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# New Additive Results for Cauchy Dual and MP−Inverse of Weighted Composition Operators

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**Abstract.** In this paper, we prove some basic results for Cauchy dual of weighted composition operators. Also we introduce some new classes of operators, called †-hyponormal, †-quasi-hyponormal, and we provide necessary and sufficient conditions for Cauchy dual and MP−inverse of weighted composition operators on $L^2(\Sigma)$ to belong to these classes . In addition, we study the complex symmetry of these types of operators. Moreover, some examples are provided to illustrate the obtained results.

## 1. Introduction and Preliminaries

Let $(X, \Sigma, \mu)$ be a sigma finite measure space and let $\varphi : X \to X$ be a measurable transformation such that $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to $\mu$. It is assumed that the Radon-Nikodym derivative $h = d\mu \circ \varphi^{-1}/d\mu$ is finite-valued or equivalently $(X, \varphi^{-1}(\Sigma), \mu)$ is sigma finite. We use the notation $L^2(\varphi^{-1}(\Sigma))$ for $L^2(X, \varphi^{-1}(\Sigma), \mu_{|\varphi^{-1}(\Sigma)})$ and henceforth we write $\mu$ in place of $\mu_{|\varphi^{-1}(\Sigma)}$. All comparisons between two functions or two sets are to be interpreted as holding up to a $\mu$-null set. We denote that the linear space of all complex-valued $\Sigma$-measurable functions on $X$ by $L^0(\Sigma)$. The support of $f \in L^0(\Sigma)$ is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. For a finite valued function $u \in L^0(\Sigma)$, the weighted composition operator $T_{u,\varphi}$ on $L^2(\Sigma)$ induced by $\varphi$ and $u$ is given by $T_{u,\varphi} = M_u \circ C_\varphi$ where $M_u$ is a multiplication operator and $C_\varphi$ is a composition operator on $L^2(\Sigma)$ defined by $M_u f = uf$ and $C_\varphi f = f \circ \varphi$, respectively.

The associated $\varphi^{-1}(\Sigma) \subseteq \Sigma$, there exists an operator $E := E^{\varphi^{-1}(\Sigma)} : L^p(\Sigma) \to L^p(\mathcal{A})$, which is called conditional expectation operator. $\mathcal{D}(E)$, the domain of $E$, contains the set of all non-negative measurable functions and each $f \in L^p(\Sigma)$ with $1 \leq p \leq \infty$, which satisfies

$$\int f d\mu = \int_A E(f) d\mu, \quad A \in \varphi^{-1}(\Sigma).$$

Recall that $E : L^2(\Sigma) \to L^2(\mathcal{A})$ is a surjective, positive and contractive orthogonal projection. For more details on the properties of $E$ see [19, 25, 27]. Since by the change of variable formula,

$$\int_X f \circ \varphi d\mu = \int_X h f d\mu, \quad f \in L^1(\Sigma),$$

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then $\|T_{u,\varphi}f\|_2 = \|\sqrt{hE(|u|^2)\circ\varphi^{-1}}f\|_2$. Put $J = hE(|u|^2)\circ\varphi^{-1}$. It follows that $T_{u,\varphi}$ is bounded on $L^2(\Sigma)$ if and only if $J \in L^\infty(\Sigma)$ (see [20] and also [6] for a discussion of $E(\cdot)\circ\varphi^{-1}$ when $\varphi$ is not invertible). Thus, $T^n_{u,\varphi}$ is a bounded operator precisely when $J_n := h_nE_n(|u_n|^2)\circ\varphi^{-n} \in L^\infty(\Sigma)$, where $n \geq 0$, $h_n = d\mu\circ\varphi^{-n}/d\mu$, $u_n = u(u\circ\varphi)(u\circ\varphi^2)\cdots(u\circ\varphi^{n-1})$ and $E_n = E^{\varphi^{-n}(\Sigma)}$. From now on, we assume that $J \in L^\infty(\Sigma)$ and $u \geq 0$. Put $h_0 = 1$, $J_1 = J$, $h_1 = h$ and $E_1 = E$.

Composition operators as an extension of shift operators are a good tool for separating weak hyponormal classes. Classic seminormal (weighted) composition operators have been extensively studied by Harrington and whitley [18], Lambert [20, 25], Singh [29], Campbell [6–8] and Stochel [13]. In [4] and [5] some weak hyponormal classes of composition operators are studied. In those work, examples were given which show that composition operators can be used to separate each partial normality class from quasinormal through $w$-hyponormal. But in some cases composition operators can not be separated some of these classes. Hence, it is better that we consider the weighted case of composition operators. In [10] and [22], the authors generalized the work done in [4] and have obtained some characterizations of related $p$-hyponormal weighted composition operators as separately. In [22] some examples were presented to illustrate that weighted composition operators lie between those classes. This note is a continuation of the work done in [22].

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators on the infinite dimensional complex Hilbert space $\mathcal{H}$. Let $T = U|T|$ be the polar decomposition for $T \in B(\mathcal{H})$, where $U$ is a partial isometry and $|T| = \sqrt{T^*T}$. Associated with $T \in B(\mathcal{H})$ there is a useful related operator $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$, called the Aluthge transform of $T$.

**Definition 1.1.** *Let $m \in \mathbb{N}$, $n \in \mathbb{N}\cup\{0\}$ and let $p > 0$. We denote by $\mathcal{K}(p, m, n)$ the set of all operators such as T on $\mathcal{H}$ that $T^{*n}(T^{*m}T^m)^pT^n \geq T^{*n}(T^mT^{*m})^pT^n$.*

Note that $\mathcal{K}(p, m, n_1) \subseteq \mathcal{K}(p, m, n_2)$, for all $n_1 \leq n_2$. An operator $T$ is $p$-paranormal if $\||T|^pU|T|^px\| \geq \||T|^px\|^2$ and $T$ is absolute-$p$-paranormal if $\||T|^pTx\| \geq \|Tx\|^{p+1}$, for all unit vectors $x \in \mathcal{H}$.

Let $B_C(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$ with closed range. For $T \in B_C(\mathcal{H})$, the Moore-Penrose inverse of $T$, denoted by $T^\dagger$, is the unique operator $T^\dagger$ that satisfies following:

$$TT^\dagger T = T,\ \ T^\dagger TT^\dagger = T^\dagger,\ \ (TT^\dagger)^* = TT^\dagger,\ \ (T^\dagger T)^* = T^\dagger T.$$

We recall that $T^\dagger$ exists if and only if $T \in B_C(\mathcal{H})$. Note that if $T \in B_C(\mathcal{H})$, then $T^*$, $|T|$ and $T^\dagger$ have closed range. The Moore-Penrose inverse is designed as a measure for the invertibility of an operator. If $T = U|T|$ is invertible, then $T^{-1} = T^\dagger$, $U$ is unitary and so $|T| = \sqrt{T^*T}$ is invertible. It is a classical fact that the polar decomposition of $T^*$ is $U^*|T^*|$. It is easy to check that $U^\dagger|T^\dagger|$ and $|T^\dagger|^{\frac{1}{2}}U^\dagger|T^\dagger|^{\frac{1}{2}}$ are the polar decomposition and Aluthge transform of $T^\dagger$, respectively. For other important properties of $T^\dagger$ see [1, 12].

Put $S_{u,\varphi} = M_{\frac{\chi_{\sigma(J)}}{J}}T^*_{u,\varphi}$. Then $S_{u,\varphi} \in B_C(L^2(\Sigma))$, because $\frac{\chi_{\sigma(J)}}{J}$ bounded away from zero on $X$. Since for each $f \in L^2(\Sigma)$, $T^*_{u,\varphi}f = hE(uf)\circ\varphi^{-1}$, then we have

$$\begin{aligned}
T_{u,\varphi}S_{u,\varphi}T_{u,\varphi}f &= u(S_{u,\varphi}T_{u,\varphi}f)\circ\varphi \\
&= u(\frac{\chi_{\sigma(J)}}{J}hE(u^2)\circ\varphi^{-1}f)\circ\varphi \\
&= u(\frac{\chi_{\sigma(J)}}{J}hE(u^2)\varphi^{-1}f)\circ\varphi \\
&= u\chi_{\sigma(J\circ\varphi)}f\circ\varphi.
\end{aligned}$$

Since $u \geq 0$ and $\sigma(h\circ\varphi) = X$, $\sigma(J\circ\varphi)) = \sigma(h\circ\varphi E(u^2)) = \sigma(E(u^2)) \supseteq \sigma(u)$, and so

$$\begin{aligned}
T_{u,\varphi}S_{u,\varphi}T_{u,\varphi}f &= (u\chi_{\sigma(u)})\chi_{\sigma(E(u^2))}f\circ\varphi \\
&= (u\chi_{\sigma(u)})f\circ\varphi = T_{u,\varphi}f.
\end{aligned}$$

And

$$\begin{aligned}S_{u,\varphi}T_{u,\varphi}S_{u,\varphi}f &= \chi_{\frac{\sigma(J)}{J}}hE(uT_{u,\varphi}S_{u,\varphi}f)\circ\varphi^{-1}\\ &= \chi_{\frac{\sigma(J)}{J}}hE(u^2(S_{u,\varphi}f)\circ\varphi)\circ\varphi^{-1}\\ &= \chi_{\frac{\sigma(J)}{J}}h(E(u^2)(S_{u,\varphi}f)\circ\varphi)\circ\varphi^{-1}\\ &= \chi_{\frac{\sigma(J)}{J}}(hE(u^2)\circ\varphi^{-1})S_{u,\varphi}f\\ &= \chi_{\sigma(J)}S_{u,\varphi}f = S_{u,\varphi}f.\end{aligned}$$

Similar computations show that

$$T_{u,\varphi}S_{u,\varphi} = M_{\frac{u\chi_{\sigma(E(u))}}{E(u^2)}}EM_u = (T_{u,\varphi}S_{u,\varphi})^*$$

and $S_{u,\varphi}T_{u,\varphi} = M_{\chi_{\sigma(J)}} = (S_{u,\varphi}T_{u,\varphi})^*$. These observations establish the following theorem.

**Theorem 1.2.** *Let $T_{u,\varphi} \in B_C(L^2(\Sigma))$. Then $T^\dagger_{u,\varphi} = M_{\frac{\chi_{\sigma(J)}}{J}}T^*_{u,\varphi}$ and $(T^\dagger_{u,\varphi})^* = M_{\frac{\chi_{\sigma(J\circ\varphi)}}{J\circ\varphi}}T_{u,\varphi}$.*

**Corollary 1.3.** *Let $C_\varphi \in B_C(L^2(\Sigma))$. Then $C^\dagger_\varphi = M_{\frac{\chi_{\sigma(h)}}{h}}C_\varphi{}^*$.*

The Cauchy dual of left invertible operators is introduced in [30] as a powerful tool in the model theory of left-invertible operators. To be precise, if T is left invertible, it easy to see that $T^*T$ is invertible and the operator given by $L_T := (T^*T)^{-1}T^*$ is a canonical left inverse of $T$. The Cauchy dual of T is then defined as

$$\omega(T) := T(T^*T)^{-1} = L^*_T,$$

which is a right inverse of $T^*$. For more details on the properties of Cauchy dual see [9, 30, 32].
We introduce now the notion of Cauchy dual for Moore−Penrose inverse.

**Definition 1.4.** *Let $T \in B_C(\mathcal{H})$. The Cauchy dual T is is defined as*

$$\omega(T) = T(T^*T)^\dagger.$$

This article has been organized in two sections. In section 2, we study Cauchy dual of operators with closed range. We use the notion of the Cauchy dual to give the some characterizations of weak $p$-hyponormal and weak $p$-paranormal classes for Cauchy dual of weighted composition operators on $L^2(\Sigma)$, also we give several basic properties such as complex symmetry of these types of operators with a special conjugation. In section 3, the concept of †-hyponormal and †-quasi-hyponormal are defined, and we provide necessary and sufficient conditions for weighted composition operators to be †-hyponormal, †-quasi-hyponormal, weak $p$-hyponormal and weak $p$-paranormal. Finally, some specific examples is provided to illustrate the obtained results.

## 2. Cauchy dual and complex symmetry

The main goal of this section is to study the Cauchy dual of weighted composition operators. Also we investigate which combinations of weight u and self-maps $\varphi$ on $X$ give rise to complex symmetric for Cauchy dual of weighted composition operators with a special conjugation. Moreover, the class of $\mathcal{K}(p, m, n)$ as a generalization of the classes of weak p-hyponormal is introduced and we investigate some characterizations of the classes of $\mathcal{K}(p, m, n)$ for Cauchy dual of weighted composition operators on $L^2(\Sigma)$. We start with the following results that extend the case of left invertible operators and are easy to obtain.

**Proposition 2.1.** *Let $T_{u,\varphi} \in B_C(L^2(\Sigma))$. we have*

*(a)* $\omega(T_{u,\varphi}) = \frac{\chi_{\sigma(J\circ\varphi)}}{J\circ\varphi} T_{u,\varphi} = (T^{\dagger}_{u,\varphi})^*$.

*(b)* $\omega(T^*_{u,\varphi}) = \frac{\chi_{\sigma(J)}}{J} T^*_{u,\varphi} = T^{\dagger}_{u,\varphi} = (\omega(T_{u,\varphi}))^*$.

*(c)* $\omega(T_{u,\varphi})^*\omega(T_{u,\varphi}) = \frac{\chi_{\sigma(J)}}{J}$, $\omega(T_{u,\varphi})\omega(T_{u,\varphi})^* = \frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)E(u^2)} uE(uf)$.

*(d)* $\omega(\omega(T_{u,\varphi})) = T_{u,\varphi}$.

*(e)* $\omega(T^{\dagger}_{u,\varphi}) = \omega(T_{u,\varphi})^{\dagger}$.

*Proof.* (a) we know that $(T^*T)^{\dagger} = T^{\dagger}T^{*\dagger}$. So, for each $f \in L^2(\Sigma)$, we have

$$\omega(T_{u,\varphi})f = T_{u,\varphi}T^{\dagger}_{u,\varphi}T^{*\dagger}_{u,\varphi}f = \frac{\chi_{\sigma(J\circ\varphi)}}{J\circ\varphi} T_{u,\varphi}f = (T^{\dagger}_{u,\varphi})^* f.$$

(b) We have

$$\omega(T^*_{u,\varphi})f = T^*_{u,\varphi}T^{*\dagger}_{u,\varphi}T^{\dagger}_{u,\varphi}f = T^*_{u,\varphi}\left(\frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)E(u^2)} uE(uf)\right)$$

$$= \frac{\chi_{\sigma(J)}}{J} T^*_{u,\varphi}f = (\omega(T_{u,\varphi}))^* f.$$

(c)

$$\omega(T_{u,\varphi})^*\omega(T_{u,\varphi})f = \frac{\chi_{\sigma(J)}}{J} T^*_{u,\varphi}\left(\frac{\chi_{\sigma(J\circ\varphi)}}{J\circ\varphi} T_{u,\varphi}f\right) = \frac{\chi_{\sigma(J)}}{J} f$$

and

$$\omega(T_{u,\varphi})\omega(T_{u,\varphi})^* f = \frac{\chi_{\sigma(J\circ\varphi)}}{J\circ\varphi} T_{u,\varphi}\left(\frac{\chi_{\sigma(J)}}{J} T^*_{u,\varphi}f\right) = \frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)E(u^2)} uE(uf).$$

(d) $\omega(\omega(T_{u,\varphi})) = \omega((T^{\dagger}_{u,\varphi})^*) = T_{u,\varphi}f$.
(e) By direct computations, we get that

$$\omega(T^{\dagger}_{u,\varphi})f = T^{\dagger}_{u,\varphi}T_{u,\varphi}T^*_{u,\varphi}f = hE(uf)\circ\varphi^{-1} = T^*_{u,\varphi}f = (\omega(T_{u,\varphi}))^{\dagger} f.$$

□

**Proposition 2.2.** *Let $T_{u,\varphi} \in B_C(L^2(\Sigma))$. Then $\omega(T_{u,\varphi}) = V|\omega(T_{u,\varphi})|$ is the polar decomposition of $\omega(T_{u,\varphi})$, such that*

$$|\omega(T_{u,\varphi})|(f) = \frac{1}{\sqrt{J}}(f);$$

$$V(f) = \frac{\chi_{\sigma(J\circ\varphi)}}{\sqrt{J\circ\varphi}} T_{u,\varphi}(f).$$

We know that $\widetilde{T_{u,\varphi}}f = \sqrt[4]{\frac{J\chi_{\sigma(E(u))}}{J\circ\varphi}} uf\circ\varphi$. Now turn to the computation of $\omega(\widetilde{T_{u,\varphi}})$ and $\widetilde{\omega(T_{u,\varphi})}$. By combining the previous results we obtain the following proposition.

**Proposition 2.3.** *Let $T_{u,\varphi} \in B_C(L^2(\Sigma))$. Then*

(a) $\omega(\widetilde{T_{u,\varphi}}) = u\frac{\chi_J}{\sqrt[4]{J(J\circ\varphi)^3}} f \circ \varphi$.

(b) $\omega(\widetilde{T_{u,\varphi}}) = u\frac{\chi_{\sigma(E(u^2\sqrt{J}))}}{h\circ\varphi E(u^2\sqrt{J})}\sqrt[4]{J(J\circ\varphi)} f \circ \varphi$.

**Corollary 2.4.** *Let $T_{u,\varphi} \in B_C(L^2(\Sigma))$. Then $\omega(\widetilde{T_{u,\varphi}}) = \omega(\widetilde{T_{u,\varphi}})$ if and only if $E(u^2\sqrt{J}) = \sqrt{J}E(u^2)$.*

We recall that a conjugation on a Hilbert space $\mathcal{H}$ is an anti-linear operator $S : \mathcal{H} \to \mathcal{H}$ which that is conjugate linear, involutive and isometric. By involutive and isometric we mean that $S^2 = I$ and $\langle Sx, Sy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$, respectively. An operator $T \in \mathcal{H}$ is said to be complex symmetric if there exists a conjugation $S$ on $\mathcal{H}$ such that $T = ST^*S$.The class of complex symmetric operators includes all normal and binormal operators, Hankel operators, truncated Toeplitz operators, and Volterra integration operators, see[14–16]. The problem of describing all complex symmetric weighted composition operators on various analytic function spaces is very active recently (see [2, 17, 33]). In the following, we show that the Cauchy dual of a weighted composition operator is complex symmetric.

**Proposition 2.5.** *Let $\sigma(J) = \sigma(h) = X$, $\varphi^2 = I$, the identify transformation. If $h(h \circ \varphi) = 1$, $u \in L^0(\mathcal{A})$, where $\mathcal{A} = \varphi^{-1}(\Sigma)$. Then $\omega(T_{u,\varphi})$ is complex symmetric.*

*Proof.* Define $S(f) = \frac{u}{|u|}\frac{\bar{f}\circ\varphi}{\sqrt{h\circ\varphi}}$. Then $S$ is conjugate linear, $S^2 = I$ and for each $f \in L^2(\Sigma)$, we have

$$S\omega(T_{u,\varphi})^*S(f) = S(T^{\dagger}_{u,\varphi})S(f) = S(\frac{\chi_{\sigma(J)}}{J}hE(uS(f))\circ\varphi^{-1}) = \frac{u}{|u|}\frac{\sqrt{h\circ\varphi}}{J\circ\varphi}u\bar{S}(f)$$

$$= \frac{uf\circ\varphi}{J\circ\varphi} = \omega(T_{u,\varphi})f.$$

Also

$$\langle Sf, Sg\rangle = \int_X \frac{(\bar{f}\circ\varphi)(g\circ\varphi)}{h\circ\varphi}d\mu = \int_X \frac{h\bar{f}g}{h}d\mu = \langle g, f\rangle.$$

So, $\omega(T_{u,\varphi})$ is complex symmetric. □

**Example 2.6.** *Suppose that $1 < a < \infty$. Let $X = [\frac{1}{a}, a]$, $d\mu = dx$ and $\Sigma$ be the Lebesgue sets. Define the non-singular transformation $\varphi : X \to X$ by $\varphi(x) = \frac{1}{x}$. Put $u(x) = x^2$ and $\mathcal{A} = \varphi^{-1}(\Sigma)$. Then $h(x) = \frac{1}{x^2}$ and $E = I$. Simple computations show that $\sigma(J) = \sigma(h) = X$, $\varphi^2 = I$ and $h(h\circ\varphi) = 1$. Define $S(f) = \frac{\bar{f}\circ\varphi}{\sqrt{h\circ\varphi}}$. It is clear that $S$ is conjugate linear. By direct computation we get that $\omega(T_{u,\varphi}) = \frac{1}{x^4}f(\frac{1}{x})$ and*

$$S\omega(T_{u,\varphi})^*S(f) = \omega(T_{u,\varphi})f.$$

*Thus $\omega(T_{u,\varphi})$ is complex symmetric operator on $L^2(\Sigma)$.*

In the following two theorems, to avoid tedious calculations, we investigate only $\mathcal{K}(p, 1, n)$ and $\mathcal{K}(p, m, 0)$ classes of weighted composition operators. Note that $T \in \mathcal{K}(p, 1, 0)$ if and only if $T$ is $p$-hyponormal and $T \in \mathcal{K}(p, 1, 1)$ if and only if $T$ is $p$-quasihyponormal. Recall that $T \in \mathcal{K}(p, 1, n)$ if and only if $T^{*n}(T^*T)^pT^n \geq T^{*n}(TT^*)^pT^n$, and $T \in \mathcal{K}(p, m, 0)$ if and only if $(T^{*m}T^m)^p \geq (T^mT^{*m})^p$.

**Theorem 2.7.** *Let $T_{u,\varphi} \in B_C(L^2(\Sigma))$, $n \in \mathbb{N}$. Then $\omega(T_{u,\varphi}) \in \mathcal{K}(p, 1, n)$ if and only if*

$$E_n(\frac{u_n^2}{J^p}) \geq E_n(\frac{u_nu(h^p\circ varphi)(E(u^2))^{p-1}E(uu_n)}{(J\circ\varphi)^{2p}}), \quad on\ \sigma(J_n).$$

*Proof.* Let $f \in L^2(\Sigma)$. Direct computations show that

$$(\omega(T_{u,\varphi})^*)^n(\omega(T_{u,\varphi})^*\omega(T_{u,\varphi}))^p(\omega(T_{u,\varphi}))^n(f)$$

$$= \frac{1}{J_n^2}h_nE_n(\frac{u_n}{J^p}(u_nf\circ\varphi^n))\circ\varphi^{-k}f$$

$$= \frac{1}{J_n^2}h_nE_n(\frac{u_n^2}{J^p})\circ\varphi^{-n}f$$

and

$$(\omega(T_{u,\varphi})^*)^n(\omega(T_{u,\varphi})\omega(T_{u,\varphi})^*)^p(\omega(T_{u,\varphi}))^n(f)$$

$$= \frac{1}{J_n^2}h_nE_n(\frac{u_nu(h^p\circ\varphi)(E(u^2))^{p-1}E(\frac{uu_nf\circ\varphi^n}{J_n\circ\varphi^n})}{(J\circ\varphi)^{2p}})\circ\varphi^{-n}f$$

$$= \frac{1}{J_n^2}h_nE_n(\frac{u_nu(h^p\circ\varphi)(E(u^2))^{p-1}E(uu_n)}{(J\circ\varphi)^{2p}})\circ\varphi^{-n}f.$$

Then $\omega(T_{u,\varphi}) \in \mathcal{K}(p,1,n)$ if and only if

$$E_n(\frac{u_n^2}{J^p}) \geq E_n(\frac{u_nu(h^p\circ\varphi)(E(u^2))^{p-1}E(uu_n)}{(J\circ\varphi)^{2p}}), \quad \text{on } \sigma(J_n).$$

□

**Corollary 2.8.** *Let $C_\varphi \in B_C(L^2(\Sigma))$. Then $\omega(C_\varphi) \in \mathcal{K}(p,1,n)$ if and only if $E_n(\frac{1}{h^p}) \geq E_n(\frac{1}{h^p\circ\varphi})$ on $\sigma(h_n)$. In particular, if $n = 1$, then $\omega(C_\varphi)$ is p-quasihyponormal if and only if $E(\frac{1}{h^p}) \geq \frac{1}{h^p\circ\varphi}$ on $\sigma(h)$.*

**Lemma 2.9.** *[26] Let $\alpha$ and $\beta$ be nonnegative and measurable functions. Then for every $f \in L^2(\Sigma)$,*

$$\int_X \alpha|f|^2d\mu \geq \int_X |E_n(\beta f)|^2d\mu$$

*if and only if $\sigma(\beta) \subseteq \sigma(\alpha)$ and $E_n(\frac{\beta^2}{\alpha}\chi_{\sigma(\alpha)}) \leq 1$.*

**Theorem 2.10.** *Let $T_{u,\varphi} \in B_C(L^2(\Sigma))$. Then $\omega(T_{u,\varphi}) \in \mathcal{K}(p,m,0)$ if and only if if and only if $\sigma(u_m) \subseteq \sigma(J_m)$ and*

$$E_m(u_m^2J_m^p) \leq (J_m^p\circ\varphi^m)E_m({u_m}^2).$$

*Proof.* Let $f \in L^2(\Sigma)$. Then we have

$$\langle(\omega(T_{u,\varphi})^{*m})\omega(T_{u,\varphi})^m)^pf,f\rangle = \int_X \frac{\chi_{\sigma(J_m)}}{J_m^p}|f|^2d\mu,$$

and

$$\langle(\omega(T_{u,\varphi})^m\omega(T_{u,\varphi})^{*m}))^pf,f\rangle$$

$$= \int_X \frac{\chi_{\sigma(J_m\circ\varphi^m)}}{(J_m\circ\varphi^m)^{2p}}(h_m\circ\varphi^m)^p(E_m(u_m^2))^{p-1}u_mE_m(u_mf)\bar{f}d\mu$$

$$= \int_X |E_m(\frac{\chi_{\sigma(J_m\circ\varphi^m)}}{J_m^p\circ\varphi^m}(h_m^{\frac{p}{2}}\circ\varphi^m)(E_m(u_m^2))^{\frac{p-1}{2}}u_mf)|^2d\mu.$$

Put $\alpha = \frac{1}{J_m^p}$ and $\beta = \frac{1}{J_m^p\circ\varphi^m}(h_m^{\frac{p}{2}}\circ\varphi^m)(E_m(u_m^2))^{\frac{p-1}{2}}u_m$. Then $\sigma(\alpha) = \sigma(J_m)$ and $\sigma(\beta) = \sigma(u_m)$. Now, the desired conclusion follows from Lemma 2.9.

□

**Corollary 2.11.** *$\omega(T_{u,\varphi})$ is p-hyponormal if and only if $\sigma(u) \subseteq \sigma(J)$ and*

$$E\left((\frac{J}{J\circ\varphi})^p\frac{u^2}{E(u^2)}\right) \leq 1.$$

**Lemma 2.12.** *[31] Let $T \in B(\mathcal{H})$ and let $U|T|$ be its polar decomposition. Suppose p be a positive real numbers. Then the following hold:*

*(a) T is p-paranormal if and only if for each $\lambda > 0$,*

$$|T^*|^p|T|^{2p}|T^*|^p - 2\lambda|T^*|^{2p} + \lambda^2 \geq 0.$$

*(b) T is absolute-p-paranormal if and only if for each $\lambda > 0$,*

$$|T^*||T|^{2p}|T^*| - (p+1)\lambda^p|T^*|^2 + p\lambda^{p+1} \geq 0.$$

**Theorem 2.13.** *Let $T_{u,\varphi} \in B_C(L^2(\Sigma))$. Then The following statements are hold.*

*(a) $\omega(T_{u,\varphi})$ is p-paranormal if and only if*

$$E(\frac{u}{J^p}) \geq \frac{\chi_{\sigma(J)}}{h^p\circ\varphi E(u^2)^{p+1}}(E(u))^3, \quad on\ \sigma(J)\cap\sigma(E(u)).$$

*(b) $\omega(T_{u,\varphi})$ is absolute-p-paranormal if and only if*

$$E(\frac{u}{J^p}) \geq \frac{\chi_{\sigma(J)}}{h^p\circ\varphi E(u^2)^{2p}}(E(u))^{2p+1}, \quad on\ \sigma(J)\cap\sigma(E(u)).$$

*Proof.* (a) Let $f \in L^2(\Sigma)$. It is easy to check that

$$|\omega(T_{u,\varphi})^*|^{2p}f = \frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)^pE(u^2)}uE(uf),$$

$$|\omega(T_{u,\varphi})^*|^pf = \frac{\chi_{\sigma(J\circ\varphi)}}{\sqrt{(J\circ\varphi)^p}E(u^2)}uE(uf),$$

$$|\omega(T_{u,\varphi})|^{2p}f = \frac{1}{J^p}f.$$

It follows that

$$|{T^{\dagger}}^*_{u,\varphi}|^p|T^{\dagger}_{u,\varphi}|^{2p}|{T^{\dagger}}^*_{u,\varphi}|^pf = \frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)^pE(u^2)}uE(\frac{uf}{J^p}).$$

Now, by Lemma 2.12(a), $\omega(T_{u,\varphi})$ is $p$-paranormal if and only if

$$\langle\frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)^pE(u^2)}uE(\frac{uf}{J^p}) - 2\lambda\frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)^pE(u^2)}uE(uf) + \lambda^2, f\rangle \geq 0, \tag{1}$$

for each $\lambda \in (0, \infty)$. Put $f = \chi_{\varphi^{-1}B}$ with $\mu(\varphi^{-1}B) < \infty$. Hence, (2.1) holds if and only if

$$\int_{\varphi^{-1}B} \{ \frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)^p E(u^2)} u E(\frac{u\chi_B \circ \varphi}{J^p}) - 2\lambda \frac{\chi_{\sigma(J\circ\varphi)} u E(u\chi_B \circ \varphi)}{(J\circ\varphi)^p E(u^2)} + \lambda^2 \} d\mu \geq 0.$$

Equivalently,

$$\int_B \{ \frac{\chi_{\sigma(J)}(E(u)\circ\varphi^{-1})}{J^p E(u^2)\circ\varphi^{-1}} E(\frac{u}{J^p})\circ\varphi^{-1} - 2\lambda \frac{\chi_{\sigma(J)}(E(u)\circ\varphi^{-1})^2}{J^p E(u^2)\circ\varphi^{-1}} + \lambda^2 \} h d\mu \geq 0.$$

But, This is equivalent to

$$\frac{\chi_{\sigma(J)}(E(u)\circ\varphi^{-1})}{J^p E(u^2)\circ\varphi^{-1}} E(\frac{u}{J^p})\circ\varphi^{-1} - 2\lambda \frac{\chi_{\sigma(J)}}{J^p E(u^2)\circ\varphi^{-1}} (E(u)\circ\varphi^{-1})^2 + \lambda^2 \geq 0.$$

Set

$$\frac{\chi_{\sigma(J)}}{J^p E(u^2)\circ\varphi^{-1}} (E(u)\circ\varphi^{-1}) E(\frac{u}{J^p})\circ\varphi^{-1} := a$$

and

$$b := \frac{\chi_{\sigma(J)}}{J^p E(u^2)\circ\varphi^{-1}} (E(u)\circ\varphi^{-1})^2$$

Then $\omega(T_{u,\varphi})$ is $p$-paranormal if and only if

$$D(\lambda) := a - 2b\lambda + \lambda^2 \geq 0, \quad \lambda \in (0, \infty).$$

Since

$$\min_{\lambda \in (0,\infty)} D(\lambda) = D(b),$$

it follows that

$$\begin{aligned} D(b) \geq 0 &\Longleftrightarrow a \geq b^2 \\ &\Longleftrightarrow \frac{\chi_{\sigma(J)}(E(u)\circ\varphi^{-1})}{J^p E(u^2)\circ\varphi^{-1}} E(\frac{u}{J^p})\circ\varphi^{-1} \geq \frac{\chi_{\sigma(J)}(E(u)\circ\varphi^{-1})^4}{J^{2p}(E(u^2)\circ\varphi^{-1})^2} \\ &\Longleftrightarrow \frac{\chi_{\sigma(J)}}{J^p \circ \varphi E(u^2)} E(u) E(\frac{u}{J^p}) \geq \frac{\chi_{\sigma(J)}}{J^{2p}\circ\varphi E(u^2)^2} (E(u))^4, \\ &\Longleftrightarrow E(\frac{u}{J^p}) \geq \frac{\chi_{\sigma(J)}}{h^p \circ \varphi E(u^2)^{p+1}} (E(u))^3, \quad \text{on } \sigma(J) \cap \sigma(E(u)). \end{aligned}$$

(b) The proof is similar to part(a). □

## 3. MP− inverse of weighted composition operators

In this section, we define the new classes of operators, called †-hyponormal and †-quasi-hyponormal. In addition, we discuss measure theoretic characterizations for Moore-Penrose inverse of weighted composition operators in this new classes. We then give some examples illustrating these classes. From now on, we assume that $T_{u,\varphi}$ has closed range. Before our main results are presented, we state the following lemma.

**Lemma 3.1.** *[21] Let $0 \leq v \in L^0(\mathcal{A})$, $0 \leq \omega \in L^0(\Sigma)$ and let $A := M_{v\omega} E M_\omega \in B(L^2(\Sigma))$. Then for each $p \in (0, \infty)$, $A^p = M_{v^p \omega E(\omega^2)^{p-1}} E M_\omega$.*

**Theorem 3.2.** *Let $T_{u,\varphi} \in B_C(L^2(\Sigma))$, $n \in \mathbb{N}$. Then $T^{\dagger}_{u,\varphi} \in \mathcal{K}(p,1,n)$ if and only if*

$$\frac{E_n(u)E_n(E(\frac{uK_n}{J_n}))}{E_n(E(u^2))} \geq \frac{E_n(J^p \circ \varphi)}{E_n(J^p)}\frac{K_n}{J_n}, \quad on\ \sigma(h_n).$$

*Where $K_n = h_nE_n(u_n) \circ \varphi^n$.*

*Proof.* Let $f \in L^2(\Sigma)$. Direct computations show that

$$(T^{\dagger}_{u,\varphi})^{*n}(T^{\dagger^*}_{u,\varphi}T^{\dagger}_{u,\varphi})^p(T^{\dagger}_{u,\varphi})^n f$$

$$= \frac{\chi_{\sigma(J_n\circ\varphi)}u_n(u\circ\varphi^n)E(u\frac{\chi_{\sigma(J_n)}}{J_n}h_nE_n(u_nf)\circ\varphi^{-n})\circ\varphi^n}{(J_n\circ\varphi^n)(J\circ\varphi^{n+1})^pE(u^2)\circ\varphi^n} := af,$$

$$(T^{\dagger}_{u,\varphi})^{*n}(T^{\dagger}_{u,\varphi}T^{\dagger^*}_{u,\varphi})^p(T^{\dagger}_{u,\varphi})^n f$$

$$= \frac{\chi_{\sigma(J_n\circ\varphi)}}{(J_n\circ\varphi^n)^2(J\circ\varphi^n)^p}u_n(h_n\circ\varphi^n)E_n(u_nf) := bf.$$

Then, $W_{\dagger} \in \mathcal{K}(p,1,n)$ if and only if

$$\langle (a-b)f, f\rangle \geq 0, \tag{2}$$

for each $\lambda \in (0,\infty)$. Put $f = \chi_{\varphi^{-n}B}$ with $\mu(\varphi^{-n}B) < \infty$. Hence, (3.1) holds if and only if

$$\int_{\varphi^{-n}B}\{\frac{\chi_{\sigma(J_n\circ\varphi)}u_n(u\circ\varphi^n)E(u\frac{\chi_{\sigma(J_n)}}{J_n}h_nE_n(u_nf)\circ\varphi^{-n})\circ\varphi^n}{(J_n\circ\varphi^n)(J\circ\varphi^{n+1})^pE(u^2)\circ\varphi^n}$$

$$-\frac{\chi_{\sigma(J_n\circ\varphi)}}{(J_n\circ\varphi^n)^2(J\circ\varphi^n)^p}u_n(h_n\circ\varphi^n)E_n(u_nf)\}d\mu \geq 0.$$

Equivalently,

$$\int_B\{\frac{\chi_{\sigma(J_n)}E_n(u_n)\circ\varphi^{-n}E_n(u)E_n(E(u\frac{\chi_{\sigma(J_n)}}{J_n}K_n))}{(J_n)E_n(J^p\circ\varphi)E_n(E(u^2))}$$

$$-\frac{\chi_{\sigma(J_n)}}{(J_n)^2E_n(J^p)}h_n(E_n(u_n)\circ\varphi^{-n})^2\}h_nd\mu \geq 0.$$

But, this is equivalent to

$$\frac{\chi_{\sigma(J_n)}}{E_n(J^p\circ\varphi)E_n(E(u^2))}E_n(u)E_n(E(u\frac{\chi_{\sigma(J_n)}}{J_n}K_n)) - \frac{\chi_{\sigma(J_n)}}{E_n(J^p)}\frac{K_n}{J_n} \geq 0,$$

on $\sigma(J_n)$. Then $T^{\dagger}_{u,\varphi} \in \mathcal{K}(p,1,n)$ if and only if

$$\frac{E_n(u)E_n(E(\frac{uK_n}{J_n}))}{E_n(E(u^2))} \geq \frac{E_n(J^p\circ\varphi)}{E_n(J^p)}\frac{K_n}{J_n}, \quad \text{on } \sigma(h_n).$$

□

**Corollary 3.3.** *Let $C_\varphi \in B_C(L^2(\Sigma))$. Then $C_\varphi^\dagger \in \mathcal{K}(p,1,n)$ if and only if $E_n(h^p) \geq E_n(h^p \circ \varphi)$ on $\sigma(h_n)$. In particular, if $n = 1$, then $C_\varphi^\dagger$ is p-quasihyponormal if and only if $E(h^p) \geq h^p \circ \varphi$, on $\sigma(h)$.*

**Theorem 3.4.** *Let $T_{u,\varphi} \in B_C(L^2(\Sigma))$. Then $T_{u,\varphi}^\dagger \in \mathcal{K}(p,m,0)$ if and only if*

$$\frac{(E_m(u_m))^2}{E_m(u_m^2)} \geq \frac{J_m^p \circ \varphi^m}{E_m(J_m^p)} \quad on\ \sigma(J_m).$$

*Proof.* Let $f \in L^2(\Sigma)$. By direct computations and Lemma 3.1, we get that

$$((T^{\dagger^*}_{u,\varphi})^m)(T^\dagger_{u,\varphi})^m)^p f = \frac{\chi_{\sigma(J_m\circ\varphi^m)}}{(J_m \circ \varphi^m)^{2p}}(h_m \circ \varphi^m)^p (E_m(u_m^2))^{p-1} u_m E_m(u_m f),$$
$$((T^\dagger_{u,\varphi})^m (T^{\dagger^*}_{u,\varphi})^m))^p f = \frac{\chi_{\sigma(J_m)}}{J_m^p} f.$$

Then, $T_{u,\varphi}^\dagger \in \mathcal{K}(p,m,0)$ if and only if

$$\langle \frac{\chi_{\sigma(J_m\circ\varphi^m)}}{(J_m \circ \varphi^m)^{2p}}(h_m \circ \varphi^m)^p (E_m(u^2))^{p-1} u_m E_m(u_m f) - \frac{\chi_{\sigma(J_m)}}{J_m^p} f, f\rangle \geq 0, \tag{3}$$

for each $\lambda \in (0,\infty)$. Put $f = \chi_{\varphi^{-m}B}$ with $\mu(\varphi^{-m}B) < \infty$. Hence, (3.2) holds if and only if

$$\int_{\varphi^{-m}B} \{\frac{\chi_{\sigma(J_m\circ\varphi^m)}}{(J_m \circ \varphi^m)^{2p}}(h_m \circ \varphi^m)^p (E_m(u^2))^{p-1} u_m E_m(u_m) - \frac{1}{J_m^p}\} d\mu \geq 0.$$

Equivalently,

$$\int_B \{\frac{\chi_{\sigma(J_m)}}{J_m^p (E_m(u_m^2) \circ \varphi^{-m})}(E_m(u_m) \circ \varphi^{-m})^2 - \frac{1}{E_m(J_m^p) \circ \varphi^{-m}}\} h_m d\mu \geq 0.$$

But, this is equivalent to

$$\frac{\chi_{\sigma(J_m)}}{J_m^p (E_m(u_m^2) \circ \varphi^{-m})}(E_m(u_m) \circ \varphi^{-m})^2 - \frac{1}{E_m(J_m^p) \circ \varphi^{-m}} \geq 0,$$

on $\sigma(J_m)$. Then, $T_{u,\varphi}^\dagger \in \mathcal{K}(p,m,0)$ if and only if

$$\frac{(E_m(u_m))^2}{E_m(u_m^2)} \geq \frac{J_m^p \circ \varphi^m}{E_m(J_m^p)} \quad \text{on } \sigma(J_m).$$

□

**Corollary 3.5.** *Let $T_{u,\varphi} \in B_C(L^2(\Sigma))$. Then $T_{u,\varphi}^\dagger$ is p-hyponormal if and only if*

$$\frac{(E(u))^2}{E(u^2)} \geq \frac{J^p \circ \varphi}{E(J^p)} \quad on\ \sigma(J).$$

**Definition 3.6.** *Let $T \in B_C(\mathcal{H})$. We said that T is $\dagger-$hyponormal if $T^\dagger T^* \geq T^* T^\dagger$ and T is $\dagger-$quasi hyponormal if $T^\dagger(T^*T) \geq (T^*T)T^\dagger$.*

**Theorem 3.7.** *Let $T_{u,\varphi} \in B_C(L^2(\Sigma))$. Then The following statements are hold.*

*(a) $T_{u,\varphi}^\dagger$ is $\dagger-$hyponormal if and only if $E(u^2) \leq 1$.*

*(b) $T_{u,\varphi}^\dagger$ is $\dagger-$quasi hyponormal if and only if*

$$E(uJ) \geq J \circ \varphi E(u), \quad on\ \sigma(E(u)).$$

*Proof.* (a) Let $f \in L^2(\Sigma)$. Direct computations show that

$$T^{\dagger}_{u,\varphi}T_{u,\varphi}f = \int_X \chi_{\sigma(J)}|f|^2 d\mu$$
$$T_{u,\varphi}T^{\dagger}_{u,\varphi}f = \int_X \frac{\chi_{\sigma(J\circ\varphi)}}{J\circ\varphi}(h\circ\varphi)[E(u^2)]E(uf)d\mu$$

$$= \int_X |E(\frac{\chi_{\sigma(J\circ\varphi)}}{J^{\frac{1}{2}}\circ\varphi}(h^{\frac{1}{2}}\circ\varphi)[E(u^2)]^{\frac{1}{2}}uf)|^2 d\mu.$$

Hence by Lemma 2.9, $T^{\dagger}_{u,\varphi}$ is $\dagger-$hyponormal if and only if

$$E(\frac{(h\circ\varphi)[E(u^2)]u^2}{J\circ\varphi})\chi_{\sigma(J\circ\varphi)} \leq 1.$$

Equivalently, $E(u^2) \leq 1$.
(b) By simple calculations we get that,

$$T^{\dagger}_{u,\varphi}(T^*_{u,\varphi}T_{u,\varphi})f = \frac{\chi_{\sigma(J)}hE(uJf)\circ\varphi^{-1}}{J}$$
$$(T^*_{u,\varphi}T_{u,\varphi})T^{\dagger}_{u,\varphi}f = hE(uf)\circ\varphi^{-1}.$$

So, by the same of proof of Theorem 3.7, $T^{\dagger}_{u,\varphi}$ is $\dagger-$quasi hyponormal if and only if

$$\frac{\chi_{\sigma(J)}E(uJ)\circ\varphi^{-1}}{J} \geq E(u)\circ\varphi^{-1} \Longleftrightarrow$$
$$\Longleftrightarrow \frac{\chi_{\sigma(J\circ\varphi)}E(uJ)}{J\circ\varphi} \geq E(u)$$
$$\Longleftrightarrow E(uJ) \geq J\circ\varphi E(u), \quad \text{on } \sigma(E(u)).$$

□

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators on the infinite dimensional complex Hilbert space $\mathcal{H}$. Let $T = U|T|$ be the polar decomposition for $T \in B(\mathcal{H})$, where $U$ is a partial isometry and $|T| = (T^*T)^{1/2}$. In the following we concentrate on the polar decomposition of $T^{\dagger}_{u,\varphi}$.

Let $f \in L^2(\Sigma)$. Then $({T^{\dagger}}^*_{u,\varphi}T^{\dagger}_{u,\varphi})(f) = \frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)^2}u(h\circ\varphi)E(uf)$ and so $|T^{\dagger}_{u,\varphi}|$ follows from Lemma 3.1. Moreover, by direct computations we have

$$U^{\dagger}_{u,\varphi} = M_{\frac{\chi_{\sigma(J)}}{\sqrt{J}}}T^*_{u,\varphi}.$$

Moreover, it is easy to check that $U^{\dagger}_{u,\varphi}|T^{\dagger}_{u,\varphi}| = T^{\dagger}_{u,\varphi}$, $U^{\dagger}_{u,\varphi}U_{u,\varphi}U^{\dagger}_{u,\varphi} = U^{\dagger}_{u,\varphi}$ and $\mathcal{N}(U^{\dagger}_{u,\varphi}) = \mathcal{N}(T^*_{u,\varphi}) = \mathcal{N}(T^{\dagger}_{u,\varphi})$. Consequently, we have the following proposition.

**Proposition 3.8.** *Let $T_{u,\varphi} \in B_C(L^2(\Sigma))$. Then $T^{\dagger}_{u,\varphi} = U^{\dagger}_{u,\varphi}|T^{\dagger}_{u,\varphi}|$ is the polar decomposition of $T^{\dagger}_{u,\varphi}$, such that*

$$|T^{\dagger}_{u,\varphi}|(f) = \frac{\chi_{\sigma(J\circ\varphi)}}{\sqrt{J\circ\varphi[E(u^2)]}}uE(uf);$$
$$U^{\dagger}_{u,\varphi}(f) = (\frac{\chi_{\sigma(j)}}{j})^{\frac{1}{2}}T^*_{u,\varphi}f.$$

**Theorem 3.9.** *Let $T_{u,\varphi} \in B_C(L^2(\Sigma))$. Then The following statements are hold.*

*(a) $T^{\dagger}_{u,\varphi}$ is p-paranormal if and only if*

$$\frac{E(u)}{E(u^2)}E(\frac{u}{\sqrt{J^p}}) \geq (\frac{J \circ \varphi}{J\sqrt{J}})^p, \quad on\ \sigma(J).$$

*(b) $T^{\dagger}_{u,\varphi}$ is absolute-p-paranormal if and only if*

$$\sqrt{J}\frac{E(u)}{E(u^2)}E(\frac{u}{\sqrt{J}}) \geq (\frac{J \circ \varphi}{J})^p, \quad on\ \sigma(J).$$

*Proof.* (a) Let $f \in L^2(\Sigma)$. It is easy to check that

$$|T^{\dagger}_{u,\varphi}|^{2p} f = \frac{\chi_{\sigma(J\circ\varphi)}}{(J \circ \varphi)^p E(u^2)} uE(uf),$$

$$|T^{\dagger}_{u,\varphi}|^{p} f = \frac{\chi_{\sigma(J\circ\varphi)}}{\sqrt{(J \circ \varphi)^p} E(u^2)} uE(uf),$$

$$|T^{\dagger^*}_{u,\varphi}|^{2p} f = \frac{1}{J^p} f.$$

It follows that

$$|T^{\dagger^*}_{u,\varphi}|^{p}|T^{\dagger}_{u,\varphi}|^{2p}|T^{\dagger^*}_{u,\varphi}|^{p} f = \frac{\chi_{\sigma(J\circ\varphi)}}{\sqrt{J^p}(J \circ \varphi)^p E(u^2)} uE(\frac{uf}{\sqrt{J^p}}).$$

Now, by Lemma 2.12(a), $T^{\dagger}{}_{u,\varphi}$ is $p$-paranormal if and only if

$$\langle \frac{\chi_{\sigma(J)}}{\sqrt{J^p}(J \circ \varphi)^p E(u^2)} uE(\frac{uf}{\sqrt{J^p}}) - 2\lambda \frac{\chi_{\sigma(J)}}{J^p} f + \lambda^2, f\rangle \geq 0, \tag{4}$$

for each $\lambda \in (0, \infty)$. Put $f = \chi_{\varphi^{-1}B}$ with $\mu(\varphi^{-1}B) < \infty$. Hence, (3.3) holds if and only if

$$\int_{\varphi^{-1}B} \{\frac{\chi_{\sigma(J)}}{\sqrt{J^p}(J \circ \varphi)^p E(u^2)} uE(\frac{u\chi_B \circ \varphi}{\sqrt{J^p}}) - 2\lambda \frac{\chi_{\sigma(J)}}{J^p}(\chi_B \circ \varphi) + \lambda^2\} d\mu \geq 0.$$

Equivalently,

$$\int_B \{\frac{\chi_{\sigma(J)} E(u) \circ \varphi^{-1}}{\sqrt{J^p} \circ \varphi^{-1} J^p E(u^2) \circ \varphi^{-1}} E(\frac{u}{\sqrt{J^p}}) \circ \varphi^{-1} - 2\lambda \frac{\chi_{\sigma(J)}}{J^p \circ \varphi^{-1}} + \lambda^2\} h d\mu \geq 0.$$

But, This is equivalent to

$$\frac{\chi_{\sigma(J)} E(u) \circ \varphi^{-1}}{\sqrt{J^p} \circ \varphi^{-1} J^p E(u^2) \circ \varphi^{-1}} E(\frac{u}{\sqrt{J^p}}) \circ \varphi^{-1} - 2\lambda \frac{\chi_{\sigma(J)}}{J^p \circ \varphi^{-1}} + \lambda^2 \geq 0.$$

Set

$$\frac{E(u) \circ \varphi^{-1}}{\sqrt{J^p} \circ \varphi^{-1} J^p E(u^2) \circ \varphi^{-1}} E(\frac{u}{\sqrt{J^p}}) \circ \varphi^{-1} := a$$

and

$$b := \frac{1}{J^p \circ \varphi^{-1}}.$$

Then, $T^{\dagger}{}_{u,\varphi}$ is $p$-paranormal if and only if

$$D(\lambda) := a - 2b\lambda + \lambda^2 \geq 0, \qquad \lambda \in (0, \infty).$$

Since

$$\min_{\lambda\in(0,\infty)} D(\lambda) = D(b),$$

it follows that

$$\begin{aligned} D(b) \geq 0 &\Longleftrightarrow a \geq b^2 \\ &\Longleftrightarrow \frac{E(u)\circ\varphi^{-1}}{\sqrt{J^p}\circ\varphi^{-1}J^pE(u^2)\circ\varphi^{-1}}E(\frac{u}{\sqrt{J^p}})\circ\varphi^{-1} \geq (\frac{1}{J^p\circ\varphi^{-1}})^2 \\ &\Longleftrightarrow \frac{E(u)}{\sqrt{J^p}(J^p\circ\varphi)E(u^2)}E(\frac{u}{\sqrt{J^p}}) \geq \frac{1}{J^{2p}}, \\ &\Longleftrightarrow \frac{E(u)}{E(u^2)}E(\frac{u}{\sqrt{J^p}}) \geq (\frac{J\circ\varphi}{J\sqrt{J}})^p, \quad \text{on } \sigma(J) \end{aligned}$$

(b) The proof is similar to part(a). □

**Corollary 3.10.** *Let $C_\varphi \in B_C(L^2(\Sigma)$. Then The following statements are hold.*

*(a) $C^{\dagger}_{\varphi}$ is p-paranormal if and only if*

$$E(\frac{1}{\sqrt{h^p}}) \geq (\frac{h\circ\varphi}{h\sqrt{h}})^p, \quad \text{on } \sigma(h).$$

*(b) $C^{\dagger}_{\varphi}$ is absolute-p-paranormal if and only if*

$$\sqrt{h}E(\frac{1}{\sqrt{h}}) \geq (\frac{h\circ\varphi}{h})^p, \quad \text{on } \sigma(h).$$

In the following, we prove that the weighted composition operators are complex symmetric

**Corollary 3.11.** *Let $\sigma(h) = X$, $\varphi^2 = I$, the identity transformation. If $h(h\circ\varphi) = 1$, $u \in L^0(\mathcal{A})$, where $\mathcal{A} = \varphi^{-1}(\Sigma)$. Then $T_{u,\varphi}$ is complex symmetric.*

*Proof.* By the same argument in the proof of Proposition 2.5, we define

$$S : L^2(\Sigma) \to L^2(\Sigma)$$

$$S(f) = \frac{u}{|u|}\frac{\bar{f}\circ\varphi}{\sqrt{h\circ\varphi}}.$$

According to assumptions it is clear that $S$ is a conjugation and $S(T^*_{u,\varphi})S = T_{u,\varphi}$. □

**Example 3.12.** *Let $X = (0, 1)$ equipped with the Lebesgue measure $d\mu = dx$ on the Lebesgue measurable subsets of $X$ and let $\varphi : X \to X$ be a non-singular measurable transformation defined by and*

$$\varphi(x) = \begin{cases} 2x & 0 < x \leq \frac{1}{2}, \\ 2 - 2x & \frac{1}{2} \leq x < 1. \end{cases}$$

*Then for each $f \in L^2(\Sigma)$ and $x \in X$ we have*

$$h(x) = \left|\frac{d}{dx}\left(\frac{x}{2}\right)\right| + \left|\frac{d}{dx}\left(\frac{2-x}{2}\right)\right| = 1;$$
$$(Ef)(x) = \frac{f(x) + f(1-x)}{2};$$
$$(E(f) \circ \varphi^{-1})(x) = \frac{1}{2}\left(f\left(\frac{x}{2}\right) + f\left(1 - \frac{x}{2}\right)\right).$$

*Put $u(x) = \sqrt{x}$. Direct computation shows that*

$$E(u) = \frac{\sqrt{x} + \sqrt{1-x}}{2},$$

$$E(u^2) = \frac{1}{2}, \quad E(u^2\sqrt{J}) = \frac{1}{2\sqrt{2}},$$

$$J = \frac{1}{2}, \quad J \circ \varphi = \frac{1}{2}.$$

*Then we have*

$$\omega(\widetilde{T_{u,\varphi}}) = \omega(\widetilde{T_{u,\varphi}}) = \omega(T_{u,\varphi}) = \begin{cases} 2\sqrt{x} f(2x) & 0 < x \leq \frac{1}{2}, \\ 2\sqrt{x} f(2-2x) & \frac{1}{2} \leq x < 1. \end{cases}$$

*In this case by Theorem 2.7 and Theorem 2.10, $\omega(T_{u,\varphi}) \in \mathcal{K}(p,1,1)\backslash\mathcal{K}(p,1,0)$. Also by Theorem 3.2, $T^{\dagger}_{u,\varphi} \in \mathcal{K}(p,1,0) \Leftrightarrow x = \frac{1}{2}$.*

**Example 3.13.** *Let $X = (1,\infty)$ equipped with the Lebesgue measure $d\mu$ on the Lebesgue measurable subsets. The transformation $\varphi$ and the weighted function $u(x)$ are given by $\varphi(x) = \sqrt{x}$ and $u(x) = \frac{1}{\sqrt{1+x}}$. Then $h(x) = 2x$, $E = I$, $J(x) = \frac{2x}{1+x^2}$, $h \circ \varphi(x) = 2\sqrt{x}$, $J \circ \varphi(x) = \frac{2\sqrt{x}}{1+x}$. In this case by Theorems 2.7 and 2.10, $\omega(C_\varphi)$ is in $\mathcal{K}(p,1,0)\backslash\mathcal{K}(p,1,1)$. Also by Theorem 3.7, $T^{\dagger}_{u,\varphi}$ is $\dagger-$hyponormal but it is not $\dagger-$quasi hyponormal.*

Note that the following example was used in [11] to show that the $p$-hyponormal classes are distinct for $p$ with $0 < p < 1$. Now we will show that block matrix operators can separate weak $p$-hyponormal classes.

**Example 3.14.** *Let $M := [A_{ij}]_{1 \leq i,j \leq \infty}$ be a block matrix operator whose blocks are $6 \times 3$ matrices such that $A_{ij} = 0$, $i \neq j$ and $A = A_0 = A_1 = A_2 = ...$, where $A_n = A_{nn}$ for each $n \in \mathbb{N}_0$ and*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \sqrt{x_1} & 0 \\ 0 & 0 & \sqrt{x_2} \end{pmatrix}.$$

*Note that $x_1$ and $x_2$ are fixed positive real numbers. Now, let $\ell^2(m)$ be the weighted Hilbert sequence space on $(\mathbb{N}_0, 2^{\mathbb{N}_0}, \mu)$. Also let $\mu$ be a measure on $\Sigma$ defined by $\mu(\{n\}) = m_n$. Define $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ by*

$$\varphi(n) = \begin{cases} 3k & n = 6k, 6k+1, 6k+2, 6k+3; \\ 3k+1 & n = 6k+4; \\ 3k+2 & n = 6k+5. \end{cases}$$

*Then by [11, Proposition 2.2], $C_\varphi$ is unitarily equivalent to the block matrix $M$ such that for every $k \in \mathbb{N}_0$*

$$\sqrt{\frac{m_{6k+i-1}}{m_{3k}}} = 1 \quad for \ \ 1 \leq i \leq 4, \ \ k \in \mathbb{N}_0,$$

$$\sqrt{\frac{m_{6k+4}}{m_{3k+1}}} = \sqrt{x_1} \quad and \quad \sqrt{\frac{m_{6k+5}}{m_{3k+2}}} = \sqrt{x_2}, \ \ k \in \mathbb{N}_0.$$

*Moreover, by the same argument in the proof of [11, Proposition 3.1] $M$ is in $\mathcal{K}(p,1,0) \cap \mathcal{K}(p,1,1)$ if and only if $C_\varphi \in \mathcal{K}(p,1,0) \cap \mathcal{K}(p,1,1)$. But this equivalent to $C_\varphi^\dagger \in \mathcal{K}(p,1,0) \cap \mathcal{K}(p,1,1)$. By Theorem 3.2 and Theorem 3.4, $C_\varphi^\dagger \in \mathcal{K}(p,1,0) \cap \mathcal{K}(p,1,1)$ if and only if $E(h^p) \geq h^p \circ \varphi$, this condition is equivalent to*

$$\left(\frac{m(\varphi^{-1}(\varphi(n)))}{m_{\varphi(n)}}\right)^p \leq \frac{1}{m(\varphi^{-1}(\varphi(n)))} \sum_{j \in \varphi^{-1}(\varphi(n))} m_j \left(\frac{m(\varphi^{-1}(j))}{m_j}\right)^p \tag{5}$$

*Now by the same argument in the proof of [11, Proposition 3.3], we deduce that (5) is equivalent to*

$$\left(\frac{x_1}{4}\right)^p + \left(\frac{x_2}{4}\right)^p \geq 2 \tag{6}$$

*Let $0 < q < p$ and $M$ be in $\mathcal{K}(p,1,0) \cap \mathcal{K}(p,1,1)$. Then by using (6), we can find $x_1$ and $x_2$ such that $M$ is not in $\mathcal{K}(q,1,0) \cap \mathcal{K}(q,1,1)$. Namely for $x_1 = 3.5$ and $x_2 = 4.25$, by using (6) it is easy to see that $M$ is in $\mathcal{K}(9,1,0) \cap \mathcal{K}(9,1,1)$, but it is not in $\mathcal{K}(8,1,0) \cap \mathcal{K}(8,1,1)$. Also by the same argument $M$ is is p-paranormal if and only if $\omega(C_\varphi)$ is p-paranormal and by Theorem (2.13)(a), $\omega(C_\varphi)$ is p-paranormal if and only if $E(\frac{1}{h^p}) \geq \frac{1}{h^p \circ \varphi}$. This is equivalent to*

$$\left(\frac{4}{x_1}\right)^p + \left(\frac{4}{x_2}\right)^p \geq 2, \tag{7}$$

*particular in the above relations if we take $x_1 = 3.5$ and $x_2 = 4.25$, then by using (7) it is easy to see that for $p \geq 6$, $M$ is p-paranormal but for $p \leq 5$, it is not p-paranormal.*


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