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Periodic Wave Solutions of Non-Newtonian Filtration Equations with Nonlinear Sources and Singularities

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Abstract. This paper is concerned with a kind of non-Newtonian filtration equation with nonlinear sources and singularities. Based on the mountain pass theorem and variational methods, a sufficient criterion for the new results on the periodic wave solutions has been provided. Here not only the structure is more general and practical than the existing works but the conditions imposed are concise. Consequently, compared with the previous results on the singular equations and non-Newtonian filtration equations, the results we established are more generalized and some previous ones can been complemented and improved. Finally, the effectiveness of the established results are validated via two numerical examples and simulations.

1. Introduction

In this paper, we consider the periodic wave solutions for the following non-Newtonian filtration equations with nonlinear sources and singularities

$$\frac{\partial q}{\partial t} = \frac{\partial}{\partial x}\left(|\frac{\partial q}{\partial x}|^{p-2}\frac{\partial q}{\partial x}\right) - \frac{1}{q^{\alpha}} - g(t,x),\ t \geq 0, \tag{1.1}$$

where $p > 1$, $\alpha \geq 1$, $g \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$.

In the past few decades, the non-Newtonian filtration equation

$$\frac{\partial q}{\partial t} = \mathrm{div}(|\nabla q|^{p-2}\nabla q),\ x \in \mathbb{R}^N,\ t \geq 0, \tag{1.2}$$

has been widely studied in literature, see for example, [5], [15], [16], [17], [18], [21], [22]. Equation (1.2) is also called evolutionary p-Laplacian which is one of most widely researched equations in the class of nonlinear

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degenerate parabolic equations. The particular feature of evolutionary p-Laplacian is its gradient-dependent diffusivity. Suppose a compressible fluid flows in a homogeneous isotropic rigid porous medium. Then the volumetric moisture content θ, the seepage velocity $\overrightarrow{V}$ and the density of the fluid are governed by the continuity equation

$$\theta\frac{\partial\rho}{\partial t}+\mathrm{div}(\rho\overrightarrow{V})=0.$$

For non-Newtonian fluid, the linear Darcy's law is no longer valid, because of the influences of many factors such as the molecular and ion effects. Instead, one has the following nonlinear relation:

$$\rho\overrightarrow{V}=-\delta|\nabla P|^{\gamma-2}\nabla P,$$

where $\rho\overrightarrow{V}$ and $P=\eta\rho$ denote the momentum velocity and pressure respectively, $\delta>0$ and $\gamma>1$ are some physical constants. After changing variables and notations, the non-Newtonian filtration equation is derived. Such equations and their stationary counterparts appear in different models in non-Newtonian fluids, turbulent flows in porous media, certain diffusion or heat transfer processes and recently in image processing (for a more detailed physical background see [1], [4]).

Recently, Jin and Yin in [10] studied the traveling wavefronts for the one-dimensional non-Newtonian filtration equation with Hodgkin-Huxley source as follows:

$$\frac{\partial q}{\partial t}=\frac{\partial q}{\partial x}\left(|\frac{\partial q}{\partial x}|^{p-2}\frac{\partial q}{\partial x}\right)+f(q,q_\tau),\ x\in\mathbb{R},\ \ t\geq 0, \tag{1.3}$$

where $p>1$, $f(q,v)=q^r(1-q)(v-a)$, $r>0$, a is a constant with $a\in(0,1)$, and $q_\tau=q(x,t-\tau)$ for $\tau>0$. By establishing some necessary and sufficient conditions, the authors obtained some existence results of monotone nondecreasing and nonincreasing traveling wavefronts of equation (1.3).

After that, on the basis of work of [10], the authors in [11] furthered studied solitary wave and periodic wave solutions for the following non-Newtonian filtration equations with nonlinear sources and a time-varying delay:

$$\frac{\partial q}{\partial t}=\frac{\partial}{\partial x}\left(|\frac{\partial q}{\partial x}|^{p-2}\frac{\partial q}{\partial x}\right)+f(q_{\delta(t)})+g(t,x),\ t\geq 0,\ x\in\mathbb{R}. \tag{1.4}$$

where $p>1$, $f\in C(\mathbb{R},\mathbb{R})$, $g\in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$, $q_{\delta(t)}(t,x)=q(t-\delta(t),x)$ and $\delta\in C(\mathbb{R},\mathbb{R})$. By using an extension of Mawhin's continuation theorem and some analysis methods, the authors proved that (1.4) has at least one periodic wave solution and one solitary wave solution.

Singular equations appear in a great deal of physical models. For example, in paper [20], the singularity models in which the restoring force caused by a compressed perfect gas; in paper [23], [24], [25], the singular term can be regarded as a generalized Lennard-Jones potential or Van der Waals force and it is widely found in molecular dynamics to model the interaction between atomic particles. During the last few decades, different kinds of singular equations have been proposed by many authors, see for example [3], [6], [7], [9], [12], [13], [14].

Compared with the classical non-Newtonian filtration equations or singular equations, non-Newtonian filtration equations with singular effects have been scarcely studied. Therefore at this stage, it is crucial and necessary to further research the dynamical relationship between the two models.

Motivated by the above fact, in this paper, by applying the mountain pass theorem and variational methods, we discuss the existence of periodic wave solutions for Eq.(1.1).

Looking for wave solutions, *i.e.*, functions of the form $q(x,t)=u(\xi)$ with $\xi=x-ct$, Eq.(1.1) leads to the following equation

$$\left(\varphi_p(u'(\xi))\right)'+cu'(\xi)-\frac{1}{u^\alpha(\xi)}=e(\xi),\ \ \xi\in\mathbb{R}, \tag{1.5}$$

where $\varphi_p(u'(\xi)) = |u'(\xi)|^{p-2}u'(\xi)$, $p > 1$, $c \in \mathbb{R}$ is a constant.

It is well-known that periodic solutions of Eq.(1.5) correspond to periodic wave solutions of Eq.(1.1). Hence, in order to study the existence of periodic wave solutions for Eq.(1.1), it reduce to prove the existence of periodic solutions for Eq.(1.5).

2. Preliminaries

Define the space

$$H = W^{1,p}([0,T]) = \Big\{u : [0,T] \to \mathbb{R} | u \text{ is absolutely continuous}, u' \in L^p([0,T],\mathbb{R}) \text{ and } u(\xi) = u(\xi+T) \text{ for } \xi \in \mathbb{R}\Big\}$$

equipped with the norm

$$\|u\|_H = \left(\int_0^T |u'(\xi)|^p d\xi + \int_0^T |u(\xi)|^p d\xi\right)^{1/p}.$$

Note that H is a Banach space.

Throughout this paper, we let $q \in (1,+\infty)$ such that $1/p + 1/q = 1$.

Define

$$\|u\|_p = \left(\int_0^T |u(\xi)|^p d\xi\right)^{1/p} \text{ and } \|u\|_{L^\infty} = \sup_{\xi\in[0,T]} |u(\xi)|.$$

In order to study the periodic solutions for the Eq.(1.5), for any $\lambda \in (0,1)$, we consider the following modified problem:

$$\Big(\varphi_p(u'(\xi))\Big)' + W'(u(\xi)) + f_\lambda(u(\xi)) = e(\xi), \tag{2.1}$$

where $W \in C^1(\mathbb{R},\mathbb{R})$ and $W(u) = cu$, $f_\lambda : \mathbb{R} \to \mathbb{R}$ is defined by

$$f_\lambda(s) = \begin{cases} -\frac{1}{s^\alpha}, & s \geq \lambda \\ -\frac{1}{\lambda^\alpha}, & s < \lambda. \end{cases}$$

In the following, we introduce the following concept of a weak solution for problem (2.1).

Definition 2.1. *We say that a function $u \in H$ is a weak solution of problem* (2.1) *if*

$$\int_0^T \varphi_p(u'(\xi))v'(\xi)d\xi - \int_0^T W'(u(\xi))v(\xi)d\xi - \int_0^T f_\lambda(u(\xi))v(\xi)d\xi + \int_0^T e(\xi)v(\xi)d\xi = 0,$$

holds for any $v \in H$.

Let $F_\lambda \in C^1(\mathbb{R},\mathbb{R})$ be defined by

$$F_\lambda(s) = \int_1^s f_\lambda(\xi)d\xi,$$

and consider the functional

$$\varphi_\lambda : H \to \mathbb{R},$$

which is defined by

$$\varphi_\lambda(u) := \frac{1}{p}\int_0^T |u'(\xi)|^p d\xi - \int_0^T W(u(\xi))d\xi - \int_0^T F_\lambda(u(\xi))d\xi + \int_0^T e(\xi)u(\xi)d\xi. \tag{2.2}$$

Clearly, φ_λ is well defined on H, and is a continuously Gáteaux differentiable functional whose Gáteaux derivative is the functional $\varphi'_\lambda(u)$, given by

$$\varphi'_\lambda(u)v = \int_0^T \varphi_p(u'(\xi))v'(\xi)d\xi - \int_0^T W'(u(\xi))v(\xi)d\xi - \int_0^T f_\lambda(u(\xi))v(\xi)d\xi + \int_0^T e(\xi)v(\xi)d\xi \tag{2.3}$$

for any $u, v \in H$. Moreover, it is easy to verify that φ_λ is weakly lower semi-continuous.

Definition 2.2. *Suppose that H is a Banach space and $\varphi \in C^1(H,\mathbb{R})$. If any sequence $\{u_n\}_{n\in N} \subset H$ for which $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$ as $n \to +\infty$ possesses a convergent subsequence in H, we say that φ satisfies (PS) condition.*

The following theorem will be used to prove the main results in the next section.

Theorem 2.3. *[19] [Theorem 4.10] Let H be a Banach space and let $\varphi \in C^1(H,\mathbb{R})$. Assume that there exist $x_0, x_1 \in H$ and a bounded open neighbourhood Ω of x_0 such that $x_1 \in H \setminus \overline{\Omega}$ and*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{x\in\partial\Omega} \varphi(x).$$

Let

$$\Gamma = \Big\{h \in C([0,1],H) : h(0) = x_0,\ h(1) = x_1\Big\},$$

and

$$\widetilde{c} = \inf_{h\in\Gamma} \max_{s\in[0,1]} \varphi(h(s)).$$

If φ satisfies the (PS)-condition, then $\widetilde{c}$ is a critical value of φ and $\widetilde{c} > \max\{\varphi(x_0), \varphi(x_1)\}$.

3. Main results

Now, we state our main results.

Theorem 3.1. *Assume that $c < 0$ and the following condition holds:*

(A) $e \in L^1([0,T],\mathbb{R})$ is T-periodic and $\int_0^T \Big(e(\xi) - c\Big)d\xi < 0$.

Then Eq.(1.1) possesses at least one periodic wave solution.

Proof. **Step 1.** We verify that the functional φ_λ satisfies the Palais-Smale condition.
Let a sequence $\{u_n\}_{n\in N} \subset H$ satisfy that $\varphi_\lambda(u_n)$ is bounded and

$$\varphi'_\lambda(u_n) \to 0,$$

i.e., there exist a constant $c_1 > 0$ and a sequence $\{\epsilon_n\}_{n\in N} \subset \mathbb{R}^+$ with $\epsilon_n \to 0$ as $n \to +\infty$ such that, for all n,

$$\left|\int_0^T \Big[\frac{1}{p}|u'_n(\xi)|^p - W(u_n(\xi)) - F_\lambda(u_n(\xi)) + e(\xi)u_n(\xi)\Big]d\xi\right| \le c_1, \tag{3.1}$$

and, for every $v \in H$,

$$\left|\int_0^T \Big[\varphi_p(u'_n(\xi))v'(\xi) - W'(u_n(\xi))v(\xi) - f_\lambda(u_n(\xi))v(\xi) + e(\xi)v(\xi)\Big]d\xi\right| \le \epsilon_n \|v\|_H. \tag{3.2}$$

Now we show that $\{u_n\}$ is bounded in H.

Let $v(t) \equiv -1$ in (3.2), one has

$$\left|\int_0^T \Big[W'(u_n(\xi)) + f_\lambda(u_n(\xi)) - e(\xi)\Big]d\xi\right| \leq T^{1/p}\epsilon_n, \ \text{ for all } n.$$

It follows from (A) that

$$\left|\int_0^T f_\lambda(u_n(\xi))d\xi\right| \leq T^{1/p}\epsilon_n + |c|\int_0^T |u_n'(\xi)|d\xi + \int_0^T |e(\xi)|d\xi.$$

From the definition of f_λ, we can see that $f_\lambda(u_n(\xi)) < 0$ for any $\xi \in [0, T]$. Thus

$$\int_0^T |f_\lambda(u_n(\xi))|d\xi = \left|\int_0^T f_\lambda(u_n(\xi))d\xi\right| \leq T^{1/p}\epsilon_n + |c|\int_0^T |u'(\xi)|d\xi + \int_0^T |e(\xi)|d\xi.$$

Let

$$v(\xi) \equiv \varpi_n(\xi) := u_n(\xi) - \overline{u}_n$$

in (3.2), where $\overline{u}_n = \frac{1}{T}\int_0^T u_n(\xi)d\xi$. By Proposition 1.1 of [19] and the Hölder inequality, we have

$$\begin{aligned}
c_2\|\varpi_n\|_H &\geq \int_0^T \Big[\varphi_p(u_n'(\xi))\varpi_n'(\xi) - W'(u_n(\xi))\varpi_n(\xi) - f_\lambda(u_n(\xi))\varpi_n(\xi) + e(\xi)\varpi_n(\xi)\Big]d\xi \\
&= \int_0^T \Big[\varphi_p(\varpi_n'(\xi))\varpi_n'(\xi) - c\varpi_n'(\xi)\varpi_n(\xi) - f_\lambda(u_n(\xi))\varpi_n(\xi) + e(\xi)\varpi_n(\xi)\Big]d\xi \\
&= \int_0^T \Big[|\varpi_n'(\xi)|^p - f_\lambda(u_n(\xi))\varpi_n(\xi) + e(\xi)\varpi_n(\xi)\Big]d\xi \\
&\geq \|\varpi_n'\|_{L^p}^p - \left(T^{1/p}\epsilon_n + |c|\int_0^T |u'(\xi)|d\xi + \int_0^T |e(\xi)|d\xi\right)\|\varpi_n\|_{L^\infty} - \|e\|_{L^1}\|\varpi_n\|_{L^\infty} \\
&\geq \|\varpi_n'\|_{L^p}^p - \left(T^{1/p}\epsilon_n + |c|T^{\frac{p-1}{p}}\|\varpi'\|_p + \|e\|_{L^1}\right)\|\varpi_n\|_{L^\infty} - \|e\|_{L^1}\|\varpi_n\|_{L^\infty} \\
&= \|\varpi_n'\|_{L^p}^p - |c|T^{\frac{p-1}{p}}\|\varpi'\|_p\|\varpi_n\|_{L^\infty} - \left(T^{1/p}\epsilon_n + 2\|e\|_{L^1}\right)\|\varpi_n\|_{L^\infty} \\
&\geq \|\varpi_n'\|_{L^p}^p - c_3\|\varpi'\|_p\|\varpi_n\|_H - c_4\|\varpi_n\|_H,
\end{aligned}$$

where c_2, c_3 and c_4 are positive constants. Thus,

$$\|\varpi_n'\|_{L^p} \leq c_5\|\varpi_n\|_H.$$

Consequently, using the Wirtinger inequality, we have that there exists $c_6 > 0$ such that

$$\|u_n'\|_{L^p} \leq c_6. \tag{3.3}$$

Now, suppose that

$$\|u_n\|_H \to +\infty \ \text{ as } \ n \to +\infty.$$

Since (3.3) holds, we have, passing to a subsequence if necessary, that either

$$M_n := \max u_n \to +\infty \ \text{ as } \ n \to +\infty, \tag{3.4}$$

or

$$m_n := \min u_n \to -\infty \ \text{ as } \ n \to -\infty. \tag{3.5}$$

We shall discuss the following cases:

case 1. Assume that (3.4) holds. By (A) and the fact that $f_\lambda < 0$, we have

$$\begin{aligned}
&\int_0^T \Big[W(u_n(\xi)) + F_\lambda(u_n(\xi)) - e(\xi)u_n(\xi)\Big]d\xi \\
&= \int_0^T \Bigg[\Big(\int_1^{M_n} f_\lambda(s)ds - \int_{u_n(\xi)}^{M_n} f_\lambda(s)ds\Big) + cu_n(\xi) - e(\xi)u_n(\xi)\Bigg]d\xi \\
&= \int_0^T F_\lambda(M_n)d\xi - \int_0^T \Bigg[\int_{u_n(\xi)}^{M_n} \Big(f_\lambda(s) - e(\xi) + c\Big)ds\Bigg]d\xi - M_n\int_0^T \Big(e(\xi) - c\Big)d\xi \\
&\geq \int_0^T F_\lambda(M_n)d\xi - \max_{\xi\in[0,T]} |M_n - u_n(\xi)| \cdot \Big(\int_0^T |e(\xi)|d\xi + |c|T\Big) - M_n\int_0^T \Big(e(\xi) - c\Big)d\xi \\
&\geq F_\lambda(M_n)T - \|M_n - m_n\|_C \cdot \Big(\|e\|_{L^1} + |c|T\Big) - M_n\int_0^T \Big(e(\xi) - c\Big)d\xi \\
&= F_\lambda(M_n)T - \Big(\|e\|_{L^1} + |c|T\Big) \cdot \left|\int_{\bar{\xi}_n}^{\hat{\xi}_n} u_n'(\xi)d\xi\right| - M_n\int_0^T \Big(e(\xi) - c\Big)d\xi \\
&\geq F_\lambda(M_n)T - \Big(\|e\|_{L^1} + |c|T\Big) \cdot \int_0^T |u_n'(\xi)|d\xi - M_n\int_0^T \Big(e(\xi) - c\Big)d\xi,
\end{aligned}$$

where $u(\hat{\xi}_n) = M_n$ and $u(\bar{\xi}_n) = m_n$. Thus, by using Sobolev and Poincaré's inequalities, we can have

$$\begin{aligned}
-M_n\int_0^T \Big(e(\xi) - c\Big)d\xi &\leq \int_0^T \Big[W(u_n(\xi)) + F_\lambda(u_n(\xi)) - e(\xi)u_n(\xi)\Big]dt - F_\lambda(M_n)T \\
&\quad + T^{1/q}\Big(\|e\|_{L^1} + |c|T\Big)\|u_n'\|_{L^p} \\
&= \int_0^T \Big[W(u_n(\xi)) + F_\lambda(u_n(\xi)) - e(\xi)u_n(\xi)\Big]d\xi \\
&\quad + T^{1/q}\Big(\|e\|_{L^1} + |c|T\Big)\|u_n'\|_{L^p} - \frac{T}{\alpha - 1}\left(\frac{1}{M_n^{\alpha-1}} - 1\right).
\end{aligned}$$

Furthermore,

$$\frac{T}{\alpha - 1}\left(\frac{1}{M_n^{\alpha-1}} - 1\right) \to 0 \text{ as } n \to +\infty,$$

then from (3.1) and (3.3), we see that the right-hand side of the above inequality is bounded, which is a contradiction.

case 2. Assume that (3.5) holds, that is, $m_n \to -\infty$ as $n \to +\infty$. We replace M_n by m_n in the preceding arguments, and we also get a contradiction. So $\{u_n\}$ is bounded in H.

Since H is a reflexive Banach space, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ for simplicity, and $u \in H$ such that $u_n \rightharpoonup u$ in H; then, by the Sobolev embedding theorem, we get $u_n \to u$ in $C([0,T])$ and $u_n \to u$ in $L^p([0,T])$. Thus, we get

$$\begin{aligned}
&\Big(\varphi_\lambda'(u_n) - \varphi_\lambda'(u)\Big)(u_n - u) \to 0, \quad \text{as } n \to \infty, \\
&\int_0^T e(\xi)(u_n(\xi) - u(\xi))d\xi \to 0, \quad \text{as } n \to \infty, \\
&\int_0^T \Big(f_\lambda(u_n(\xi)) - f_\lambda(u(\xi))\Big)(u_n(\xi) - u(\xi))d\xi \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Then, by (2.3), we have

$$\begin{aligned}\Big(\varphi'_\lambda(u_n)-\varphi'_\lambda(u)\Big)(u_n-u) &= \int_0^T \Big(|u_n'(\xi)|^{p-2}u_n'(\xi)-|u'(\xi)|^{p-2}u'(\xi)\Big)(u_n'(\xi)-u'(\xi))d\xi \\ &\quad - \int_0^T c(u_n(\xi)-u(\xi))d\xi \\ &\quad - \int_0^T \Big(f_\lambda(u_n(\xi))-f_\lambda(u(\xi))\Big)(u_n(\xi)-u(\xi))d\xi \\ &\quad + \int_0^T e(\xi)(u_n(\xi)-u(\xi))d\xi.\end{aligned}$$

Then, we can have

$$\int_0^T \Big(|u_n'(\xi)|^{p-2}u_n'(\xi)-|u'(\xi)|^{p-2}u'(\xi)\Big)(u_n'(\xi)-u'(\xi))d\xi \to 0, \quad \text{as } n\to\infty.$$

Moreover, since $u_n \to u$ in $L^p([0,T])$, we have $\|u_n-u\|_H \to 0$ as $n\to\infty$, thus $\{u_n\}_{n\in\mathbb{N}}$ strongly converges to u in H. Therefore, φ_λ satisfies the Palais-Smale condition.

Step 2 Let

$$\Omega=\left\{u\in H \,\middle|\, \min_{\xi\in[0,T]} u(\xi)>1\right\},$$

and

$$\partial\Omega=\Big\{u\in H \Big| u(\xi)\geq 1 \text{ for all } \xi\in(0,T),\ \exists \xi_u\in(0,T): u(\xi_u)=1\Big\}.$$

We show that there exists $d>0$ such that $\inf\limits_{u\in\partial\Omega}\varphi_\lambda(u)\geq -d$ whenever $\lambda\in(0,1)$.

For any $u\in\partial\Omega$, there exists some $\xi_u\in(0,T)$ such that

$$\min_{\xi\in[0,T]} u(\xi)=u(\xi_u)=1.$$

By (2.2) and extending the functions by T-periodicity, we have

$$\begin{aligned}\varphi_\lambda(u) &= \int_{\xi_u}^{\xi_u+T}\left[\frac{1}{p}|u'(\xi)|^p - cu(\xi) - F_\lambda(u(\xi)) + e(\xi)u(\xi)\right]d\xi \\ &\geq \frac{1}{p}\int_{\xi_u}^{\xi_u+T}|u'(\xi)|^p + \int_{\xi_u}^{\xi_u+T}(e(\xi)-c)(u(\xi)-1)d\xi + \int_{\xi_u}^{\xi_u+T}(e(\xi)-c)d\xi \\ &\geq \frac{1}{p}\int_{\xi_u}^{\xi_u+T}|u'(\xi)|^p - (\|e\|_{L^1}+cT)\cdot\max_{\xi\in[0,T]}(u(\xi)-1) + (\|e\|_{L^1}-cT) \\ &= \frac{1}{p}\|u'\|_{L^p}^p - (\|e\|_{L^1}+cT)\cdot\int_{\xi_u}^{\check{\xi}_u}u'(\xi)d\xi + (\|e\|_{L^1}-cT) \\ &\geq \frac{1}{p}\|u'\|_{L^p}^p - (\|e\|_{L^1}+cT)\cdot\int_{\xi_u}^{\xi_u+T}u'(\xi)d\xi + (\|e\|_{L^1}-cT),\end{aligned}$$

where $\check{\xi}_u\in[0,T]$ and $\max\limits_{\xi\in[0,T]}u(\xi)=u(\check{\xi}_u)$. By applying the Hölder inequality, we get

$$\varphi_\lambda(u)\geq \frac{1}{p}\|u'\|_{L^p}^p - (\|e\|_{L^1}+cT)T^{1/q}\|u'\|_{L^p} + (\|e\|_{L^1}-cT),$$

which implies that

$$\varphi_\lambda(u) \to +\infty \text{ as } \|u'\|_{L^p} \to +\infty.$$

Since $\min_{\xi\in[0,T]} u(\xi) = 1$, we have

$$u(\xi) - 1 = \int_{\xi_u}^{\xi} u'(s)ds \le \int_0^T |u'(s)|ds \le T^{1/q}\left(\int_0^T |u'(s)|^p ds\right)^{1/p},$$

then we can see that $\|u(\cdot) - 1\|_H \to +\infty$ is equivalent to $\|u'\|_{L^p} \to +\infty$. Hence

$$\varphi_\lambda(u) \to +\infty \text{ as } \|u\|_H \to +\infty, \ \forall u \in \partial\Omega,$$

which shows that φ_λ is coercive. The weak lower semicontinuity of φ_λ yields

$$\inf_{u\in\partial\Omega} \varphi_\lambda(u) > -\infty.$$

It follows that there exists $d > 0$ such that $\inf_{u\in\partial\Omega} \varphi_\lambda(u) > -d$ for all $\lambda \in (0, 1)$.

Step 3. We prove that there exists $\lambda_0 \in (0, 1)$ with the property that, for every $\lambda \in (0, \lambda_0)$, any solution u of problem (2.1) satisfying $\varphi_\lambda(u) > -d$ is such that $\min_{\xi\in[0,T]} u(\xi) \ge \lambda_0$, and hence u is a solution of problem (1.5).

Assume on the contrary that there are sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ such that
(1) $\lambda_n \le \frac{1}{n}$;
(2) u_n is a solution of (2.1) with $\lambda = \lambda_n$;
(3) $\varphi_{\lambda_n}(u_n) \ge -d$;
(4) $\min_{\xi\in[0,T]} u_n(\xi) < \frac{1}{n}$.
Moreover, from (2.1), we can know that

$$\int_0^T \Big[cu_n'(\xi) + f_{\lambda_n}(u_n(\xi)) - e(\xi)\Big]d\xi = \int_0^T (\varphi_p(u_n'(\xi)))'d\xi = 0.$$

Since $f_{\lambda_n} < 0$, we have that

$$\|f_{\lambda_n}(u_n(\cdot))\|_{L^1} \le c_7, \text{ for some constant } c_7 > 0.$$

Hence

$$\|u_n'\|_{L^\infty} \le c_8, \text{ for some constant } c_8 > 0. \tag{3.6}$$

From $\varphi_{\lambda_n}(u_n) \ge -d$, we can see that there must exist two constants h_1 and h_2 with $0 < h_1 < h_2$, such that

$$\max\Big\{u_n(\xi) : \xi \in [0, T]\Big\} \subset [h_1, h_2].$$

If not, u_n would tend uniformly to 0. In both cases, by (A) and (3.6), we have

$$\varphi_{\lambda_n}(u_n) \to -\infty \text{ as } n \to +\infty,$$

which contradicts $\varphi_{\lambda_n}(u_n) \ge -d$.
Let τ_n^1, τ_n^2 be such that, for n large enough,

$$u_n(\tau_n^1) = \frac{1}{n} < h_1 = u_n(\tau_n^2).$$

Multiplying the the following equation

$$\Big(\varphi_p(u'(\xi))\Big)' + W'(u(\xi)) + f_\lambda(u(\xi)) = e(\xi)$$

by u_n' and integrating it on $[\tau_n^1, \tau_n^2]$, or on $[\tau_n^2, \tau_n^1]$, we can obtain

$$\begin{aligned}
\Psi &:= \int_{\tau_n^1}^{\tau_n^2} (\varphi_p(u_n'(\xi)))' u_n'(\xi) d\xi + \int_{\tau_n^1}^{\tau_n^2} c(u_n'(\xi))^2 d\xi + \int_{\tau_n^1}^{\tau_n^2} f_{\lambda_n}(u_n(\xi)) u_n'(\xi) d\xi \\
&= \int_{\tau_n^1}^{\tau_n^2} (\varphi_p(u_n'(\xi)))' d(u_n(\xi)) + \int_{\tau_n^1}^{\tau_n^2} c(u_n'(\xi))^2 d\xi + \int_{\tau_n^1}^{\tau_n^2} f_{\lambda_n}(u_n(\xi)) u_n'(\xi) d\xi \\
&= \int_{u_n(\tau_n^1)}^{u_n(\tau_n^2)} (\varphi_p(s'))' ds + \int_{\tau_n^1}^{\tau_n^2} c(u_n'(\xi))^2 d\xi + \int_{\tau_n^1}^{\tau_n^2} f_{\lambda_n}(u_n(\xi)) u_n'(\xi) d\xi \\
&= \int_{\tau_n^1}^{\tau_n^2} e(\xi) u_n'(\xi) d\xi.
\end{aligned} \tag{3.7}$$

It is easy to verify that

$$\Psi = \Psi_1 + \Psi_2 + \int_{u_n(\tau_n^1)}^{u_n(\tau_n^2)} (\varphi_p(s'))' ds,$$

where

$$\Psi_1 = \int_{\tau_n^1}^{\tau_n^2} c(u_n'(\xi))^2 d\xi,$$

and

$$\Psi_2 = \int_{\tau_n^1}^{\tau_n^2} f_{\lambda_n}(u_n(\xi)) u_n'(\xi) d\xi.$$

From (A), (3.6) and (3.7), it follows that Ψ is bounded, thus Ψ_2 is bounded.

On the other hand, it is easy to see that

$$f_{\lambda_n}(u_n(\xi)) u_n'(\xi) = \frac{d}{d\xi}\Big[F_{\lambda_n}(u_n(\xi))\Big].$$

Thus, we have

$$\Psi_2 = F_{\lambda_n}(h_1) - F_{\lambda_n}\left(\frac{1}{n}\right).$$

From the fact that $F_{\lambda_n}\left(\frac{1}{n}\right) \to +\infty$ as $n \to +\infty$, we obtain $\Psi_2 \to -\infty$, *i.e.*, Ψ_2 is unbounded. This is a contradiction.

Step 4. φ_λ has a mountain-pass geometry for $\lambda \le \lambda_0$.

Fixing $\lambda \in (0, \lambda_0]$, one has

$$F_\lambda(0) = \int_1^0 f_\lambda(s) ds = -\int_0^1 f_\lambda(s) ds = -\int_0^\lambda f_\lambda(s) ds - \int_\lambda^1 f_\lambda(s) ds = \frac{1}{\lambda^{\alpha-1}} - \int_\lambda^1 f_\lambda(s) ds,$$

which implies that

$$F_\lambda(0) > -\int_\lambda^1 f_\lambda(s) ds = \int_1^\lambda f_\lambda(s) ds = F_\lambda(\lambda).$$

Thus we have

$$\varphi_\lambda(0) = -TF_\lambda(0) < -TF_\lambda(\lambda) = \begin{cases} -\frac{T}{\alpha-1}\left(\frac{1}{\lambda^{\alpha-1}} - 1\right), & \alpha > 1, \\ T\ln\lambda, & \alpha = 1. \end{cases} \tag{3.8}$$

Consider $\lambda \in (0, \lambda_0] \cap (0, e^{-d}) \cap \left(0, \left(\frac{T}{T+d(\alpha-1)}\right)^{1/(\alpha-1)}\right)$, then it follows from (3.8) that $\varphi_\lambda(0) < -d$.

Also, using (A), we can choose a constant $R > 1$ large enough that

$$-\int_0^T \Big(e(t) - c\Big)dt \cdot R - \frac{T}{\alpha-1}\left(1 - \frac{1}{R^{\alpha-1}}\right) > d,\ \alpha > 1,$$

and

$$-\int_0^T \Big(e(t) - c\Big)dt \cdot R - T\ln R > d,\ \alpha = 1.$$

Then, by (2.2) and (3.8), we have

$$\varphi_\lambda(R) = -TcR - TF_\lambda(R) + R\int_0^T e(t)dt \le \int_0^T \Big(e(t) - c\Big)dt \cdot R + \frac{T}{\alpha-1}\left(1 - \frac{1}{R^{\alpha-1}}\right) < -d,$$

and

$$\varphi_\lambda(R) = -TcR - TF_\lambda(R) + R\int_0^T e(t)dt \le \int_0^T \Big(e(t) - c\Big)dt \cdot R + T\ln R < -d.$$

Thus the set Ω is a neighborhood of the constant function $\mathbb{R}$, $0 \notin \Omega$ and

$$\max\Big\{\varphi_\lambda(0), \varphi_\lambda(R)\Big\} < \inf_{x\in\partial\Omega} \varphi_\lambda(u).$$

Steps 1 and **Steps 2** imply that φ_λ has a critical point u_λ such that

$$\varphi_\lambda(u_\lambda) = \inf_{h\in\Gamma}\max_{s\in[0,1]} \varphi_\lambda(h(s)) \ge \inf_{x\in\partial\Omega} \varphi_\lambda(u),$$

where

$$\Gamma = \Big\{h \in C([0,1], H) : h(0) = 0, h(1) = R\Big\}.$$

Since $\inf_{u\in\partial\Omega} \varphi_\lambda(u_\lambda) \ge -d$, it follows from **Step 3** that u_λ is a solution of problem (1.1). Therefore, we can conclude that Eq.(1.1) has at least one periodic wave solution.

Theorem 3.2. *Assume that $c > 0$ and the following condition holds:*

(A) $e \in L^1([0,T], \mathbb{R})$ *is T-periodic and* $\int_0^T \Big(e(\xi) - c\Big)d\xi > 0$.

Then Eq.(1.1) *possesses at least one periodic wave solution.*

Proof. **Step 1.** We verify that the functional φ_λ satisfies the Palais-Smale condition.

Let a sequence $\{u_n\}_{n\in\mathbb{N}}$ in H satisfy that $\varphi_\lambda(u_n)$ is bounded and

$$\varphi'_\lambda(u_n) \to 0,$$

i.e., there exist a constant $\overline{c_1} > 0$ and a sequence $\{\epsilon_n\}_{n\in N} \subset \mathbb{R}^+$ with $\epsilon_n \to 0$ as $n \to +\infty$ such that, for all n,

$$\left|\int_0^T \Big[\frac{1}{p}|u_n'(\xi)|^p - W(u_n(\xi)) - F_\lambda(u_n(\xi)) + e(\xi)u_n(\xi)\Big]d\xi\right| \le \overline{c_1}, \tag{3.9}$$

and, for every $v \in H$,

$$\left|\int_0^T \Big[\varphi_p(u_n'(\xi))v'(\xi) - W'(u_n(\xi))v(\xi) - f_\lambda(u_n(\xi))v(\xi) + e(\xi)v(\xi)\Big]d\xi\right| \le \epsilon_n\|v\|_H. \tag{3.10}$$

Now we show that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in H.

Let $v(t) \equiv -1$ in (3.10), then we can have

$$\left|\int_0^T \Big[W'(u_n(\xi)) + f_\lambda(u_n(\xi)) - e(\xi)\Big]d\xi\right| \le T^{1/p}\epsilon_n, \text{ for all } n.$$

It follows from (A) that

$$\left|\int_0^T f_\lambda(u_n(\xi))d\xi\right| \le T^{1/p}\epsilon_n + c\int_0^T |u_n'(\xi)|d\xi + \int_0^T e(\xi)d\xi.$$

From the definition of f_λ, we can see that $f_\lambda < 0$ for any $\xi \in [0, T]$. Thus,

$$\int_0^T |f_\lambda(u_n(\xi))|d\xi = \left|\int_0^T f_\lambda(u_n(\xi))d\xi\right| \le T^{1/p}\epsilon_n + c\int_0^T |u'(\xi)|d\xi + \int_0^T e(\xi)d\xi.$$

Taking $v(\xi) \equiv \varpi_n(\xi) := u_n(\xi) - \overline{u}_n$ in (3.10), where $\overline{u}_n = \frac{1}{T}\int_0^T u_n(\xi)d\xi$. By Proposition 1.1 of [19] and the Hölder inequality, we have

$$\begin{aligned}
\overline{c_2}\|\varpi_n\|_H &\ge \int_0^T \Big[\varphi_p(u_n'(\xi))\varpi_n'(\xi) - W'(u_n(\xi))\varpi_n(\xi) - f_\lambda(u_n(\xi))\varpi_n(\xi) + e(\xi)\varpi_n(\xi)\Big]d\xi \\
&= \int_0^T \Big[\varphi_p(\varpi_n'(\xi))\varpi_n'(\xi) - c\varpi_n'(\xi)\varpi_n(\xi) - f_\lambda(u_n(\xi))\varpi_n(\xi) + e(\xi)\varpi_n(\xi)\Big]d\xi \\
&= \int_0^T \Big[|\varpi_n'(\xi)|^p - f_\lambda(u_n(\xi))\varpi_n(\xi) + e(\xi)\varpi_n(\xi)\Big]d\xi \\
&\ge \|\varpi_n'\|_{L^p}^p - \Big(T^{1/p}\epsilon_n + c\int_0^T |u'(\xi)|d\xi + \int_0^T e(\xi)d\xi\Big)\|\varpi_n\|_{L^\infty} - \|e\|_{L^1}\|\varpi_n\|_{L^\infty} \\
&\ge \|\varpi_n'\|_{L^p}^p - \Big(T^{1/p}\epsilon_n + cT^{\frac{p-1}{p}}\|\varpi'\|_p + \|e\|_{L^1}\Big)\|\varpi_n\|_{L^\infty} - \|e\|_{L^1}\|\varpi_n\|_{L^\infty} \\
&= \|\varpi_n'\|_{L^p}^p - cT^{\frac{p-1}{p}}\|\varpi'\|_p\|\varpi_n\|_{L^\infty} - \Big(T^{1/p}\epsilon_n + 2\|e\|_{L^1}\Big)\|\varpi_n\|_{L^\infty} \\
&\ge \|\varpi_n'\|_{L^p}^p - \overline{c_3}\|\varpi'\|_p\|\varpi_n\|_H - \overline{c_4}\|\varpi_n\|_H,
\end{aligned}$$

where $\overline{c_2}$, $\overline{c_3}$ and $\overline{c_4}$ are positive constants. Thus,

$$\|\varpi_n'\|_{L^p} \le \overline{c_5}\|\varpi_n\|_H.$$

Consequently, using the Wirtinger inequality, we get the existence of $\overline{c_6} > 0$ such that

$$\|u_n'\|_{L^p} \le \overline{c_6}. \tag{3.11}$$

Now, suppose that

$$\|u_n\|_H \to +\infty \ \text{as} \ n \to +\infty.$$

Since (3.11) holds, we have, passing to a subsequence if necessary, that either

$$M_n := \max u_n \to +\infty \ \text{as} \ n \to +\infty, \tag{3.12}$$

or

$$m_n := \min u_n \to -\infty \ \text{as} \ n \to -\infty. \tag{3.13}$$

(i) Assume that (3.12) occurs. By (A) and $f_\lambda < 0$, we can have

$$\begin{aligned}
&\int_0^T \Big[W(u_n(\xi)) + F_\lambda(u_n(\xi)) - e(\xi)u_n(\xi)\Big]d\xi \\
&= \int_0^T \Bigg[\Big(\int_1^{M_n} f_\lambda(s)ds - \int_{u_n(\xi)}^{M_n} f_\lambda(s)ds\Big) + cu_n(\xi) - e(\xi)u_n(\xi)\Bigg]d\xi \\
&= \int_0^T F_\lambda(M_n)d\xi - \int_0^T \Bigg[\int_{u_n(\xi)}^{M_n}\Big(f_\lambda(s) - e(\xi) + c\Big)ds\Bigg]d\xi - M_n\int_0^T e(\xi)d\xi + cTM_n \\
&\geq \int_0^T F_\lambda(M_n)dt - \max_{\xi\in[0,T]}|M_n - u_n(\xi)|\Big(\int_0^T |e(\xi)|d\xi + cT\Big) - M_n\int_0^T \Big(e(\xi) - c\Big)d\xi \\
&\geq F_\lambda(M_n)T - \|M_n - m_n\|_C \cdot \Big(\|e\|_{L^1} + cT\Big) - M_n\int_0^T \Big(e(\xi) - c\Big)d\xi \\
&= F_\lambda(M_n)T - \Big(\|e\|_{L^1} + cT\Big)\left|\int_{\bar{\xi}_n}^{\hat{\xi}_n} u_n'(\xi)d\xi\right| - M_n\int_0^T \Big(e(\xi) - c\Big)d\xi \\
&\geq F_\lambda(M_n)T - \Big(\|e\|_{L^1} + cT\Big)\int_0^T |u_n'(\xi)|d\xi - M_n\int_0^T \Big(e(\xi) - c\Big)d\xi,
\end{aligned}$$

where $u(\hat{\xi}_n) = M_n$ and $u(\bar{\xi}_n) = m_n$. Thus, using Sobolev and Poincaré's inequalities,

$$\begin{aligned}
-M_n\int_0^T \Big(e(\xi) - c\Big)d\xi &\leq \int_0^T \Big[W(u_n(\xi)) + F_\lambda(u_n(\xi)) - e(\xi)u_n(\xi)\Big]dt - F_\lambda(M_n)T + T^{1/q}\Big(\|e\|_{L^1} + cT\Big)\|u_n'\|_{L^p} \\
&= \int_0^T \Big[W(u_n(\xi)) + F_\lambda(u_n(\xi)) - e(\xi)u_n(\xi)\Big]d\xi \\
&\quad + T^{1/q}\Big(\|e\|_{L^1} + c\Big)\|u_n'\|_{L^p} - \frac{T}{\alpha-1}\left(\frac{1}{M_n^{\alpha-1}} - 1\right)
\end{aligned}$$

From (3.9), (3.11) and the fact that $1/M_n^{\alpha-1} \to 0$ as $n \to +\infty$, we see that the right-hand side of the above inequality is bounded, which is a contradiction.

(ii) Assume that (3.13) occurs, that is, $m_n \to -\infty$ as $n \to +\infty$. We replace M_n by m_n in the preceding arguments, and we also get a contradiction. Then, by a similar argument in proof of Theorem 3.3, we can see that φ_λ satisfies the Palais-Smale condition.

Step 2 Let

$$\Omega = \left\{u \in H \Big| \min_{\xi\in[0,T]} u(\xi) > 1\right\},$$

and

$$\partial\Omega = \Big\{u \in H \Big| u(\xi) \geq 1 \text{ for all } \xi \in (0,T),\ \exists \xi_u \in (0,T) : u(\xi_u) = 1\Big\}.$$

We show that there exists $d > 0$ such that $\inf\limits_{u\in\partial\Omega} \varphi_\lambda(u) \geq -d$ whenever $\lambda \in (0,1)$.

For any $u \in \partial\Omega$, there exists some $\xi_u \in (0,T)$ such that

$$\min_{\xi\in[0,T]} u(\xi) = u(\xi_u) = 1.$$

By (2.2) and extending the functions by T-periodicity, combining with (A)′, we have

$$\begin{aligned}
\varphi_\lambda(u) &= \int_{\xi_u}^{\xi_u+T} \left[\frac{1}{p}|u'(\xi)|^p - cu(\xi) - F_\lambda(u(\xi)) + e(\xi)u(\xi)\right] d\xi \\
&\geq \frac{1}{p}\int_{\xi_u}^{\xi_u+T} |u'(\xi)|^p + \int_{\xi_u}^{\xi_u+T} (e(\xi)-c)(u(\xi)-1)d\xi + \int_{\xi_u}^{\xi_u+T} (e(\xi)-c)d\xi \\
&\geq \frac{1}{p}\int_{\xi_u}^{\xi_u+T} |u'(\xi)|^p + (\|e\|_{L^1} - cT)\cdot \max_{\xi\in[0,T]}(u(\xi)-1) \\
&= \frac{1}{p}\|u'\|_{L^p}^p + (\|e\|_{L^1} - cT)\cdot \int_{\xi_u}^{\check{\xi}_u} u'(\xi)d\xi \\
&\geq \frac{1}{p}\|u'\|_{L^p}^p + (\|e\|_{L^1} - cT)\cdot \int_{\xi_u}^{\xi_u+T} u'(\xi)d\xi,
\end{aligned}$$

where $\check{\xi}_u \in [0,T]$ and $\max\limits_{\xi\in[0,T]} u(\xi) = u(\check{\xi}_u)$. Applying the Hölder inequality, we get

$$\varphi_\lambda(u) \geq \frac{1}{p}\|u'\|_{L^p}^p - (\|e\|_{L^1} - cT)\cdot T^{\frac{p-1}{p}} \cdot \|u'\|_{L^p},$$

which implies that

$$\varphi_\lambda(u) \to +\infty \ \text{ as } \ \|u'\|_{L^p} \to +\infty.$$

Since $\min\limits_{\xi\in[0,T]} u(\xi) = 1$, we have

$$u(\xi) - 1 = \int_{\xi_u}^{\xi} u'(s)ds \leq \int_0^T |u'(s)|ds \leq T^{1/q}\left(\int_0^T |u'(s)|^p ds\right)^{1/p},$$

then we can see that $\|u(\cdot)-1\|_H \to +\infty$ is equivalent to $\|u'\|_{L^p} \to +\infty$. Hence

$$\varphi_\lambda(u) \to +\infty \ \text{ as } \ \|u\|_H \to +\infty,\ \forall u \in \partial\Omega,$$

which shows that φ_λ is coercive. The weak lower semicontinuity of φ_λ yields

$$\inf_{u\in\partial\Omega} \varphi_\lambda(u) > -\infty.$$

It follows that there exists $d > 0$ such that $\inf\limits_{u\in\partial\Omega} \varphi_\lambda(u) > -d$ for all $\lambda \in (0,1)$.

The rest of proof is similar to Theorem 3.3. Therefore, we can conclude that Eq.(1.1) has at least one periodic wave solution.

Theorem 3.3. *Assume that $c < 0$ and the following condition holds:*

(A) $e \in L^1([0,T],\mathbb{R})$ is T-periodic and $\int_0^T (e(\xi)-c)d\xi = 0$.

Then Eq.(1.1) possesses at least one periodic wave solution.

Proof. The proof is similar with that in Theorem 3.2. We omit it.

4. Examples and numerical simulations

In this section, we consider two numerical examples, with which the non-Newtonian filtration equation with nonlinear sources and a singularity, to show the effectiveness of the theoretical results given in the previous sections. We will see that it is not hard to verify the conditions stated in our main theorems.

Example 4.1. *We consider the following one-dimensional non-Newtonian filtration equation with nonlinear sources and a singularity as follows:*

$$\frac{\partial q}{\partial t} = \frac{\partial q}{\partial x}\left(|\frac{\partial q}{\partial x}|^{p-2}\frac{\partial q}{\partial x}\right) - \frac{1}{q^{\alpha}} + \frac{2}{T}\cos^2\left(x+\frac{t}{T}\right),\ x \in \mathbb{R},\ \ t \geq 0. \tag{4.1}$$

Eq.(4.1) *can be regarded as a problem of the form* (1.1), *where* $g(x,t) = -\frac{2}{T}\cos^2\left(x+\frac{t}{T}\right)$. *Let* $q(x,t) = u(\xi)$ *with* $\xi = x + \frac{t}{T}$, $g(t,x) = e(\xi) = e(x-\frac{t}{T})$, *then Eq.*(4.1) *is transformed into the following ordinary differential equation*

$$(\varphi_p(u'(\xi)))' + \frac{1}{T}u'(\xi) - \frac{1}{u^{\alpha}(\xi)} = -\frac{2}{T}\cos^2(\xi), \tag{4.2}$$

where $\varphi_p(u'(\xi)) = |u'(\xi)|^{p-2}u'(\xi)$. *Then, we can have* $e(\xi) = -\frac{2}{T}\cos^2(\xi)$, $c = \frac{1}{T}$, $T = \pi$. *Furthermore, by a simple calculation, we can see that* $\int_0^T \left(e(\xi) - c\right)d\xi < 0$. *Thus, Theorem 3.3 guarantees the existence of at least one periodic wave solutions for equation* (4.1). *Let Eq.*(4.2) *be rewritten equivalently to the following equations:*

$$\begin{cases} u'(\xi) = |v'(\xi)|^{q-2}v'(\xi), \\ v'(\xi) = -\frac{1}{T}u'(\xi) + \frac{1}{u^{\alpha}(\xi)} - \frac{2}{T}\cos^2(\xi), \end{cases}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. *Without loss of generality, let* $p = \frac{3}{2}$ *and* $\alpha = 3$, *then* (4.1) *possesses at least one* π*-periodic wave solutions. This fact can be presented in the following Figures 1.*

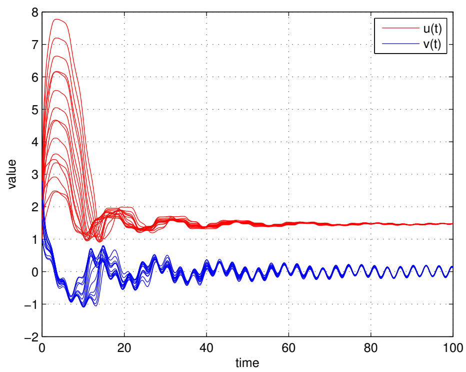

Figure 1: Time-domain behavior of the state variables u and v of system (4.1).

Example 4.2. *We consider the following one-dimensional non-Newtonian filtration equation with nonlinear sources and a singularity as follows:*

$$\frac{\partial q}{\partial t} = \frac{\partial q}{\partial x}\left(|\frac{\partial q}{\partial x}|^{p-2}\frac{\partial q}{\partial x}\right) - \frac{1}{q^{\alpha}} - \frac{1}{T}\sin^2\left(x-\frac{2t}{T}\right) - 3,\ x \in \mathbb{R},\ \ t \geq 0. \tag{4.3}$$

Eq.(4.3) *can be regarded as a problem of the form* (1.1), *where*

$$g(x,t) = \frac{1}{T}\sin^2\left(x-\frac{2t}{T}\right) + 3.$$

Let $q(x,t) = u(\xi)$ *with* $\xi = x - \frac{2t}{T}$, $g(t,x) = e(\xi) = e(x - \frac{2t}{T})$, *then Eq.*(4.3) *is transformed into the following ordinary differential equation*

$$(\varphi_p(u'(\xi)))' + \frac{2}{T}u'(\xi) - \frac{1}{u^\alpha(\xi)} = \frac{1}{T}\sin^2(\xi) - 3, \tag{4.4}$$

where $\varphi_p(u'(\xi)) = |u'(\xi)|^{p-2}u'(\xi)$. *Then, we can have* $e(\xi) = \frac{1}{T}\sin^2(\xi) - 3$, $c = \frac{2}{T}$, $T = \pi$. *Furthermore, by a simple calculation, we can see that* $\int_0^T \Big(e(\xi) - c\Big)d\xi > 0$. *Thus, Theorem 3.2 guarantees the existence of at least one periodic wave solutions for Eq.*(4.3). *Let Eq.*(4.4) *be rewritten equivalently to the following equations:*

$$\begin{cases} u'(\xi) = |v'(\xi)|^{q-2}v'(\xi), \\ v'(\xi) = -\frac{2}{T}u'(\xi) + \frac{1}{u^\alpha(\xi)} + \frac{1}{T}\sin^2(\xi) - 3, \end{cases}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. *Without loss of generality, let* $p = \frac{3}{2}$ *and* $\alpha = 3$, *then* (4.3) *possesses at least one* π*-periodic wave solutions. This fact can be presented in the following Figures 2.*

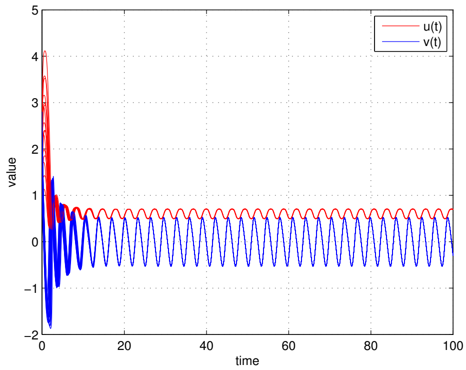

Figure 2: Time-domain behavior of the state variables u and v of system (4.3).

5. Conclusion

In this paper, the existence of periodic wave solutions for non-Newtonian filtration equation with nonlinear sources and singularities is discussed. By using variational methods and mountain pass theorem, we obtain the existence of at least one periodic wave solution for the considered equation. The novelties of this paper is that it is the first time to discuss the existence of periodic wave solutions for the singular non-Newtonian filtration equations by using variational methods and mountain pass theorem. Our results enrich and extend some corresponding results in the literature.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the manuscript.

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