^s}$-cover [5] (see also [6]) of X, if it is infinite and for every $B \in \mathfrak{B}$ there exist a $n_0 \in \mathbb{N}$ and a sequence $\{\delta_n : n \geq n_0\}$ of positive real numbers satisfying $B^{\delta_n} \subseteq U_n$ for all $n \geq n_0$. The collection of all $\gamma_{\mathfrak{B}^s}$-covers is denoted by $\Gamma_{\mathfrak{B}^s}$. An open cover $\mathcal{U}$ of X is said to be $\mathfrak{B}^s$-groupable [8] if it can be expressed as a union of countably many finite pairwise disjoint sets $\mathcal{U}_n$ such that for each $B \in \mathfrak{B}$ there exist a $n_0 \in \mathbb{N}$ and a sequence $\{\delta_n : n \geq n_0\}$ of positive real numbers with $B^{\delta_n} \subseteq U$ for some $U \in \mathcal{U}_n$ for all $n \geq n_0$. X is said to have the $\mathfrak{B}^s$-Hurewicz property [8] if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open $\mathfrak{B}^s$-covers of X, there is a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ where $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ for each $n \in \mathbb{N}$, such that for every $B \in \mathfrak{B}$ there exist a $n_0 \in \mathbb{N}$ and a sequence $\{\delta_n : n \geq n_0\}$ of positive real numbers satisfying $B^{\delta_n} \subseteq U$ for some $U \in \mathcal{V}_n$ for all $n \geq n_0$.

For two metric spaces X and Y, Y^X ($C(X, Y)$) stands for the set of all functions (continuous functions) from X to Y. The commonly used topologies on $C(X, Y)$ are the compact-open topology τ_k, and the topology of pointwise convergence τ_p. The corresponding spaces are, in general, respectively denoted by $(C(X, Y), \tau_k)$ (resp. $C_k(X)$ when $Y = \mathbb{R}$), and $(C(X, Y), \tau_p)$ (resp. $C_p(X)$ when $Y = \mathbb{R}$).

Let $\mathfrak{B}$ be a bornology on X with closed base. Then the topology of strong uniform convergence $\tau^s_{\mathfrak{B}}$ is determined by a uniformity on Y^X with a base consisting of all sets of the form

$$[B, \varepsilon]^s = \{(f, g) : \exists\delta > 0 \text{ for every } x \in B^\delta, \rho'(f(x), g(x)) < \varepsilon\},$$

for $B \in \mathfrak{B}, \varepsilon > 0$.

The topology of strong uniform convergence $\tau^s_{\mathfrak{B}}$ is finer than the topology of pointwise convergence τ_p if $\mathfrak{B} = \mathcal{F}$.

Throughout we use the convention that if $\mathfrak{B}$ is a bornology on X, then $X \notin \mathfrak{B}$.

3. Statistical Variations of Certain Bornological Notions

3.1. The s-$\gamma_{\mathfrak{B}^s}$-Cover and Related Selection Principles

First we introduce the following definition which plays a central role in our paper.

Definition 3.1. A countable open cover $\mathcal{U}$ is said to be a statistical-$\gamma_{\mathfrak{B}^s}$-cover (in short, s-$\gamma_{\mathfrak{B}^s}$-cover) if there is an enumeration of $\mathcal{U}$, say $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ such that for $B \in \mathfrak{B}$ there is a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers such that $d(\{n \in \mathbb{N} : B^{\delta_n} \not\subseteq U_n\}) = 0$.

In contrast to the classical definition, this definition depends on the enumeration of pieces. A s-$\gamma_{\mathfrak{B}^s}$-cover may not still be a s-$\gamma_{\mathfrak{B}^s}$-cover under a changed enumeration (see Example 3.2 below). Throughout we follow the convention that whenever we consider a countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$, we always consider a fixed enumeration.

The collection of all s-$\gamma_{\mathfrak{B}^s}$-covers of X is denoted by s-$\Gamma_{\mathfrak{B}^s}$. It is clear from the context that every $\gamma_{\mathfrak{B}^s}$-cover is a s-$\gamma_{\mathfrak{B}^s}$-cover i.e. $\Gamma_{\mathfrak{B}^s} \subset s$-$\Gamma_{\mathfrak{B}^s}$. The following example shows that the inclusion is proper.

Example 3.1. Consider $X = \mathbb{R}$ and a bornology $\mathfrak{B}$ on X generated by $\{(-x, x) : x \in \mathbb{R}\}$. Now consider an open $\mathfrak{B}^s$-cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$, where $U_n = (0, n)$ when $n = k^2$ and $U_n = (-n, n)$ when $n \neq k^2$ for each $k \in \mathbb{N}$. We show that $\mathcal{U}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover. Let $B \in \mathfrak{B}$. Say, $B = (-x_0, x_0)$. Now for a $\delta > 0$ there is a $n_0 \in \mathbb{N}$ such that $B^\delta \subseteq U_n$ for all $n \geq n_0$ and $n \neq k^2$ for any $k \in \mathbb{N}$. Define $\delta_n = \delta$ for each n, then for this sequence $\{\delta_n : n \in \mathbb{N}\}$ we have $\{n \in \mathbb{N} : B^{\delta_n} \not\subseteq U_n\} \subseteq \{n \in \mathbb{N} : n = k^2 \text{ for } k \in \mathbb{N}\} \cup \{1, 2, \ldots, n_0 - 1\}$. Clearly $\mathcal{U}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover, as $d(\{n \in \mathbb{N} : n = k^2 \text{ for } k \in \mathbb{N}\}) = 0$. It is also clear that for any $\delta > 0$, $B^\delta \not\subseteq U_n$ for infinitely many n. Thus $\mathcal{U}$ can not be a $\gamma_{\mathfrak{B}^s}$-cover of X.

Example 3.2. Under a changed enumeration the s-$\gamma_{\mathfrak{B}^s}$-cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of Example 3.1 may not remain a s-$\gamma_{\mathfrak{B}^s}$-cover of X.

First consider a partition $\{\mathcal{P}_j : j \in \mathbb{N}\}$ of $A = \{1^2, 2^2, \ldots\}$, where $\mathcal{P}_1 = \{1^2, 2^2\}$ and $\mathcal{P}_j = \{(j^2 - j + 1)^2, (j^2 - j + 2)^2, \ldots, (j^2 + j)^2\}$ for $j > 1$.

Let $\sigma : \mathbb{N} \to \mathbb{N}$ be the bijection given by

$$\sigma(n) = \begin{cases} k + i & \text{if } n = k^2 \text{ and } n \in \mathcal{P}_i \text{ for some } k, i \in \mathbb{N} \\ (n - k)^2 & \text{if } k^2 < n < (k + 1)^2 \text{ for some } k \in \mathbb{N} \end{cases}$$

and consider the enumeration $\{U_{\sigma(n)} : n \in \mathbb{N}\}$ of $\mathcal{U}$. Clearly $U_{\sigma(n)} = (-\sigma(n), \sigma(n))$ if $n = k^2$ for $k \in \mathbb{N}$ and $U_{\sigma(n)} = (0, \sigma(n))$ if $n \neq k^2$. Let $B = (-1, 1)$. It is clear that for any sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers $\mathbb{N} \setminus A \subseteq \{n \in \mathbb{N} : B^{\delta_n} \not\subseteq U_{\sigma(n)}\}$. Also $d(\{n \in \mathbb{N} : B^{\delta_n} \not\subseteq U_{\sigma(n)}\}) \neq 0$ as $d(\mathbb{N} \setminus A) = 1$. Thus $\{U_{\sigma(n)} : n \in \mathbb{N}\}$ is not a s-$\gamma_{\mathfrak{B}^s}$-cover of X.

It is also interesting to observe in Example 3.1 that $\{U_{k^2} : k \in \mathbb{N}\}$ is an infinite subset of $\mathcal{U}$ which is not a s-$\gamma_{\mathfrak{B}^s}$-cover (not even a cover) of X. Generally an infinite subset of a s-$\gamma_{\mathfrak{B}^s}$-cover is not necessarily a s-$\gamma_{\mathfrak{B}^s}$-cover. However, on the positive side, the result holds if we consider any statistically dense subset of this cover.

Lemma 3.1. *A statistically dense subset of a s-$\gamma_{\mathfrak{B}^s}$-cover of X is again a s-$\gamma_{\mathfrak{B}^s}$-cover of X.*

Proof. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a s-$\gamma_{\mathfrak{B}^s}$-cover of X. Let $\{U_{n_k} : k \in \mathbb{N}\}$ be a statistically dense subset of $\mathcal{U}$. We aim to show that this is again a s-$\gamma_{\mathfrak{B}^s}$-cover. Assume on the contrary that there is a $B \in \mathfrak{B}$ such that for any sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers, $d(\{k \in \mathbb{N} : B^{\delta_{n_k}} \not\subseteq U_{n_k}\}) \neq 0$. Since $d(\{k : B^{\delta_{n_k}} \not\subseteq U_{n_k}\}) \leq d(\{n \in \mathbb{N} : B^{\delta_n} \not\subseteq U_n\})$, it follows that $d(\{n \in \mathbb{N} : B^{\delta_n} \not\subseteq U_n\}) \neq 0$. Which in turn contradicts that $\mathcal{U}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X. □

The next two observations about the s-$\gamma_{\mathfrak{B}^s}$-cover will be useful in what follows.

Lemma 3.2. *Let $\mathfrak{B}$ be a bornology on X with closed base and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of s-$\gamma_{\mathfrak{B}^s}$-covers of X, where $\mathcal{U}_n = \{U_k^n : k \in \mathbb{N}\}$. Then for each n the collection $\mathcal{V}_n = \{U_k^1 \cap U_k^2 \ldots \cap U_k^n : U_k^i \in \mathcal{U}_i, 1 \leq i \leq n, k \in \mathbb{N}\}$ is also a s-$\gamma_{\mathfrak{B}^s}$-cover of X.*

Proof. Let $B \in \mathfrak{B}$ and fix a positive integer n. For each $i = 1, 2, \ldots, n$ choose a sequence $\{\delta_k^i : k \in \mathbb{N}\}$ of positive real numbers such that $d(T_i) = 0$, where $T_i = \{k \in \mathbb{N} : B^{\delta_k^i} \not\subseteq U_k^i\}$). Choose $V_k^n = U_k^1 \cap U_k^2 \ldots \cap U_k^n$ and take $\delta_k = \min\{\delta_k^i : i = 1, 2, \ldots, n\}$. We show that $d(S) = 0$, where $S = \{k \in \mathbb{N} : B^{\delta_k} \not\subseteq V_k^n\}$. If $k \in S$, then $B^{\delta_k} \not\subseteq V_k^n$ i.e. $B^{\delta_k^i} \not\subseteq U_k^i$ for some $i \in \{1, 2, \ldots, n\}$. Clearly $S \subseteq \cup_{i=1}^n T_i$ and $d(S) = 0$. Hence $\mathcal{V}_n$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X. □

Since every $\gamma_{\mathfrak{B}^s}$-cover is a s-$\gamma_{\mathfrak{B}^s}$-cover, the next result follows from [8, Lemma 3.4].

Lemma 3.3. *Let $\mathfrak{B}$ be a bornology on X with closed base and $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be an open $\mathfrak{B}^s$-cover of X. If $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ where $V_n = \cup_{i=1}^n U_i$, then $\mathcal{V}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X.*

In the next result we show that s-α_4 property lies between S_1 and S_{fin}-type selection properties for some suitable classes of covers. We will further investigate these types of statistical selection properties in the final section.

Proposition 3.1. *Let $\mathfrak{B}$ be a bornology on X with closed base. Consider the following statements:*
(1) *X satisfies $S_1(s\text{-}\Gamma_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$;*
(2) *X satisfies $s\text{-}\alpha_4(s\text{-}\Gamma_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$;*
(3) *X satisfies $S_{fin}(s\text{-}\Gamma_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$.*
Then (1) ⇒ (2) ⇒ (3) *holds.*

Proof. We only give proof of (2) ⇒ (3). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of s-$\gamma_{\mathfrak{B}^s}$-covers of X and let $\mathcal{U}_n = \{U_m^n : m \in \mathbb{N}\}$. By Lemma 3.1 and using (2), there is a subset $K = \{n_1 < n_2 < \cdots\}$ of $\mathbb{N}$ with $d(K) = 1$ and a s-$\gamma_{\mathfrak{B}^s}$ cover $\{U_{m_i}^{n_i} : i \in \mathbb{N}\}$ such that $U_{m_i}^{n_i} \in \mathcal{U}_{n_i}$, for each $i \in \mathbb{N}$ (see [12, Theorem 6.1]). If $n = n_i$, choose $\mathcal{W}_n = \{U_{m_i}^{n_i}\}$ and choose $\mathcal{W}_n = \emptyset$ otherwise. Clearly $\cup_{n \in \mathbb{N}} \mathcal{W}_n$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X and $\mathcal{W}_n$ is a finite subset of $\mathcal{U}_n$ for each n. Hence (3) holds. □

We now present certain implications among the selection principles in the next few results.

Theorem 3.1. *Let $\mathfrak{B}$ be a bornology on X with closed base. The following statements hold:*
(1) *$S_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s}) = S_{fin}(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$;*
(2) *$S_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma) = S_{fin}(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma)$;*
(3) *$S_1(s\text{-}\Gamma, \Gamma_{\mathfrak{B}^s}) = S_{fin}(s\text{-}\Gamma, \Gamma_{\mathfrak{B}^s})$.*

Proof. We only give proof of (1) as the other proofs are analogous. Suppose that X satisfies $S_{fin}(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of s-$\gamma_{\mathfrak{B}^s}$-covers of X. Let $\mathcal{U}_n = \{U_k^n : k \in \mathbb{N}\}$ for each $n \in \mathbb{N}$. For each n, consider the collection $\mathcal{V}_n = \{V_k^n : k \in \mathbb{N}\}$, where $V_k^n = U_k^1 \cap U_k^2 \cap \ldots \cap U_k^n$ and $U_k^i \in \mathcal{U}_i$, $i = 1, 2, \ldots, n$. By Lemma 3.2, $\mathcal{V}_n$'s are s-$\gamma_{\mathfrak{B}^s}$-covers of X. Now applying $S_{fin}(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ to $\{\mathcal{V}_n : n \in \mathbb{N}\}$ to choose a finite subset $\mathcal{W}_n \subseteq \mathcal{V}_n$ for each n such that $\cup_{n \in \mathbb{N}} \mathcal{W}_n$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Choose a sequence of positive integers $n_1 < n_2 < \cdots$ such that $\mathcal{W}_{n_j} \setminus \cup_{i<j} \mathcal{W}_{n_i} \neq \emptyset$ for $j \in \mathbb{N}$.

Now for each j, choose a $V_{k_j}^{n_j} \in \mathcal{W}_{n_j} \setminus \cup_{i<j} \mathcal{W}_{n_i}$. As an infinite subset of a $\gamma_{\mathfrak{B}^s}$-cover of X is a $\gamma_{\mathfrak{B}^s}$-cover, $\{V_{k_j}^{n_j} : j \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. For $1 \leqslant n \leqslant n_1$, define $U_n = U_{k_1}^n$, where $V_{k_1}^{n_1} = U_{k_1}^1 \cap U_{k_1}^2 \cap \cdots \cap U_{k_1}^{n_1}$ and for each $n \in (n_j, n_{j+1}]$, define $U_n = U_{k_{j+1}}^n$, where $V_{k_{j+1}}^{n_{j+1}} = U_{k_{j+1}}^1 \cap U_{k_{j+1}}^2 \cap \cdots \cap U_{k_{j+1}}^{n_{j+1}}$. We show that $\{U_n : n \in \mathbb{N}\} \in \Gamma_{\mathfrak{B}^s}$. Let $B \in \mathfrak{B}$. Since $\{V_{k_j}^{n_j} : j \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X, there exist a $j_0 \in \mathbb{N}$ and a sequence $\{\delta_j : j \geq j_0\}$ of positive real numbers such that $B^{\delta_{j+1}} \subseteq V_{k_{j+1}}^{n_{j+1}}$ for all $j \geq j_0$ i.e. $B^{\delta_{j+1}} \subseteq U_{k_{j+1}}^1 \cap U_{k_{j+1}}^2 \cap \cdots \cap U_{k_{j+1}}^{n_{j+1}}$. For each $n \in (n_j, n_{j+1}]$, define $\delta_n = \delta_{j+1}$. Thus we have $B^{\delta_n} \subseteq U_n$ for all $n \geq n_{j_0}$. Consequently $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X and hence X satisfies $S_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. The other direction is straightforward. □

Theorem 3.2. *Let $\mathfrak{B}$ be a bornology on X with closed base. If X is $\mathfrak{B}^s$-Lindelöf, then the following statements hold:*
(1) $\mathrm{S}_{\mathrm{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, \Lambda) = \mathrm{U}_{\mathrm{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Lambda)$;
(2) $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Lambda) = \mathrm{S}_{\mathrm{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, \Lambda)$;
(3) $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Lambda) = \mathrm{U}_{\mathrm{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, \Lambda)$.

Proof. We prove only (3). Let X satisfy $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Lambda)$. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of s-$\gamma_{\mathfrak{B}^s}$-covers of X. Apply $\mathrm{S}_{\mathrm{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Lambda)$ to $\{\mathcal{U}_n : n \in \mathbb{N}\}$ to choose a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$ for each n such that $\cup_{n\in\mathbb{N}}\mathcal{V}_n$ is a large cover of X. Choose a sequence $1 = k_1 < k_2 < \dots$ of positive integers and enumerate $\cup_{n\in\mathbb{N}}\mathcal{V}_n$ as $\{V_i : i \in \mathbb{N}\}$, where $\mathcal{V}_n = \{V_i : k_n \leqslant i < k_{n+1}\}$. Since each x belongs to infinitely many V_i's, it follows that each x belongs to $\cup\mathcal{V}_n$ for infinitely many n. Clearly $\{\cup\mathcal{V}_n : n \in \mathbb{N}\}$ is a large cover of X and also X satisfies $\mathrm{U}_{\mathrm{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, \Lambda)$.

In the other direction, assume that X satisfies $\mathrm{U}_{\mathrm{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, \Lambda)$. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X. Enumerate each $\mathcal{U}_n$ bijectively as $\{U^n_k : k \in \mathbb{N}\}$ and for each n consider the collection $\mathcal{V}_n = \{V^n_k : k \in \mathbb{N}\}$ where $V^n_k = U^n_1 \cup \dots \cup U^n_k$. By Lemma 3.3, each $\mathcal{V}_n$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X. Apply $\mathrm{U}_{\mathrm{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, \Lambda)$ to $\{\mathcal{V}_n : n \in \mathbb{N}\}$ to choose a finite subset $\mathcal{W}_n$ of $\mathcal{V}_n$ for each n such that $\{\cup\mathcal{W}_n : n \in \mathbb{N}\}$ is a large cover of X. By deconstructing the members of $\mathcal{W}_n$, we can find a finite subset $\mathcal{Z}_n$ of $\mathcal{U}_n$ for each n. The proof will be complete if we show that $\cup_{n\in\mathbb{N}}\mathcal{Z}_n$ is a large cover of X. Let $x \in X$. Now $x \in \cup\mathcal{W}_n$ for infinitely many n i.e. for infinitely many n there is a $V^n_k \in \mathcal{W}_n$ such that $x \in V^n_k = U^n_1 \cup \dots \cup U^n_k$. Thus there is a $U^n_j \in \mathcal{Z}_n$ such that $x \in U^n_j$ for infinitely many n and consequently $\cup_{n\in\mathbb{N}}\mathcal{Z}_n$ is a large cover of X. □

Theorem 3.3. *Let $\mathfrak{B}$ be a bornology on X with closed base. If X is $\mathfrak{B}^s$-Lindelöf, then the following statements hold:*
(1) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s}) = \mathrm{U}_{\mathrm{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$;
(2) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s}) = \mathrm{U}_{\mathrm{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$;
(3) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}) = \mathrm{U}_{\mathrm{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, \mathcal{O})$;
(4) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma) = \mathrm{U}_{\mathrm{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, s\text{-}\Gamma)$;
(5) $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, s\text{-}\Gamma_{\mathfrak{B}^s}) = \mathrm{U}_{\mathrm{fin}}(s\text{-}\Gamma, s\text{-}\Gamma_{\mathfrak{B}^s})$.

Proof. We prove only (2). Suppose that X satisfies $\mathrm{U}_{\mathrm{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X where $\mathcal{U}_n = \{U^n_k : k \in \mathbb{N}\}$ for each n. Now for each $n \in \mathbb{N}$ consider the collection $\mathcal{V}_n = \{V^n_k : k \in \mathbb{N}\}$, where $V^n_k = U^n_1 \cup \dots \cup U^n_k$. By Lemma 3.3, $\mathcal{V}_n$'s are s-$\gamma_{\mathfrak{B}^s}$-covers of X. Apply $\mathrm{U}_{\mathrm{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$ to $\{\mathcal{V}_n : n \in \mathbb{N}\}$ to find a finite subset $\mathcal{W}_n$ of $\mathcal{V}_n$ for each n such that $\{\cup\mathcal{W}_n : n \in \mathbb{N}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X. By deconstructing members of $\mathcal{W}_n$, we find a finite subset $\mathcal{Z}_n$ of $\mathcal{U}_n$ for each n. Clearly $\cup\mathcal{W}_n = \cup\mathcal{Z}_n$ for each n. We show that $\{\cup\mathcal{Z}_n : n \in \mathbb{N}\} \in s\text{-}\Gamma_{\mathfrak{B}^s}$. Let $B \in \mathfrak{B}$. Since $\{\cup\mathcal{W}_n : n \in \mathbb{N}\} \in s\text{-}\Gamma_{\mathfrak{B}^s}$, there is a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers such that $d(\{n \in \mathbb{N} : B^{\delta_n} \not\subseteq \cup\mathcal{W}_n\}) = 0$ i.e. $d(\{n \in \mathbb{N} : B^{\delta_n} \not\subseteq \cup\mathcal{Z}_n\}) = 0$. Consequently $\{\cup\mathcal{Z}_n : n \in \mathbb{N}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X and hence X satisfies $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$. The other direction is straightforward. □

Extending [8, Theorem 3.5], we obtain the following game theoretic characterization of $\mathrm{S}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

Theorem 3.4. *Let $\mathfrak{B}$ be a bornology on X with closed base. The following conditions are equivalent:*
(1) *X satisfies* $\mathrm{S}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$;
(2) *ONE has no wining strategy in the game* $\mathrm{G}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

Proof. It is enough to prove (1) ⇒ (2). Let F be a strategy for ONE in the game $\mathrm{G}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. Let the first move of ONE be $F(X)$, a s-$\gamma_{\mathfrak{B}^s}$-cover of X enumerated bijectively as $\{U_{(n)} : n \in \mathbb{N}\}$. Let for each finite sequence τ of natural numbers of length at most m, U_τ have been already defined. Now define $\{U_{(n_1,\dots,n_k,m)} : m \in \mathbb{N}\}$ to be $F(U_{(n_1)}, \dots, U_{(n_1,\dots,n_k)}) \setminus \{U_{(n_1)}, \dots, U_{(n_1,\dots,n_k)}\}$, where the enumeration $\{U_{(n_1,\dots,n_k,m)} : m \in \mathbb{N}\}$ is bijective. It is clear that for each finite sequence τ of natural numbers, $\{U_{\tau\frown(m)} : m \in \mathbb{N}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X. Now using (1) and proceeding as in [8, Theorem 3.5], we can conclude that F is not a wining strategy for ONE. □

Likewise the following characterization can be obtained.

Theorem 3.5. *Let $\mathfrak{B}$ be a bornology on X with closed base. The following conditions are equivalent:*
(1) *X satisfies* $\mathrm{S}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma)$;
(2) *ONE has no wining strategy in the game* $\mathrm{G}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma)$.

Combining with Theorem 3.4, we obtain the following characterization related to the α_i-properties.

Theorem 3.6. *Let $\mathfrak{B}$ be a bornology on X with closed base. The following conditions are equivalent:*
(1) *X satisfies $\alpha_2(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$;*
(2) *X satisfies $\alpha_3(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$;*
(3) *X satisfies $\alpha_4(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$;*
(4) *X satisfies $S_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$;*
(5) *ONE does not have a wining strategy in $G_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*

Proof. We give only proof of the following implications. The other implications follow from the standard argument.

(3) $\Rightarrow$ (4) Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of s-$\gamma_{\mathfrak{B}^s}$-covers of X and $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$, $n \in \mathbb{N}$.

By Lemma 3.2, $\mathcal{V}_n = \{V^n_m : m \in \mathbb{N}\} \in s\text{-}\Gamma_{\mathfrak{B}^s}$ for each n where $V^n_m = U^1_m \cap U^2_m \cap \ldots \cap U^n_m$. Apply (3) to obtain a sequence $1 = n_0 < n_1 < n_2 < \cdots$ of positive integers such that $\mathcal{V} = \{V^{n_i}_{m_i} : i \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover, where $V^{n_i}_{m_i} \in \mathcal{V}_{n_i}$ for each i. Now for each $i \geqslant 0$, each j with $n_i < j \leq n_{i+1}$, consider $V^{n_{i+1}}_{m_{i+1}} = U^1_{m_{i+1}} \cap \ldots \cap U^{n_{i+1}}_{m_{i+1}}$ and let $U_j = U^j_{m_{i+1}}$. Clearly $\{U_j : j \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X.

(5) $\Rightarrow$ (1) Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of s-$\gamma_{\mathfrak{B}^s}$-covers of X. Let $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$. We define a strategy σ for ONE in the game $G_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ as follows. Let the first move of ONE be $\sigma(\emptyset) = \mathcal{U}_1$. TWO chooses $U^1_{m_{i_1}} \in \mathcal{U}_1$. Now by Lemma 3.2, $\{U^1_m \cap U^2_m : m \geq m_{i_1}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X. Let the second move of ONE be $\sigma(U^1_{m_{i_1}}) = \{U^1_m \cap U^2_m : m \geq m_{i_1}\}$. TWO chooses $U^1_{m_{i_2}} \cap U^2_{m_{i_2}}$ and so on.

Since the play $\sigma(\emptyset)$, $U^1_{m_{i_1}}$, $\sigma(U^1_{m_{i_1}})$, $U^1_{m_{i_2}} \cap U^2_{m_{i_2}}$ $\ldots$ in $G_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ is lost by ONE, the collection $\{U^1_{m_{i_1}}, U^1_{m_{i_2}}, U^2_{m_{i_2}} \ldots\}$ is a $\gamma_{\mathfrak{B}^s}$-cover, which contains infinitely many elements of $\mathcal{U}_n$ for each n. Hence X satisfies $\alpha_2(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. $\square$

Quite similarly the following characterization can be obtained.

Theorem 3.7. *Let $\mathfrak{B}$ be a bornology on X with closed base. The following conditions are equivalent:*
(1) *X satisfies $\alpha_2(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma)$;*
(2) *X satisfies $\alpha_3(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma)$;*
(3) *X satisfies $\alpha_4(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma)$;*
(4) *X satisfies $S_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma)$;*
(5) *ONE does not have a wining strategy in $G_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma)$.*

3.2. The s-$\mathfrak{B}^s$-Hurewicz Property

We now define the statistically-strong-$\mathfrak{B}$-Hurewicz property and the corresponding game.

Definition 3.2. Let $\mathfrak{B}$ be a bornology on X with closed base. X is said to have the statistically-strong-$\mathfrak{B}$-Hurewicz property (in short, s-$\mathfrak{B}^s$-Hurewicz property) if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open $\mathfrak{B}^s$-covers of X, there is a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ where $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ for each $n \in \mathbb{N}$ such that for every $B \in \mathfrak{B}$ there exists a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers satisfying $d(\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U$ for any $U \in \mathcal{V}_n\}) = 0$.

Definition 3.3. The statistically-strong-$\mathfrak{B}$-Hurewicz game (in short, s-$\mathfrak{B}^s$-Hurewicz game) on X is defined as follows. Two players named ONE and TWO play an infinite long game. In the n-th inning ONE selects an open $\mathfrak{B}^s$-cover $\mathcal{U}_n$ of X, TWO responds by choosing a finite set $\mathcal{V}_n \subseteq \mathcal{U}_n$. TWO wins the play: $\mathcal{U}_1$, $\mathcal{V}_1$, $\mathcal{U}_2$, $\mathcal{V}_2, \ldots, \mathcal{U}_n, \mathcal{V}_n, \ldots$ if for each $B \in \mathfrak{B}$ there exists a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers satisfying $d(\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U$ for any $U \in \mathcal{V}_n\}) = 0$. Otherwise ONE wins.

We now present an example of a space having the s-$\mathfrak{B}^s$-Hurewicz property.

Example 3.3. The space $X = \mathbb{R}^2$ together with the Euclidean metric d and the bornology $\mathfrak{B}$ generated by $\{S(0,r) : r > 0\}$, where $S(0,r)$ is an open ball centred at 0 with radius r, has the s-$\mathfrak{B}^s$-Hurewicz property. To see this, let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X. Then clearly for each $k \in \mathbb{N}$, there is a $U \in \mathcal{U}_n$ such that $S(0,k) \subseteq U$, where $S(0,k) \in \mathfrak{B}$.

Consider a sequence of positive integers $k_1 < k_2 < \cdots$. For each $n \in \mathbb{N}$, choose a $U_n \in \mathcal{U}_n$ such that $S(0,k_n) \subseteq U_n$ and define $\mathcal{W}_n = \{U_n\}$. We show that $\{\mathcal{W}_n : n \in \mathbb{N}\}$ witnesses the s-$\mathfrak{B}^s$-Hurewicz property. Let $B \in \mathfrak{B}$. Since $\mathcal{U} = \{S(0,n) : n \in \mathbb{N}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover, for $B \in \mathfrak{B}$ there is a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers satisfying $d(\{n \in \mathbb{N} : B^{\delta_n} \not\subseteq S(0,n)\}) = 0$. We show that $d(\{n \in \mathbb{N} : B^{\delta_n} \not\subseteq U \text{ for any } U \in \mathcal{W}_n\}) = 0$. Observe that if for $n \in \mathbb{N}$, $B^{\delta_n} \not\subseteq U$ for any $U \in \mathcal{W}_n$ then $B^{\delta_n} \not\subseteq S(0,k_n)$ i.e. $B^{\delta_n} \not\subseteq S(0,n)$ as $n \leq k_n$. Clearly $\{n \in \mathbb{N} : B^{\delta_n} \not\subseteq U \text{ for any } U \in \mathcal{W}_n\} \subseteq \{n \in \mathbb{N} : B^{\delta_n} \not\subseteq S(0,n)\}$ and $d(\{n \in \mathbb{N} : B^{\delta_n} \not\subseteq U \text{ for any } U \in \mathcal{W}_n\}) = 0$. The conclusion now follows.

We now give an example of a space without the s-$\mathfrak{B}^s$-Hurewicz property (see [8, Example 4.2]).

Example 3.4. Let X be the Baire space with the bornology $\mathfrak{B} = \mathcal{F}$. Choose a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open $\mathfrak{B}^s$-covers of X, where $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$ and $U^n_m = \{f \in X : f(n) \leq m\}$. In order to show that the sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ fails to witness the s-$\mathfrak{B}^s$-Hurewicz property, let $\{\mathcal{V}_n : n \in \mathbb{N}\}$ be any sequence, where $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ for each n. Let $h \in X$ be such that $h(n) > 2 \cdot \max\{m : U^n_m \in \mathcal{V}_n\}$. Choose $f, g \in X$ in such a way that $h = f + g$. Clearly $\max\{f(n), g(n)\} \geq \frac{1}{2}h(n)$ and hence $\{f, g\} \not\subseteq U^n_m$ for each $U^n_m \in \mathcal{V}_n$ and each n. Let $B = \{f, g\} \in \mathfrak{B}$. Thus for any sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers $\{n \in \mathbb{N} : B^{\delta_n} \not\subseteq U^n_m \text{ for any } U^n_m \in \mathcal{V}_n\} = \mathbb{N}$. Since $d(\mathbb{N}) = 1$, it follows that X does not have the s-$\mathfrak{B}^s$-Hurewicz property.

Next we introduce the notion of statistically-strong-$\mathfrak{B}$-groupable cover.

Definition 3.4. Let $\mathfrak{B}$ be a bornology on X with closed base. An open cover $\mathcal{U}$ of X is said to be statistically-strong-$\mathfrak{B}$-groupable (in short, s-$\mathfrak{B}^s$-groupable) if it can be expressed as a union of countably many finite pairwise disjoint sets $\mathcal{U}_n$ such that for each $B \in \mathfrak{B}$ there exists a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers satisfying $d(\{n \in \mathbb{N} : B^{\delta_n} \not\subseteq U \text{ for any } U \in \mathcal{U}_n\}) = 0$.

The collection of all s-$\mathfrak{B}^s$-groupable covers is denoted by s-$\mathcal{O}^{gp}_{\mathfrak{B}^s}$. Clearly every $\mathfrak{B}^s$-groupable cover is s-$\mathfrak{B}^s$-groupable.

Using the techniques of [9, Note 2.2] and [12, Theorem 3.5], we show that under certain condition the s-$\mathfrak{B}^s$-Hurewicz property is equivalent to the selection hypothesis $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\mathcal{O}^{gp}_{\mathfrak{B}^s})$ (see also [10]).

Theorem 3.8. *Let $\mathfrak{B}$ be a bornology on X with closed base and $\mathrm{CDR_{Sub}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ hold. The following statements are equivalent:*

(1) *X has the s-$\mathfrak{B}^s$-Hurewicz property;*

(2) *X satisfies $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\mathcal{O}^{gp}_{\mathfrak{B}^s})$.*

Proof. (1) ⇒ (2) Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X. Since X satisfies $\mathrm{CDR_{Sub}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$, we can assume that $\mathcal{U}_n$'s are pairwise disjoint. Since X has the s-$\mathfrak{B}^s$-Hurewicz property, there is a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of finite sets with $\mathcal{V}_n \subseteq \mathcal{U}_n$ for each n such that for $B \in \mathfrak{B}$ there exists a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers with $d(\{n \in \mathbb{N} : B^{\delta_n} \not\subseteq U \text{ for any } U \in \mathcal{V}_n\}) = 0$. Since $\mathcal{V}_n$'s are pairwise disjoint, $\{\mathcal{V}_n : n \in \mathbb{N}\}$ witnesses the s-$\mathfrak{B}^s$-groupability of $\cup_{n\in\mathbb{N}}\mathcal{V}_n$. Hence X satisfies $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\mathcal{O}^{gp}_{\mathfrak{B}^s})$.

(2) ⇒ (1) Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X and $\mathcal{U}_n = \{U^n_l : l \in \mathbb{N}\}$ for each n. Consider $\mathcal{V}_n = \{U^1_{l_1} \cap U^2_{l_2} \cap \ldots \cap U^n_{l_n} : n < l_1 < \cdots < l_n\}$ for each n. By [8, Lemma 3.1] and [8, Proposition 3.1], $\mathcal{V}_n$ is an open $\mathfrak{B}^s$-cover of X for each n. Now we apply $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\mathcal{O}^{gp}_{\mathfrak{B}^s})$ to the sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ and obtain a sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of pairwise disjoint finite sets such that $\mathcal{F}_n \subseteq \mathcal{V}_n$ for each n and $\cup_{n\in\mathbb{N}}\mathcal{F}_n$ is a s-$\mathfrak{B}^s$-groupable cover of X. So there is a sequence $\{\mathcal{H}_k : k \in \mathbb{N}\}$ of pairwise disjoint finite subsets of $\cup_{n\in\mathbb{N}}\mathcal{F}_n$ such that for $B \in \mathfrak{B}$ there exists a sequence $\{\delta_k : k \in \mathbb{N}\}$ of positive real numbers for which $d(\{k \in \mathbb{N} : B^{\delta_k} \not\subseteq H \text{ for any } H \in \mathcal{H}_k\}) = 0$.

Define $A_i = \{k \in \mathbb{N} : \mathcal{H}_k \subseteq \cup_{j>i}\mathcal{F}_j\}$ for each $i \in \mathbb{N}$. Since each $\mathcal{F}_j$'s are finite and $\mathcal{H}_k$'s are pairwise disjoint sets, each A_i is cofinite and so $d(A_i) = 1$. Also $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$. Choose a $m_1 \in A_1$ such that $m_1 > 1$.

Choose a $m_2 \in A_2$ with $m_2 > m_1$ in such a way that for all $n \geqslant m_2$, $\frac{|A_1(n)|}{n} > \frac{1}{2}$ and so on. Thus we obtain a sequence $m_1 < m_2 < \dots$ of positive integers such that $m_i \in A_i$ and $\frac{|A_i(n)|}{n} > \frac{i-1}{i}$ for every $n \geqslant m_i$. We now define a subset K of positive integers as follows. $k \in K$ if $k \in (1, m_1] \cap A_1$. Also for $i > 1$, $k \in K$ if and only if $k \in (m_i, m_{i+1}] \cap A_i$. We write $K = \{k_1 < k_2 < \cdots\}$. It is easy to verify that $d(K) = 1$. For the finite number of elements k of K coming from A_1, we take the set of all U_l^1, the first components in the representation of elements of $\mathcal{H}_k$ and denote that collection by $\mathcal{G}_1$. Again for the next finite numbers of element k of K coming from A_2, we take the set of all U_l^2, the second component in the representation of the elements of $\mathcal{H}_k$ as $\mathcal{H}_k \subseteq \cup_{j>2}\mathcal{F}_j$. We denote that collection by $\mathcal{G}_2$. Continuing in this way we obtain a sequence $\{\mathcal{G}_n : n \in \mathbb{N}\}$ where $\mathcal{G}_n$ is a finite subset of $\mathcal{U}_n$ for each n.

We observe that $d(\{k_n \in K : B^{\delta_{k_n}} \nsubseteq H \text{ for any } H \in \mathcal{H}_{k_n}\}) = 0$ and hence $d(\{k_n \in K : B^{\delta_{k_n}} \subseteq H \text{ for some } H \in \mathcal{H}_{k_n}\}) = 1$ as $d(K) = 1$. Also it follows that $d(\{k_n \in K : B^{\delta_{k_n}} \subseteq H \text{ for some } H \in \mathcal{H}_{k_n}\}) \leqslant d(\{n \in \mathbb{N} : B^{\delta_{k_n}} \subseteq H \text{ for some } H \in \mathcal{H}_{k_n}\})$, i.e. $d(\{n \in \mathbb{N} : B^{\delta_{k_n}} \subseteq H \text{ for some } H \in \mathcal{H}_{k_n}\}) = 1$.

We now choose $\sigma_n = \delta_{k_n}$ for each n. Clearly $d(\{n \in \mathbb{N} : B^{\sigma_n} \subseteq G \text{ for some } G \in \mathcal{G}_n\}) = 1$ and hence $d(\{n \in \mathbb{N} : B^{\sigma_n} \nsubseteq G \text{ for any } G \in \mathcal{G}_n\}) = 0$. Consequently $\{\mathcal{G}_n : n \in \mathbb{N}\}$ witnesses the s-$\mathfrak{B}^s$-Hurewicz property of X for $\{\mathcal{U}_n : n \in \mathbb{N}\}$. Thus (1) holds. □

Remark 3.1. We do not require the assumption $\mathrm{CDR}_{\mathrm{Sub}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ to prove the implication (2) ⇒ (1). We do not know whether the other direction can be obtained without this assumption.

The intent of the next result is to show that under certain condition, a countable open $\mathfrak{B}^s$-cover becomes s-$\mathfrak{B}^s$-groupable. In the following result we use the fact that if $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover, then for each $B \in \mathfrak{B}$ there are positive real numbers $\delta_n > 0$ and $U_n \in \mathcal{U}$ such that $B^{\delta_n} \subseteq U_n$ for infinitely many n and conversely (see [8, Proposition 3.1]). Thus for any finite subset $\mathcal{V}$ of $\mathcal{U}$, $\mathcal{U} \setminus \mathcal{V}$ is also an open $\mathfrak{B}^s$-cover of X.

Proposition 3.2. *Let $\mathfrak{B}$ be a bornology on X with closed base. If ONE has no wining strategy in the s-$\mathfrak{B}^s$-Hurwicz game on X, then every countable open $\mathfrak{B}^s$-cover of X is s-$\mathfrak{B}^s$-groupable.*

Proof. Let $\mathcal{U}$ be a countable open $\mathfrak{B}^s$-cover of X. Consider the following strategy σ for ONE in the s-$\mathfrak{B}^s$-Hurewicz game on X. Let the first move of ONE be $\sigma(\emptyset) = \mathcal{U}$. TWO responds with a finite set $\mathcal{V}_1 \subseteq \mathcal{U}$. The second move of ONE is $\sigma(\mathcal{V}_1) = \mathcal{U} \setminus \mathcal{V}_1$. TWO responds with a finite set $\mathcal{V}_2 \subseteq \sigma(\mathcal{V}_1)$ and so on.

By our assumption, there is a play $\sigma(\emptyset), \mathcal{V}_1, \sigma(\mathcal{V}_1), \mathcal{V}_2, \dots$ which is lost by ONE. So for each $B \in \mathfrak{B}$ there exists a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers such that $d(\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U \text{ for any } U \in \mathcal{V}_n\}) = 0$.

By construction, $\mathcal{V}_n$'s are pairwise disjoint finite sets and hence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ witnesses the s-$\mathfrak{B}^s$-groupability of $\cup_{n\in\mathbb{N}}\mathcal{V}_n$. Since $\mathcal{U}$ is countable, we can show that $\mathcal{U}$ is a s-$\mathfrak{B}^s$-groupable cover of X. □

We end this section with the following game-theoretic implication.

Proposition 3.3. *Let $\mathfrak{B}$ be a bornology on X with closed base. If ONE has no wining strategy in the s-$\mathfrak{B}^s$-Hurewicz game, then ONE has no wining strategy in the game $\mathrm{G}_{\mathrm{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\mathcal{O}^{gp}_{\mathfrak{B}^s})$.*

Proof. Let τ be a strategy for ONE in $\mathrm{G}_{\mathrm{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\mathcal{O}^{gp}_{\mathfrak{B}^s})$. We define a strategy σ for ONE in the s-$\mathfrak{B}^s$-Hurewicz game. Let the first move of ONE in the s-$\mathfrak{B}^s$-Hurewicz game be $\sigma(\emptyset) = \tau(\emptyset)$. TWO responds with a finite set $\mathcal{V}_1 \subset \sigma(\emptyset)$. Let the second move of ONE be $\sigma(\mathcal{V}_1) = \tau(\mathcal{V}_1) \setminus \mathcal{V}_1$. TWO responds with a finite set $\mathcal{V}_2 \subseteq \sigma(\mathcal{V}_1)$ and so on.

By our assumption, there is a play $\sigma(\emptyset), \mathcal{V}_1, \sigma(\mathcal{V}_1), \mathcal{V}_2, \dots$ which is lost by ONE. Thus for each $B \in \mathfrak{B}$ there is a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers such that $d(\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U \text{ for any } U \in \mathcal{V}_n\}) = 0$.

Clearly $\cup_{n\in\mathbb{N}}\mathcal{V}_n$ is a s-$\mathfrak{B}^s$-groupable open cover of X. Now for the strategy τ, consider the play $\tau(\emptyset), \mathcal{V}_1, \sigma(\mathcal{V}_1), \mathcal{V}_2, \dots$ in $\mathrm{G}_{\mathrm{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\mathcal{O}^{gp}_{\mathfrak{B}^s})$. As $\cup_{n\in\mathbb{N}}\mathcal{V}_n$ is a s-$\mathfrak{B}^s$-groupable open cover of X, τ is not a wining strategy for ONE in $\mathrm{G}_{\mathrm{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\mathcal{O}^{gp}_{\mathfrak{B}^s})$. □

4. Results in Function Spaces

Let $\mathfrak{B}$ be a bornology on (X,ρ) with closed base and (Y,ρ') be another metric space. For $f \in C(X,Y)$, the neighbourhood of f with respect to the topology $\tau^s_{\mathfrak{B}}$ of strong uniform convergence on a bornology $\mathfrak{B}$ is denoted by

$$[B,\varepsilon]^s(f) = \{g \in C(X,Y) : \exists \delta > 0, \rho'(f(x),g(x)) < \varepsilon, \forall x \in B^\delta\},$$

for $B \in \mathfrak{B}$, $\varepsilon > 0$ ([4, 5]).

The symbol $\underline{0}$ denotes the zero function in $(C(X),\tau^s_{\mathfrak{B}})$ (where $C(X) = C(X,\mathbb{R})$). Since the function space $(C(X),\tau^s_{\mathfrak{B}})$ is homogeneous, it suffices to focus at the point $\underline{0}$ when we deal with local properties of this space.

Recall that a sequence $\{f_n : n \in \mathbb{N}\}$ of functions in $(C(X),\tau^s_{\mathfrak{B}})$ s-converges to $\underline{0}$ with respect to $\tau^s_{\mathfrak{B}}$ if for any neighbourhood $[B,\varepsilon]^s(\underline{0})$, $(B \in \mathfrak{B}, \varepsilon > 0)$, $d(\{n \in \mathbb{N} : f_n \notin [B,\varepsilon]^s(\underline{0})\}) = 0$.

The following lemmas are very useful in our subsequent results.

Lemma 4.1. ([12, Lemma 2.3]) *Let $\mathfrak{B}$ be a bornology on X with closed base. A sequence of functions in $(C(X),\tau^s_{\mathfrak{B}})$ is s-convergent if and only if any of its statistically dense subsequence is s-convergent.*

Lemma 4.2. ([6, Lemma 2.2]) *Let $\mathfrak{B}$ be a bornology on the metric space (X,ρ). Consider the following statements:*

(a) Let $\mathcal{U}$ be an open $\mathfrak{B}^s$-cover of X. If $A = \{f \in C(X) : \exists U \in \mathcal{U}, f(x) = 1$ for all $x \in X \setminus U\}$. Then $\underline{0} \in \overline{A} \setminus A$ in $(C(X),\tau^s_{\mathfrak{B}})$;

(b) Let $A \subseteq (C(X),\tau^s_{\mathfrak{B}})$ and let $\mathcal{U} = \{f^{-1}(-\frac{1}{n},\frac{1}{n}) : f \in A\}$, where $n \in \mathbb{N}$. If $\underline{0} \in \overline{A}$ and $X \notin \mathcal{U}$, then $\mathcal{U}$ is an open $\mathfrak{B}^s$-cover of X.

Lemma 4.3. *Let $\mathfrak{B}$ be a bornology on X with closed base. Let $\{f_n : n \in \mathbb{N}\}$ be a sequence in $(C(X),\tau^s_{\mathfrak{B}})$ that s-converge to $\underline{0}$. If for each n there is an open set U_n in X such that $f_n(X \setminus U_n) = \{1\}$, then $\{U_n : n \in \mathbb{N}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X.*

Proof. For the neighbourhood $[B,1]^s(\underline{0})$, we have $d(T) = 0$, where $T = \{n \in \mathbb{N} : f_n \notin [B,1]^s(\underline{0})\}$. We need to show that for $B \in \mathfrak{B}$, there is a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers for which $d(S) = 0$, where $S = \{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U_n\}$. First note that $f_n^{-1}(-1,1) \subseteq U_n$ for each n.

If $n \notin T$, then $f_n \in [B,1]^s(\underline{0})$ i.e. there is a $\eta_n > 0$ with $B^{\eta_n} \subseteq f_n^{-1}(-1,1) \subseteq U_n$. Fix a $\delta > 0$. Define $\delta_n = \delta$ if $n \in T$ and $\delta_n = \eta_n$ otherwise. Then $n \notin T$ implies $n \notin S$. Thus we obtain a sequence $\{\delta_n : n \in \mathbb{N}\}$ for which $S = T$ and $d(S) = 0$. Hence $\{U_n : n \in \mathbb{N}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X. □

Lemma 4.4. *Let $\mathfrak{B}$ be a bornology on X with closed base. Let $\{f_n : n \in \mathbb{N}\}$ be a sequence of functions in $(C(X),\tau^s_{\mathfrak{B}})$. If $\{f_n^{-1}(-\frac{1}{n},\frac{1}{n}) : n \in \mathbb{N}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X, then $\{f_n : n \in \mathbb{N}\}$ s-converges to $\underline{0}$ with respect to $\tau^s_{\mathfrak{B}}$.*

Proof. For $B \in \mathfrak{B}$ there is a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers for which $d(S) = 0$, where $S = \{n \in \mathbb{N} : B^{\delta_n} \nsubseteq f_n(-\frac{1}{n},\frac{1}{n})\}$. We now show that for any neighbourhood $[B,\varepsilon]^s(\underline{0})$, $d(T) = 0$, where $T = \{n \in \mathbb{N} : f_n \notin [B,\varepsilon]^s(\underline{0})\}$. For $\varepsilon > 0$, choose a n_1 such that $\frac{1}{n_1} < \varepsilon$. Now if $n \in T \setminus \{1,2,\ldots,n_1-1\}$, then $f_n \notin [B,\varepsilon]^s(\underline{0})$ i.e. for any $\zeta > 0$, $B^\zeta \nsubseteq f_n^{-1}(-\varepsilon,\varepsilon)$ i.e. $B^\zeta \nsubseteq f_n^{-1}(-\frac{1}{n},\frac{1}{n})$. Choose $\zeta = \delta_n$, then $B^{\delta_n} \nsubseteq f_n^{-1}(-\frac{1}{n},\frac{1}{n})$. Clearly $n \in S$ and hence $T \setminus \{1,2,\ldots,n_1-1\} \subseteq S$. It is now evident that $d(T) = 0$. Hence $\{f_n : n \in \mathbb{N}\}$ s-converges to $\underline{0}$ with respect to $\tau^s_{\mathfrak{B}}$. □

Under certain assumption on a subset of $C(X)$, in the following result we show that every open $\mathfrak{B}^s$-cover of X has a s-$\gamma_{\mathfrak{B}^s}$-subcover.

Proposition 4.1. *Let $\mathfrak{B}$ be a bornology on X with closed base. For $A \subseteq (C(X),\tau^s_{\mathfrak{B}})$ with $\underline{0} \in \overline{A}$, if there is a sequence in A which s-converges to $\underline{0}$, then every open $\mathfrak{B}^s$-cover of X contains a s-$\gamma_{\mathfrak{B}^s}$-cover of X.*

Proof. Let $\mathcal{U}$ be an open $\mathfrak{B}^s$-cover of X. For $B \in \mathfrak{B}$ there exist a $U_B \in \mathcal{U}$ and a $\delta > 0$ such that $B^{2\delta} \subseteq U_B$. Choose a $f_B \in C(X)$ such that $f_B(B^\delta) = \{0\}$ and $f_B(X \setminus U_B) = \{1\}$. Consider the set $A = \{f_B : B \in \mathfrak{B}\}$. Clearly $\underline{0} \in \overline{A}$. By our assumption, there is a sequence $\{f_{B_n} : n \in \mathbb{N}\}$ s-converging to $\underline{0}$. Clearly $\{U_{B_n} : n \in \mathbb{N}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover by Lemma 4.3. □

The statistically strictly Frèchet-Urysohn property of $C(X)$ can be characterized by a S_1-type selective property of X.

Theorem 4.1. *Let $\mathfrak{B}$ be a bornology on X with closed base. The following statements are equivalent:*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ *is s-SFU;*
(2) *X satisfies* $S_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$.

Proof. (1) $\Rightarrow$ (2) Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X. For $B \in \mathfrak{B}$ and each $n \in \mathbb{N}$, there are $U_{B,n} \in \mathcal{U}_n$ and $\delta > 0$ such that $B^{2\delta} \subseteq U_{B,n}$. Consider the collection $\mathcal{U}_{B,n} = \{U \in \mathcal{U}_n : B^{2\delta} \subseteq U\}$. For each $U \in \mathcal{U}_{B,n}$ choose a $f_{B,U} \in C(X)$ such that $f_{B,U}(B^\delta) = \{0\}$ and $f_{B,U}(X \setminus U) = \{1\}$. Now define $A_n = \{f_{B,U} : B \in \mathfrak{B}, U \in \mathcal{U}_{B,n}\}$ for each $n \in \mathbb{N}$. Clearly $\underline{0} \in \overline{A_n} \setminus A_n$. Apply (1) to the sequence $\{A_n : n \in \mathbb{N}\}$ to find a $f_{B_n,U_n} \in A_n$ for each n such that $\{f_{B_n,U_n} : n \in \mathbb{N}\}$ s-converges to $\underline{0}$ with respect to $\tau^s_{\mathfrak{B}}$, where $U_n \in \mathcal{U}_n$ for each n. Clearly $\{U_n : n \in \mathbb{N}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover by Lemma 4.3. Hence X satisfies $S_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$.

(2) $\Rightarrow$ (1) Let $\{A_n : n \in \mathbb{N}\}$ be a sequence of subsets of $C(X)$ such that $\underline{0} \in \overline{A_n} \setminus A_n$ for each $n \in \mathbb{N}$. By Lemma 4.2, $\mathcal{U}_n = \{f^{-1}(-\frac{1}{n}, \frac{1}{n}) : f \in A_n\}$ is an open $\mathfrak{B}^s$-cover of X for each $n \in \mathbb{N}$. Apply (2) to the sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ to obtain a $U_n \in \mathcal{U}_n$ for each n such that $\{U_n : n \in \mathbb{N}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X. Now $U_n = f_n^{-1}(-\frac{1}{n}, \frac{1}{n})$. By Lemma 4.4, it follows that the sequence $\{f_n : n \in \mathbb{N}\}$ s-converges to $\underline{0}$ with respect to $\tau^s_{\mathfrak{B}}$ and hence $(C(X), \tau^s_{\mathfrak{B}})$ is s-SFU. □

Similar to Proposition 3.1, we can compare the following selective properties in $C(X)$.

Proposition 4.2. *Let $\mathfrak{B}$ be a bornology on X with closed base. Consider the following statements:*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ *satisfies* $S_1(s\text{-}\Sigma_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$;
(2) $(C(X), \tau^s_{\mathfrak{B}})$ *satisfies* $s\text{-}\alpha_4(s\text{-}\Sigma_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$;
(3) $(C(X), \tau^s_{\mathfrak{B}})$ *satisfies* $S_{\text{fin}}(s\text{-}\Sigma_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$.
Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *holds.*

Proof. The proof of (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3) Let $\{S_n : n \in \mathbb{N}\}$ be a sequence of elements in s-$\Sigma_{\underline{0}}$. For each n take $S_n = \{f_{n,m} : m \in \mathbb{N}\}$. By (2), there is a $T \in s\text{-}\Sigma_{\underline{0}}$ and a (statistically dense) subset $K = \{n_1 < n_2 < \cdots\}$ of $\mathbb{N}$ with $d(K) = 1$ such that the set $S_{n_i} \cap T$ is non empty for each $n_i \in K$. Let $f_{n_i,m_i} \in S_{n_i} \cap T$ for each $n_i \in K$. By Lemma 4.1, the subsequence $\{f_{n_i,m_i} : i \in \mathbb{N}\}$ is s-convergent to $\underline{0}$. Define $F_n = \{f_{n_i,m_i}\}$ if $n = n_i$ and $F_n = \emptyset$ otherwise. The conclusion now follows from the fact that $\cup_{n\in\mathbb{N}} F_n$ is s-convergent to $\underline{0}$ and each F_n is a finite subset of S_n. □

We now give another sufficient condition for a countable open $\mathfrak{B}^s$-cover of X to be s-$\mathfrak{B}^s$-groupable (compare with Proposition 3.2) with the help of s-Reznichenko property of $C(X)$. In the following result we use the fact that a collection $\mathcal{U}$ of open subsets of X is an open $\mathfrak{B}^s$-cover if and only if for each $U \in \mathcal{U}$ there is a closed set (may be empty) $C(U) \subseteq U$ such that $\{C(U) : U \in \mathcal{U}\}$ is a $\mathfrak{B}^s$-cover of X (see [8, Theorem 5.1]).

Proposition 4.3. *Let $\mathfrak{B}$ be a bornology on X with closed base. If $(C(X), \tau^s_{\mathfrak{B}})$ has the s-Reznichenko property, then every countable open $\mathfrak{B}^s$-cover of X is s-$\mathfrak{B}^s$-groupable.*

Proof. Let $\mathcal{U}$ be a countable open $\mathfrak{B}^s$-cover of X. For each $U \in \mathcal{U}$ there is a closed set $C(U)$ with $C(U) \subseteq U$ such that $\{C(U) : U \in \mathcal{U}\}$ is a $\mathfrak{B}^s$-cover of X. Define a continuous function f_U on X such that $f_U(C(U)) = \{0\}$ and $f_U(X \setminus U) = \{1\}$. Clearly $\{f_U^{-1}(\{0\}) : U \in \mathcal{U}\}$ is a $\mathfrak{B}^s$-cover of X. We can assume that f_U and f'_U are distinct whenever U and U' are distinct. Consider the set $A = \{f_U : U \in \mathcal{U}\}$. Evidently $\underline{0} \in \overline{A} \setminus A$. By our assumption, there is a sequence $\{A_n : n \in \mathbb{N}\}$ of pairwise disjoint finite subsets of A such that for the neighbourhood $[B, 1]^s(\underline{0})$, $d(T) = 0$, where $T = \{n \in \mathbb{N} : [B, 1]^s(\underline{0}) \cap A_n = \emptyset\}$. Let $\mathcal{U}_n = \{U : f_U \in A_n\}$ for each n. Clearly $\mathcal{U}_n$'s are pairwise disjoint and finite. Using the fact that $d(T) = 0$, we can show that $\{\mathcal{U}_n : n \in \mathbb{N}\}$ witnesses the s-$\mathfrak{B}^s$-groupability of $\mathcal{U}$. This completes the proof. □

Theorem 4.2. *Let $\mathfrak{B}$ be a bornology on X with closed base. The following statements are equivalent:*
(1) *TWO has a wining strategy in the game* $G_1(\Omega_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$ *on* $(C(X), \tau^s_{\mathfrak{B}})$;
(2) *TWO has a wining strategy in the game* $G_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$ *on X.*

Proof. (1) ⇒ (2) Let ψ be a wining strategy for TWO in $G_1(\Omega_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$. We define a strategy σ for ONE in the game $G_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$. In the n-th inning, the move of ONE in $G_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$ is $\sigma(U_1, \ldots, U_{n-1}) = \mathcal{U}_n$. For each $B \in \mathfrak{B}$, there exist a $\delta > 0$ and a $U \in \mathcal{U}_n$ such that $B^{2\delta} \subseteq U$. Choose a $f_{B,U} \in C(X)$ such that $f_{B,U}(B^\delta) = \{0\}$ and $f_{B,U}(X \setminus U) = \{1\}$. Then the collection $A_n = \{f_{B,U} : B \in \mathfrak{B}, U \in \mathcal{U}_n\} \in \Omega_{\underline{0}}$. The corresponding move of ONE in $G_1(\Omega_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$ is $\psi(f_1, \ldots, f_{n-1}) = A_n$. TWO responds with $f_{B_n,U_n} \in A_n$ in $G_1(\Omega_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$. Correspondingly in $G_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$, TWO's response is U_n. The play in $G_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$ is $\sigma(\emptyset), U_1, \sigma(U_1), U_2, \ldots, \sigma(U_1, \ldots, U_{n-1}), U_n \ldots$ and the play in $G_1(\Omega_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$ is $\psi(\emptyset), f_1, \psi(f_1), f_2, \ldots, \psi(f_1, \ldots, f_{n-1}), f_n \ldots$, where $f_n = f_{B_n,U_n}$. Since ψ is a wining strategy for TWO in the play in $G_1(\Omega_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$, $\{f_n : n \in \mathbb{N}\} \in s\text{-}\Sigma_{\underline{0}}$. Now by Lemma 4.3, σ a wining strategy for TWO in the game $G_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$.

(2) ⇒ (1) Let σ be a wining strategy for TWO in the game $G_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$. For each n let $I_n = (-\frac{1}{n}, \frac{1}{n})$.

Now define a strategy ψ for ONE in $G_1(\Omega_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$. In the n-th inning ONE's move in $G_1(\Omega_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$ is $\psi(f_1, \ldots, f_{n-1}) = A_n \in \Omega_{\underline{0}}$. Then n-th move of ONE in $G_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$ is $\mathcal{U}(A_n) = \{f^{-1}(I_n) : f \in A_n\}$, by Lemma 4.2. TWO responds with $U_n \in \mathcal{U}(A_n)$. TWO's move in $G_1(\Omega_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$ is f_n, where $U_n = f_n^{-1}(I_n)$.

Similarly, by using Lemma 4.4, we can show that ψ is a wining strategy for TWO in $G_1(\Omega_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$. □

Translating [8, Theorem 5.6] into the language of statistical convergence, we obtain the following result.

Proposition 4.4. *Let $\mathfrak{B}$ be a bornology on X with closed base.*
(1) *If ONE has no wining strategy in the game* $G_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\mathcal{O}^{gp}_{\mathfrak{B}^s})$ *on X, then* $(C(X), \tau^s_{\mathfrak{B}})$ *satisfies* $S_1(\Omega_{\underline{0}}, s\text{-}\Omega^{gp}_{\underline{0}})$;
(2) *If ONE has no wining strategy in the game* $G_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\mathcal{O}^{gp}_{\mathfrak{B}^s})$ *on X, then* $(C(X), \tau^s_{\mathfrak{B}})$ *satisfies* $S_{\text{fin}}(\Omega_{\underline{0}}, s\text{-}\Omega^{gp}_{\underline{0}})$.

With the help of Propositions 3.3 and 4.4, we mention our final result without proof.

Proposition 4.5. *Let $\mathfrak{B}$ be a bornology on X with closed base. If ONE has no wining strategy in the s-$\mathfrak{B}^s$-Hurewicz game on X, then* $(C(X), \tau^s_{\mathfrak{B}})$ *satisfies* $S_{\text{fin}}(\Omega_{\underline{0}}, s\text{-}\Omega^{gp}_{\underline{0}})$ *i.e. countable fan tightness and the s-Reznichenko property.*

Acknowledgement

The authors are thankful to the Referee(s) for careful reading of the paper and valuable suggestions.

References

[1] A.V. Arhangel'skii, Hurewicz spaces, analytic sets and fan tightness of function spaces, Soviet Math. Dokl. 33 (1986) 396–399.
[2] A.V. Arhangel'skii, Topological Function Spaces, Kluwer, 1992.
[3] L.F. Aurichi, R.M. Mezabarba, Bornologies and filters applied to selection principles and function spaces, Topology Appl. 258 (2019) 187–201.
[4] G. Beer, S. Levi, Strong uniform continuity, J. Math. Anal. Appl. 350 (2009) 568–589.
[5] A. Caserta, G. Di Maio, Ľ. Holá, Arzelà's theorem and strong uniform convergence on bornologies, J. Math. Anal. Appl. 371 (2010) 384–392.
[6] A. Caserta, G. Di Maio, Lj.D.R. Kočinac, Bornologies, selection principles and function spaces, Topology Appl. 159 (2012) 1847–1852.
[7] A. Caserta, Lj.D.R. Kočinac, On statistical exhaustiveness, Appl. Math. Lett. 25 (2012) 1447–1451.
[8] D. Chandra, P. Das, S. Das, Applications of bornological covering properties in metric spaces, Indag. Math. 31 (2020) 43–63.
[9] P. Das, Certain types of open covers and selection principles using ideals, Houston J. Math. 39 (2) (2013) 637–650.
[10] P. Das, Lj.D.R. Kočinac, D. Chandra, Some remarks on open covers and selection principles using ideals, Topology Appl. 202 (2016) 183–193.
[11] G. Di Maio, D. Djurčić, Lj.D.R. Kočinac, M.R. Žižović, Statistical convergence, selection principles and asymptotic analysis, Chaos Solitons Fractals 42 (2009) 2815–2821.
[12] G. Di Maio, Lj.D.R. Kočinac, Statistical convergence in topology, Topology Appl. 156 (2008) 28–45.
[13] G. Di Maio, Lj.D.R. Kočinac, Statistical convergence in function spaces, Abstr. Appl. Anal. 2011 (2011) Article ID 420419.
[14] R. Engelking, General Topology, (2nd edition), Sigma Ser. Pure Math., Vol. 6, Heldermann, Berlin, 1989.
[15] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244.
[16] H. Hogbe-Nlend, Bornologies and Functional Analysis, North-Holland, Amsterdam, 1977.
[17] W. Just, A.W. Miller, M. Scheepers, P.J. Szeptycki, The combinatorics of open covers (II), Topology Appl. 73 (1996) 241–266.
[18] Lj.D.R. Kočinac, Selected results on selection principles, in: Proc. Third Seminar on Geometry and Topology, Tabriz, Iran, July 15-17, (2004) 71–104.

[19] Lj.D.R. Kočinac, Selection principles related to α_i -properties, Taiwanese J. Math. 12 (2008) 561–571.
[20] Lj.D.R. Kočinac, M. Scheepers, Combinatorics of open covers (VII), Fund. Math. 179 (2003) 131–155.
[21] P. Kostyrko, M. Mačaj, T. Šalát, O. Strauch, On statistical limit points, Proc. Amer. Math. Soc. 129 (2000) 2647–2654.
[22] R.A. McCoy, I. Ntantu, Topological Properties of Spaces of Continuous Functions, Lecture Notes in Math., Vol. 1315, Springer-Verlag, Berlin, 1988.
[23] M. Sakai, The Pytkeev property and the Reznichenko property in function spaces, Note Mat. 22 (2003) 43–52.
[24] M. Scheepers, Combinatorics of open covers (I): Ramsey theory, Topology Appl. 69 (1996) 31–62.
[25] M. Scheepers, Selection principles and covering properties in topology, Note Mat. 22 (2) (2003/2004) 3–41.
[26] I.J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly 66 (1959) 361–375.
[27] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math. 2 (1951) 73–74.
[28] A. Zygmund, Trigonometric Series, (2nd edition), Cambridge University Press, Cambridge, 1979.