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Ricci Recurrent Almost Kenmotsu 3-Manifolds

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Abstract. In this paper, we obtain that a Ricci recurrent 3-dimensional almost Kenmotsu manifold with constant scalar curvature satisfying $\nabla_\xi h = 0, h \neq 0$, is locally isometric to the Riemannian product $\mathbb{H}^2(-4)\times\mathbb{R}$.

1. Introduction

It is well known that, symmetric spaces play a significant role in differential geometry. The study of Riemannian symmetric spaces was initiated in the last twenties by Cartan [1], who obtained classification of those spaces. Let M be a manifold with Riemannian metric g and let ∇ be the Levi-Civita connection of M. A Riemannian manifold is called locally symmetric [1] if $\nabla R = 0$, where R is the Riemannian curvature tensor of M. The class of Riemannian symmetric manifolds is a very natural generalization of the class of manifolds of constant curvaure.

The class of semi-symmetric manifolds includes the set of locally symmetric manifolds as a proper subset and is defined by $R \cdot R = 0$. A fundamental study on such Riemannian manifolds was made by Szabo [12] and Kowalski [9]. Also, it is well known that M is locally symmetric if and only if Ricci tensor S is parallel, i.e.,

$$\nabla S = 0.$$

Generalizing the notion of Ricci parallel manifolds, Patterson [11] introduced the notion of Ricci recurrent manifolds and is defined by

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z), \tag{1}$$

where A is a 1-form on M. In addition, if the vector fields X, Y, Z are orthogonal to ξ, then the manifold is called locally Ricci recurrent manifold. The equation (1) can also be written as

$$(\nabla_X Q)Y = A(X)QY, \tag{2}$$

where Q denotes the associated Ricci operator with respect to g.

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On the other hand, Kenmotsu in [8] obtained that locally symmetric Kenmotsu manifold is of constant sectional curvature −1. Extending this for dimension 3, the authors Wang [15] and Cho [3] independently proved that a 3-dimensional almost Kenmotsu manifold is locally symmetric if and only if it is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the product space $\mathbb{H}^2(-4) \times \mathbb{R}$. In [17], the author generalizing the result of Wang [15] and Cho [3], and proved that the Ricci tensor of non-Kenmotsu almost Kenmotsu 3-h-manifold is η-parallel if and only if the manifold is locally isometric to either the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$ or a non-unimodular Lie group equipped with a left invariant non-Kenmotsu almost Kenmotsu structure. The above cited works inspires us to study the Ricci recurrent condition on almost Kenmotsu 3-manifolds.

In this paper, we aim to study a non-Kenmotsu almost Kenmotsu 3-manifold M satisfying $\nabla_\xi h = 0$. After collecting basic definition and properties of almost Kenmotsu 3-manifold in section 2, we prove that a non-Kenmotsu almost Kenmotsu 3-h-manifold with constant scalar curvature is locally isometric to the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$. Also, Ricci recurrent almost Kenmotsu 3-manifolds with constant scalar curvature and $Q\xi = S(\xi,\xi)\xi$ are classified. Before concluding this paper, we present an example to justify our main results.

2. Almost Kenmotsu 3-Manifolds

Let M be a smooth differentiable manifold of dimension $2n+1$. On M, if there exist a $(1,1)$-tensor field φ, a characterstic vector field ξ, a 1-form η and a Riemannian metric g such that

$$\begin{aligned} &\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \\ &g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \end{aligned} \tag{3}$$

for any vector fields X, Y, we say that M admits an almost contact metric structure. We call ξ as a Reeb vector field. As a result of (3) we have $\varphi(\xi) = 0$, $\eta(\varphi) = 0$. One can define an almost complex structure J on $M \times \mathbb{R}$ by

$$J\left(X, u\frac{d}{dt}\right) = \left(\varphi X - u\xi, \eta(X)\frac{d}{dt}\right),$$

where t is coordinate of $\mathbb{R}$ and u is a smooth function. If the aforementioned structure J is integrable, then we say that an almost contact structure is normal, and this is equivalent to require

$$[\varphi, \varphi] = -2d\eta \otimes \xi,$$

where $[\varphi, \varphi]$ indicates the Nijenhuis tensor of φ.

According to Janssens and Vanhecke [6], an almost Kenmotsu manifold is an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ satisfying $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, where the fundamental 2-form Φ is defined by $\Phi(X,Y) = g(X, \varphi Y)$ for any vector fields X and Y. A normal almost Kenmotsu manifold is called Kenmotsu manifold [6, 8]. It is well known that almost Kenmotsu manifold is Kenmotsu if and only if

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(X)\varphi X,$$

for any vector fields X and Y.

We consider three tensor fields $\ell = R(\cdot, \xi)\xi$, $2h = \pounds_\xi \varphi$ and $h' = h \circ \varphi$ on M, where R is the Riemannian curvature tensor of g and £ is the Lie derivative. Following [4, 5], it is seen that three (1,1)-type tensor fields ℓ, h and h' are symmetric and satisfies the following formula

$$h\xi = \ell\xi = 0, \quad tr(h) = tr(h') = 0, \quad h\varphi + \varphi h = 0, \tag{4}$$

$$\nabla \xi = h' + id - \eta \otimes \xi, \tag{5}$$

$$\nabla_\xi h = -\varphi - 2h - \varphi h^2 - \varphi \ell, \tag{6}$$

$$tr(\ell) = S(\xi,\xi) = g(Q\xi,\xi) = -2n - tr(h^2), \tag{7}$$

where tr, S and Q denotes trace operator, Ricci tensor and Ricci operator respectively. Since 3-dimensional almost Kenmotsu manifold M is CR-integrable, we obtain that M is Kenmotsu if and only if h is identically zero.

Let us remember some useful formula listed in [2, 3]. Let $\mathcal{U}_1$ be the maximal open subset of 3-dimensional almost Kenmotsu manifold M on which $h \neq 0$ and $\mathcal{U}_2$ is the maximal open subset of M which is defined by $\mathcal{U}_2 = \{p \in M : h = 0$ in a neighborhood of $p\}$. Then $\mathcal{U}_1 \cup \mathcal{U}_2$ is open and dense in M and there exists a local orthonormal basis $\{\xi, e, \varphi e\}$ of three smooth unit eigenvectors of h for any point $p \in \mathcal{U}_1 \cup \mathcal{U}_2$. On $\mathcal{U}_1$, we set $h(e) = \mu e$ and hence $h\varphi e = -\mu\varphi e$, where μ is a positive function on $\mathcal{U}_1$. The eigenvalue function μ is continuous on M and smooth on $\mathcal{U}_1 \cup \mathcal{U}_2$.

Cho and Kimura [2] obtained the following lemma;

Lemma 2.1. *On $\mathcal{U}_1$ we have*

$$\begin{aligned} &\nabla_\xi \xi = 0, \quad \nabla_\xi e = a\varphi e, \quad \nabla_\xi \varphi e = -ae, \\ &\nabla_e \xi = e - \mu\varphi e, \quad \nabla_e e = -\xi - b\varphi e, \quad \nabla_e \varphi e = \mu\xi + be, \\ &\nabla_{\varphi e} \xi = -\mu e + \varphi e, \quad \nabla_{\varphi e} e = \mu\xi + c\varphi e, \quad \nabla_{\varphi e}\varphi e = -\xi - ce, \end{aligned} \tag{8}$$

where a, b, c are smooth functions.

As a result of above lemma, the following Jacobi identity

$$[[\xi, e], \varphi e] + [[e, \varphi e], \xi] + [[\varphi e, \xi], e] = 0,$$

yields

$$\begin{aligned} &e(\mu) - \xi(b) - e(a) + c(\mu - a) - b = 0, \\ &\varphi e(\mu) - \xi(c) + \varphi e(a) + b(\mu + a) - c = 0 \end{aligned} \tag{9}$$

In view of Lemma 2.1, the Ricci operator can be written as

$$\begin{aligned} Q\xi &= -2(\mu^2 + 1)\xi - (\varphi e(\mu) + 2\mu b)e - (e(\mu) + 2\mu c)\varphi e, \\ Qe &= -(\varphi e(\mu) + 2\mu b)\xi - (f + 2\mu a)e + (\xi(\mu) + 2\mu)\varphi e, \\ Q\varphi e &= -(e(\mu) + 2\mu c)\xi + (\xi(\mu) + 2\mu)e - (f - 2\mu a)\varphi e, \end{aligned} \tag{10}$$

with respect to local basis $\{\xi, e, \varphi e\}$, where we take $f = e(c) + \varphi e(b) + b^2 + c^2 + 2$. From (10) we find that the scalar curvature of M is

$$r = -2(f + \mu^2 + 1). \tag{11}$$

3. Ricci Recurrent Almost Kenmotsu 3-Manifolds

An almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = 0, h \neq 0$ was said to be almost Kenmotsu 3-h-manifold by Wang in [14]. Notice that on a Kenmotsu 3-manifold such a condition is meaningless because of $h = 0$. In ([4], Proposition 6), Dileo and Pastore showed that even on a locally symmetric non-Kenmotsu almost Kenmotsu manifold there still holds $\nabla_\xi h = 0$. By making use of this conditon, Wang [16, 17, 19] and Wang [14] gave some local classification theorem on almost Kenmotsu 3-manifold and also presented some examples of almost Kenmotsu 3-manifold on which $\nabla_\xi h = 0$.

Example 3.1. *Let $\mathcal{G}$ be a 3-dimensional non-unimodular Lie group (see [5]) admitting a left invariant orthonormal frame fields $\{e_1, e_2, e_3\}$ satisfying*

$$[e_1, e_2] = -e_2 - e_3, \quad [e_2, e_3] = 0, \quad [e_1, e_3] = -e_2 - e_3.$$

We consider the metric g on $\mathcal{G}$ by $g(e_i, e_j) = \delta_{ij}$ for $1 \leq i, j \leq 3$. We set $\xi = e_1$ and denote by η its dual 1-form. We define (1,1)-tensor field φ by $\varphi(e_1) = 0$, $\varphi(e_2) = e_3$ and $\varphi(e_3) = -e_2$. Then, one can verify that (φ, ξ, η, g) makes $\mathcal{G}$ a left invariant non-Kenmotsu almost Kenmotsu structure satisfying $\nabla_\xi h = 0$.

Now, we prove the following main result.

Theorem 3.2. *If an almost Kenmotsu 3-h-manifold M with constant scalar curvature is Ricci recurrent, then it is locally isometric to the Riemannian product* $\mathbb{H}^2(-4)\times\mathbb{R}$.

Proof. We know that an almost Kenmotsu 3-manifold satisfies $\nabla_\xi h = 0$, $h \neq 0$, that is non Kenmotsu. Then, we see that $\mathcal{U}_1$ is non empty subset, and hence, Lemma 2.1 is applicable. From this lemma, a direct calculation gives

$$(\nabla_\xi h)e = \xi(\mu)e + 2a\mu\varphi e, \quad (\nabla_\xi h)\varphi e = -\xi(\mu) + 2a\mu e. \tag{12}$$

Also we assumed that M satisfies the condition $\nabla_\xi h = 0$ and $h \neq 0$. By using this condition in (12) we have

$$\xi(\mu) = a = 0, \tag{13}$$

where we taken an account of μ is a positive on $\mathcal{U}_1$. Due to (13) and Lemma 2.1, the relation (10) can be written as

$$(\nabla_\xi Q)\xi = -\xi(\varphi e(\mu) + 2\mu b)e - \xi(e(\mu) + 2\mu c)\varphi e. \tag{14}$$

$$(\nabla_\xi Q)e = -\xi(\varphi e(\mu) + 2\mu b)\xi - \xi(f)e. \tag{15}$$

$$(\nabla_\xi Q)\varphi e = -\xi(e(\mu) + 2\mu c)\xi - \xi(f)\varphi e. \tag{16}$$

$$\begin{aligned}(\nabla_e Q)\xi =& 2(\varphi e(\mu) - 3\mu e(\mu) + 2\mu b - 2\mu^2 c)\xi\\ &+ (f - 2 - e(\varphi e(\mu) + 2\mu b) - b(e(\mu) + 2\mu c))e\\ &+ (2\mu^3 + b(\varphi e(\mu) + 2\mu b) - e(e(\mu) + 2\mu c) - \mu f)\varphi e.\end{aligned} \tag{17}$$

$$\begin{aligned}(\nabla_e Q)e =& (f - 2 - e(\varphi e(\mu) + 2\mu b) - b(e(\mu) + 2\mu c))\xi\\ &- (e(f) + 2\varphi e(\mu))e + (e(\mu) + \mu\varphi e(\mu) + 2\mu^2 b - 2\mu c)\varphi e.\end{aligned} \tag{18}$$

$$\begin{aligned}(\nabla_e Q)\varphi e =& (2\mu^3 - f\mu + b(\varphi e(\mu) + 2\mu b) - e(e(\mu) + 2\mu c))\xi\\ &- (e(\mu) + \mu\varphi e(\mu) - 2\mu c + 2\mu^2 b)e\\ &+ (2\mu(e(\mu) + 2\mu c) - e(f) - 4\mu b)\varphi e.\end{aligned} \tag{19}$$

$$\begin{aligned}(\nabla_{\varphi e} Q)\xi =& 2(e(\mu) - 3\mu\varphi e(\mu) + 2\mu c - 2\mu^2 b)\xi\\ &+ (2\mu^3 + c(e(\mu) + 2\mu c) - \varphi e(\varphi e(\mu) + 2\mu b) - \mu f)e\\ &+ (f - 2 - \varphi e(e(\mu) + 2\mu c) - c(\varphi e(\mu) + 2\mu b))\varphi e.\end{aligned} \tag{20}$$

$$\begin{aligned}(\nabla_{\varphi e} Q)e =& (2\mu^3 - f\mu + c(e(\mu) + 2\mu c) - \varphi e(\varphi e(\mu) + 2\mu b))\xi\\ &- (\varphi e(f) + 4\mu c - 2\mu(\varphi e(\mu) + 2\mu b))e\\ &+ (\varphi e(\mu) + \mu e(\mu) + 2\mu^2 c - 2\mu b)\varphi e.\end{aligned} \tag{21}$$

$$\begin{aligned}(\nabla_{\varphi e} Q)\varphi e =& (f - 2 - \varphi e(e(\mu) + 2\mu c) - c(\varphi e(\mu) + 2\mu b))\xi\\ &+ (\varphi e(\mu) + \mu e(\mu) + 2\mu^2 c - 2\mu b)e - (\varphi e(f) + 2e(\mu))\varphi e.\end{aligned} \tag{22}$$

Putting X and Y by ξ into (2) we have $(\nabla_\xi Q)\xi = A(\xi)Q\xi$. In this relation, applying (14) and first term of (10) we obtain

$$A(\xi) = 0, \quad \xi(\varphi e(\mu) + 2\mu b) = 0, \quad \xi(e(\mu) + 2\mu c) = 0. \tag{23}$$

Simillarly, taking X and Y by e into (2) we have $(\nabla_e Q)e = A(e)Qe$. This relation together with (18) and second term of (10) gives

$$\begin{aligned} f - 2 - e(\varphi e(\mu) + 2\mu b) - b(e(\mu) + 2\mu c) + A(e)(\varphi e(\mu) + 2\mu b) = 0, \\ A(e)f - (e(f) + 2\varphi e(\mu)) = 0, \\ e(\mu) + \mu\varphi e(\mu) + 2\mu^2 b - 2\mu c - 2A(e)\mu = 0. \end{aligned} \tag{24}$$

Substituting X and Y by φe into (2) we obtain $(\nabla_{\varphi e} Q)\varphi e = A(\varphi e)Q\varphi e$. As a result of this, we have from (22) and third term of (10) that

$$\begin{aligned} f - 2 - \varphi e(e(\mu) + 2\mu c) - c(\varphi e(\mu) + 2\mu b) + A(\varphi e)(e(\mu) + 2\mu c) = 0, \\ \varphi e(\mu) + \mu e(\mu) + 2\mu^2 c - 2\mu b - 2A(\varphi e)\mu = 0, \\ A(\varphi e)f - (\varphi e(f) + 2e(\mu)) = 0. \end{aligned} \tag{25}$$

On the other hand, setting X and Y in (2) by e and ξ respectively, we obtain $(\nabla_e Q)\xi = A(e)Q\xi$. In this relation, taking into account of (17), first term of (10) we obtain

$$\begin{aligned} 2(\varphi e(\mu) - 3\mu e(\mu) + 2\mu b - 2\mu^2 c) + 2A(e)(\mu^2 + 1) = 0, \\ 2\mu^3 + b(\varphi e(\mu) + 2\mu b) - e(e(\mu) + 2\mu c) - \mu f + A(e)(e(\mu) + 2\mu c) = 0, \end{aligned} \tag{26}$$

where we utilized the first term of (24). Similarly, switching X and Y in (2) with φe and ξ respectively, we obtain $(\nabla_{\varphi e} Q)\xi = A(\varphi e)Q\xi$. As a result of this, we have from (22) and first term of (10) that

$$\begin{aligned} 2(e(\mu) - 3\mu\varphi e(\mu) + 2\mu c - 2\mu^2 b) + 2A(\varphi e)(\mu^2 + 1) = 0, \\ 2\mu^3 + c(e(\mu) + 2\mu c) - \varphi e(\varphi e(\mu) + 2\mu b) - \mu f + A(\varphi e)(\varphi e(\mu) + 2\mu b) = 0, \end{aligned} \tag{27}$$

where we utilized the first term of (25). Agian, putiing X and Y by e and φe respectively into (2) we have $(\nabla_e Q)\varphi e = A(e)Q\varphi e$. In this relation, applying (19) and third term of (10) we obtain

$$2\mu(e(\mu) + 2\mu c) - e(f) - 4\mu b + A(e)f = 0, \tag{28}$$

where we utilized the third term of (24) and second term of (26). Finally, substituting X and Y in (2) by φe and e respectively, we obtain $(\nabla_{\varphi e} Q)e = A(\varphi e)Qe$. This relation together with (21) and second term of (10) yields

$$A(\varphi e)f - \varphi e(f) + 4\mu c - 2\mu(\varphi e(\mu) + 2\mu b) = 0, \tag{29}$$

where we used the second terms of (25) and (27).

Since M is of constant scalar curvature, we have from (11) that

$$\xi(f) = 0, \quad e(f) = -2\mu e(\mu), \quad \varphi e(f) = -2\mu\varphi e(\mu). \tag{30}$$

Applying the constancy of r, (14), (18) and (22) in the well known formula $div\ Q = \frac{1}{2} grad\ r$ we obtain

$$\begin{aligned} A(e)(\varphi e(\mu) + 2\mu b) + A(\varphi e)(e(\mu) + 2\mu c) = 0, \\ -\xi(\varphi e(\mu) + 2\mu b) + 3\mu e(\mu) - \varphi e(\mu) + 2\mu^2 c - 2\mu b = 0, \\ -\xi(e(\mu) + 2\mu c) + 3\mu\varphi e(\mu) - 3(\mu) + 2\mu^2 b - 2\mu c = 0, \end{aligned} \tag{31}$$

where we used the equation (30) and first terms of (24) and (25). Utilization of second equation of (23) and third equation of (23) in second and third term of (31) respectively, we obtain

$$\begin{aligned} &3\mu e(\mu) - \varphi e(\mu) + 2\mu^2 c - 2\mu b = 0, \\ &3\mu\varphi e(\mu) - 3(\mu) + 2\mu^2 b - 2\mu c = 0. \end{aligned} \tag{32}$$

It is easily obtain from second and third term of (23) that

$$\begin{aligned} &\xi(e(\mu)) + 2\mu\xi(c) = 0, \\ &\xi(\varphi e(\mu)) + 2\mu\xi(b) = 0, \end{aligned} \tag{33}$$

where we used the equation (13). Differentiate (32) along ξ, recalling (33) and (13) one gets

$$\xi(b) = \xi(c) = 0, \tag{34}$$

where we taken an account of μ is a positive on $\mathcal{U}_1$. As a result of (34) and (13), we have from (9) that

$$\begin{aligned} &e(\mu) = b - \mu c, \\ &\varphi e(\mu) = c - \mu b. \end{aligned} \tag{35}$$

Employing (35) into (32) gives that

$$\begin{aligned} &2\mu b - \mu^2 c - c = 0, \\ &2\mu c - \mu^2 b - b = 0. \end{aligned} \tag{36}$$

It follows from (36) that $(\mu^2 + 1)(b^2 - c^2) = 0$. Because of μ is positive on $\mathcal{U}_1$, we have $(b^2 - c^2) = 0$ and hence we get either $b - c = 0$ or $b + c = 0$. We continue the discussion with the following two cases.

Case i: Making use of $b = c$ into (36) we obtain either $b = c = 0$ or $\mu = 1$. First we consider that $b = c = 0$ and using this subcase in (35) we see that μ is a positive constant. In this context, we have from (8) that

$$[\xi, e] = \mu\varphi e - e, \quad [e, \varphi e] = 0, \quad [\varphi e, \xi] = -\mu e + \varphi e.$$

As a result of Milnor [10], we now say that M is locally isometric to a 3-dimensional non-unimodular Lie group equipped with a left invariant strictly almost Kenmotsu structure. Moreover, employing $b = c = 0$ in equations (14)-(22) we obatin

$$\nabla Q = 0.$$

It is known that the Ricci tensor is parallel if and only if the curvature tensor of the manifold is parallel, i.e., the manifold is locally symmetric. Also we seen from Wang [15] and Cho [3] that a locally symmetric 3-dimensional non-Kenmotsu almost Kenmotsu manifold is locally isometric to the Riemmanian product $\mathbb{H}^2(-4) \times \mathbb{R}$.

Otherwise, if $\mu = 1$, then from (36) we seen that $b = c$. Using this in first term of (24) and second term of (26) gives that

$$e(b) = A(e)b.$$

Again using $\mu = 1$ and $b = c$ in first term of (25) and second term of (27) we obtain

$$\varphi e(b) = A(\varphi e)b.$$

Making use of aforementioned last two equations in first term of (31) gives

$$e(b) + \varphi e(b) = 0. \tag{37}$$

It is easily find from (30) that f is constant, where we applied μ is positive constant. As a result of (37), we have $f = 2(b^2 + 1)$. Since f is constant, thus we have b is constant and hence c is also constant. Using these relations in (14)-(22) we get $\nabla Q = 0$ and this is equivalent to the locally symmetry. Then the proof follows from Wang [15] or Cho [3].

Case ii: Here we consider the other case $b + c = 0$. This together with (36) yields $b = c = 0$. Taking $b = c = 0$ in (35) and applying (13) we see that μ is constant. Therfore, the proof follows from **Case i**. □

In [7], Jun et al proved that a Ricci recurrent Riemannian manifold is Ricci semi-symmetry and this is more weaker than semi-symmetry. In this connection, it worth to note that Wang in [18] studied semi-symmetry on almost Kenmotsu 3-manifolds in which $\nabla_\xi h = 2a\varphi h$.

Next, we prove the following;

Theorem 3.3. *Let M be a strictly almost Kenmotsu 3-manifold with constant scalar curvature such that ξ is an eigenvector field of the Ricci operator. If M is Ricci recurrent, then it is locally isometric to the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$.*

Proof. From Lemma 2.1 and relation (10), a direct calculation gives that

$$\begin{aligned}(\nabla_e Q)e =&(f - 2 + 2\mu a + \mu\xi(\mu) - e(\varphi e(\mu) + 2\mu b) - b(e(\mu) + 2\mu c))\xi\\ &+ (2b(\xi(\mu) + 2\mu) - 2(\varphi e(\mu) + 2\mu b) - e(f + 2\mu a))e\\ &+ (4\mu ab - (e(\mu) + 2\mu c) + \mu(\varphi e(\mu) + 2\mu b) + e(\xi(\mu) + 2\mu))\varphi e.\end{aligned} \tag{38}$$

$$\begin{aligned}(\nabla_{\varphi e} Q)\varphi e =&(f - 2 - 2\mu a + \mu\xi(\mu) - \varphi e(e(\mu) + 2\mu c) - c(\varphi e(\mu) + 2\mu b))\xi\\ &+ (\mu(e(\mu) + 2\mu c) - 4\mu ac + \varphi e(\xi(\mu) + 2\mu) - (\varphi e(\mu) + 2\mu b))e\\ &+ (2c(\xi(\mu) + 2\mu) - 2(e(\mu) + 2\mu c) - \varphi e(f - 2\mu a))\varphi e.\end{aligned} \tag{39}$$

$$\begin{aligned}(\nabla_e Q)\varphi e =&(2\mu^3 + 2\mu^2 a - f\mu - \xi(\mu) - e(e(\mu) + 2\mu c) + b(\varphi e(\mu) + 2\mu b))\xi\\ &+ (4\mu ab - (e(\mu) + 2\mu c) + \mu(\varphi e(\mu) + 2\mu b) + e(\xi(\mu) + 2\mu))e\\ &+ (2\mu(e(\mu) + 2\mu c) - 2b(\xi(\mu) + 2\mu) - e(f - 2\mu a))\varphi e.\end{aligned} \tag{40}$$

Suppose that M is Ricci recurrent. Then, putting $X = Y = e$ and $Z = \xi$ into (1) we have $g((\nabla_e Q)e, \xi) = A(e)g(Qe, \xi)$. In this relation, taking into account of (38) and second relation of (10) we have

$$f - 2 + 2\mu a + \mu\xi(\mu) - e(\varphi e(\mu) + 2\mu b) - b(e(\mu) + 2\mu c) + A(e)(\varphi e(\mu) + 2\mu b) = 0. \tag{41}$$

Similarly, putting $X = Y = \varphi e$ and $Z = \xi$ into (1) we have $g((\nabla_{\varphi e} Q)\varphi e, \xi) = A(\varphi e)g(Q\varphi e, \xi)$. Utilization of (39) and third term of (10) in this relation gives

$$f - 2 - 2\mu a + \mu\xi(\mu) - \varphi e(e(\mu) + 2\mu c) - c(\varphi e(\mu) + 2\mu b) + A(\varphi e)(e(\mu) + 2\mu c) = 0. \tag{42}$$

Finally, substituting $X = e$, $Y = \varphi e$ and $Z = \xi$ into (1) we have $g((\nabla_e Q)\varphi e, \xi) = A(e)g(Q\varphi e, \xi)$. This together with (40) and third term of (10) we obatin

$$2\mu^3 + 2\mu^2 a - f\mu - \xi(\mu) - e(e(\mu) + 2\mu c) + b(\varphi e(\mu) + 2\mu b) + A(e)(e(\mu) + 2\mu c) = 0. \tag{43}$$

Since ξ is an eigenvector field of the Ricci operator, we have

$$\varphi e(\mu) + 2\mu b = 0, \quad e(\mu) + 2\mu c = 0. \tag{44}$$

As a result of (44), we obtain from (41) and (42) that

$$\begin{aligned}&f - 2 + 2\mu a + \mu\xi(\mu) = 0,\\ &f - 2 - 2\mu a + \mu\xi(\mu) = 0.\end{aligned} \tag{45}$$

It follows from (45) that

$$f - 2 + \mu\xi(\mu) = 0 \quad \text{and} \quad a = 0, \tag{46}$$

where we utilized μ is positive on $\mathcal{U}_1$. Using (46) in (43), recalling (44) we obtain $\xi(\mu) = -2\mu$. This together with first term of (45) gives that

$$f = 2(\mu^2 + 1). \tag{47}$$

Since M is of constant scalar curvature, we have from (11) that

$$\xi(f) = -2\mu\xi(\mu), \quad e(f) = -2\mu e(\mu), \quad \varphi e(f) = -2\mu\varphi e(\mu). \tag{48}$$

In view of (47) and (48), one can easily see that μ is positive constant. Making use of this constancy in (44) we obtain $b = c = 0$. At this stage, we omit the remaining proof since it is very similar with that of **Case i** of Theorem 3.2. This completes the proof. □

Remark 3.4. *From the conclusion of Theorem 3.2 and 3.3, one can see that $Q\xi = S(\xi,\xi)\xi$ is more stronger than $\nabla_\xi h = 0$.*

Finally we show one example of non-Kenmotsu almost Kenmotsu 3-manifolds which verify our results.

Example 3.5. *We consider 3-dimensional manifold*

$$M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$$

and linearly independent vector fields

$$e_1 = (y + z)\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right), \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.$$

One can easily check that

$$[e_1, e_2] = -e_2 - e_3, \quad [e_2, e_3] = 0, \quad [e_1, e_3] = -e_2 - e_3.$$

On M, we consider (1,1)-tensor field φ as $\varphi(e_1) = 0$, $\varphi(e_2) = e_3$ and $\varphi(e_3) = -e_2$, and we define Riemannian metric g by $g(e_i, e_j) = \delta_{ij}$, for $1 \leq i, j \leq 3$. We denote by $\xi = e_1$ and η its dual 1-form with respect to the metric g.

On the other hand, it is not hard to verify that the structure (φ, ξ, η, g) makes M an almost Kenmotsu structure, which is not Kenmotsu, since the operator h' does not vanish. In fact, it is given by

$$h'(e_1) = 0, \quad h'(e_2) = e_3, \quad h'(e_3) = e_2.$$

Utilization of Koszul formula gives

$$\begin{aligned} \nabla_{e_1} e_1 &= 0, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0, \\ \nabla_{e_2} e_1 &= e_2 + e_3, \quad \nabla_{e_2} e_2 = -e_1, \quad \nabla_{e_2} e_3 = -e_1, \\ \nabla_{e_3} e_1 &= e_2 + e_3, \quad \nabla_{e_3} e_2 = -e_1, \quad \nabla_{e_3} e_3 = -e_1. \end{aligned} \tag{49}$$

By the above results, we can easily obtain the components of the curvature tensor R as follows;

$$\begin{aligned} R(e_1, e_2)e_1 &= 2(e_2 + e_3), \quad R(e_1, e_2)e_2 = -2e_1, \quad R(e_1, e_2)e_3 = -2e_1, \\ R(e_2, e_3)e_1 &= 0, \quad R(e_2, e_3)e_2 = 0, \quad R(e_2, e_3)e_3 = 0, \\ R(e_1, e_3)e_1 &= 2(e_2 + e_3), \quad R(e_1, e_3)e_2 = -2e_1, \quad R(e_1, e_3)e_3 = -2e_1. \end{aligned} \tag{50}$$

One can easily see that ξ on M belongs to the $(-2, -2)'$-nullity distribution (see [13]). In this case, by Example 3.1 we have $\nabla_\xi h = 0$. With the help of (50), one can easily see that

$$Qe_1 = -4e_1, \quad Qe_2 = -2(e_2 + e_3), \quad Qe_3 = -2(e_2 + e_3). \tag{51}$$

Then the scalar curvature $r = -8$, a constant. As a result of (49) and (51), a direct calculation gives $(\nabla_{e_i} Q)e_j = 0$, for $1 \leq i, j \leq 3$, that is, Ricci tensor is parallel and so locally symmetric. This example verifies our both Theorem 3.2 and Theorem 3.3.

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