Filomat 35:7 (2021), 2257–2266
https://doi.org/10.2298/FIL2107257A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A New Result of Prešić Type Theorems with Applications to Second Order Boundary Value Problems

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Abstract. In this paper, taking into account the recent contractive technique we present a new result of Prešić type fixed point theorems. Then, we provide a comparative example to put forth the validity of our theoretical result. Finally, considering a special case of the main theorem, we give some existence results for the second order two point boundary value problems.

1. Introduction and preliminaries

There is absolutely no doubt that Banach contraction principal which was introduced in 1922 [3] is a fundamental and powerful result to ensure the existence of solution in linear, non-linear, ordinary differential, partial differential, integral and difference equations. This principal has been extended by many researchers in several different ways over last few decades.

In 1965, Prešić [8] generalized Banach contraction mapping principle as follows:

Theorem 1.1. *Let (M, d) be a complete metric space, k be any positive integer, and let $T : M^k \to M$ be a mapping satisfying the following contraction condition: for all $\zeta_1, \zeta_2, \cdots, \zeta_{k+1} \in M$,*

$$d(T(\zeta_1, \zeta_2, \cdots, \zeta_k), T(\zeta_2, \zeta_3, \cdots, \zeta_{k+1})) \leq q_1 d(\zeta_1, \zeta_2) + q_2 d(\zeta_2, \zeta_3) + \cdots + q_k d(\zeta_k, \zeta_{k+1}), \tag{1}$$

where $q_1, q_2, \cdots, q_k$ are positive constants such that $q_1 + q_2 + \cdots + q_k < 1$. Then there exists a unique point $\zeta \in M$ such that $\zeta = T(\zeta, \zeta, \cdots, \zeta)$. Moreover, if $\zeta_1, \zeta_2, \cdots, \zeta_k$ are arbitrary points in M for $n \in \mathbb{N}$,

$$\zeta_{n+k} = T(\zeta_n, \zeta_{n+1}, \cdots, \zeta_{n+k-1}),$$

then the sequence $\{\zeta_n\}$ is convergent and $\lim \zeta_n = T(\lim \zeta_n, \lim \zeta_n, \cdots, \lim \zeta_n)$.

Note that for $k = 1$, Theorem 1.1 reduces to Banach contraction principle.

Later on, in 2007, Ćirić and Prešić [4] further generalized Prešić type contraction for complete metric space which is stated as follows:

2010 *Mathematics Subject Classification*. Primary 54H25 ; Secondary 47H10
Keywords. Fixed point, Prešić contraction, θ-contraction, boundary value problem.
Received: 03 June 2020; Accepted: 18 June 2020
Communicated by Vladimir Rakočević
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Theorem 1.2. *Let (M, d) be a complete metric space, k be any positive integer, and let $T : M^k \to M$ be a mapping satisfying the following contraction condition: for all $\zeta_1, \zeta_2, \cdots, \zeta_{k+1} \in M$,*

$$d(T(\zeta_1, \zeta_2, ..., \zeta_k), T(\zeta_2, \zeta_3, \cdots, \zeta_{k+1})) \leq \lambda \max\{d(\zeta_i, \zeta_{i+1}) : 1 \leq i \leq k\}, \tag{2}$$

where $\lambda \in (0, 1)$. Then there exists a point $\zeta \in M$ such that $\zeta = T(\zeta, \zeta, \cdots, \zeta)$. Moreover, if $\zeta_1, \zeta_2, \cdots, \zeta_k$ are arbitrary points in M for $n \in \mathbb{N}$,

$$\zeta_{n+k} = T(\zeta_n, \zeta_{n+1}, \cdots, \zeta_{n+k-1}),$$

then the sequence $\{\zeta_n\}$ is convergent and $\lim \zeta_n = T(\lim \zeta_n, \lim \zeta_n, \cdots, \lim \zeta_n)$. In addition, if for all $\upsilon, \nu \in M$ with $\upsilon \neq \nu$, the condition

$$d(T(\upsilon, \upsilon, ..., \upsilon), T(\nu, \nu, \cdots, \nu)) < d(\upsilon, \nu)$$

holds, then ζ is the unique point in M such that $\zeta = T(\zeta, \zeta, \cdots, \zeta)$.

Some important applications of above stated result such as studying asymptotic stability of the equilibrium for non-linear difference equation and global attractivity of matrix difference equations can be found in [1, 5].

On the other hand, Jleli and Samet [7] introduced a new class of contraction named as θ-contraction which generalizes the Banach contraction. Let $\theta : (0, \infty) \to (1, \infty)$ be a function. We will consider the following properties for θ:

(θ_1) θ is nondecreasing;

(θ_2) for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ and $\lim_{n\to\infty} t_n = 0^+$ are equivalent;

(θ_3) there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$;

We denote by Θ the set of all function satisfying (θ_1- θ_3). Some examples of the functions belonging Θ are $\theta_1(t) = e^{\sqrt{t}}$ and $\theta_2(t) = e^{\sqrt{te^t}}$.

Jleli and Samet [7] obtained the following result by considering the class Θ.

Theorem 1.3. *Let (M, d) be a complete metric space and $T : M \to M$ be a mapping. Suppose that there exist $\theta \in \Theta$ and $\lambda \in (0, 1)$ such that for all $\zeta, \eta \in M$, $d(T\zeta, T\eta) > 0$ implies that*

$$\theta(d(T\zeta, T\eta)) \leq [\theta(d(\zeta, \eta))]^\lambda.$$

Then T has a unique fixed point in M.

The organization of this paper as follows: In section 2, we have put forth the conditions on θ to obtain the θ version of the Ćirić-Prešić type contractive inequality given in (2). Hence, we introduced the concept of Ćirić-Prešić type θ-contraction and then presented a generalization of Theorem 1.2. Also, we provided a suitable example for our main result to show the validity and to compare with some previous results. In Section 3, considering a special case of the main theorem, we gave some existence results for the second order two point boundary value problems.

2. Main result

We will denote the class of all functions belonging to Θ satisfying the following property by Θ^*:

(θ_4) $\theta(t) \leq \left[\theta\left(\frac{t}{\beta}\right)\right]^{\sqrt{\beta}}$ for all $\beta \in (0, 1)$ and $t > 0$.

For example, if we consider the function $\theta \in \Theta$ defined by $\theta(t) = e^{\sqrt{te^t}}$, then for all $\beta \in (0, 1)$ and $t > 0$ we have

$$\left[\theta\left(\frac{t}{\beta}\right)\right]^{\sqrt{\beta}} = \left[e^{\sqrt{\frac{t}{\beta}e^{\frac{t}{\beta}}}}\right]^{\sqrt{\beta}} = e^{\sqrt{te^{\frac{t}{\beta}}}} \geq e^{\sqrt{te^t}} = \theta(t).$$

Therefore $\theta \in \Theta^*$. Similarly, we can see that the function θ defined by $\theta(t) = e^{\sqrt{t}}$ belongs to Θ^*.

Definition 2.1. *Let (M, d) be a metric space, k be any positive integer and $\theta \in \Theta$. A mapping $T : M^k \to M$ is called Ćirić-Prešić type θ-contraction if there exists $\lambda \in (0, 1)$ such that for all $\zeta_1, \zeta_2, \cdots, \zeta_k, \zeta_{k+1} \in M$, $d(T(\zeta_1, \zeta_2, \cdots, \zeta_k), T(\zeta_2, \zeta_3, \cdots, \zeta_{k+1})) > 0$ implies that*

$$\theta(d(T(\zeta_1, \zeta_2, \cdots, \zeta_k), T(\zeta_2, \zeta_3, \cdots, \zeta_{k+1}))) \leq \theta(\max\{d(\zeta_i, \zeta_{i+1}) : i \in \{1, 2, \cdots, k\}\})^{\sqrt{\lambda}}. \tag{3}$$

If we consider $\theta(t) = e^{\sqrt{t}}$, then the inequality (3) turns to (2).

Here, we present our main theorem.

Theorem 2.2. *Let (M, d) be a complete metric space, k a positive integer and $T : M^k \to M$ be a mapping satisfying Ćirić-Prešić type θ-contraction condition with $\theta \in \Theta^*$. Then there exists a point $\zeta \in M$ such that $\zeta = T(\zeta, \zeta, \cdots, \zeta)$. Moreover, if $\zeta_1, \zeta_2, \cdots, \zeta_k$ are arbitrary points in M, for $n \in \mathbb{N}$*

$$\zeta_{n+k} = T(\zeta_n, \zeta_{n+1}, \cdots, \zeta_{n+k-1}),$$

then the sequence $\{\zeta_n\}$ converges to ζ. In addition, if for all $\upsilon, \nu \in M$ with $\upsilon \neq \nu$, the condition

$$d(T(\upsilon, \upsilon, ..., \upsilon), T(\nu, \nu, \cdots, \nu)) < d(\upsilon, \nu) \tag{4}$$

holds, then ζ is the unique point in M such that $\zeta = T(\zeta, \zeta, \cdots, \zeta)$.

Proof. Let $\zeta_1, \zeta_2, \cdots, \zeta_k$ be arbitrary points in M. Define a sequence $\{\zeta_n\}$ by using these points as follows

$$\zeta_{n+k} = T(\zeta_n, \zeta_{n+1}, \cdots, \zeta_{n+k-1}).$$

Let $d_n = d(\zeta_n, \zeta_{n+1})$ for simplicity. We will prove

$$\theta(d_n) \leq [\theta(\kappa)]^{\sqrt[2k]{\lambda^n}} \tag{5}$$

for all $n \in \mathbb{N}$, where

$$\kappa = \max\left\{d_i \lambda^{-\frac{i}{k}} : i \in \{1, 2, \cdots, k\}\right\}.$$

First let $i \in \{1, 2, \cdots, k\}$ then by (θ_4) we have

$$\theta(d_i) \leq \left[\theta\left(d_i \lambda^{-\frac{i}{k}}\right)\right]^{\sqrt[2k]{\lambda^i}} \leq [\theta(\kappa)]^{\sqrt[2k]{\lambda^i}}.$$

Therefore the inequality (5) is satisfied for $n = 1, 2, \cdots, k$. Let the inequalities

$$\theta(d_{n+i-1}) \leq [\theta(\kappa)]^{\sqrt[2k]{\lambda^{n+i-1}}}$$

holds for $i \in \{1, 2, \cdots, k\}$ be the induction hypotheses. Then we have

$$\begin{aligned}
\theta(d_{n+k}) &= \theta(d(\zeta_{n+k}, \zeta_{n+k+1})) \\
&= \theta(d(T(\zeta_n, \zeta_{n+1}, \cdots, \zeta_{n+k-1}), T(\zeta_{n+1}, \zeta_{n+2}, \cdots, \zeta_{n+k}))) \\
&\leq \theta(\max\{d(\zeta_i, \zeta_{i+1}) : i \in \{n, n+1, \cdots, n+k-1\}\})^{\sqrt{\lambda}} \\
&= \theta(\max\{d_i : i \in \{n, n+1, \cdots, n+k-1\}\})^{\sqrt{\lambda}} \\
&= \theta(\max\{d_{n+i-1} : i \in \{1, 2, \cdots, k\}\})^{\sqrt{\lambda}} \\
&= (\max\{\theta(d_{n+i-1}) : i \in \{1, 2, \cdots, k\}\})^{\sqrt{\lambda}} \\
&\leq \left(\max\left\{[\theta(\kappa)]^{\sqrt[2k]{\lambda^{n+i-1}}} : i \in \{1, 2, \cdots, k\}\right\}\right)^{\sqrt{\lambda}} \\
&\leq \left([\theta(\kappa)]^{\sqrt[2k]{\lambda^n}}\right)^{\sqrt{\lambda}} \\
&= [\theta(\kappa)]^{\sqrt[2k]{\lambda^{n+k}}}.
\end{aligned}$$

Therefore, the inequality (5) is true for all $n \in \mathbb{N}$. Now, by our claim, for any k, we have

$$\theta(d_{k+1}) \leq [\theta(\kappa)]^{\sqrt[2k]{\lambda^{k+1}}}$$

and

$$\theta(d_{k+2}) \leq [\theta(\kappa)]^{\sqrt[2k]{\lambda^{k+2}}}$$

and so on. Hence, for all $n \in \mathbb{N}$ we have

$$\theta(d_{k+n}) \leq [\theta(\kappa)]^{\sqrt[2k]{\lambda^{k+n}}}. \tag{6}$$

By taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \theta(d_{k+n}) = 1$$

which follows by (θ_2)

$$\lim_{n\to\infty} d_{k+n} = 0.$$

Now by (θ_3), there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that

$$\lim_{n\to 0^+} \frac{\theta(d_{n+k}) - 1}{[d_{n+k}]^r} = l.$$

Suppose $l < \infty$, and $B = \frac{l}{2} > 0$. By definition of limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta(d_{k+n}) - 1}{[d_{k+n}]^r} - l \right| \leq B, \forall n \geq n_0.$$

This implies that, for all $n \geq n_0$,

$$\frac{\theta(d_{k+n}) - 1}{[d_{k+n}]^r} \geq l - B = B.$$

Then, for all $n \geq n_0$,

$$Bn\,[d_{k+n}]^r \leq n\,[\theta(d_{k+n}) - 1].$$

Now, let $l = \infty$ and B be any arbitrary positive number. By definition of limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n > n_0$

$$\frac{\theta(d_{k+n}) - 1}{[d_{k+n}]^r} \geq B.$$

It follows that for all $n \geq n_0$,

$$Bn\,[d_{k+n}]^r \leq n\,[\theta(d_{k+n}) - 1].$$

Considering these two cases and inequality (6), we get

$$n\,[d_{k+n}]^r \leq \frac{n}{B}\,[\theta(d_{k+n}) - 1] \leq \frac{n}{B}\left[\left[[\theta(\kappa)]^{\frac{k}{2k}}\right]^{\frac{n}{2k}} - 1\right] \tag{7}$$

for some $B > 0$. Letting $n \to \infty$ in (7), we get

$$\lim_{n\to\infty} n[d_{k+n}]^r = 0. \tag{8}$$

Thus from (8), there exists $n_0 \in \mathbb{N}$ such that $n[d_{k+n}]^r \leq 1$ for all $n \geq n_0$ and consequently, we get

$$d_{k+n} \leq \frac{1}{n^{1/r}} \text{ for all } n \geq n_0. \tag{9}$$

In order to show that $\{\zeta_n\}$ is a Cauchy sequence, consider $m \geq n \geq n_0$, we have

$$\begin{aligned} d(\zeta_{k+n}, \zeta_{k+m}) &= d(T(\zeta_n, \zeta_{n+1}, \cdots, \zeta_{n+k-1}), T(\zeta_m, \zeta_{m+1}, \cdots, \zeta_{m+k-1})) \\ &\leq d(T(\zeta_n, \cdots, \zeta_{n+k-1}), T(\zeta_{n+1}, \cdots, \zeta_{n+k})) \\ &\quad + d(T(\zeta_{n+1}, \cdots, \zeta_{n+k}), T(\zeta_{n+2}, \cdots, \zeta_{n+k+1})) \\ &\quad + \cdots + d(T(\zeta_{m-1}, \cdots, \zeta_{m+k-2}), T(\zeta_m, \cdots, \zeta_{m+k-1})) \\ &= d(\zeta_{n+k}, \zeta_{n+k+1}) + d(\zeta_{n+k+1}, \zeta_{n+k+2}) + \cdots + d(\zeta_{m+k-1}, \zeta_{m+k}) \\ &= d_{n+k} + d_{n+k+1} + \cdots + d_{m+k-1} \\ &\leq \sum_{i=n}^{\infty} d_{i+k} \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore $\{\zeta_n\}$ is a Cauchy sequence in (M, d). Since (M, d) is a complete metric space, there exists $\upsilon \in M$ such that

$$\lim_{m,n\to\infty} d(\zeta_n, \zeta_m) = \lim_{n\to\infty} d(\zeta_n, \upsilon) = 0.$$

On other hand, by (θ_1) and inequality (3), we get

$$d(T(\zeta_1, \zeta_2, \cdots, \zeta_k), T(\zeta_2, \zeta_3, \cdots, \zeta_{k+1})) \leq \max\{d(\zeta_i, \zeta_{i+1}) : i \in \{1, 2, \cdots, k\}\}$$

for all $\zeta, \eta \in M$. Therefore, we have

$$\begin{aligned} d(\upsilon, T(\upsilon, \upsilon, \cdots, \upsilon)) &\leq d(\upsilon, \zeta_{n+k}) + d(T(\upsilon, \upsilon, \cdots, \upsilon), \zeta_{n+k}) \\ &\leq d(\upsilon, \zeta_{n+k}) + d(T(\upsilon, \upsilon, \cdots, \upsilon), T(\zeta_n, \zeta_{n+1}, \cdots, \zeta_{n+k-1})) \\ &\leq d(\upsilon, \zeta_{n+k}) + d(T(\upsilon, \upsilon, \cdots, \upsilon), T(\upsilon, \upsilon, \cdots, \upsilon, \zeta_n)) \\ &\quad + d(T(\upsilon, \upsilon, \cdots, \upsilon, \zeta_n), T(\upsilon, \upsilon, \cdots, \zeta_n, \zeta_{n+1})) + \cdots \\ &\quad + d(T(\upsilon, \zeta_n, \cdots, \zeta_{n+k-2}), T(\zeta_n, \zeta_{n+1}, \cdots, \zeta_{n+k-1})) \\ &\leq d(\upsilon, \zeta_{n+k}) + d(\upsilon, \zeta_n) + \max\{d(\upsilon, \zeta_n), d(\zeta_n, \zeta_{n+1})\} + \cdots \\ &\quad + \max\{d(\upsilon, \zeta_n), d(\zeta_n, \zeta_{n+1}), \cdots, d(\zeta_{n+k-2}, \zeta_{n+k-1})\}. \end{aligned}$$

Taking limit as $n \to \infty$, we have $d(\upsilon, T(\upsilon, \upsilon, \cdots, \upsilon)) = 0$ and hence $\upsilon = T(\upsilon, \upsilon, \cdots, \upsilon)$. The uniqueness follows by the condition (4). □

Remark 2.3. *Note that for $k = 1$, Theorem 2.2 reduces to Theorem 1.3. Also, considering $\theta(t) = e^{\sqrt{t}}$, we can see that Theorem 1.2 is a special case of Theorem 2.2.*

Further, we obtain the following Corollaries from Theorem 2.2.

Corollary 2.4. *Let (M, d) be a complete metric space and $T : M^2 \to M$ be a mapping satisfying Ćirić-Prešić type θ-contraction condition with $\theta \in \Theta^*$. Then there exists a point $\zeta \in M$ such that $\zeta = T(\zeta, \zeta)$. Moreover, if $\zeta_1, \zeta_2, \zeta_3$ are arbitrary points in M, for $n \in \mathbb{N}$*

$$\zeta_{n+2} = T(\zeta_n, \zeta_{n+1}),$$

then the sequence $\{\zeta_n\}$ converges to ζ. In addition, if for all $\upsilon, \nu \in M$ with $\upsilon \neq \nu$, the condition

$$d(T(\upsilon, \upsilon), T(\nu, \nu)) < d(\upsilon, \nu)$$

holds, then ζ is the unique point in M such that $\zeta = T(\zeta, \zeta)$.

Corollary 2.5. *Let (M, d) be a complete metric space and $T : M^2 \to M$ be a mapping. Suppose there exists $\lambda \in (0, 1)$ such that*

$$\frac{d(T(\zeta, \eta), T(\eta, \xi))}{\max\{d(\zeta, \eta), d(\eta, \xi)\}} \exp\{d(T(\zeta, \eta), T(\eta, \xi)) - \max\{d(\zeta, \eta), d(\eta, \xi)\}\} \leq \lambda$$

for all $\zeta, \eta, \xi \in M$ with $d(T(\zeta, \eta), T(\eta, \xi)) > 0$. Then there exists a point $\zeta \in M$ such that $\zeta = T(\zeta, \zeta)$. Moreover, if $\zeta_1, \zeta_2, \zeta_3$ are arbitrary points in M, for $n \in \mathbb{N}$

$$\zeta_{n+2} = T(\zeta_n, \zeta_{n+1}),$$

then the sequence $\{\zeta_n\}$ converges to ζ. In addition, if for all $\upsilon, \nu \in M$ with $\upsilon \neq \nu$, the condition

$$d(T(\upsilon, \upsilon), T(\nu, \nu)) < d(\upsilon, \nu)$$

holds, then ζ is the unique point in M such that $\zeta = T(\zeta, \zeta)$.

Proof. It is enough to take $k = 2$ and $\theta(t) = e^{\sqrt{te^t}}$ in Theorem 2.2. □

Corollary 2.6. *Let (M, d) be a complete metric space and $T : M^2 \to M$ be a mapping. Suppose there exists $\lambda \in (0, 1)$ such that*

$$d(T(\zeta, \eta), T(\eta, \xi)) \leq \lambda \max\{d(\zeta, \eta), d(\eta, \xi)\}$$

for all $\zeta, \eta, \xi \in M$. Then there exists a point $\zeta \in M$ such that $\zeta = T(\zeta, \zeta)$. Moreover, if $\zeta_1, \zeta_2, \zeta_3$ are arbitrary points in M, for $n \in \mathbb{N}$

$$\zeta_{n+2} = T(\zeta_n, \zeta_{n+1}),$$

then the sequence $\{\zeta_n\}$ converges to ζ. In addition, if for all $\upsilon, \nu \in M$ with $\upsilon \neq \nu$, the condition

$$d(T(\upsilon, \upsilon), T(\nu, \nu)) < d(\upsilon, \nu)$$

holds, then ζ is the unique point in M such that $\zeta = T(\zeta, \zeta)$.

Now, we provide an easy but effective example to show the validity of our results.

Example 2.7. *Let $M = \mathbb{N}$ be endowed with the metric d defined by $d(\zeta, \zeta) = 0$ and $d(\zeta, \eta) = \zeta + \eta$ for $\zeta \neq \eta$. It is easy to see that (M, d) is a complete metric space. Define $T : M^2 \to M$ by*

$$T(\zeta, \eta) = \begin{cases} 0, & \zeta = \eta \\ \max\{\zeta, \eta\} - 1, & \zeta \neq \eta \end{cases}.$$

Then, for $\xi > 1$, we have

$$d(T(0, 1), T(1, \xi)) = d(0, \xi - 1) = \xi - 1$$

and

$$\max\{d(0, 1), d(1, \xi)\} = \xi + 1.$$

Therefore, since $\sup_{\xi \in M} \frac{\xi - 1}{\xi + 1} = 1$, then there isn't any $\lambda \in (0, 1)$ such that

$$d(T(\zeta, \eta), T(\eta, \xi)) \leq \lambda \max\{d(\zeta, \eta), d(\eta, \xi)\}$$

for all $\zeta, \eta, \xi \in M$. Thus, T is not a Prešić type contraction. So neither Theorem 1.1 nor Theorem 1.2 can be applied to this example.

Now we claim that T is Ćirić-Prešić type θ-contraction with $\theta(t) = e^{\sqrt{te^t}}$ *and* $\lambda = \exp(-1)$*. To see the inequality (3), we have to show that*

$$\frac{d(T(\zeta,\eta),T(\eta,\xi))}{\max\{d(\zeta,\eta),d(\eta,\xi)\}}\exp\{d(T(\zeta,\eta),T(\eta,\xi))-\max\{d(\zeta,\eta),d(\eta,\xi)\}\}\leq\exp(-1) \tag{10}$$

for all $\zeta,\eta,\xi\in M$ *with* $d(T(\zeta,\eta),T(\eta,\xi))>0$*. For the simplicity, we will denote the left side of inequality (10) by* $E(\zeta,\eta,\xi)$*. By taking into account* $d(T(\zeta,\eta),T(\eta,\xi))>0$*, we have the following cases and without loss of generality we will assume* $\zeta\leq\eta\leq\xi$ *in these cases:*

Case 1. If $T(\zeta,\eta)=0$ *and* $T(\eta,\xi)>0$*, then* $\zeta=\eta<\xi$ *or* $\zeta=0$, $\eta=1<\xi$ *and so we have*

$$E(\zeta,\eta,\xi)=\frac{\xi-1}{\xi+\eta}\exp(-1-\eta)\leq\exp(-1),$$

Case 2. If $T(\zeta,\eta)>0$ *and* $T(\eta,\xi)=0$*, then* $\zeta<\eta=\xi$ *and so we have*

$$E(\zeta,\eta,\xi)=\frac{\eta-1}{\eta+\zeta}\exp(-1-\zeta)\leq\exp(-1),$$

Case 3. If $T(\zeta,\eta)>0$ *and* $T(\eta,\xi)>0$*, then* $\zeta<\eta<\xi$ *and so*

$$E(\zeta,\eta,\xi)=\frac{\eta+\xi-2}{\zeta+2\eta+\xi}\exp(-2-\zeta-\eta)\leq\exp(-1).$$

Hence, by Corollary 2.5 or Theorem 2.2, there exists $\zeta\in M$ *such that* $\zeta=T(\zeta,\zeta)$*. Moreover, for all* $\upsilon,\nu\in M$ *with* $\upsilon\neq\nu$*, we have*

$$d(T(\upsilon,\upsilon),T(\nu,\nu))=0<\upsilon+\nu=d(\upsilon,\nu)$$

and so the point satisfying $\zeta=T(\zeta,\zeta)$ *is unique.*

3. Application to second order boundary value problem

Now, by considering Corollary 2.6, we present two results about the existence of solution of the second order two point boundary value problem as follows:

$$\begin{cases} -\frac{d^2\upsilon}{dt^2}=f(t,\upsilon(t)),\ t\in[0,1] \\ \upsilon(0)=\upsilon(1)=0 \end{cases}, \tag{11}$$

where $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous function. By considering some certain conditions on the function f, many existence results provided for problem (11) in the literature (see [2, 6, 9, 10]). Here, we will consider some different conditions on f, we provide two new theorems. By considering the Green's function defined as

$$G(t,s)=\begin{cases} t(1-s) & , \quad 0\leq t\leq s\leq 1 \\ s(1-t) & , \quad 0\leq s\leq t\leq 1 \end{cases}$$

we can see that the problem (11) is equivalent to the integral equation

$$\upsilon(t)=\int_0^1 G(t,s)f(s,\upsilon(s))ds,\ t\in[0,1]. \tag{12}$$

Therefore, $\upsilon\in C^2[0,1]$ is a solution of (11) if and only if it is a solution of (12). It is clear that

$$\int_0^1 G(t,s)ds=\frac{t(1-t)}{2}$$

and thus

$$\sup_{t\in[0,1]} \int_0^1 G(t,s)ds = \frac{1}{8}.$$

Consider $M = C[0,1]$, which is the space of all continuous real valued functions defined on $[0,1]$, with uniform metric d_∞, that is,

$$d_\infty(\upsilon,\nu) = \|\upsilon - \nu\|_\infty = \sup\{|\upsilon(t) - \nu(t)| : t \in [0,1]\}.$$

It is well known that the space (M, d_∞) is complete.

Theorem 3.1. *The second order two point boundary value problem given by (11) has a solution under the following assumptions:*

(i) there exist two continuous functions $g, h : [0,1] \times \mathbb{R} \to \mathbb{R}$ such that $f(t,\upsilon) = g(t,\upsilon) + h(t,\upsilon)$ and there exists a continuous function $p : [0,1] \to [0,\infty)$ satisfying, for all $\upsilon, \nu, \omega \in \mathbb{R}$

$$\left|g(t,\upsilon) + h(t,\nu) - g(t,\nu) - h(t,\omega)\right| \le p(t) \max\{|\upsilon - \nu|, |\nu - \omega|\},$$

(ii) there exists $k < 1$ such that $\int_0^1 G(t,s)p(s)ds \le k$.

Remark 3.2. *Note that if $\max_{s\in[0,1]} p(s) \le 8k$ for $k < 1$ we have $\int_0^1 G(t,s)p(s)ds \le k$.*

Proof. [Proof of Theorem 3.1] Consider the operator $T : C[0,1] \times C[0,1] \to C[0,1]$ defined by

$$T(\upsilon(t), \nu(t)) = \int_0^1 G(t,s)\{g(s,\upsilon(s)) + h(s,\nu(s))\}ds.$$

Then for any $\upsilon, \nu, \omega \in C[0,1]$ and $t \in [0,1]$ we have

$$\begin{aligned} |T(\upsilon(t),\nu(t)) - T(\nu(t),\omega(t))| &= \left|\int_0^1 G(t,s)\{g(s,\upsilon(s)) + h(s,\nu(s)) - g(s,\nu(s)) - h(s,\omega(s))\}ds\right| \\ &\le \int_0^1 G(t,s)\left|g(s,\upsilon(s)) + h(s,\nu(s)) - g(s,\nu(s)) - h(s,\omega(s))\right| ds \\ &\le \int_0^1 G(t,s)p(s)\max\{|\upsilon(s) - \nu(s)|, |\nu(s) - \omega(s)|\}ds \\ &\le \max\{\|\upsilon - \nu\|_\infty, \|\nu - \omega\|_\infty\} \int_0^1 G(t,s)p(s)ds \\ &\le k\max\{\|\upsilon - \nu\|_\infty, \|\nu - \omega\|_\infty\}. \end{aligned}$$

Hence we have

$$\|T(\upsilon,\nu) - T(\nu,\omega)\|_\infty \le k\max\{\|\upsilon - \nu\|_\infty, \|\nu - \omega\|_\infty\}.$$

Therefore the contractive condition of Corollary 2.6 is satisfied and so there exist $\upsilon \in C[0,1]$ such that

$$\upsilon(t) = T(\upsilon(t), \upsilon(t))$$

or equivalently

$$\begin{aligned} \upsilon(t) &= \int_0^1 G(t,s)\{g(s,\upsilon(s)) + h(s,\upsilon(s))\}ds \\ &= \int_0^1 G(t,s)f(s,\upsilon(s))ds. \end{aligned}$$

□

Example 3.3. *Consider the boundary value problem*

$$\begin{cases} -\frac{d^2\upsilon}{dt^2} = q(t) + r(t)\arctan\left(\frac{2\upsilon(t)}{1-\upsilon^2(t)}\right), \ t \in [0,1] \\ \upsilon(0) = \upsilon(1) = 0 \end{cases}, \tag{13}$$

where q and r are continuous functions on $[0,1]$ such that $0 \le r(t) \le 4k < 4$ for all $t \in [0,1]$. Define $g, h : [0,1]\times\mathbb{R} \to \mathbb{R}$ by $g(t,\upsilon) = q(t) + r(t)\arctan\upsilon$ and $h(t,\upsilon) = r(t)\arctan\upsilon$, then

$$g(t,\upsilon(t)) + h(t,\upsilon(t)) = q(t) + r(t)\arctan\left(\frac{2\upsilon(t)}{1-\upsilon^2(t)}\right)$$

and for all $\upsilon, \nu, \omega \in \mathbb{R}$ we have

$$\begin{aligned} \left|g(t,\upsilon) + h(t,\nu) - g(t,\nu) - h(t,\omega)\right| &= r(t)\left|\arctan\upsilon + \arctan\nu - \arctan\nu - \arctan\omega\right| \\ &\le r(t)\left(\left|\arctan\upsilon - \arctan\nu\right| + \left|\arctan\nu - \arctan\omega\right|\right) \\ &\le r(t)\left(|\upsilon - \nu| + |\nu - \omega|\right) \\ &\le 2r(t)\max\left\{|\upsilon - \nu|, |\nu - \omega|\right\}. \end{aligned}$$

Also we have

$$\int_0^1 G(t,s)2r(s)ds \le k < 1.$$

Therefore by Theorem 3.1 the boundary value problem (13) has a solution.

Theorem 3.4. *The second order two point boundary value problem given by (11) has a solution under the following assumptions:*

(i) there exist two continuous functions $g, h : [0,1]\times\mathbb{R} \to \mathbb{R}$ such that $f(t,\upsilon) = g(t,\upsilon)h(t,\upsilon)$ and there exists a continuous function $p : [0,1] \to [0,\infty)$ satisfying, for all $\upsilon, \nu, \omega \in \mathbb{R}$

$$\left|g(t,\upsilon)h(t,\nu) - g(t,\nu)h(t,\omega)\right| \le p(t)\max\left\{|\upsilon - \nu|, |\nu - \omega|\right\},$$

(ii) there exists $k < 1$ such that $\int_0^1 G(t,s)p(s)ds \le k$.

Proof. Consider the operator $T : C[0,1]\times C[0,1] \to C[0,1]$ defined by

$$T(\upsilon(t),\nu(t)) = \int_0^1 G(t,s)g(s,\upsilon(s))h(s,\nu(s))ds.$$

Then for any $\upsilon, \nu, \omega \in C[0,1]$ and $t \in [0,1]$ we have

$$\begin{aligned} |T(\upsilon(t),\nu(t)) - T(\nu(t),\omega(t))| &= \left|\int_0^1 G(t,s)\{g(s,\upsilon(s))h(s,\nu(s)) - g(s,\nu(s))h(s,\omega(s))\}ds\right| \\ &\le \int_0^1 G(t,s)\left|g(s,\upsilon(s))h(s,\nu(s)) - g(s,\nu(s))h(s,\omega(s))\right|ds \\ &\le \int_0^1 G(t,s)p(s)\max\{|\upsilon(s) - \nu(s)|, |\nu(s) - \omega(s)|\}ds \\ &\le \max\{\|\upsilon - \nu\|_\infty, \|\nu - \omega\|_\infty\}\int_0^1 G(t,s)p(s)ds \\ &\le k\max\{\|\upsilon - \nu\|_\infty, \|\nu - \omega\|_\infty\}. \end{aligned}$$

Hence we have

$$\|T(\upsilon,\nu) - T(\nu,\omega)\|_\infty \le k\max\{\|\upsilon - \nu\|_\infty, \|\nu - \omega\|_\infty\}.$$

Therefore the contractive condition of Corollary 2.6 is satisfied and so there exist $\upsilon \in C[0,1]$ such that

$$\upsilon(t) = T(\upsilon(t), \upsilon(t))$$

or equivalently

$$\begin{aligned} \upsilon(t) &= \int_0^1 G(t,s) g(s, \upsilon(s)) h(s, \upsilon(s)) ds \\ &= \int_0^1 G(t,s) f(s, \upsilon(s)) ds. \end{aligned}$$

□

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