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On G-Mappings Defined by G-Methods and G-Topological Groups

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Abstract. In this note, the concepts of (G_1, G_2)-open, (G_1, G_2)-closed, (G_1, G_2)-quotient and (G_1, G_2)–perfect mappings on arbitrary sets are introduced and some theorems on them are established firstly. In particular, some results improve the corresponding results in [17]. Secondly, we give a partial answer to the question posed by L. Liu [14]. Finally, some properties of G-topological groups, G-connectedness and totally G-disconnectedness in G-topological groups are discussed.

1. Introduction

The concept of G-convergence and any concept related to G-convergence are very important research objects in topology and analysis. G-convergence is closely related to G-sequentially compactness, G-continuity and other related properties, which play a fundamental role in mathematics and its applications. On the basis of ordinary convergence of sequences, there exists a variety of convergence types which play an important role not only in pure mathematics but also in other branches of science involving mathematics, especially in information theory, biological science and dynamical systems.

J. Connor and K.G. Grosse-Erdmann [11] introduced G-methods and G-convergence in the linear subspace of the set of real sequences spaces, and studied the continuity in the sense of G-methods. Huseyin Çakalli [2] extended the above concepts to the first countable Hausdorff topological groups and introduced the concept of G-sequential compactness. By G-sequential closures and G-sequentially closed sets, he discussed G-continuity further in [4]. The notion of G-sequential connectedness in topological groups was introduced by Huseyin Çakalli [6] and some properties of the G-continuity were studied in [9]. Some other types of continuities which can not be given by any sequential method can be found in [3, 5, 7, 9, 10].

In [8], H. Çakalli and P. Das extended the concept of G-sequential compactness to a fuzzy topological group and introduced the notion of G-fuzzy sequential compactness, where G is a function from a suitable subset of the set of all sequences of fuzzy points in a fuzzy first countable topological space X. In [1], Açikgoz, Çakalli, Esenbel and Kočinac introduced neutrosophic G-continuity and investigated its properties in neutrosophic topological spaces.

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In 2016, S. Lin and L. Liu introduced the concepts of G-method and G-convergence on arbitrary sets and the related notion of G-continuity. Several results for G-methods on first-countable topological groups are improved [13].

In 2019, L. Liu [14] defined G-sequentially compact subsets in arbitrary sets and obtained some basic properties. They also introduced the definition of G-method on a Cartesian product of an arbitrary family of sets and posed the following problem:

Problem 1.1. ([14, Problem 3.12]) *Whether the G-sequentially compact is closed under finite product?*

Y.X. Wu and F.C. Lin [18] introduced the concept of G-topological groups and proved that the G-connectedness is preserved by countable product.

Inspired by O. Mucuk, T. Şahan's work [17], in this paper we introduce the concepts of G-open mappings, G-closed mappings and G-quotient mappings on arbitrary sets under G-method and some results about them are presented. We also discuss the properties of G-connectedness and totally G-disconnectedness in G-topological groups. Readers may refer to [12] for some terminology unstated here.

2. Preliminaries

In this paper, $\mathbb{N}$ denotes the set of all positive integers. Let X be a set, $s(X)$ denote the set of all X-valued sequences, i.e., $\boldsymbol{x}\in s(X)$ if and only if $\boldsymbol{x}=\{x_n\}_{n\in\mathbb{N}}$ is a sequence with each $x_n \in X$. If X is a topological space, the set of all X-valued convergent sequences is denoted by $c(X)$, and we put $\lim \boldsymbol{x} = \lim_{n\to\infty} x_n$ for any $\boldsymbol{x}=\{x_n\}_{n\in\mathbb{N}} \in c(X)$[13].

Definition 2.1. ([13, Definition 1.1]) Let X be a set. A method on X is a function $G : c_G(X) \to X$ defined on a subset $c_G(X)$ of $s(X)$. A sequence $\boldsymbol{x}=\{x_n\}_{n\in\mathbb{N}}$ in X is said to be G-convergent to $l \in X$ if $\boldsymbol{x}\in c_G(X)$ and $G(\boldsymbol{x})=l$.

Definition 2.2. Let X be a topological space.

(1) A method $G : c_G(X) \to X$ is called *regular* [13, Definition 1.1 (2.1)] if $c(X) \subset c_G(X)$ and $G(\boldsymbol{x})=\lim \boldsymbol{x}$ for each $\boldsymbol{x}\in c(X)$.

(2) A method G: $c_G(X) \to X$ is called *subsequential* [13, Definition 1.1 (2.2)] if, whenever $\boldsymbol{x}\in c_G(X)$ is G-convergent to $l \in X$, then there exists a subsequence $\boldsymbol{x}' \in c(X)$ of $\boldsymbol{x}$ with $\lim \boldsymbol{x}'=l$.

(3) We say a method G *preserves the G-convergence of subsequences* [16, pp.1083] if, whenever a sequence $\boldsymbol{x}$ is G-convergent with $G(\boldsymbol{x})=l$, then any subsequence of $\boldsymbol{x}$ is G-convergent to the same point l.

The G-closures, as a generalization of the concept of closures in topological spaces, are essential concepts in G-methods, see [13].

Definition 2.3. Let X be a set, and G be a method on X. A subset A of X is called a *G-closed set* [13, Definition 2.1] of X if, whenever $\boldsymbol{x} \in s(A) \cap c_G(X)$, then $G(\boldsymbol{x}) \in A$. A subset A of X is called *G-open* [13, Definition 3.1 (2)] if $X \setminus A$ is G-closed in X.

Definition 2.4. Let X be a set, G be a method on X and $A \subset X$.

(1) The *G-closure* [13, Definition 2.4 (2)] of A is defined as the intersection of all G-closed sets containing A, and the G-closure of A is denoted by $\overline{A}^G$.

(2) The *G-interior* [13, Definition 3.3 (2)] of A is defined as the union of all G-open sets contained in A, and the G-interior of A is denoted by $A^{\circ G}$.

(3) A subset A of X is called a *G-neighborhood* [13, Definition 3.1 (1)] of a point $x \in X$ if there exists a G-open set U with $x \in U \subset A$.

(4) A subset F of A is called *G-closed* [18, Definition 2.6 (7)] in A if there exists a G-closed subset K of X such that $F = K \cap A$.

Remark 2.5. (1) The empty set $\emptyset$ and the whole space X are G-closed. It is clear that $\overline{\emptyset}^G = \emptyset$ and $\overline{X}^G = X$ for a regular method G.

(2) The union of any family of G-open subsets of X is G-open. Thus the G-interior $A^{\circ G}$ of a set A is the largest G-open set contained in A.

(3) A subset U of A is G-open in A if and only if there exists a G-open subset V of X such that $U = A \cap V$.

Definition 2.6. ([13, Definition 7.1]) Let G_1, G_2 be methods on sets X and Y, respectively. A mapping $f : X \to Y$ is called (G_1, G_2)-*continuous* if $f(\mathbf{x}) \in c_{G_2}(Y)$ and $G_2(f(\mathbf{x})) = f(G_1(\mathbf{x}))$ for each $\mathbf{x} \in c_{G_1}(X)$. (G_1, G_2)-continuity is called the *G-continuity* if G_1 and G_2 are the same method G.

Definition 2.7. ([16, Definition 4.1]) A non-empty subset A of X is called *G-connected* if there are no non-empty disjoint G-closed subsets F and K of A such that $A = F \cup K$. In particular, X is called G-connected if there are no non-empty, disjoint G-closed subsets of X whose union is X.

Now we recall the concepts of G-topological groups.

Definition 2.8. ([18, Definition 5.4]) Let X be a group with operations. A topology τ on the set X is a *G-topological group* provided that the following statements hold:

(1) The multiplication mapping $M : (X, \tau) \times (X, \tau) \to (X, \tau)$ is G-continuous;

(2) The inverse mapping $In : (X, \tau) \to (X, \tau)$ is G-continuous.

Let X be a G-topological group and $a \in X$. A family $\mathcal{B}_a$ of G-open neighborhoods of a is called a *fundamental system* [16, pp.1087] of G-open neighborhoods of a if for each G-open neighborhood U of a, there is a $V \in \mathcal{B}_a$ such that $V \subset U$.

3. Mappings Defined by *G*-Methods on Arbitrary Sets

As a generalization of the concepts of G-open and G-closed mappings in the class of Hausdorff topological groups [17], we extend these concepts to arbitrary sets. Similarly to the corresponding concepts in topological spaces, we give the concepts of (G_1, G_2)-quotient and (G_1, G_2)-perfect mappings under G-methods on arbitrary sets. **All the mappings in this section are onto mappings.**

Definition 3.1. Let G_1, G_2 be methods on sets X and Y, respectively. $f : X \to Y$ be a mapping.

(1) f is called (G_1, G_2)-*quotient* if, $F \subset Y$ and $f^{-1}(F)$ is a G_1-closed subset of X, then F is a G_2-closed subset of Y.

(2) f is called (G_1, G_2)-*open* if, V is a G_1-open subset of X, then $f(V)$ is a G_2-open subset of Y.

(3) f is called (G_1, G_2)-*closed* if, F is a G_1-closed subset of X, then $f(F)$ is a G_2-closed subset of Y.

Now, we recall an important result which is due to S. Lin and L. Liu[13].

Proposition 3.2. ([13, Lemma 7.2, Theorem 7.3]) *Let $f : X \to Y$ be a mapping, where G_1, G_2 are methods on sets X and Y, respectively. Then (1)⇒(2)⇔(3)⇔(4) in the following conditions.*

(1) *f is a (G_1, G_2)-continuous mapping.*

(2) *$f(\overline{A}^{G_1}) \subset \overline{f(A)}^{G_2}$ for each $A \subset X$.*

(3) *$f^{-1}(W)$ is a G_1-open set of X for each G_2-open set W of Y.*

(4) *$f^{-1}(F)$ is a G_1-closed set of X for each G_2-closed set F of Y.*

By modifying very slightly the proofs of the work of O. Mucuk, T. Şahan [17], we improve the corresponding result by replacing topological groups with arbitrary sets.

Theorem 3.3. *Let G be a method on a set X and $f, g : X \to X$ be mappings on X. Then the following are satisfied:*

(1) *If f and g are G-continuous, then so also is gf.*

(2) *If f and g are G-open (closed), then so also is gf.*

(3) *If gf is G-open (closed) and f is onto and G-continuous, then g is G-open (closed).*

(4) *If gf is G-open (closed) and g is one to one and G-continuous, then f is G-open (closed).*

Proof. (1) For each $\mathbf{x}\in c_G(X)$, we have $f(\mathbf{x})\in c_G(X)$ and $G(f(\mathbf{x}))=f(G(\mathbf{x}))$ by f is G-continuous. Since g is G-continuous, $gf(\mathbf{x})=g(f(\mathbf{x}))\in c_G(X)$ and $G(g(f(\mathbf{x})))=g(G(f(\mathbf{x})))$. Thus $G(gf(\mathbf{x}))=G(g(f(\mathbf{x})))=g(f(G(\mathbf{x})))=gf(G(\mathbf{x}))$, which proves that gf is G-continuous.

(2) It is obvious.

(3) Let A be a G-open subset of X. Since f is G-continuous, $f^{-1}(A)$ is G-open. Since gf is G-open and f is onto, we conclude that $(gf)(f^{-1}(A)) = g(A)$ is G-open. Similarly, we can prove that g is G-closed.

(4) Let A be a G-open subset of X. It follows from gf is G-open that $gf(A)$ is G-open. Since g is G-continuous and one to one, we can conclude that $g^{-1}gf(A) = f(A)$ is G-open. The proof is similar when A is G-closed. □

The following theorem indicates that each (G_1, G_2)-continuous mapping is (G_1, G_2)-quotient if $c_{G_1}(X) = s(X)$.

Theorem 3.4. *Let G_1, G_2 be methods on sets X and Y, respectively, where $c_{G_1}(X) = s(X)$. If $f : X \to Y$ is a (G_1, G_2)-continuous mapping, then f is (G_1, G_2)-quotient.*

Proof. Assume $B \subset Y$ and $f^{-1}(B)$ is a G_1-closed set of X, we show that B is a G_2-closed set of Y. If $\mathbf{y}=\{y_n\}_{n\in\mathbb{N}} \in c_{G_2}(Y) \cap s(B)$. For every $n \in \mathbb{N}$, we chose a point $x_n \in X$ such that $x_n \in f^{-1}(y_n)$, then $\mathbf{x}=\{x_n\}_{n\in\mathbb{N}} \in s(f^{-1}(B)) \cap c_{G_1}(X)$.

Put $G_1(\mathbf{x})= x$. It follows from $f^{-1}(B)$ is a G_1-closed set of X that $x \in f^{-1}(B)$. Thus $f(x) \in B$. By (G_1, G_2)-continuity of the mapping f , we have $f(x) = f(G_1(\mathbf{x}))=G_2(f(\mathbf{x}))= G_2(\mathbf{y})$, and hence $G_2(\mathbf{y})\in B$. Therefore, B is a G_2-closed set of Y, it follows that f is (G_1, G_2)-quotient. □

It is natural to ask whether each (G_1, G_2)-quotient mapping is (G_1, G_2)-continuous. In the general case, the next example tells us that it is not true. Now, let us recall the related definition which will be used in the following example .

Let $G : c_G(X) \to X$ be a method on a set X, and $Y \subset X$. Put $c_{G|_Y}(Y) = \{\mathbf{x}\in s(Y) \cap c_G(X) : G(\mathbf{x})\in Y\}$, and define a function $G|_Y : c_{G|_Y}(Y) \to Y$ by $G|_Y(\mathbf{x})=G(\mathbf{x})$. The function $G|_Y : c_{G|_Y}(Y) \to Y$ is called the *submethod* [13, Definition 4.1] of G on the subset Y of X, or the method on the subset Y induced by G.

Example 3.5. ([13, Example 7.4 (1)]) There exists a (G_1, G_2)-quotient mapping which is not (G_1, G_2)-continuous.

Let X be the set $\mathbb{Z}$ of all integers endowed with the discrete topology. Put

$$c_{G_1}(X) = \{\{x_n\}_{n\in\mathbb{N}} \in s(X) : \text{there exists an } m \in \mathbb{N} \text{ such that}$$
$$\{x_n - x_{n-1}\}_{n>m} \text{ is a constant sequence}\}.$$

Define $G_1 : c_{G_1}(X) \to X$ by $G_1(\mathbf{x})=\lim_{n\to\infty}(x_{n+1} - x_n)$ for each $\mathbf{x}=\{x_n\}_{n\in\mathbb{N}} \in c_{G_1}(X)$. Then G_1 is a method on X.

Let $Y = \{0, 1\}$. Define a mapping $f : X \to Y$ as follows: $f(x) = 0$ if and only if $x = 2k$, $k \in \mathbb{Z}$.

Let $G_2 = G_1|_Y$. Then G_2 is a method on the set Y. Let F be a non-empty subset of Y, then F is equal to $\{0\}$, $\{1\}$ or Y. Thus $f^{-1}(F)=f^{-1}(\{0\}) = \{2k\}_{k\in Z}$, $f^{-1}(F)= f^{-1}(\{1\}) = \{2k + 1\}_{k\in Z}$, or$f^{-1}(F)=f^{-1}(Y) = X$. It is easy to verify that if $f^{-1}(F)$ is G_1-closed in X, then F is equal to $\{0\}$ or Y which are G_2-closed in Y. Therefore, f is (G_1, G_2)-quotient. It follows from the proof of [13, Example 7.4 (1)] that f is not (G_1, G_2)-continuous.

It is easy to check the validity of the next theorem.

Theorem 3.6. *Let G_1, G_2 be methods on sets X and Y, respectively. Then:*

(1) f is (G_1, G_2)-quotient if f is a (G_1, G_2)-open mapping from X onto Y.

(2) f is (G_1, G_2)-quotient if f is a (G_1, G_2)-closed mapping from X onto Y..

Proof. (1) Let $F \subset Y$ and $f^{-1}(F)$ is a G_1-closed subset of X, then $X \setminus f^{-1}(F)$ is a G_1-open subset of X. It follows from f is (G_1, G_2)-open that $Y \setminus f(X \setminus f^{-1}(F)) = F$ is a G_2-closed subset of Y. Therefore, f is (G_1, G_2)-quotient.

(2) Let $F \subset Y$ and $f^{-1}(F)$ is a G_1-closed subset of X, then $F = f(f^{-1}(F))$ is a G_2-closed subset of Y. It follows that f is (G_1, G_2)-quotient. □

Corollary 3.7. *Let G_1, G_2 be methods on sets X and Y, respectively. If $f : X \to Y$ is a one to one and (G_1, G_2)-continuous mapping, then the followings are equivalent:*
(1) *f is (G_1, G_2)-open.*
(2) *f is (G_1, G_2)-closed.*
(3) *f is (G_1, G_2)-quotient.*

It is well known that the next lemma is true.

Lemma 3.8. *Let $f : X \to Y$ be a mapping, U and F be subsets of sets X and Y, respectively. Then*

$$f^{-1}(F) \subset U \Leftrightarrow F \subset Y - f(X - U)$$

The next theorem is useful when we discuss some properties of (G_1, G_2)-closedness.

Theorem 3.9. *Let G_1, G_2 be methods on sets X and Y, respectively. A mapping $f : X \to Y$ is (G_1, G_2)-closed if and only if for each $y \in Y$ and every G_1-open set $U \subset X$ with $f^{-1}(y) \subset U$, there exists a G_2-open $V \subset Y$ containing y such that $f^{-1}(V) \subset U$.*

Proof. Suppose f is (G_1, G_2)-closed, then for every $y \in Y$ and a G_1-open set $U \subset X$ such that $f^{-1}(y) \subset U$. Put $V = Y - f(X - U)$, then V is G_2-open in Y. It follows from Lemma 3.8 that $y \in V$ and $f^{-1}(V) \subset U$.

Conversely, suppose F is G_1-closed in X. Take an arbitrary $y \in Y - f(F)$, then $f^{-1}(y) \subset X - F$. By assumption, there exists a G_2-open set W in Y such that $y \in W$ and $f^{-1}(W) \subset X - F$. Therefore, W is a G_2-open subset of Y containing y and $W \cap f(F) = \emptyset$ by Lemma 3.8. Thus $f(F)$ is G_2-closed in Y which implies that f is (G_1, G_2)-closed. □

Making use of Theorem 3.9, it is easy to deduce the corollary below.

Corollary 3.10. *Let G_1, G_2 be methods on sets X and Y, respectively and $f : X \to Y$ be a (G_1, G_2)-continuous mapping. Then the following are equivalent:*
(1) *f is (G_1, G_2)-closed.*
(2) *For each $y \in Y$ and each G_1-open set $U \subset X$ which contains $f^{-1}(y)$, there exists a G_1-open set V in X such that $f^{-1}(y) \subset V \subset U$, $V = f^{-1}(f(V))$ and $f(V)$ is G_2-open in Y.*

Proof. (1) ⇒ (2) Assume that f is (G_1, G_2)-closed. For each $y \in Y$ and each G_1-open set $U \subset X$ which contains $f^{-1}(y)$, by Theorem 3.9, there exists a G_2-open set W_y in Y such that $y \in W_y$ and $f^{-1}(W_y) \subset U$. Put $V = f^{-1}(W_y)$, then $f(V) = W_y$. It is easy to see that the set V is required.

(2) ⇒ (1) For each $y \in Y$ and each G_1-open set $U \subset X$ which contains $f^{-1}(y)$, by (2), there is a G_1-open set V in X such that $f^{-1}(y) \subset V \subset U$, $V = f^{-1}(f(V))$ and $f(V)$ is G_2-open set in Y. Take $W = f(V)$, then W is a G_2-open set of Y, $y \in W$ and $f^{-1}(W) \subset U$. It follows from Theorem 3.9 that f is (G_1, G_2)-closed. □

In order to characterize (G_1, G_2)-closed mappings by means of G-closures and G- closed sets, we recall two important propositions below.

Proposition 3.11. ([13, Proposition 2.6]) *Let G be a method on a set X and $A \subset X$. The following are equivalent:*
(1) *A is G-closed.*
(2) *$\overline{A}^G \subset A$, i.e., $\overline{A}^G = A$.*

Proposition 3.12. ([13, Corollary 3.6]) *Let G be a method on a set X and $A \subset X$. The following are equivalent.*
(1) *A is G-open.*
(2) *$A \subset A^{\circ G}$, i.e., $A^{\circ G} = A$.*

Making use of Proposition 3.11 and the definition of (G_1, G_2)-closed mappings , it is easy to see that the following theorem is true.

Theorem 3.13. *Let G_1, G_2 be methods on sets X and Y, respectively. A mapping $f : X \to Y$ is (G_1, G_2)-closed if $\overline{f(A)}^{G_2} \subset f(\overline{A}^{G_1})$ for every subset A of X.*

Proof. Let K be a G_1-closed subset of X. By assumption $\overline{f(K)}^{G_2} \subset f(\overline{K}^{G_1})$. It follows from Proposition 3.11 that $\overline{K}^{G_1} = K$. So we have $\overline{f(K)}^{G_2} \subset f(K)$, and $f(K)$ is a G_2-closed subset of Y. □

It follows from Theorem 3.13 that the next corollary is true.

Corollary 3.14. ([17, Theorem 2.9]) *Let G be a method on a set X. A function $f : X \to X$ is G-closed if $\overline{f(A)}^{G} \subset f(\overline{A}^{G})$ for every subset A of X.*

Similarly, we can characterize (G_1, G_2)-open mappings by means of G-interiors.

Theorem 3.15. *Let G_1, G_2 be methods on sets X and Y, respectively. A mapping $f : X \to Y$ is (G_1, G_2)-open if and only if $f(A^{\circ G_1}) \subset f(A)^{\circ G_2}$ for every subset A of X.*

Proof. Since $A^{\circ G_1} \subset A$, we have that $f(A^{\circ G_1}) \subset f(A)$. Therefore, $f(A^{\circ G_1})^{\circ G_2} \subset f(A)^{\circ G_2}$. Since f is (G_1, G_2)-open, we conclude that $f(A^{\circ G_1})$ is a G_2-open subset in Y. It follows from Proposition 3.12 that $f(A^{\circ G_1}) \subset f(A)^{\circ G_2}$.

Conversely, suppose $f(A^{\circ G_1}) \subset f(A)^{\circ G_2}$ for any subset A of X. For any G_1-open subset U of X, it follows from Proposition 3.12 that $U^{\circ G_1} = U$. We have $f(U) \subset f(U)^{\circ G_2}$. As a consequence, $f(U)$ is a G_2-open subset of Y. □

Corollary 3.16. ([17, Theorem 2.12]) *Let G be a method on set X. A mapping $f : X \to X$ is G-open if and only if $f(A^{\circ G}) \subset f(A)^{\circ G}$ for every subset A of X.*

At the end of this section, firstly we define G-compact sets and (G_1, G_2)-perfect mappings under G-methods. Subsequently we discuss (G_1, G_2)-perfect mappings. Finally, we give a partial answer to the question about G-sequentially compact sets posed in [14].

Definition 3.17. Let G be a method on a set X, a family $\{A_s\}_{s\in S}$ of G-open subsets of X is called *G-open cover* if, $\bigcup_{s\in S} A_s = X$.

Definition 3.18. Let G be a method on a set X, X is called *G-compact* if, every G-open cover of X has a finite subcover.

Definition 3.19. Let G_1, G_2 be methods on sets X and Y, respectively. A mapping $f : X \to Y$ is called a *(G_1, G_2)-perfect mapping* if f is (G_1, G_2)-continuous, (G_1, G_2)-closed and $f^{-1}(y)$ is a G_1-compact subset of X for each $y \in Y$.

We can use the concept of G-compact to prove that the image of a G-compact set under a G-continuous mapping is G-compact.

Theorem 3.20. *Let G_1, G_2 be methods on sets X and Y, respectively. If the mapping $f : X \to Y$ is (G_1, G_2)-continuous and X is G_1-compact, then Y is G_2-compact.*

Proof. Let $\{U_s\}_{s\in S}$ be a G_2-open cover of Y. The family $\{f^{-1}(U_s)\}_{s\in S}$ is a G_1-open cover of X. Thus there exists a finite set $\{s_1, s_2, \cdots, s_k\} \subset S$ such that

$$f^{-1}(U_{s_1}) \cup f^{-1}(U_{s_2}) \cup \cdots \cup f^{-1}(U_{s_k}) = X$$

and this implies that $U_{s_1} \cup U_{s_2} \cup \cdots \cup U_{s_k} = Y$. □

In 2019, L. Liu [14] introduced the definition of G-method on a Cartesian product of an arbitrary family of sets. L. Liu and Z. Ping[15] also introduce the concept of product G-methods on sets which lead to a G-generalized topology on the Cartesian products.

Definition 3.21. Suppose $\{G_\alpha : c_{G_\alpha}(X_\alpha) \to X_\alpha\}_{\alpha\in\Lambda}$ is a family of G-methods. Put

$$c_G(\prod_{\alpha\in\Lambda} X_\alpha) = \{\boldsymbol{z}\in s(\prod_{\alpha\in\Lambda} X_\alpha) : p_\alpha(\boldsymbol{z})\in c_{G_\alpha}(X_\alpha), \alpha \in \Lambda\}.$$

The product of the family $\{G_\alpha\}_{\alpha\in\Lambda}$ denoted by G is defined as follows:

$$G : c_G(\prod_{\alpha\in\Lambda} X_\alpha) \to \prod_{\alpha\in\Lambda} X_\alpha$$

$$G(\boldsymbol{z})=(G_\alpha(p_\alpha(\boldsymbol{z})))_{\alpha\in\Lambda},$$

where $p_\alpha : \prod_{\alpha\in\Lambda} X_\alpha \to X_\alpha$ is a projection for each $\alpha \in \Lambda$.

Theorem 3.22. *Let G_1, G_2 be methods on sets X and Y, respectively, where G_2 is a regular method. Then the mapping f from X to $X \times \{y\}$ defined by $f(x) = (x, y)$ for some $y \in Y$ is $(G_1, G_1 \times G_2|_{\{y\}})$-continuous.*

Proof. Let $\boldsymbol{x}=\{x_n\}_{n\in\mathbb{N}} \in c_{G_1}(X)$, then $f(\boldsymbol{x})=\{(x_n, y)\}_{n\in\mathbb{N}}$. Put $\boldsymbol{y}=\{(x_n, y)\}_{n\in\mathbb{N}}$ and $G = G_1 \times G_2|_{\{y\}}$, then $\boldsymbol{y}\in c_G(X \times \{y\})$. Take $G_1(\boldsymbol{x})= x$, then $f(G_1(\boldsymbol{x}))= f(x) = (x, y) = G(\{(x_n, y)\}_{n\in\mathbb{N}}) = G(\boldsymbol{y})=G(f(\boldsymbol{x}))$. We have shown that f is $(G_1, G_1 \times G_2|_{\{y\}})$-continuous. □

Theorem 3.23. *Let G_1, G_2 be methods on sets X and Y, respectively. If $f : X \to Y$ is a (G_1, G_2)-perfect mapping, then for every G_2-compact subset $Z \subset Y$, $f^{-1}(Z)$ is a G_1-compact subset of X.*

Proof. It suffices to show that for any family $\{U_s\}_{s\in S}$ of G_1-open subsets of X whose union contains $f^{-1}(Z)$ there exists a finite set $S_0 \subset S$ such that $f^{-1}(Z) \subset \bigcup_{s\in S_0} U_s$. Let $\mathcal{J}$ be the family of all finite subsets of S and $U_J = \bigcup_{s\in J} U_s$ for $J \in \mathcal{J}$. For each $z \in Z$, $f^{-1}(z)$ is G_1-compact and thus is contained in the set U_J for some $J \in \mathcal{J}$. It follows that $z \in Y \setminus f(X \setminus U_J)$, and thus $Z \subset \bigcup_{J\in\mathcal{J}}(Y \setminus f(X \setminus U_J))$. Since $Y \setminus f(X \setminus U_J)$ is G_2-open, there exist $J_1, J_2, \cdots, J_k \in \mathcal{J}$ such that $Z \subset \bigcup_{i=1}^{k}(Y \setminus f(X \setminus U_{J_i}))$. Hence

$$f^{-1}(Z) \subset \bigcup_{i=1}^{k} f^{-1}(Y \setminus f(X \setminus U_{J_i}))=\bigcup_{i=1}^{k}(X \setminus f^{-1}f(X \setminus U_{J_i})) \subset \bigcup_{i=1}^{k}(X \setminus (X \setminus U_{J_i})) = \bigcup_{i=1}^{k} U_{J_i} = \bigcup_{s\in S_0} U_s,$$

where $S_0 = J_1 \cup J_2 \cup \cdots \cup J_k$. □

It is easy to conclude that the next corollary is true.

Corollary 3.24. *Let G_1, G_2 be methods on sets X and Y, respectively. If $f : X \to Y$ is a (G_1, G_2)-perfect mapping and Y is G_2-compact, then X is G_1-compact.*

The next problem about the product of G-sequentially compact was posed by L. Liu in [14] .

Problem 3.25. *Whether G-sequentially compact sets is closed under finite product*?

First, we recall the related definition.

Definition 3.26. ([14]) Let G be a method on X and $F \subset X$. F is called a *G-sequentially compact subset of X* if, whenever $\boldsymbol{x} \in s(F)$ there is a subsequence $\boldsymbol{x'}$ of $\boldsymbol{x}$ with $\boldsymbol{x'}\in c_G(X)$ and $G(\boldsymbol{x'}) \in F$.

It is natural to consider the relation between the concepts of G-compact and G-sequentially compact.

Example 3.27. There is a G-sequentially compact set X which is not G-compact.

Let $X = [0, +\infty)$, $c_G(X) = s(X)$. Define $G : s(X) \to X$ by $G(\boldsymbol{x})=x_1$ for each $\boldsymbol{x}=\{x_n\}_{n\in\mathbb{N}} \in s(X)$. It is evident that every sequence in X is G-convergent, thus X is G-sequentially compact. For each subset A of X, it is easy to see that A is both G-closed and G-open in X. Thus X is not G-compact.

Now, we give a partial answer to the above Problem 3.25.

Theorem 3.28. *Let G_1 be a method on a set X preserving the G_1-convergence of subsequences and G_2 be a method on a set Y. If X is G_1-sequentially compact and Y is G_2-sequentially compact, then $X \times Y$ is G-sequentially compact with respect to the product method G.*

Proof. Let $\{z_n\}_{n\in\mathbb{N}} \in s(X\times Y)$, where $z_n = (x_n, y_n)$ for every $n \in \mathbb{N}$. Since X is G_1-sequentially compact, there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $\{x_{n_k}\}_{k\in\mathbb{N}} \in c_{G_1}(X)$. Since Y is G_2-sequentially compact, there exists a subsequence $\{y_{n_{k_j}}\}_{j\in\mathbb{N}}$ of $\{y_{n_k}\}_{k\in\mathbb{N}}$ such that $\boldsymbol{y}=\{y_{n_{k_j}}\}_{j\in\mathbb{N}} \in c_{G_2}(Y)$. It follows from G_1 is a method preserving the G_1-convergence of subsequences that $\boldsymbol{x}=\{x_{n_{k_j}}\}_{j\in\mathbb{N}}$ satisfies $\boldsymbol{x}\in c_{G_1}(X)$. Put $G_1(\boldsymbol{x})=x$ and $G_2(\boldsymbol{y})=y$, then $G(\boldsymbol{x},\boldsymbol{y})=(x, y) \in X \times Y$. □

It is easy to check that if G_2 is a method on a set Y preserving the G_2-convergence of subsequences and G_1 is a method on a set X, then $X \times Y$ is G-sequentially compact with respect to the product method G whenever X is G_1-sequentially compact and Y is G_2-sequentially compact.

4. Some Basic Properties of *G*-Topological Groups

In this part, **we assume that** G **is a regula method preserving the subsequence.** Y. Wu and F. Lin [18] showed that the right translation and the left translation are all G-continuous in G-topological groups. Making use of Theorem 3.3, we have next two propositions.

Proposition 4.1. *Let X be a G-topological group. Then for any $a \in X$, the left translation $\lambda_a : X \to X$, $x \mapsto ax$ is G-closed and G-open.*

Proof. Since X is a G-topological group, λ_a is G-continuous for each $a \in X$. Since $\lambda_{a^{-1}} \circ \lambda_a = \lambda_a \circ \lambda_{a^{-1}}$ is an identity mapping which is G-open and G-closed, it follows from $\lambda_{a^{-1}}$ is one to one, onto and G-continuous that λ_a is G-open and G-closed by Theorem 3.3. □

Proposition 4.2. *Let X be a G-topological group. Then the inverse mapping $In : X \to X$, $x \mapsto x^{-1}$ is G-closed and G-open.*

Proof. Since X is a G-topological group, then the inverse mapping In is G-continuous. Thus F^{-1} is a G-closed set in X for each G-closed set F of X which implies that the inverse mapping In is G-closed. By the same way, we can conclude that the inverse mapping In is G-open. □

S. Lin and L. Liu [13] induced naturally a method on the subset Y of a set X with a G-method.
Put

$$c_{G|_Y}(Y) = \{\boldsymbol{x}\in s(Y) \cap c_G(X) : G(\boldsymbol{x})\in Y\},$$

and define a function $G|_Y : c_{G|_Y}(Y) \to Y$ by

$$G|_Y(\boldsymbol{x})=G(\boldsymbol{x}),\ \boldsymbol{x}\in c_{G|_Y}(Y).$$

Then $G|_Y$ is a method on the subset Y of X.

Recall that if G is a method on a set X, and $Y \subset X$. The function $G|_Y : c_{G|_Y}(Y) \to Y$ is called the *submethod* of G on the subset Y of X, or the method on the subset Y induced by G [13, Definition 4.1].

In 2018, O. Mucuk, H. Çakalli [16] give a remark which say that if G is a method defined on a Hausdorff topological group X, then we can also obtain a similar method on $X \times X$ defined by $G(\boldsymbol{x},\boldsymbol{y}) = (G(\boldsymbol{x}), G(\boldsymbol{y}))$ when $\boldsymbol{x}$ and $\boldsymbol{y}$ are G-convergent sequences in X. By Definition 3.21, if G is a method defined on a set X, then we denote the method $G \times G$ by G on $X \times X$ in the following results .

Theorem 4.3. *If X is a G-topological group and H is a subgroup of X. Then H is a $G|_H$-topological group.*

Proof. It is only need to prove that the multiplication mapping $M|_{H\times H} : (H, \tau|_H) \times (H, \tau|_H) \to (H, \tau|_H)$ and the inverse mapping $In|_H : (H, \tau|_H) \to (H, \tau|_H)$ are $G|_H$-continuous.

Indeed, if $\boldsymbol{y}\in c_{G|_H}(H) \subset c_G(X)$, then $In|_H(\boldsymbol{y})=In(\boldsymbol{y})\in c_G(X)$ and $G(In(\boldsymbol{y}))=In(G(\boldsymbol{y}))$ by G-continuity of the inverse mapping In . Since H is a subgroup of X, $In_H(\boldsymbol{y})=In(\boldsymbol{y})$ is a sequence in H. Note that $G(In|_H(\boldsymbol{y}))= G(In(\boldsymbol{y}))=In(G(\boldsymbol{y}))$ and $G(\boldsymbol{y})\in H$, so we have $G(In|_H(\boldsymbol{y})) \in H$. Thus $In|_H(\boldsymbol{y}) \in c_{G|_H}(H)$. It is easy to conclude that $G|_H(In|_H(\boldsymbol{y}))=G(In(\boldsymbol{y}))=In(G(\boldsymbol{y}))=In|_H(G|_H(\boldsymbol{y}))$. This proves that $In|_H$ is $G|_H$-continuous.

In fact, if z=(x,y)∈ $c_{G|_H}(H\times H)$, then x∈ $c_{G|_H}(H)$ and $y \in c_{G|_H}(H)$ by Definition 3.21. Thus z=(x,y)∈ $c_G(X\times X)$. Since $M : X \times X \to X$ is G-continuous, $M|_{H\times H}(z)$=$M(z)$∈ $c_G(X)$ and $G(M(z))$=$M(G(z))$. It follows from H is a subgroup of X that $M|_{H\times H}(z)$ is a sequence in H. Since $G(M|_{H\times H}(z))$= $G(M(z))$=$M(G(z))$ and $G(z)$=($G(x)$, $G(y)$)∈ $H \times H$, it follows that $G(M|_{H\times H}(z))$∈ H. Thus $M|_{H\times H}(z)$∈ $c_{G|_H}(H)$. It follows from $G|_H(M|_{H\times H}(z))$=$G(M(z))$=$M(G(z))$=$M|_{H\times H}(G|_H(z))$ that the multiplication mapping $M|_{H\times H}$ is $G|_H$-continuous. □

In [16], the authors proved the following fact.

Proposition 4.4. ([16, Proposition 3.15]) *Let G be a method on X, A and B be subsets of X. If A and B are G-closed, then A × B is G-closed.*

A delicate situation arise when we discuss G-closures of G-topological subgroup in G-topological group.

Proposition 4.5. ([18, Proposition 5.14]) *Let X be a G-topological group and $A \subset X$. Then $x \in \overline{A}^G$ if and only if for each G-open neighborhood U of x ones have $A \cap U \neq \emptyset$.*

Theorem 4.6. *If H is a subgroup of a G-topological group X, then $\overline{H}^G$ is a subgroup of X.*

Proof. We claim that $\overline{H}^G H \subset \overline{H}^G$. In fact, for each $y \in \overline{H}^G$ and $x \in H$, we have $yx \in \overline{H}^G x \subset \overline{Hx}^G$ since X is a G-topological group. Since H is a subgroup, $Hx \subset H$ and hence $yx \in \overline{H}^G$. Thus $\overline{H}^G H \subset \overline{H}^G$. Next we verify that for each $z \in \overline{H}^G$ if $y \in \overline{H}^G$, then $yz \in \overline{H}^G$. Indeed, for every G-open neighborhood U of the neutral element e, there is $y' \in zU \cap H$, then $yy' \in yzU \cap \overline{H}^G H \subset \overline{H}^G$, which proves that $\overline{H}^G\overline{H}^G \subset \overline{H}^G$.

Since the inverse mapping In is G-continuous, we conclude that $In(\overline{H}^G) \subset \overline{In(H)}^G$, that is $\overline{H}^{G^{-1}} \subset \overline{H^{-1}}^G$. It follows that $\overline{H}^{G^{-1}} \subset \overline{H}^G$, and thus the set $\overline{H}^G$ is also closed under taking inverses. We have shown that $\overline{H}^G$ is a subgroup of X. □

Corollary 4.7. *If H is a subgroup of a G-topological group X, then $\overline{H}^G$ is a $G|_{\overline{H}^G}$ topological group.*

Theorem 4.8. *Suppose that X is a G-topological group. If X × X is the Cartesian product and the product operation is defined coordinatewise, then X × X is a G-topological group.*

Proof. Let $Y = X \times X$. It suffices to prove that the multiplication mapping $M : Y \times Y \to Y$ and the inverse mapping $In : Y \to Y$ are G-continuous.

In fact, if (x,y)∈ $c_G(Y \times Y)$, then x=(x^1,x^2)∈ $c_G(Y) = c_G(X \times X)$ and y=(y^1,y^2)∈ $c_G(Y) = c_G(X \times X)$ by Definition 3.21. Note that X is a G-topological group, so we have $M((x^1,x^2))$∈ $c_G(X)$ and $M((y^1,y^2))$∈ $c_G(X)$. Thus $M((x,y))$=($M(x^1,x^2)$, $M(y^1,y^2)$)∈ $c_G(Y)$. It follows from $G(M(x,y))$=$G(M(x^1,x^2), M(y^1,y^2))$=($G(M(x^1,x^2))$, $G(M(y^1,y^2))$)=($M(G(x^1,x^2))$, $M(G(y^1,y^2))$) = $M(G(x^1,x^2), G(y^1,y^2))$=$M(G(x,y))$. This proves that the multiplication mapping M is G-continuous.

If z=(z^1,z^2)∈ $c_G(Y) = c_G(X \times X)$, then z^1∈ $c_G(X)$ and z^2∈ $c_G(X)$. Since X is a G-topological group, we can conclude that $In(z)$=($In(z^1)$, $In(z^2)$)∈ $c_G(Y)$. It is easy to verify that $G(In(z))$=($G(In(z^1))$, $G(In(z^2))$)=($In(G(z^1))$, $In(G(z^2))$)=$In(G(z))$ This proves that In is G-continuous. □

5. *G*-Connectedness and Total *G*-Disconnectedness in *G*-Topological Groups

In this section, we mainly discuss G-connectedness and total G-disconnectedness in G-topological groups. **we assume that** G **is a regular method preserving the subsequence.**

The following results play an important role in this section.

Proposition 5.1. ([18, Theorem 5.5]) *A fundamental system $\mathcal{B}_e$ of G-open neighborhoods of the neutral element e of a G-topological group X satisfies the following conditions:*

(1) If $U \in \mathcal{B}_e$, then there exists $V \in \mathcal{B}_e$ such that $V^{-1} \subset U$.

(2) The right translation and the left translation are all G-continuous.

Proposition 5.2. ([18, Theorem 5.7]) *Let $\mathcal{B}_e$ be a fundamental system at e of a G-topological group X. Then, for each $x \in X$, the family $\mathcal{B}_x = \{xU : U \in \mathcal{B}_e\}$ is a fundamental system of x.*

Proposition 5.3. ([18, Proposition 5.11]) *Let A be a subgroup with operations of a G-topological group X. If A is G-open, then it is G-closed.*

Proposition 5.4. ([18, Proposition 5.15]) *Let X be a G-topological group, U a G-open subset of X, and A any subset of X. Then the set AU (respectively, UA) is G-open in X.*

Proposition 5.5. ([18, Corollary 5.9]) *Suppose that X is a G-topological group, and assume that H is a G-topological subgroup with operations of X. If H contains a non-empty G-open subset of X then H is G-open in X.*

Proposition 5.6. ([6, Theorem 1]) *A G-continuous image of any G-connected subset of X is G-connected.*

Proposition 5.7. ([18, Proposition 4.4]) *If X and Y are G-connected, then $X \times Y$ is G-connected.*

In what follows we often use the next obvious statement.

Lemma 5.8. *Let X be a G-topological group. Let A and B be G-connected subsets of X, then the subsets A^{-1} and AB are G-connected.*

Proof. By Proposition 5.6, A^{-1} is G-connected as the image of the G-connected set A under the G-continuous mapping $x \mapsto x^{-1}$.

Since A and B are G-connected in X, $A \times B$ is G-connected in $X \times X$ by Proposition 5.7. The set AB is the image of $A \times B$ under the multiplication mapping which is G-continuous , and hence AB is G-connected. □

If $x \in X$, the G-connected component of x in X is denoted by $C_G(x)$. By [16, Theorem 4.4], we have the following proposition.

Proposition 5.9. *The G-connected component of $e \in X$ are closed, normal subgroup of X.*

Theorem 5.10. *Let X be a G-topological group and let e be the neutral element of X. Then $C_G(y) = yC_G(e) = C_G(e)y$ for every $y \in X$.*

Proof. Let y be a point of X. We show that $C_G(y) = yC_G(e) = C_G(e)y$. It follows from Proposition 5.1 that the mappings $x \mapsto yx$ and $x \mapsto y^{-1}x$ are G-continuous. Thus the sets $yC_G(e)$ and $y^{-1}C_G(y)$ are G-connected. By $y \in yC_G(e)$ and $e \in y^{-1}C_G(y)$, it is clear that $yC_G(e) \subset C_G(y)$ and $y^{-1}C_G(y) \subset C_G(e)$. Moreover, the inclusion $y^{-1}C_G(y) \subset C_G(e)$ is equivalent with the inclusion $C_G(y) \subset yC_G(e)$. By the foregoing, $yC_G(e) = C_G(y)$. Since the subgroup $C_G(e)$ is normal, $yC_G(e) = C_G(e)y$. □

The following lemma is clear by [6, Lemma 5].

Lemma 5.11. *Let X be a G-topological group, $A \subset X$, and U a G-open and G-closed subset of X. If A is G-connected, then either $A \subset U$ or $A \subset X \setminus U$.*

Lemma 5.12. *Let G_1, G_2 be methods on sets X and Y, respectively. Let f be a (G_1, G_2)-open mapping from X onto Y such that $f^{-1}(y)$ is G_1-connected for every $y \in Y$. Then $f^{-1}(D)$ is G_1-connected for every G_2-connected subset D in Y.*

Proof. Let $D \subset Y$ be G_2-connected. Put $E = f^{-1}(D)$ and $g = f|_E$. First, g is $(G_1|_E, G_2|_D)$-open. Let A be a $G_1|_E$-open subset of E, there exists a G_1-open B of X such that $A = B \cap E$. It follows that $g(A) = f(A) = f(B \cap E) = f(B \cap f^{-1}(D)) = f(B) \cap D$. Hence $f(B) \cap D$ is $G_2|_D$-open in D, since f is (G_1, G_2)-open. Next we shall prove that E is G_1-connected. Let U be a non-empty $G_1|_E$-clopen subset of E. We show that $U = E$. For every $x \in E$, the set $g^{-1}g(x) = f^{-1}f(x)$ is G_1-connected, and it follows from Lemma 5.11 that either $g^{-1}g(x) \subset U$ or $g^{-1}g(x) \cap U = \emptyset$. Thus $g(U) \cap g(E \setminus U) = \emptyset$. Since the set U is $G_1|_E$-clopen in E and the mapping g is $(G_1|_E, G_2|_D)$-open, the set $g(U)$ and $g(E \setminus U)$ are $(G_1|_E, G_2|_D)$-open in $g(E) = D$ of Y. We have $g(U) \cap g(E \setminus U) = \emptyset$ and $g(U) \cup g(E \setminus U) = D$. Since D is G_2-connected and $g(U) \neq \emptyset$, it follows that $g(E \setminus U) = \emptyset$. As a consequence, we have $E \setminus U = \emptyset$ and hence $U = E$. □

Proposition 5.13. ([18, Theorem 5.12]) *Every G-topological group X has a G-open base at the identity consisting of symmetric neighborhoods.*

Theorem 5.14. *Let X be a G-topological group, the following are equivalent:*

(1) *X has no proper G-open subgroup.*

(2) *$X = \bigcup_{n=1}^{\infty} V^n$ for every $V \in \mathcal{B}_e$, where $\mathcal{B}_e$ is a fundamental system at e.*

Proof. (1) ⇒ (2) For each $V \in \mathcal{B}_e$, there is a symmetric G-open neighborhood $U$of e such that $U \subset V$. Put $H = \bigcup_{n=1}^{\infty} U^n$. It is easy to verify that H is a G-open subgroup of X . Since X has no proper G-open subgroup, $X = H = \bigcup_{n=1}^{\infty} U^n \subset \bigcup_{n=1}^{\infty} V^n$, which completes our proof.

(2) ⇒ (1) Assume that (2) holds. To prove (1), let H be a G-open subgroup of X, there exists $V \in \mathcal{B}_e$ such that $V \subset H$. It follows from $\bigcup_{n=1}^{\infty} V^n = X$ that $H = X$. □

Corollary 5.15. *Let X be a G-connected G-topological group and $V \in \mathcal{B}_e$. Then $X = \bigcup_{n=1}^{\infty} V^n$.*

Recall that X/H denotes the set of all left cosets aH of H in H when X is a group and H is a subgroup of X.

The next corollary follows from Theorem 3.4.

Corollary 5.16. *Suppose that X is a G_1-topological group and $c_{G_1}(X) = s(X)$. Assume that H is a G_1-topological subgroup with operations of X. Let G_2 be a method on X/H. If the natural mapping $\pi : X \to X/H$ is (G_1, G_2)-quotient, then π is (G_1, G_2)-open.*

Proof. Let U be a G_1-open of X, then $UH = \pi^{-1}(\pi(U))$ be a G_1-open of X. thus $\pi(U)$ is a G_2-open of X/H, so π is (G_1, G_2)-open. □

Theorem 5.17. *Let X be a G_1-topological group, $c_{G_1}(X) = s(X)$ and H be a G_1-connected subgroup of X. If the natural mapping $\pi : X \to X/H$ is (G_1, G_2)-quotient and X/H is G_2-connected, then X is G_1-connected.*

Proof. By Corollary 5.16, π is (G_1, G_2)-open. For every $y \in X/H$, $\pi^{-1}(y) = yH$. Since X is a G_1-topological group, the right translation is G_1-continuous, thus $\varrho_y(H) = yH$ is G_1-connected. Since X/H is G_2-connected, by Lemma 5.12, it follows that X is G_1-connected. □

X is called totally G-disconnected when $C_G(x) = \{x\}$ for every $x \in X$.

Theorem 5.18. *Let X be a G_1-topological group and $c_{G_1}(x) = s(X)$. Suppose C is the G_1-connected component of the neutral element of X and G_2 is a method on X/C. If natural mapping $\pi_C : X \to X/C$ is (G_1, G_2)-continuous, then X/C is totally G_2-disconnected.*

Proof. By Proposition 5.9, C is a normal subgroup of X. We shall show that X/C is totally G_2-disconnected. Denote by D the G_2-connected component of the neutral element C of X/C. By Theorem 5.10, totally G_2-disconnectedness of X/C follows once we show that $D = \{C\}$.

By Corollary 5.16, π_C is (G_1, G_2)-open. Since the subgroup C of X is G_1-connected, it follows from Lemma 5.12 that the subset $\pi_C^{-1}(D)$ of X is G_1-connected. Since $C \in D$, we have $e \in \pi_C^{-1}(D)$. From the foregoing it follows that the set $\pi_C^{-1}(D)$ is contained in the G_1-connected component C of e. As a consequence, we have $\pi_C(\pi_C^{-1}(D)) \subset \{C\}$, in other words, $D \subset \{C\}$. It follows, since $C \in D$, that $D = \{C\}$. □

Theorem 5.19. *Every totally G-disconnected normal subgroup of a G-connected G-topological group is contained in the center of the G-topological group.*

Proof. Let H be a totally G-disconnected normal subgroup of a G-connected G-topological group X, and let $h \in H$. The following we shall prove that $xh = hx$ for every $x \in X$. We define a mapping $f : X \to X$ by the formula $f(x) = xhx^{-1}$, and we note that f is G-continuous. Since H is a normal subgroup, we have $xhx^{-1} \in H$ for every $x \in X$. As a consequence, $f(X) \subset H$. The G-continuous image $f(X)$ of the G-connected set X is G-connected. Since H is totally G-disconnected, it follows that the set $f(X)$ is a singleton. Since $f(e) = h$, it follows from the foregoing, that $f(X) = \{h\}$. Hence we have, for every $x \in X$, that $xhx^{-1} = h$, i.e. that $xh = hx$. □

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