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Some Inequalities in Quasi-Banach Algebra of Non-Newtonian Bicomplex Numbers

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Abstract. In this paper, we construct the quasi-Banach algebra $\mathbb{BC}(N)$ of non-Newtonian bicomplex numbers and we generalize some topological concepts and inequalities as Schwarz's, Hölder's and Minkowski's in the set of bicomplex numbers in the sense of non-Newtonian calculus.

1. Introduction and Preliminaries

In 1972, Grossman and Katz [1] pointed out to different calculus, called non-Newtonian calculus consisting of some special calculus such as geometric, bigeometric, quadratic, biquadratic calculus, and so forth, which modify the calculus created by Isaac Newton and Gottfried Wilhelm Leibnitz in the 17th century. The non-Newtonian calculus provides a wide diversity of mathematical tools for use in technology and mathematics. Also, it has wonderful applications in various areas including engineering, physics, finance, approximation theory, dynamical systems, tumor therapy, weighted calculus etc. There is a provision in each member of non-Newtonian calculus class of all concepts used in classic calculus.

Recently, Tekin and Başar [2] obtained some sequence spaces over non-Newtonian complex field by defining non-Newtonian complex field. Çakmak and Başar [3] constructed the space of continuous functions on the non-Newtonian complex field and gave some important features. Kadak and Efe [4] studied Hilbert spaces and examined Cauchy-Schwarz and triangle inequalities in terms of $*-$calculus. For further works we refer the reader to [5–7].

In 1892, Corrado Segre defined the concept of bicomplex numbers [8]. There are many works in the bicomplex setting. They can be found in [9–11]. Sager and Sağır [12] obtained well - known Hölder's and Minkowski's inequalities for sums in the bicomplex numbers, introduce bicomplex sequence spaces with Euclidean norm in the set of bicomplex numbers and study completeness property of the spaces.

In the literature, it has been observed that bicomplex numbers have not been defined according to non-Newtonian calculus. This idea enabled us to apply $*-$calculus to definition and the algebraic and topological properties of the set of bicomplex numbers.

Following the same line, the main aim of our this study is to establish the quasi-Banach algebra $\mathbb{BC}(N)$ by defining non-Newtonian bicomplex numbers as a generalization of both bicomplex numbers and non-Newtonian complex numbers and also, examine the validity of the non-Newtonian bicomplex version of the well-known Hölder's and Minkowski's inequalities for sums.

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The followings will be needed in the sequel.

Let i and j be independent imaginary units such that $i^2 = j^2 = -1$, $ij = ji$ and $\mathbb{C}(i)$ be the set of complex numbers with the imaginary unit i. The set of bicomplex numbers $\mathbb{BC}$ is defined by

$$\mathbb{BC} = \{z = z_1 + jz_2 : z_1, z_2 \in \mathbb{C}(i)\}.$$

The set $\mathbb{BC}$ forms a Banach algebra with respect to the addition, scalar multiplication, multiplication and norm for all $z, w \in \mathbb{BC}$, $\lambda \in \mathbb{R}$ defined as [13]

$$\begin{aligned} z + w &= (z_1 + jz_2) + (w_1 + jw_2) = (z_1 + w_1) + j(z_2 + w_2), \\ \lambda.z &= \lambda.(z_1 + jz_2) = \lambda z_1 + j\lambda z_2, \\ z \times w &= (z_1 + jz_2).(w_1 + jw_2) = (z_1w_1 - z_2w_2) + j(z_1w_2 + z_2w_1), \\ \|.\| &: \mathbb{BC} \to \mathbb{R}, z \to \|z\| = \sqrt{|z_1|^2 + |z_2|^2}. \end{aligned}$$

A complete ordered field is called arithmetic if its realm is a subset of $\mathbb{R}$. A generator is a one-to-one function whose domain $\mathbb{R}$ and whose range is a subset of $\mathbb{R}$. Let α be a generator with range A. We denote by $\mathbb{R}_\alpha$ the range of generator α. Also, the elements of $\mathbb{R}_\alpha$ are called non-Newtonian real numbers.

Let α and β be arbitrarily chosen generators which image the set $\mathbb{R}$ to A and B respectively and $*-$(″star-″) calculus also be the ordered pair of arithmetics ($\alpha-$ arithmetic, $\beta-$ arithmetic). The following notations will be used. All definitions given for $\alpha-$arithmetic are also valid for $\beta-$arithmetic.

	$\alpha-$arithmetic	$\beta-$ arithmetic
Realm	$A\,(= \mathbb{R}_\alpha)$	$B\left(= \mathbb{R}_\beta\right)$
Summation	$y \dot{+} z = \alpha\left\{\alpha^{-1}(y) + \alpha^{-1}(z)\right\}$	$\ddot{+}$
Subtraction	$y \dot{-} z = \alpha\left\{\alpha^{-1}(y) - \alpha^{-1}(z)\right\}$	$\ddot{-}$
Multiplication	$y \dot{\times} z = \alpha\left\{\alpha^{-1}(y) \times \alpha^{-1}(z)\right\}$	$\ddot{\times}$
Division	$y \dot{/} z = \alpha\left\{\frac{\alpha^{-1}(y)}{\alpha^{-1}(z)}\right\}\ \left(z \neq \dot{0}\right)$	$\ddot{/}$
Ordering	$y \dot{\leq} z \Longleftrightarrow \alpha^{-1}(y) \leq \alpha^{-1}(z)$	$\ddot{\leq}$

There are the following three properties for the isomorphism from $a-$arithmetic to $\beta-$arithmetic that is the unique function ι(iota).

1. ι is one-to-one.
2. ι is on A and onto B.
3. For all $u,v \in A$,

$$\begin{aligned} \iota(u \dot{+} v) &= \iota(u) \ddot{+} \iota(v),\ \iota(u \dot{-} v) = \iota(u) \ddot{-} \iota(v), \\ \iota(u \dot{\times} v) &= \iota(u) \ddot{\times} \iota(v),\ \iota\left(u \dot{/} v\right) = \iota(u) \ddot{/} \iota(v), v \neq \dot{0} \\ u &\dot{<} v \Longleftrightarrow \iota(u) \ddot{<} \iota(v), \\ \iota(u) &= \beta\left\{\alpha^{-1}(u)\right\}, \end{aligned}$$

Also, for every integer n, we set $\iota(\dot{n}) = \ddot{n}$ [1].

A $\alpha-$ positive number is a number x with $\dot{0} \dot{<} x$ and a $\alpha-$negative number is a number with $x \dot{<} \dot{0}$. $\alpha-$zero and $\alpha-$one numbers are denoted by $\dot{0} = \alpha(0)$ and $\dot{1} = \alpha(1)$, and the set of $\alpha-$ positive numbers is denoted by $\mathbb{R}_\alpha^+$. Also, $\alpha(-p) = \alpha\left\{\alpha^{-1}\left(\ddot{p}\right)\right\} = \dot{-}\dot{p}$ for all $p \in \mathbb{Z}^+$. The $\alpha-$absolute value of $x \in A$ is defined by

$$\dot{|}\, x\, \dot{|} = \begin{cases} x, & if\ \dot{0} \dot{<} x \\ \dot{0}, & if\ \dot{0} = x \\ \dot{0} \dot{-} x, & if\ x \dot{<} \dot{0}. \end{cases}$$

The definitions of $\alpha-$convergence of a sequence of the elements in A and $*-$ limit, and $*-$continuity of the function $f : X \subset A \rightarrow B$ are found in [1].

The following definitions which appeared in [2] will be a crucial tool in our study.

Let $\dot{a} \in \left(A, \dot{+}, \dot{-}, \dot{\times}, \dot{/}, \dot{\leq}\right)$ and $\ddot{b} \in \left(B, \ddot{+}, \ddot{-}, \ddot{\times}, \ddot{/}, \ddot{\leq}\right)$ be arbitrarily chosen elements from corresponding arithmetics. Then, the ordered pair $\left(\dot{a}, \ddot{b}\right)$ is called as a $*-$ point. The set of all $*-$ points is called the set of $*-$complex numbers (non-Newtonian complex numbers) and is denoted by $\mathbb{C}^*$ or $\mathbb{C}(N)$ that is,

$$\mathbb{C}(N) = \left\{\left(\dot{a}, \ddot{b}\right) : \dot{a} \in A \subseteq \mathbb{R}, \ddot{b} \in B \subseteq \mathbb{R}\right\}.$$

The set $\mathbb{C}(N)$ forms a field with respect to addition $\oplus_1$ and multiplication $\otimes_1$ for all $z_1^* = \left(\dot{a_1}, \ddot{b_1}\right)$, $z_2^* = \left(\dot{a_2}, \ddot{b_2}\right) \in \mathbb{C}(N)$ defined as

$$\begin{aligned}
\oplus_1 &: \quad \mathbb{C}(N) \times \mathbb{C}(N) \rightarrow \mathbb{C}(N), \\
\left(z_1^*, z_2^*\right) &\rightarrow \quad z_1^* \oplus_1 z_2^* = \left(\dot{a_1}, \ddot{b_1}\right) \oplus_1 \left(\dot{a_2}, \ddot{b_2}\right) = \left(\dot{a_1} \dot{+} \dot{a_2}, \ddot{b_1} \ddot{+} \ddot{b_2}\right), \\
\otimes_1 &: \quad \mathbb{C}(N) \times \mathbb{C}(N) \rightarrow \mathbb{C}(N), \\
\left(z_1^*, z_2^*\right) &\rightarrow \quad z_1^* \otimes_1 z_2^* = \left(\dot{a_1}, \ddot{b_1}\right) \otimes_1 \left(\dot{a_2}, \ddot{b_2}\right) = \left(\alpha\left(a_1 a_2 - b_1 b_2\right), \beta\left(a_1 b_2 + b_1 a_2\right)\right).
\end{aligned}$$

Let $\ddot{b} \in B \subseteq \mathbb{R}$. Then, the number $\ddot{b} \ddot{\times} \ddot{b}$ is called the $\beta-$ square of $\ddot{b}$ and is denoted by $b^{\ddot{2}}$. Let $\ddot{b}$ be a nonnegative number in B. Then, $\beta\left[\sqrt{\beta^{-1}\left(\ddot{b}\right)}\right]$ is called the $\beta-$ square root of $\ddot{b}$ and is denoted by $\sqrt[\ddot{}]{\ddot{b}}$. The $*-$ distance d_* between two elements $z_1^* = \left(\dot{a_1}, \ddot{b_1}\right)$ and $z_2^* = \left(\dot{a_2}, \ddot{b_2}\right)$ of the set $\mathbb{C}(N)$ is defined by

$$\begin{aligned}
d_* &: \quad \mathbb{C}(N) \times \mathbb{C}(N) \rightarrow \left[\ddot{0}, \ddot{\infty}\right) \subset B \\
\left(z_1^*, z_2^*\right) &\rightarrow \quad d_*\left(z_1^*, z_2^*\right) = \sqrt[\ddot{}]{\left[\iota\left(\dot{a_1} \dot{-} \dot{a_2}\right)\right]^{\ddot{2}} \ddot{+} \left(\ddot{b_1} \ddot{-} \ddot{b_2}\right)^{\ddot{2}}} = \beta\left[\sqrt{\left(a_1 - a_2\right)^2 + \left(b_1 - b_2\right)^2}\right].
\end{aligned}$$

Let $z^* \in \mathbb{C}(N)$. Then, $d_*\left(z^*, 0^*\right)$ is called $*-$ norm of z^* and is denoted by $\ddot{\|} \,.\, \ddot{\|}_1$. In other words,

$$\ddot{\|}\, z^* \,\ddot{\|}_1 = d_*\left(z^*, 0^*\right) = \sqrt[\ddot{}]{\left[\iota\left(\dot{a} \dot{-} \dot{0}\right)\right]^{\ddot{2}} \ddot{+} \left(\ddot{b} \ddot{-} \ddot{0}\right)^{\ddot{2}}} = \beta\left(\sqrt{a^2 + b^2}\right),$$

where $z^* = \left(\dot{a}, \ddot{b}\right)$ and $0^* = \left(\dot{0}, \ddot{0}\right)$.

Now we are ready to present and discuss our main results.

2. Main Results

2.1. $*-$Bicomplex Numbers (Non-Newtonian Bicomplex Numbers)

In this section, we define bicomplex numbers and bicomplex sequences in the sense of non-Newtonian calculus and we prove that the set of non-Newtonian bicomplex numbers $\mathbb{BC}(N)$ is a Banach space with respect to the norm $\ddot{\|} \,.\, \ddot{\|}_2$.

Definition 2.1. *Let $\dot{a}, \dot{c} \in \left(A, \dot{+}, \dot{-}, \dot{\times}, \dot{/}, \dot{\leq}\right)$ and $\ddot{b}, \ddot{d} \in \left(B, \ddot{+}, \ddot{-}, \ddot{\times}, \ddot{/}, \ddot{\leq}\right)$. Then, $\left(\dot{a}, \ddot{b}, \dot{c}, \ddot{d}\right)$ is called as a $*-$ bicomplex point. The set of all $*-$ bicomplex points is called the set of $*-$bicomplex numbers (non-Newtonian bicomplex numbers) and is denoted by $\mathbb{BC}^*$ or $\mathbb{BC}(N)$; that is,*

$$\begin{aligned}
\mathbb{BC}(N) &= \left\{\left(\dot{a}, \ddot{b}, \dot{c}, \ddot{d}\right) : \dot{a}, \dot{c} \in A \subseteq \mathbb{R}, \ddot{b}, \ddot{d} \in B \subseteq \mathbb{R}\right\} \\
&= \left\{\left(z^*, w^*\right) : z^* = \left(\dot{a}, \ddot{b}\right), w^* = \left(\dot{c}, \ddot{d}\right), \dot{a}, \dot{c} \in A \subseteq \mathbb{R}, \ddot{b}, \ddot{d} \in B \subseteq \mathbb{R}\right\}
\end{aligned}$$

The algebraic operations addition $\oplus_2$, multiplication $\otimes_2$ and scalar multiplication $\odot_2$ defined on $\mathbb{BC}(N)$ as follows:

$$\begin{array}{rcl} \oplus_2 & : & \mathbb{BC}(N)\times\mathbb{BC}(N)\to\mathbb{BC}(N), \\ \left(\zeta_1^*,\zeta_2^*\right) & \to & \zeta_1^*\oplus_2\zeta_2^* = \left(z_1^*,w_1^*\right)\oplus_2\left(z_2^*,w_2^*\right) = \left(z_1^*\oplus_1 z_2^*, w_1^*\oplus_1 w_2^*\right), \\ \otimes_2 & : & \mathbb{BC}(N)\times\mathbb{BC}(N)\to\mathbb{BC}(N), \\ \left(\zeta_1^*,\zeta_2^*\right) & \to & \zeta_1^*\otimes_2\zeta_2^* = \left(z_1^*,w_1^*\right)\otimes_2\left(z_2^*,w_2^*\right) = \left(\left(z_1^*\otimes_1 z_2^*\right)\ominus_1\left(w_1^*\otimes_1 w_2^*\right), \left(z_1^*\otimes_1 w_2^*\right)\oplus_1\left(z_2^*\otimes_1 w_1^*\right)\right), \\ \odot_2 & : & \mathbb{C}(N)\times\mathbb{BC}(N)\to\mathbb{BC}(N), \\ \left(z^*,\zeta_1^*\right) & \to & z^*\odot_2\zeta_1^* = z^*\odot_2\left(z_1^*,w_1^*\right) = \left(z^*\otimes_1 z_1^*, z^*\otimes_1 w_1^*\right) \end{array}$$

where $\zeta_1^* = \left(z_1^*,w_1^*\right)$, $\zeta_2^* = \left(z_2^*,w_2^*\right) \in \mathbb{BC}(N)$ and $z^* \in \mathbb{C}(N)$. According to the these operations, it can simply be shown that the set $\mathbb{BC}(N)$ forms a $*-$vector space over the field $\mathbb{C}(N)$ and a ring.

Remark 2.2. *We can denote non-Newtonian complex number $z^* = \left(\dot{a},\ddot{b}\right)$ by $\left(\dot{a},\ddot{0}\right)\oplus_1 i^*\otimes_1\left(\dot{0},\ddot{b}\right) = \dot{a}\oplus_1 i^*\otimes_1\ddot{b}$ where $i^* = \left(\dot{0},\ddot{1}\right) = \left(\dot{0},\ddot{1},\dot{0},\ddot{0}\right), (i^*)^2 = \ominus_1 1^*$. Also, we can denote non-Newtonian bicomplex number $\zeta^* = (z^*,w^*)$ by $(z^*,0^*)\oplus_2 j^*\otimes_2(w^*,0^*) = z^*\oplus_2 j^*\otimes_2 w^* = \left(\dot{a},\ddot{b}\right)\oplus_2 j^*\otimes_2\left(\dot{c},\ddot{d}\right)$ where $z^* = \left(\dot{a},\ddot{b}\right), w^* = \left(\dot{c},\ddot{d}\right), j^* = \left(\dot{0},\ddot{0},\dot{1},\ddot{0}\right) = (0^*,1^*), (j^*)^2 = \ominus_2 1^*$ and also define z^* and w^* by $\Re\zeta^*$ and $\Im\zeta^*$, respectively.*

Definition 2.3. *The $*-$distance $d_{\mathbb{BC}(N)}$ between two arbitrarily elements $\zeta_1^* = z_1^*\oplus_2 j^*\otimes_2 w_1^*, \zeta_2^* = z_2^*\oplus_2 j^*\otimes_2 w_2^*$ of the set $\mathbb{BC}(N)$ is defined by*

$$\begin{array}{rcl} d_{\mathbb{BC}(N)} & : & \mathbb{BC}(N)\times\mathbb{BC}(N)\to\left[\ddot{0},\infty\right)\subset B, \\ \left(\zeta_1^*,\zeta_2^*\right) & \to & d_{\mathbb{BC}(N)}\left(\zeta_1^*,\zeta_2^*\right) = \ddot{\sqrt{\| z_1^*\ominus_1 z_2^*\|_1^{\ddot{2}}\ddot{+}\| w_1^*\ominus_1 w_2^*\|_1^{\ddot{2}}}}. \end{array}$$

Theorem 2.4. *The $*-$distance $d_{\mathbb{BC}(N)}$ is a metric on $\mathbb{BC}(N)$.*

Proof. It is trivial that $d_{\mathbb{BC}(N)}$ satisfies the metric axioms on $\mathbb{BC}(N)$. We only show that $d_{\mathbb{BC}(N)}\left(\zeta_1^*,\zeta_2^*\right)\ddot{\leq} d_{\mathbb{BC}(N)}\left(\zeta_1^*,\zeta_3^*\right)\ddot{+}d_{\mathbb{BC}(N)}\left(\zeta_3^*,\zeta_2^*\right)$ for all $\zeta_1^*,\zeta_2^*,\zeta_3^*\in\mathbb{BC}(N)$. Let $\zeta_1^* = z_1^*\oplus_2 j^*\otimes_2 w_1^*, \zeta_2^* = z_2^*\oplus_2 j^*\otimes_2 w_2^*$ and $\zeta_3^* = z_3^*\oplus_2 j^*\otimes_2 w_3^*$. Then,

$$\begin{array}{rcl} d_{\mathbb{BC}(N)}\left(\zeta_1^*,\zeta_2^*\right) & = & \ddot{\sqrt{\| z_1^*\ominus_1 z_2^*\|_1^{\ddot{2}}\ddot{+}\| w_1^*\ominus_1 w_2^*\|_1^{\ddot{2}}}} \\ & = & \ddot{\sqrt{\|\left(z_1^*\ominus_1 z_3^*\right)\oplus_1\left(z_3^*\ominus_1 z_2^*\right)\|_1^{\ddot{2}}\ddot{+}\|\left(w_1^*\ominus_1 w_3^*\right)\oplus_1\left(w_3^*\ominus_1 w_2^*\right)\|_1^{\ddot{2}}}} \\ & \ddot{\leq} & \ddot{\sqrt{\left[\| z_1^*\ominus_1 z_3^*\|_1\ddot{+}\| z_3^*\ominus_1 z_2^*\|_1\right]^{\ddot{2}}\ddot{+}\left[\| w_1^*\ominus_1 w_3^*\|_1\ddot{+}\| w_3^*\ominus_1 w_2^*\|_1\right]^{\ddot{2}}}} \\ & = & \ddot{\sqrt{\begin{array}{c}\left[\| z_1^*\ominus_1 z_3^*\|_1^{\ddot{2}}\ddot{+}\| z_3^*\ominus_1 z_2^*\|_1^{\ddot{2}}\ddot{+}\ddot{2}\ddot{\times}\| z_1^*\ominus_1 z_3^*\|_1\ddot{\times}\| z_3^*\ominus_1 z_2^*\|_1\right] \\ \ddot{+}\left[\| w_1^*\ominus_1 w_3^*\|_1^{\ddot{2}}\ddot{+}\| w_3^*\ominus_1 w_2^*\|_1^{\ddot{2}}\ddot{+}\ddot{2}\ddot{\times}\| w_1^*\ominus_1 w_3^*\|_1\ddot{\times}\| w_3^*\ominus_1 w_2^*\|_1\right]\end{array}}} \\ & \ddot{\leq} & \ddot{\sqrt{\| z_1^*\ominus_1 z_3^*\|_1^{\ddot{2}}\ddot{+}\| w_1^*\ominus_1 w_3^*\|_1^{\ddot{2}}}}\ddot{+}\ddot{\sqrt{\| z_3^*\ominus_1 z_2^*\|_1^{\ddot{2}}\ddot{+}\| w_3^*\ominus_1 w_2^*\|_1^{\ddot{2}}}} \\ & = & d_{\mathbb{BC}(N)}\left(\zeta_1^*,\zeta_3^*\right)\ddot{+}d_{\mathbb{BC}(N)}\left(\zeta_3^*,\zeta_2^*\right), \end{array}$$

as required. Therefore, the function $d_{\mathbb{BC}(N)}$ is a metric on $\mathbb{BC}(N)$ and so, $\left(\mathbb{BC}(N), d_{\mathbb{BC}(N)}\right)$ is a metric space. □

Definition 2.5. *A sequence in* $\mathbb{BC}(N)$ *is a function defined by* $s : \mathbb{N} \to \mathbb{BC}(N), n \to s_n^*$. *This sequence is called a non-Newtonian bicomplex sequence. It converges to a limit* $s^* \in \mathbb{BC}(N)$ *with respect to the metric* $d_{\mathbb{BC}(N)}$ *if and only if to each* $\varepsilon \ddot{>} \ddot{0}$ *there corresponds a natural number* $n_0(\varepsilon) \in \mathbb{N}$ *such that* $d_{\mathbb{BC}(N)}(s_n^*, s^*) \ddot{<} \varepsilon$ *for all* $n \geq n_0(\varepsilon)$. *It is denoted by* $\lim_{n\to\infty}{}^* s_n^* = s^*$. *The sequence* (s_n^*) *is a Cauchy sequence with respect to the metric* $d_{\mathbb{BC}(N)}$ *if and only if to each* $\varepsilon \ddot{>} \ddot{0}$ *there corresponds a natural number* $n_0(\varepsilon) \in \mathbb{N}$ *such that* $d_{\mathbb{BC}(N)}(s_n^*, s_m^*) \ddot{<} \varepsilon$ *for all* $n, m \geq n_0(\varepsilon)$.

Theorem 2.6. *If* $s : \mathbb{N} \to \mathbb{BC}(N), n \to s_n^*$ *is a non-Newtonian bicomplex sequence,* $s^* = z^* \oplus_2 j^* \otimes_2 w^* \in \mathbb{BC}(N)$ *and*

(1) $s_n^* = z_n^* \oplus_2 j^* \otimes_2 w_n^*$, $\lim_{n\to\infty}{}^* s_n^* = \lim_{n\to\infty}{}^* (z_n^* \oplus_2 j^* \otimes_2 w_n^*) = s^*$,

then, the following limits exist and have the values shown respectively:

(2) $\lim_{n\to\infty}{}^* z_n^* = z^*$, $\lim_{n\to\infty}{}^* w_n^* = w^*$,

Moreover, if the limits exist as indicated in (2), then $\lim_{n\to\infty}{}^* s_n^*$ *exists as stated in (1), and* $\lim_{n\to\infty}{}^* s_n^* = s^*$.

Proof. The proof is application of definition of non-Newtonian bicomplex sequences. □

Definition 2.7. *Let* (s_n^*) *be a non-Newtonian bicomplex sequence. Then, the infinite sum*

$$_{\oplus_2}\sum_{k=1}^{\infty} s_k^* = s_1^* \oplus_2 s_2^* \oplus_2 ... \oplus_2 s_n^* \oplus_2 ... \qquad (1)$$

is called a non-Newtonian bicomplex series. Define the non-Newtonian bicomplex sequence $S : \mathbb{N} \to \mathbb{BC}(N), n \to S_n^*$ *by setting* $S_n^* = {}_{\oplus_2}\sum_{k=1}^{n} s_k^*$ *for every* $n \in \mathbb{N}$. *The infinite series (2.1) converges to a limit* $S^* \in \mathbb{BC}(N)$ *with respect to the metric* $d_{\mathbb{BC}(N)}$ *if and only if* (S_n^*) *converges to a limit* $S^* \in \mathbb{BC}(N)$ *with respect to the metric* $d_{\mathbb{BC}(N)}$. *Then,* S^* *is called the sum of non-Newtonian bicomplex series, and we write*

$$_{\oplus_2}\sum_{k=1}^{\infty} s_k^* = S^*. \qquad (2)$$

Theorem 2.8. *In (2.2), let* $s : \mathbb{N} \to \mathbb{BC}(N), n \to s_n^* = z_n^* \oplus_2 j^* \otimes_2 w_n^*$ *be a non-Newtonian bicomplex sequence,* $S^* = z^* \oplus_2 j^* \otimes_2 w^* \in \mathbb{BC}(N)$. *Then, the infinite series (2.1) converges to a limit* $S^* \in \mathbb{BC}(N)$ *with respect to the metric* $d_{\mathbb{BC}(N)}$ *and has the sum* S^* *if and only if the following infinite series converge and have the sums shown:*

$$_{\oplus_1}\sum_{k=1}^{\infty} z_k^* = z^*, {}_{\oplus_1}\sum_{k=1}^{\infty} w_k^* = w^*.$$

Proof. The proof is application of definition of non-Newtonian bicomplex series. □

Theorem 2.9. *The metric space* $\mathbb{BC}(N)$ *is complete with respect to the metric* $d_{\mathbb{BC}(N)}$.

Proof. Let $(s_n^*) = (z_n^* \oplus_2 j^* \otimes_2 w_n^*)$ be a Cauchy sequence with respect to the metric $d_{\mathbb{BC}(N)}$. Then, there exists a natural number $n_0(\varepsilon) \in \mathbb{N}$ such that $d_{\mathbb{BC}(N)}(s_n^*, s_m^*) \ddot{<} \varepsilon$ for all $n, m \geq n_0(\varepsilon)$. Then,

$$d_{\mathbb{BC}(N)}(s_n^*, s_m^*) = \beta\left(\sqrt{\left|z_1^n - z_1^m\right|^2 + \left|w_1^n - w_1^m\right|^2}\right) \ddot{<} \varepsilon = \beta(\varepsilon').$$

In this case, the inequality $\sqrt{\left|z_1^n - z_1^m\right|^2 + \left|w_1^n - w_1^m\right|^2} < \varepsilon'$ holds. Thus, we obtain that $\left|z_1^n - z_1^m\right| < \varepsilon'$ and $\left|w_1^n - w_1^m\right| < \varepsilon'$. Then, $\left(z_1^n\right)$ and $\left(w_1^n\right)$ are complex Cauchy sequences. Since $\mathbb{C}$ is complete with respect to the norm $|.|$, we can see that for every $\varepsilon' > 0$ there exists a natural number $n_1(\varepsilon), n_2(\varepsilon) \in \mathbb{N}$ and $z, w \in \mathbb{C}$ such that $\left|z_1^n - z\right| < \frac{\varepsilon'}{2}$ for all $n \geq n_1(\varepsilon)$ and $\left|w_1^n - w\right| < \frac{\varepsilon'}{2}$ for all $n \geq n_2(\varepsilon)$.

Now, define $s^* = z^* \oplus_2 j^* \otimes_2 w^*$ where $z^*, w^* \in \mathbb{C}(N)$. Then, we get

$$\begin{aligned} d_{\mathbb{BC}(N)}(s_n^*, s^*) &= \beta\left(\sqrt{\left|z_1^n - z\right|^2 + \left|w_1^n - w\right|^2}\right) \\ &\ddot{\leq} \beta\left(\sqrt{\left|z_1^n - z\right|^2} + \sqrt{\left|w_1^n - w\right|^2}\right) \\ &= \beta\left(\left|z_1^n - z\right| + \left|w_1^n - w\right|\right) \\ &\ddot{\leq} \beta\left(\frac{\varepsilon'}{2} + \frac{\varepsilon'}{2}\right) \\ &= \beta(\varepsilon') = \varepsilon. \end{aligned}$$

Hence, the sequence (s_n^*) converges to $s^* \in \mathbb{BC}(N)$. Thus, $\left(\mathbb{BC}(N), d_{\mathbb{BC}(N)}\right)$ is a complete metric space. □

Remark 2.10. *Theorem 2.9 says that the space $\mathbb{BC}(N)$ is complete with the metric $d_{\mathbb{BC}(N)}$ induced by the norm $\ddot{\|}\,.\,\ddot{\|}_2$ defined by*

$$\ddot{\|}\,\zeta^*\,\ddot{\|}_2 = d_{\mathbb{BC}(N)}(\zeta^*, 0^*) = \ddot{\sqrt{\ddot{\|}\,z^*\,\ddot{\|}_1^{\ddot{2}} \ddot{+} \ddot{\|}\,w^*\,\ddot{\|}_1^{\ddot{2}}}}$$

for $\zeta^* = z^* \oplus_2 j^* \otimes_2 w^* \in \mathbb{BC}(N)$.

Corollary 2.11. $\mathbb{BC}(N)$ *is a Banach space with respect to the norm $\ddot{\|}\,.\,\ddot{\|}_2$.*

2.2. *Some inequalities in $\mathbb{BC}(N)$ with respect to the norm $\ddot{\|}\,.\,\ddot{\|}_2$*

In this section, we obtain that the system $\left(\mathbb{BC}(N), \oplus_2, \odot_2, \ddot{\|}\,.\,\ddot{\|}_2, \otimes_2\right)$ is a quasi-Banach algebra by using some inequalities in $\mathbb{BC}(N)$ with respect to the norm $\ddot{\|}\,.\,\ddot{\|}_2$ which are discussed in this part. Also, we generalize Hölder's and Minkowski's inequalities in the set of bicomplex numbers to the set of non-Newtonian bicomplex numbers.

Lemma 2.12. *Let $\zeta_1^*, \zeta_2^* \in \mathbb{BC}(N)$. Then, the following inequalities are satisfied:*

i) $\ddot{\|}\,\zeta_1^* \oplus_2 \zeta_2^*\,\ddot{\|}_2 \ddot{\leq} \ddot{\|}\,\zeta_1^*\,\ddot{\|}_2 \ddot{+} \ddot{\|}\,\zeta_2^*\,\ddot{\|}_2$.

ii) $\ddot{\|}\,\zeta_1^* \ominus_2 \zeta_2^*\,\ddot{\|}_2 \ddot{\leq} \ddot{\|}\,\zeta_1^*\,\ddot{\|}_2 \ddot{+} \ddot{\|}\,\zeta_2^*\,\ddot{\|}_2$.

iii) $\ddot{\left|\ddot{\|}\,\zeta_1^*\,\ddot{\|}_2 \ddot{-} \ddot{\|}\,\zeta_2^*\,\ddot{\|}_2\right|} \ddot{\leq} \ddot{\|}\,\zeta_1^* \ominus_2 \zeta_2^*\,\ddot{\|}_2$.

iv) $\ddot{\left|\ddot{\|}\,\zeta_1^*\,\ddot{\|}_2 \ddot{-} \ddot{\|}\,\zeta_2^*\,\ddot{\|}_2\right|} \ddot{\leq} \ddot{\|}\,\zeta_1^* \oplus_2 \zeta_2^*\,\ddot{\|}_2$.

v) $\dfrac{\ddot{\|}\zeta_1^* \oplus_2 \zeta_2^*\ddot{\|}_2}{\ddot{1} \ddot{+} \ddot{\|}\zeta_1^* \oplus_2 \zeta_2^*\ddot{\|}_2} \ddot{\leq} \dfrac{\ddot{\|}\zeta_1^*\ddot{\|}_2}{\ddot{1} \ddot{+} \ddot{\|}\zeta_1^*\ddot{\|}_2} \ddot{+} \dfrac{\ddot{\|}\zeta_2^*\ddot{\|}_2}{\ddot{1} \ddot{+} \ddot{\|}\zeta_2^*\ddot{\|}_2}$.

Proof. i) Let $\zeta_1^* = z_1^* \oplus_2 j^* \otimes_2 w_1^*, \zeta_2^* = z_2^* \oplus_2 j^* \otimes_2 w_2^* \in \mathbb{BC}(N)$. By using Lemma 2 in [2], we have

$$\begin{aligned} \ddot{\|}\,\zeta_1^* \oplus_2 \zeta_2^*\,\ddot{\|}_2 &= \ddot{\sqrt{\ddot{\|}\,z_1^* \oplus_1 z_2^*\,\ddot{\|}_1^{\ddot{2}} \ddot{+} \ddot{\|}\,w_1^* \oplus_1 w_2^*\,\ddot{\|}_1^{\ddot{2}}}} \\ &\ddot{\leq} \ddot{\sqrt{\left(\ddot{\|}\,z_1^*\,\ddot{\|}_1^{\ddot{2}} \ddot{+} \ddot{\|}\,z_2^*\,\ddot{\|}_1^{\ddot{2}}\right) \ddot{+} \left(\ddot{\|}\,w_1^*\,\ddot{\|}_1^{\ddot{2}} \ddot{+} \ddot{\|}\,w_2^*\,\ddot{\|}_1^{\ddot{2}}\right)}} \\ &\ddot{\leq} \ddot{\sqrt{\ddot{\|}\,z_1^*\,\ddot{\|}_1^{\ddot{2}} \ddot{+} \ddot{\|}\,w_1^*\,\ddot{\|}_1^{\ddot{2}}}} \ddot{+} \ddot{\sqrt{\ddot{\|}\,z_2^*\,\ddot{\|}_1^{\ddot{2}} \ddot{+} \ddot{\|}\,w_2^*\,\ddot{\|}_1^{\ddot{2}}}} \\ &= \ddot{\|}\,\zeta_1^*\,\ddot{\|}_2 \ddot{+} \ddot{\|}\,\zeta_2^*\,\ddot{\|}_2, \end{aligned}$$

as required.

ii) Let $\zeta_1^* = z_1^* \oplus_2 j^* \otimes_2 w_1^*, \zeta_2^* = z_2^* \oplus_2 j^* \otimes_2 w_2^* \in \mathbb{BC}(N)$. By using Lemma 2 in [2] in a similar way, we have

$$\begin{aligned}
\ddot{\|} \zeta_1^* \ominus_2 \zeta_2^* \ddot{\|}_2 &= \ddot{\sqrt{\ddot{\|} z_1^* \ominus_1 z_2^* \ddot{\|}_1^{\ddot{2}} \dot{+} \ddot{\|} w_1^* \ominus_1 w_2^* \ddot{\|}_1^{\ddot{2}}}} \\
&\ddot{\leq} \ddot{\sqrt{\left(\ddot{\|} z_1^* \ddot{\|}_1^{\ddot{2}} \dot{+} \ddot{\|} z_2^* \ddot{\|}_1^{\ddot{2}}\right) \dot{+} \left(\ddot{\|} w_1^* \ddot{\|}_1^{\ddot{2}} \dot{+} \ddot{\|} w_2^* \ddot{\|}_1^{\ddot{2}}\right)}} \\
&\ddot{\leq} \ddot{\sqrt{\ddot{\|} z_1^* \ddot{\|}_1^{\ddot{2}} \dot{+} \ddot{\|} w_1^* \ddot{\|}_1^{\ddot{2}}}} \dot{+} \ddot{\sqrt{\ddot{\|} z_2^* \ddot{\|}_1^{\ddot{2}} \dot{+} \ddot{\|} w_2^* \ddot{\|}_1^{\ddot{2}}}} \\
&= \ddot{\|} \zeta_1^* \ddot{\|}_2 \dot{+} \ddot{\|} \zeta_2^* \ddot{\|}_2,
\end{aligned}$$

as required.

iii) Since

$$\ddot{\|} \zeta_1^* \ddot{\|}_2 = \ddot{\|} \zeta_1^* \oplus_2 \left(\zeta_2^* \ominus_2 \zeta_2^*\right) \ddot{\|}_2 = \ddot{\|} \left(\zeta_1^* \ominus_2 \zeta_2^*\right) \oplus_2 \zeta_2^* \ddot{\|}_2 \ddot{\leq} \ddot{\|} \zeta_1^* \ominus_2 \zeta_2^* \ddot{\|}_2 \dot{+} \ddot{\|} \zeta_2^* \ddot{\|}_2,$$

we have that $\ddot{\|} \zeta_1^* \ddot{\|}_2 \ddot{-} \ddot{\|} \zeta_2^* \ddot{\|}_2 \ddot{\leq} \ddot{\|} \zeta_1^* \ominus_2 \zeta_2^* \ddot{\|}_2$. Also, since

$$\ddot{\|} \zeta_2^* \ddot{\|}_2 = \ddot{\|} \zeta_2^* \oplus_2 \left(\zeta_1^* \ominus_2 \zeta_1^*\right) \ddot{\|}_2 = \ddot{\|} \left(\zeta_2^* \ominus_2 \zeta_1^*\right) \oplus_2 \zeta_1^* \ddot{\|}_2 \ddot{\leq} = \ddot{\|} \zeta_1^* \ominus_2 \zeta_2^* \ddot{\|}_2 \dot{+} \ddot{\|} \zeta_1^* \ddot{\|}_2,$$

we have that $\ddot{-} \ddot{\|} \zeta_1^* \ominus_2 \zeta_2^* \ddot{\|}_2 \ddot{\leq} \ddot{\|} \zeta_1^* \ddot{\|}_2 \ddot{-} \ddot{\|} \zeta_2^* \ddot{\|}_2$. Thus, we can write

$$\ddot{-} \ddot{\|} \zeta_1^* \ominus_2 \zeta_2^* \ddot{\|}_2 \ddot{\leq} \ddot{\|} \zeta_1^* \ddot{\|}_2 \ddot{-} \ddot{\|} \zeta_2^* \ddot{\|}_2 \ddot{\leq} \ddot{\|} \zeta_1^* \ominus_2 \zeta_2^* \ddot{\|}_2,$$

and so, $\ddot{|}\ddot{\|} \zeta_1^* \ddot{\|}_2 \ddot{-} \ddot{\|} \zeta_2^* \ddot{\|}_2 \ddot{|} \ddot{\leq} \ddot{\|} \zeta_1^* \ominus_2 \zeta_2^* \ddot{\|}_2$, as required.

iv) Since

$$\ddot{\|} \zeta_1^* \ddot{\|}_2 = \ddot{\|} \zeta_1^* \oplus_2 \left(\zeta_2^* \ominus_2 \zeta_2^*\right) \ddot{\|}_2 = \ddot{\|} \left(\zeta_1^* \oplus_2 \zeta_2^*\right) \ominus_2 \zeta_2^* \ddot{\|}_2 \ddot{\leq} \ddot{\|} \zeta_1^* \oplus_2 \zeta_2^* \ddot{\|}_2 \dot{+} \ddot{\|} \zeta_2^* \ddot{\|}_2,$$

we have that $\ddot{\|} \zeta_1^* \ddot{\|}_2 \ddot{-} \ddot{\|} \zeta_2^* \ddot{\|}_2 \ddot{\leq} \ddot{\|} \zeta_1^* \oplus_2 \zeta_2^* \ddot{\|}_2$. Also, since

$$\ddot{\|} \zeta_2^* \ddot{\|}_2 = \ddot{\|} \zeta_2^* \oplus_2 \left(\zeta_1^* \ominus_2 \zeta_1^*\right) \ddot{\|}_2 = \ddot{\|} \left(\zeta_2^* \oplus_2 \zeta_1^*\right) \ominus_2 \zeta_1^* \ddot{\|}_2 \ddot{\leq} \ddot{\|} \zeta_1^* \oplus_2 \zeta_2^* \ddot{\|}_2 \dot{+} \ddot{\|} \zeta_1^* \ddot{\|}_2,$$

we have that $\ddot{-} \ddot{\|} \zeta_1^* \oplus_2 \zeta_2^* \ddot{\|}_2 \ddot{\leq} \ddot{\|} \zeta_1^* \ddot{\|}_2 \ddot{-} \ddot{\|} \zeta_2^* \ddot{\|}_2$. Thus, we can write

$$\ddot{-} \ddot{\|} \zeta_1^* \oplus_2 \zeta_2^* \ddot{\|}_2 \ddot{\leq} \ddot{\|} \zeta_1^* \ddot{\|}_2 \ddot{-} \ddot{\|} \zeta_2^* \ddot{\|}_2 \ddot{\leq} \ddot{\|} \zeta_1^* \oplus_2 \zeta_2^* \ddot{\|}_2,$$

and so, $\ddot{|}\ddot{\|} \zeta_1^* \ddot{\|}_2 \ddot{-} \ddot{\|} \zeta_2^* \ddot{\|}_2 \ddot{|} \ddot{\leq} \ddot{\|} \zeta_1^* \oplus_2 \zeta_2^* \ddot{\|}_2$, as required.

v) Let $\zeta_1^* = \left(\dot{a}_1, \ddot{b}_1\right) \oplus_2 j^* \otimes_2 \left(\dot{c}_1, \ddot{d}_1\right), \zeta_2^* = \left(\dot{a}_2, \ddot{b}_2\right) \oplus_2 j^* \otimes_2 \left(\dot{c}_2, \ddot{d}_2\right)$. Then, a straightforward calculation gives that

$$
\begin{aligned}
\frac{\ddot{\|}\,\zeta_1^* \oplus_2 \zeta_2^*\,\ddot{\|}_2}{\ddot{1} \ddot{+} \ddot{\|}\,\zeta_1^* \oplus_2 \zeta_2^*\,\ddot{\|}_2}
&= \frac{\beta\left[\sqrt{(a_1+a_2)^2+(b_1+b_2)^2+(c_1+c_2)^2+(d_1+d_2)^2}\right]}{\ddot{1} \ddot{+} \beta\left[\sqrt{(a_1+a_2)^2+(b_1+b_2)^2+(c_1+c_2)^2+(d_1+d_2)^2}\right]} \\
&= \beta\left[\frac{\beta^{-1}\left(\beta\sqrt{(a_1+a_2)^2+(b_1+b_2)^2+(c_1+c_2)^2+(d_1+d_2)^2}\right)}{\beta^{-1}\left[\beta\left[\beta^{-1}\left(\ddot{1}\right)+\beta^{-1}\left(\beta\sqrt{(a_1+a_2)^2+(b_1+b_2)^2+(c_1+c_2)^2+(d_1+d_2)^2}\right)\right]\right]}\right] \\
&= \beta\left[\frac{\sqrt{(a_1+a_2)^2+(b_1+b_2)^2+(c_1+c_2)^2+(d_1+d_2)^2}}{1+\sqrt{(a_1+a_2)^2+(b_1+b_2)^2+(c_1+c_2)^2+(d_1+d_2)^2}}\right] \\
&\ddot{\leq} \beta\left[\frac{\sqrt{a_1^2+b_1^2+c_1^2+d_1^2}}{1+\sqrt{a_1^2+b_1^2+c_1^2+d_1^2}}+\frac{\sqrt{a_2^2+b_2^2+c_2^2+d_2^2}}{1+\sqrt{a_2^2+b_2^2+c_2^2+d_2^2}}\right] \\
&= \beta\left\{\begin{array}{c}\frac{\beta^{-1}\left[\beta\left(\sqrt{a_1^2+b_1^2+c_1^2+d_1^2}\right)\right]}{\left[\beta^{-1}(\beta(1))+\beta^{-1}\left(\beta\left(\sqrt{a_1^2+b_1^2+c_1^2+d_1^2}\right)\right)\right]} \\ +\frac{\beta^{-1}\left[\beta\left(\sqrt{a_2^2+b_2^2+c_2^2+d_2^2}\right)\right]}{\left[\beta^{-1}(\beta(1))+\beta^{-1}\left(\beta\left(\sqrt{a_2^2+b_2^2+c_2^2+d_2^2}\right)\right)\right]}\end{array}\right\} \\
&= \beta\left\{\frac{\beta^{-1}\left[\ddot{\|}\,\zeta_1^*\,\ddot{\|}_2\right]}{\left[\beta^{-1}\left(\ddot{1}\right)+\beta^{-1}\left(\ddot{\|}\,\zeta_1^*\,\ddot{\|}_2\right)\right]}+\frac{\beta^{-1}\left[\ddot{\|}\,\zeta_2^*\,\ddot{\|}_2\right]}{\left[\beta^{-1}\left(\ddot{1}\right)+\beta^{-1}\left(\ddot{\|}\,\zeta_2^*\,\ddot{\|}_2\right)\right]}\right\} \\
&= \beta\left\{\frac{\beta^{-1}\left[\ddot{\|}\,\zeta_1^*\,\ddot{\|}_2\right]}{\beta^{-1}\left[\beta\left[\beta^{-1}\left(\ddot{1}\right)+\beta^{-1}\left(\ddot{\|}\,\zeta_1^*\,\ddot{\|}_2\right)\right]\right]}+\frac{\beta^{-1}\left[\ddot{\|}\,\zeta_2^*\,\ddot{\|}_2\right]}{\beta^{-1}\left[\beta\left[\beta^{-1}\left(\ddot{1}\right)+\beta^{-1}\left(\ddot{\|}\,\zeta_2^*\,\ddot{\|}_2\right)\right]\right]}\right\} \\
&= \beta\left\{\beta^{-1}\left\{\beta\left[\frac{\beta^{-1}\left[\ddot{\|}\,\zeta_1^*\,\ddot{\|}_2\right]}{\beta^{-1}\left[\ddot{1}\ddot{+}\ddot{\|}\,\zeta_1^*\,\ddot{\|}_2\right]}\right]\right\}+\beta^{-1}\left\{\beta\left[\frac{\beta^{-1}\left[\ddot{\|}\,\zeta_2^*\,\ddot{\|}_2\right]}{\beta^{-1}\left[\ddot{1}\ddot{+}\ddot{\|}\,\zeta_2^*\,\ddot{\|}_2\right]}\right]\right\}\right\} \\
&= \beta\left\{\beta^{-1}\left[\frac{\ddot{\|}\,\zeta_1^*\,\ddot{\|}_2}{\ddot{1}\ddot{+}\ddot{\|}\,\zeta_1^*\,\ddot{\|}_2}\right]+\beta^{-1}\left[\frac{\ddot{\|}\,\zeta_2^*\,\ddot{\|}_2}{\ddot{1}\ddot{+}\ddot{\|}\,\zeta_2^*\,\ddot{\|}_2}\right]\right\} \\
&= \frac{\ddot{\|}\,\zeta_1^*\,\ddot{\|}_2}{\ddot{1}\ddot{+}\ddot{\|}\,\zeta_1^*\,\ddot{\|}_2}\ddot{+}\frac{\ddot{\|}\,\zeta_2^*\,\ddot{\|}_2}{\ddot{1}\ddot{+}\ddot{\|}\,\zeta_2^*\,\ddot{\|}_2},
\end{aligned}
$$

as required. □

Theorem 2.13. $\left(\right.$*Schwarz's inequality in* $\mathbb{BC}((N))$ *with respect to* $\left.\ddot{\|}\,.\,\ddot{\|}_2\right)$ *Let* $s_k^*, t_k^* \in \mathbb{BC}(N)$ *for* $k \in \{1,2,...,n\}$. *Then,*

$$
\left(\ddot{\sum_{k=1}^{n}}\left(\ddot{\|}\,s_k^*\,\ddot{\|}_2 \ddot{\times} \ddot{\|}\,t_k^*\,\ddot{\|}_2\right)\right)^{\ddot{2}} \ddot{\leq} \left(\ddot{\sum_{k=1}^{n}}\ddot{\|}\,s_k^*\,\ddot{\|}_2^{\ddot{2}}\right)\ddot{\times}\left(\ddot{\sum_{k=1}^{n}}\ddot{\|}\,t_k^*\,\ddot{\|}_2^{\ddot{2}}\right).
$$

Proof. Since $\ddot{\|}\,s_k^*\,\ddot{\|}_2, \ddot{\|}\,t_k^*\,\ddot{\|}_2 \in \mathbb{R}_\beta^+$ for $s_k^*, t_k^* \in \mathbb{BC}(N)$, If we take $\beta-$arithmetic intead of $\alpha-$arithmetic and we choose $p = q = \ddot{2}$ in non-Newtonian Hölder's inequality in [16], then we derive Theorem 2.13. □

Theorem 2.14. *Let $\zeta_1^*, \zeta_2^* \in \mathbb{BC}(N)$ and $z^* \in \mathbb{C}(N)$. Then, the following statements are satisfied:*

i) $\ddot{\|} z^* \odot_2 \zeta_1^* \ddot{\|}_2 = \ddot{\|} z^* \ddot{\|}_1 \ddot{\times} \ddot{\|} \zeta_1^* \ddot{\|}_2$.

ii) $\ddot{\|} \zeta_1^* \otimes_2 \zeta_2^* \ddot{\|}_2 \ddot{\leq} \ddot{\sqrt{2}} \ddot{\times} \ddot{\|} \zeta_1^* \ddot{\|}_2 \ddot{\times} \ddot{\|} \zeta_2^* \ddot{\|}_2$.

iii) $\ddot{\|} \left(\zeta_1^*\right)^n \ddot{\|}_2 \ddot{\leq} \left(\ddot{2}\right)^{(n-1)/2} \ddot{\times} \left(\ddot{\|} \zeta_1^* \ddot{\|}_2\right)^n$ *for all* $n \in \mathbb{N}$.

Proof. Let $\zeta_1^* = z_1^* \oplus_2 j^* \otimes_2 w_1^*, \zeta_2^* = z_2^* \oplus_2 j^* \otimes_2 w_2^* \in \mathbb{BC}(N)$ and $z^* \in \mathbb{C}(N)$.

i) Since

$$\begin{aligned}
\ddot{\|} z^* \odot_2 \zeta_1^* \ddot{\|}_2 &= \ddot{\|} z^* \odot_2 \left(z_1^* \oplus_2 j^* \otimes_2 w_1^*\right) \ddot{\|}_2 \\
&= \ddot{\|} \left(z^* \otimes_1 z_1^*\right) \oplus_2 j^* \otimes_2 \left(z^* \otimes_1 w_1^*\right) \ddot{\|}_2 \\
&= \ddot{\sqrt{\ddot{\|} z^* \otimes_1 z_1^* \ddot{\|}_1^{\ddot{2}} \ddot{+} \ddot{\|} z^* \otimes_1 w_1^* \ddot{\|}_1^{\ddot{2}}}} \\
&= \ddot{\sqrt{\left(\ddot{\|} z^* \ddot{\|}_1^{\ddot{2}} \ddot{\times} \ddot{\|} z_1^* \ddot{\|}_1^{\ddot{2}}\right) \ddot{+} \left(\ddot{\|} z^* \ddot{\|}_1^{\ddot{2}} \ddot{\times} \ddot{\|} w_1^* \ddot{\|}_1^{\ddot{2}}\right)}} \\
&= \ddot{\sqrt{\ddot{\|} z^* \ddot{\|}_1^{\ddot{2}} \ddot{\times} \left(\ddot{\|} z_1^* \ddot{\|}_1^{\ddot{2}} \ddot{+} \ddot{\|} w_1^* \ddot{\|}_1^{\ddot{2}}\right)}} \\
&= \ddot{\|} z^* \ddot{\|}_1 \ddot{\times} \ddot{\sqrt{\ddot{\|} z_1^* \ddot{\|}_1^{\ddot{2}} \ddot{+} \ddot{\|} w_1^* \ddot{\|}_1^{\ddot{2}}}} \\
&= \ddot{\|} z^* \ddot{\|}_1 \ddot{\times} \ddot{\|} \zeta_1^* \ddot{\|}_2,
\end{aligned}$$

the equality $\ddot{\|} z^* \odot_2 \zeta_1^* \ddot{\|}_2 = \ddot{\|} z^* \ddot{\|}_1 \ddot{\times} \ddot{\|} \zeta_1^* \ddot{\|}_2$ holds.

ii) By using Schwarz' s inequality in $\mathbb{BC}(N)$ with respect to the norm $\ddot{\|} \,.\, \ddot{\|}_2$, we obtain that

$$\begin{aligned}
\ddot{\|} \zeta_1^* \otimes_2 \zeta_2^* \ddot{\|}_2 &= \ddot{\|} \left(z_1^* \oplus_2 j^* \otimes_2 w_1^*\right) \otimes_2 \zeta_2^* \ddot{\|}_2 \\
&= \ddot{\|} \left(z_1^* \odot_2 \zeta_2^*\right) \oplus_2 \left(j^* \otimes_2 \left(w_1^* \odot_2 \zeta_2^*\right)\right) \ddot{\|}_2 \\
&\ddot{\leq} \ddot{\|} z_1^* \odot_2 \zeta_2^* \ddot{\|}_2 \ddot{+} \ddot{\|} j^* \otimes_2 \left(w_1^* \odot_2 \zeta_2^*\right) \ddot{\|}_2 \\
&= \left(\ddot{\|} z_1^* \ddot{\|}_1 \ddot{\times} \ddot{\|} \zeta_2^* \ddot{\|}_2\right) \ddot{+} \left(\ddot{\|} w_1^* \ddot{\|}_1 \ddot{\times} \ddot{\|} \zeta_2^* \ddot{\|}_2\right) \\
&= \left(\ddot{\|} z_1^* \ddot{\|}_1 \ddot{+} \ddot{\|} w_1^* \ddot{\|}_1\right) \ddot{\times} \ddot{\|} \zeta_2^* \ddot{\|}_2 \\
&\ddot{\leq} \ddot{\sqrt{2}} \ddot{\times} \ddot{\sqrt{\ddot{\|} z_1^* \ddot{\|}_1^{\ddot{2}} \ddot{+} \ddot{\|} w_1^* \ddot{\|}_1^{\ddot{2}}}} \ddot{\times} \ddot{\|} \zeta_2^* \ddot{\|}_2 \\
&= \ddot{\sqrt{2}} \ddot{\times} \ddot{\|} z_1^* \oplus_2 j^* \otimes_2 w_1^* \ddot{\|}_2 \ddot{\times} \ddot{\|} \zeta_2^* \ddot{\|}_2 \\
&= \ddot{\sqrt{2}} \ddot{\times} \ddot{\|} \zeta_1^* \ddot{\|}_2 \ddot{\times} \ddot{\|} \zeta_2^* \ddot{\|}_2,
\end{aligned}$$

as required.

iii) For $n = 1$ and $n = 2$; the proof is clear. We assume that the inequality holds for $n = k$, that is,

$\ddot{\|} \left(\zeta_1^*\right)^k \ddot{\|}_2 \ddot{\leq} \left(\ddot{2}\right)^{(k-1)/2} \ddot{\times} \left(\ddot{\|} \zeta_1^* \ddot{\|}_2\right)^k$. Now, we want to show that it holds for $n = k + 1$. Consider

$$\begin{aligned}
\ddot{\|} \left(\zeta_1^*\right)^{k+1} \ddot{\|}_2 &= \ddot{\|} \left(\zeta_1^*\right)^k \ddot{\times} \zeta_1^* \ddot{\|}_2 \\
&\ddot{\leq} \ddot{\sqrt{2}} \ddot{\times} \ddot{\|} \left(\zeta_1^*\right)^k \ddot{\|}_2 \ddot{\times} \ddot{\|} \zeta_1^* \ddot{\|}_2 \\
&\ddot{\leq} \ddot{\sqrt{2}} \ddot{\times} \left(\ddot{2}\right)^{(k-1)/2} \ddot{\times} \left(\ddot{\|} \zeta_1^* \ddot{\|}_2\right)^k \ddot{\times} \ddot{\|} \zeta_1^* \ddot{\|}_2 \\
&= \left(\ddot{2}\right)^{((k+1)-1)/2} \ddot{\times} \left(\ddot{\|} \zeta_1^* \ddot{\|}_2\right)^{k+1}.
\end{aligned}$$

Then, above given inequality is true for $n = k + 1$. Thus, the mathematical induction principle completes the proof. □

Corollary 2.15. *The system* $\left(\mathbb{BC}(N), \oplus_2, \odot_2, \ddot{\|} . \ddot{\|}_2, \otimes_2\right)$ *is a quasi-Banach algebra.*

Proof. All properties of the system have been established in preceding theorems. In the usual definition of a Banach algebra, the norm of the product of two elements is required to be equal to or less than the product of the norms of these elements. So, $\mathbb{BC}(N)$ is a quasi-Banach algebra (see [14, 15].) □

Theorem 2.16. (*Non-Newtonian Young's inequality*) *Let* $\dot{1} \dot{<} \dot{p} \dot{<} \dot{q} \dot{<} \infty$ *be such that* $\frac{\dot{1}}{\dot{p}} \dot{+} \frac{\dot{1}}{\dot{q}} = \dot{1}$. *Then,*

$$a \dot{\times} b \dot{\leq} \frac{a^{\dot{p}}}{\dot{p}} \dot{+} \frac{b^{\dot{q}}}{\dot{q}}$$

for a and b positive non-Newtonian real numbers. The equality holds if $a^{\dot{p}} = b^{\dot{q}}$.

Proof. One can see from Young' s inequality that

$$\begin{aligned}
a \dot{\times} b &= \alpha\left\{\alpha^{-1}(a) \times \alpha^{-1}(b)\right\} \\
&\dot{\leq} \alpha\left\{\frac{\alpha^{-1}(a)^p}{p} + \frac{\alpha^{-1}(b)^q}{q}\right\} \\
&= \alpha\left\{\alpha^{-1}\alpha\left(\frac{\alpha^{-1}(a)^p}{p}\right) + \alpha^{-1}\alpha\left(\frac{\alpha^{-1}(b)^q}{q}\right)\right\} \\
&= \alpha\left\{\alpha^{-1}\left(\frac{a^{\dot{p}}}{\dot{p}}\right) + \alpha^{-1}\left(\frac{b^{\dot{q}}}{\dot{q}}\right)\right\} \\
&= \frac{a^{\dot{p}}}{\dot{p}} \dot{+} \frac{b^{\dot{q}}}{\dot{q}},
\end{aligned}$$

as required. Also, if $a^{\dot{p}} = b^{\dot{q}}$, then $a = b^{\dot{q} \dot{-} \dot{1}}$. Thus,

$$a \dot{\times} b = b^{\dot{q} \dot{-} \dot{1}} \dot{\times} b = b^{\dot{q}} = b^{\dot{q}} \dot{\times} \left(\frac{\dot{1}}{\dot{p}} \dot{+} \frac{\dot{1}}{\dot{q}}\right) = \frac{a^{\dot{p}}}{\dot{p}} \dot{+} \frac{b^{\dot{q}}}{\dot{q}}.$$

This completes the proof. □

Lemma 2.17. $\Big($*Hölder's inequality in* $\mathbb{BC}(N)$ *with respect to* $\ddot{\|} . \ddot{\|}_2\Big)$ *Let p and q be non-Newtonian real numbers with* $\ddot{1} \ddot{<} p \ddot{<} \infty$ *be such that* $\frac{\ddot{1}}{p} \ddot{+} \frac{\ddot{1}}{q} = \ddot{1}$ *and* $s_k^*, t_k^* \in \mathbb{BC}(N)$ *for* $k \in \{1, 2, ..., n\}$. *Then,*

$$\ddot{\sum_{k=1}^{n}} \ddot{\|} s_k^* \otimes_2 t_k^* \ddot{\|}_2 \ddot{\leq} \ddot{\sqrt{2}} \ddot{\times} \left(\ddot{\sum_{k=1}^{n}} \ddot{\|} s_k^* \ddot{\|}_2^{\ddot{p}}\right)^{\frac{\ddot{1}}{p}} \ddot{\times} \left(\ddot{\sum_{k=1}^{n}} \ddot{\|} t_k^* \ddot{\|}_2^{\ddot{q}}\right)^{\frac{\ddot{1}}{q}}.$$

Proof. Let us take

$$\theta = \frac{\ddot{\|}\, s_k^* \,\ddot{\|}_2}{\left(\ddot{\sum\limits_{k=1}^{n}} \ddot{\|}\, s_k^* \,\ddot{\|}_2^{p}\right)^{\frac{\ddot{1}}{p}}}, \vartheta = \frac{\ddot{\|}\, t_k^* \,\ddot{\|}_2}{\left(\ddot{\sum\limits_{k=1}^{n}} \ddot{\|}\, t_k^* \,\ddot{\|}_2^{q}\right)^{\frac{\ddot{1}}{q}}}.$$

By non-Newtonian Young's inequality, we get

$$\theta \ddot{\times} \vartheta = \frac{\ddot{\|}\, s_k^* \,\ddot{\|}_2 \ddot{\times} \ddot{\|}\, t_k^* \,\ddot{\|}_2}{\left(\ddot{\sum\limits_{k=1}^{n}} \ddot{\|}\, s_k^* \,\ddot{\|}_2^{p}\right)^{\frac{\ddot{1}}{p}} \ddot{\times} \left(\ddot{\sum\limits_{k=1}^{n}} \ddot{\|}\, t_k^* \,\ddot{\|}_2^{q}\right)^{\frac{\ddot{1}}{q}}} \ddot{\leq} \frac{\ddot{1}}{p} \ddot{\times} \frac{\ddot{\|}\, s_k^* \,\ddot{\|}_2^{p}}{\ddot{\sum\limits_{k=1}^{n}} \ddot{\|}\, s_k^* \,\ddot{\|}_2^{p}} \ddot{+} \frac{\ddot{1}}{q} \ddot{\times} \frac{\ddot{\|}\, t_k^* \,\ddot{\|}_2^{q}}{\ddot{\sum\limits_{k=1}^{n}} \ddot{\|}\, t_k^* \,\ddot{\|}_2^{q}}.$$

Termwise summation gives

$$\ddot{\sum_{k=1}^{n}} \frac{\ddot{\|}\, s_k^* \,\ddot{\|}_2 \ddot{\times} \ddot{\|}\, t_k^* \,\ddot{\|}_2}{\left(\ddot{\sum\limits_{k=1}^{n}} \ddot{\|}\, s_k^* \,\ddot{\|}_2^{p}\right)^{\frac{\ddot{1}}{p}} \ddot{\times} \left(\ddot{\sum\limits_{k=1}^{n}} \ddot{\|}\, t_k^* \,\ddot{\|}_2^{q}\right)^{\frac{\ddot{1}}{q}}} \ddot{\leq} \frac{\ddot{1}}{p} \ddot{+} \frac{\ddot{1}}{q} = \ddot{1}$$

and from this

$$\begin{aligned} \ddot{\sum_{k=1}^{n}} \left(\ddot{\|}\, s_k^* \otimes_2 t_k^* \,\ddot{\|}_2\right) &\ddot{\leq} \ddot{\sum_{k=1}^{n}} \ddot{\sqrt{2}} \ddot{\times} \ddot{\|}\, s_k^* \,\ddot{\|}_2 \ddot{\times} \ddot{\|}\, t_k^* \,\ddot{\|}_2 \\ &= \ddot{\sqrt{2}} \ddot{\times} \ddot{\sum_{k=1}^{n}} \ddot{\|}\, s_k^* \,\ddot{\|}_2 \ddot{\times} \ddot{\|}\, t_k^* \,\ddot{\|}_2 \\ &\ddot{\leq} \ddot{\sqrt{2}} \ddot{\times} \left(\ddot{\sum_{k=1}^{n}} \ddot{\|}\, s_k^* \,\ddot{\|}_2^{p}\right)^{\frac{\ddot{1}}{p}} \ddot{\times} \left(\ddot{\sum_{k=1}^{n}} \ddot{\|}\, t_k^* \,\ddot{\|}_2^{q}\right)^{\frac{\ddot{1}}{q}}. \end{aligned}$$

The proof is completed. □

Lemma 2.18. *(Minkowski's inequality in $\mathbb{BC}(N)$ with respect to $\ddot{\|}\,.\,\ddot{\|}_2$) Let p be a non-Newtonian real number with $\ddot{1} \ddot{<} p \ddot{<} \infty$ and $s_k^*, t_k^* \in \mathbb{BC}(N)$ for $k \in \{1, 2, ..., n\}$. Then,*

$$\left(\ddot{\sum_{k=1}^{n}} \ddot{\|}\, s_k^* \oplus_2 t_k^* \,\ddot{\|}_2^{p}\right)^{\frac{\ddot{1}}{p}} \ddot{\leq} \left(\ddot{\sum_{k=1}^{n}} \ddot{\|}\, s_k^* \,\ddot{\|}_2^{p}\right)^{\frac{\ddot{1}}{p}} \ddot{+} \left(\ddot{\sum_{k=1}^{n}} \ddot{\|}\, t_k^* \,\ddot{\|}_2^{p}\right)^{\frac{\ddot{1}}{p}}.$$

Proof. We have

$$\begin{aligned} \ddot{\sum_{k=1}^{n}} \ddot{\|}\, s_k^* \oplus_2 t_k^* \,\ddot{\|}_2^{p} &= \ddot{\sum_{k=1}^{n}} \ddot{\|}\, s_k^* \oplus_2 t_k^* \,\ddot{\|}_2^{p \ddot{-} \ddot{1}} \ddot{\times} \ddot{\|}\, s_k^* \oplus_2 t_k^* \,\ddot{\|}_2 \\ &\ddot{\leq} \ddot{\sum_{k=1}^{n}} \ddot{\|}\, s_k^* \oplus_2 t_k^* \,\ddot{\|}_2^{p \ddot{-} \ddot{1}} \ddot{\times} \left(\ddot{\|}\, s_k^* \,\ddot{\|}_2 \ddot{+} \ddot{\|}\, t_k^* \,\ddot{\|}_2\right) \\ &= \ddot{\sum_{k=1}^{n}} \ddot{\|}\, s_k^* \,\ddot{\|}_2 \ddot{\times} \ddot{\|}\, s_k^* \oplus_2 t_k^* \,\ddot{\|}_2^{p \ddot{-} \ddot{1}} \ddot{+} \ddot{\sum_{k=1}^{n}} \ddot{\|}\, t_k^* \,\ddot{\|}_2 \ddot{\times} \ddot{\|}\, s_k^* \oplus_2 t_k^* \,\ddot{\|}_2^{p \ddot{-} \ddot{1}}. \end{aligned}$$

Set $q = \frac{p}{p \ddot{-} \ddot{1}}$. Then, $\frac{\ddot{1}}{p} \ddot{+} \frac{\ddot{1}}{q} = \ddot{1}$, so by the Hölder's inequality in $\mathbb{BC}(N)$ with respect to the norm $\ddot{\|} \, . \, \ddot{\|}_2$, we have

$$\ddot{\sum_{k=1}^{n}} \ddot{\|} s_k^* \ddot{\|}_2 \ddot{\times} \ddot{\|} s_k^* \oplus_2 t_k^* \ddot{\|}_2^{p \ddot{-} \ddot{1}} \ddot{\le} \left(\ddot{\sum_{k=1}^{n}} \ddot{\|} s_k^* \ddot{\|}_2^{p} \right)^{\frac{\ddot{1}}{p}} \ddot{\times} \left(\ddot{\sum_{k=1}^{n}} \ddot{\|} s_k^* \oplus_2 t_k^* \ddot{\|}_2^{(p \ddot{-} \ddot{1}) q} \right)^{\frac{\ddot{1}}{q}},$$

$$\ddot{\sum_{k=1}^{n}} \ddot{\|} t_k^* \ddot{\|}_2 \ddot{\times} \ddot{\|} s_k^* \oplus_2 t_k^* \ddot{\|}_2^{p \ddot{-} \ddot{1}} \ddot{\le} \left(\ddot{\sum_{k=1}^{n}} \ddot{\|} t_k^* \ddot{\|}_2^{p} \right)^{\frac{\ddot{1}}{p}} \ddot{\times} \left(\ddot{\sum_{k=1}^{n}} \ddot{\|} s_k^* \oplus_2 t_k^* \ddot{\|}_2^{(p \ddot{-} \ddot{1}) q} \right)^{\frac{\ddot{1}}{q}}.$$

Adding these two inequalities, we obtain that

$$\ddot{\sum_{k=1}^{n}} \ddot{\|} s_k^* \oplus_2 t_k^* \ddot{\|}_2^{p} \ddot{\le} \left[\left(\ddot{\sum_{k=1}^{n}} \ddot{\|} s_k^* \ddot{\|}_2^{p} \right)^{\frac{\ddot{1}}{p}} \ddot{+} \left(\ddot{\sum_{k=1}^{n}} \ddot{\|} t_k^* \ddot{\|}_2^{p} \right)^{\frac{\ddot{1}}{p}} \right] \ddot{\times} \left(\ddot{\sum_{k=1}^{n}} \ddot{\|} s_k^* \oplus_2 t_k^* \ddot{\|}_2^{(p \ddot{-} \ddot{1}) q} \right)^{\frac{\ddot{1}}{q}}.$$

Observing that $\left(p \ddot{-} \ddot{1}\right) q = p$ by definiton, we have

$$\ddot{\sum_{k=1}^{n}} \ddot{\|} s_k^* \oplus_2 t_k^* \ddot{\|}_2^{p} \ddot{\le} \left[\left(\ddot{\sum_{k=1}^{n}} \ddot{\|} s_k^* \ddot{\|}_2^{p} \right)^{\frac{\ddot{1}}{p}} \ddot{+} \left(\ddot{\sum_{k=1}^{n}} \ddot{\|} t_k^* \ddot{\|}_2^{p} \right)^{\frac{\ddot{1}}{p}} \right] \ddot{\times} \left(\ddot{\sum_{k=1}^{n}} \ddot{\|} s_k^* \oplus_2 t_k^* \ddot{\|}_2^{p} \right)^{\frac{\ddot{1}}{q}}$$

and so,

$$\left(\ddot{\sum_{k=1}^{n}} \ddot{\|} s_k^* \oplus_2 t_k^* \ddot{\|}_2^{p} \right)^{\frac{\ddot{1}}{p}} = \left(\ddot{\sum_{k=1}^{n}} \ddot{\|} s_k^* \oplus_2 t_k^* \ddot{\|}_2^{p} \right)^{\ddot{1} \ddot{-} \frac{\ddot{1}}{q}} \ddot{\le} \left[\left(\ddot{\sum_{k=1}^{n}} \ddot{\|} s_k^* \ddot{\|}_2^{p} \right)^{\frac{\ddot{1}}{p}} \ddot{+} \left(\ddot{\sum_{k=1}^{n}} \ddot{\|} t_k^* \ddot{\|}_2^{p} \right)^{\frac{\ddot{1}}{p}} \right].$$

The proof is completed. □

3. CONCLUSION

In the present work, we have studied some basic structures in non-Newtonian bicomplex setting and obtained that the set of non-Newtonian bicomplex numbers is a quasi-Banach algebra. For the future, we will examine the validity of the bicomplex and non-Newtonian bicomplex versions of some geometric properties as convexity, strictly convexity and uniformly convexity. Also, we will construct non-Newtonian bicomplex sequence spaces as a generalization of the bicomplex sequence spaces, and then, discuss the concepts of convexity, strictly convexity and uniformly convexity in the bicomplex and non-Newtonian bicomplex setting of these spaces.

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