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Separability of Path Spaces under the Open-Point and Bi-Point-Open Topologies

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The paper is dedicated to professor Robert A. McCoy

Abstract. In [3], two new kinds of topologies called the open-point topology and the bi-point-open topology on $C(X)$, the set of all real-valued continuous functions on a Tychonoff space X, have been introduced. In this article, we study the separability of the space $P(X)$, of all continuous maps on $[0, 1]$ into a Hausdorff space X, with the open-point and bi-point-open topologies. Our result also demonstrates, the claim made in [3], that both the domain as well as the codomain play significant roles in the construction of the open-point and bi-point-open topologies.

1. Introduction

The set of all real-valued continuous functions defined on a space X is denoted by $C(X)$. The open-point and bi-point-open topologies on $C(X)$ were introduced by A. Jindal et al. in [3]. Their endeavor was to define such meaningful topologies in which both the domain X as well as the codomain $\mathbb{R}$ play significant roles in the construction of topologies on $C(X)$. Recall that the *point-open topology p on* $C(X)$ is generated by subbase consisting of sets of the form

$$[x, V]^+ = \{f \in C(X) : f(x) \in V\},$$

where $x \in X$ and V is open in $\mathbb{R}$. The *open-point topology h on* $C(X)$ has a subbase consisting of sets of the form

$$[U, r]^- = \{f \in C(X) : f^{-1}(r) \cap U \neq \emptyset\},$$

where U is an open subset of X and $r \in \mathbb{R}$. The space $C(X)$ equipped with the open-point topology is denoted by $C_h(X)$. The *bi-point-open topology ph on* $C(X)$ is the join of the point-open topology p and the open-point topology h. In other words, it is the topology having subbasic open sets of both kinds: $[x, V]^+$ and $[U, r]^-$, where $x \in X$ and V is an open subset of $\mathbb{R}$, while U is an open subset of X and $r \in \mathbb{R}$. The space $C(X)$ equipped with the bi-point-open topology is denoted by $C_{ph}(X)$. These topologies (as shown in [3]) are fundamentally different from the usual set-open topologies that we study on the space $C(X)$. The spaces $C_h(X)$ and $C_{ph}(X)$ have been studied extensively in [2–7].

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In particular, the separability of the open-point and bi-point-open topologies on $C(X)$ has been studied in [2, 3, 6, 7] under the assumption that X has a countable π-base. In [3], it has been proved that if X has a countable π-base consisting of nontrivial connected sets, then $C_h(X)$ is separable, and if X is also submetrizable, then $C_{ph}(X)$ is also separable. Osipov in [7], further studied the separability of the spaces $C_h(X)$ and $C_{ph}(X)$ in a wider perspective. He proved the following improved version of the result given in [3].

Theorem 1.1. ([7]) *Let X be a Tychonoff space with a countable π-base. Then the following statements are true:*

(i) *The space $C_h(X)$ is separable if and only if X has a countable π-network consisting of $\mathcal{I}$-sets.*
(ii) *The space $C_{ph}(X)$ is separable if and only if X is submetrizable and has a countable π-network consisting of $\mathcal{I}$-sets.*

The above theorem was also proved independently by Jindal, McCoy and Kundu in [2]. But in [2], in order to study the separability of the spaces $C_h(X)$ and $C_{ph}(X)$ the concept of an $\mathcal{R}$-set has been defined, which is equivalent to the concept of an $\mathcal{I}$-set used in [7]. But the complete characterization of the separability of $C_h(X)$ and $C_{ph}(X)$ in terms of topological properties for an arbitrary domain space X is still an open problem.

In this article, we take the codomain to be an arbitrary topological space X instead of $\mathbb{R}$. More precisely, we consider the open-point and bi-point-open topologies on the space $P(X) = \{p : [0,1] \to X : p \text{ is continuous}\}$, the space of all paths into a Hausdorff space X. We show that the separability of the open-point and bi-point-open topologies on $P(X)$ can be characterized completely in terms of topological properties of the codomain space X. This demonstrates the fact that the codomain also plays a significant role in the construction of the open-point and bi-point-open topologies.

Throughout this paper the following conventions are used. The symbols $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ denote the space of real numbers, rational numbers and natural numbers, respectively. For a subset A of a space X, $\overline{A}$ denotes the closure of A in X. Also for any two topological spaces X and Y that have the same underlying set, the expression, $X \leq Y$ means that, the topology of X is weaker than or equal to topology of Y. For other basic topological notions, refer to [1].

2. Preliminaries

Let I be the unit interval $[0,1]$ with the usual topology, and let $P(X) = C(I, X)$ be the space of all paths into X. We define the *open-point topology* on $P(X)$ having subbasic open sets of the form

$$[U, x]^- = \{f \in P(X) : f^{-1}(x) \cap U \neq \emptyset\},$$

where U is an open subset of I and $x \in X$. The space $P(X)$ equipped with the open-point topology is denoted by $P_h(X)$.

Similarly, we define the *bi-point-open topology* on $P(X)$ having subbasic open sets of the form $[a, V]^+$ and $[U, x]^-$, where $a \in I$, $x \in X$, V is open in X and U is open in I. The space $P(X)$ equipped with the bi-point-open topology is denoted by $P_{ph}(X)$.

We first give bases for the spaces $P_h(X)$ and $P_{ph}(X)$ that are useful in characterizing separability for these spaces. Let $\mathcal{U}$ be some given countable base for $I = [0,1]$ consisting of open intervals and $\mathcal{V}$ be a given base for X.

Proposition 2.1. *The space $P_h(X)$ has a base consisting of the sets of the form $[U_1, x_1]^- \cap \ldots \cap [U_n, x_n]^-$, where $n \in \mathbb{N}$, $U_i \in \mathcal{U}$, $x_i \in X$ and $\sup U_i < \inf U_j$, whenever $1 \leq i < j \leq n$.*

Proof. Let G be any open set in $P_h(X)$ of the form $[V_1, t_1]^- \cap \ldots \cap [V_n, t_n]^-$, where each V_i is an open set in $[0,1]$ and $t_i \in X$. Let $f \in G$. So there exists $r_i \in V_i$ such that $f(r_i) = t_i$ for $1 \leq i \leq n$. If for some $1 \leq i < j \leq n$, $r_i = r_j$, then $t_i = t_j$ and $r_i \in V_i \cap V_j \neq \emptyset$. So $f \in [V_i \cap V_j, t_i]^- \subseteq [V_i, t_i]^- \cap [V_j, t_i]^-$. Take $G' = [V_1, t_1]^- \cap \ldots \cap [V_{i-1}, t_{i-1}]^- \cap [V_{i+1}, t_{i+1}]^- \cap \ldots \cap [V_{j-1}, t_{j-1}]^- \cap [V_{j+1}, t_{j+1}]^- \cap \ldots \cap [V_n, t_n]^- \cap [V_i \cap V_j, t_i]^-$. Clearly $f \in G' \subseteq G$. By proceeding in this way we get a basic open set $\widetilde{G} = [W_1, z_1]^- \cap \ldots \cap [W_m, z_m]^-$

such that $m \leq n$ and $f \in \widetilde{G} \subseteq G$; and for each $1 \leq j \leq m$, there exists $y_j \in W_j$ with $f(y_j) = z_j$ and $y_1, \ldots, y_m$ are distinct points. Since $y_1, \ldots, y_m$ are distinct points in $[0,1]$, there exist pairwise disjoint open sets $\{B_1, \ldots, B_m\}$ in $\mathcal{U}$ such that $y_i \in B_i \subseteq \overline{B_i} \subseteq W_i$ and $\overline{B_i} \cap \overline{B_j} = \emptyset$ for $1 \leq i < j \leq m$. Since for each $1 \leq j \leq m$, B_j is an interval in $[0,1]$, we can arrange $B_1, \ldots, B_m$ in a manner such that there infimums are in increasing order and name them $U_1, \ldots, U_m$, and the corresponding $z_1, \ldots, z_m$ call them $x_1, \ldots, x_m$. Therefore $f \in [U_1, x_1]^- \cap \ldots \cap [U_m, x_m]^- \subseteq \widetilde{G} \subseteq G$ and for $1 \leq i < j \leq m$, we have $\sup U_i < \inf U_j$. □

The next result gives a base for the space $P_{ph}(X)$. It can be proved easily by using Proposition 2.1.

Proposition 2.2. *The space $P_{ph}(X)$ has a base consisting of sets of the form $[r_1, V_1]^+ \cap \ldots \cap [r_m, V_m]^+ \cap [U_1, x_1]^- \cap \ldots \cap [U_n, x_n]^-$, where $m, n \in \mathbb{N}$, $r_i \in [0,1]$, $V_i \in \mathcal{V}$, $U_l \in \mathcal{U}$, $x_l \in X$, and $r_i < r_j$, $\sup U_l < \inf U_s$, whenever $1 \leq i < j \leq m$ and $1 \leq l < s \leq n$.*

3. Main Result

In order to study the separability of the spaces $P_h(X)$ and $P_{ph}(X)$, first we need the following definitions.

Definition 3.1. A compact, connected, locally connected metric space is said to be a *Peano space*.

Definition 3.2. *A subset of a space is said to be Peano subspace if it is Peano with its relative topology.*

Definition 3.3. *If a space can be written as a countable union of its Peano subspaces, then it is said to be a σ-Peano space.*

The space $\mathbb{R}$ with the usual topology is an example of a σ-Peano space, which is not a Peano space.

The Urysohn's metrization theorem says that every regular second countable T_1 space is separable and metrizable. So a continuous image of a compact metric space space into a Hausdorff space is again metrizable (see corollary 23.2, page 166 in [9]). Moreover, since quotient of a locally connected space is locally connected, it follows that continuous open as well as continuous closed image of locally connected space is locally connected. Therefore, a Hausdorff space which is a continuous image of a Peano space is Peano.

, if $f \in P(X)$ and X is Hausdorff, then $f(I)$ is a Peano subspace of X. The Hahn and Mazurkiewicz theorem (Theorem 31.5, page 221 in [9]) says that the converse is also true, that is, if X is a Peano space, then $X = f(I)$ for some $f \in P(X)$. Therefore, a Hausdorff space is a Peano space if and only if it is a continuous image of a closed and bounded interval. Every Peano space is pathwise connected and locally pathwise connected. Therefore if A is a Peano subspace of a space X, then A is contained in some path component of X. For more details on these properties, see [1] and [9].

Lemma 3.4. *If X is a σ-Peano space, then X has countably many path components.*

Proof. Let $X = \cup Y_n$, where each Y_n is a Peano subspace of X and $\mathcal{K}$ be the collection of all path components of X. Since every Peano space is pathwise connected, for each $n \in \mathbb{N}$, Y_n is pathwise connected. Therefore, for each $n \in \mathbb{N}$, there exists $K_{i_n} \in \mathcal{K}$ such that $Y_n \subseteq K_{i_n}$. Thus $X = \cup_{n\in\mathbb{N}} Y_n \subseteq \cup_{n\in\mathbb{N}} K_{i_n} \subseteq X$. Hence $\mathcal{K}$ is countable. □

Lemma 3.5. *If X is a σ-Peano space, then X has a countable network consisting of pathwise connected subsets of X.*

Proof. Let $X = \cup Y_n$, where each Y_n is a Peano subspace of X. Since every Peano space is locally pathwise connected and second countable, for each $n \in \mathbb{N}$, Y_n has a countable base $\mathbb{B}_n$ consisting of pathwise connected sets. We show that the countable collection $\mathbb{B} = \cup_{n\in\mathbb{N}} \mathbb{B}_n$ forms a network for X. Let U be an open set in X and $x \in U$. So $x \in Y_n$ for some $n \in \mathbb{N}$ and $U \cap Y_n$ is open in Y_n containing x. Therefore, there exists $B \in \mathbb{B}_n$ such that $x \in B \subseteq U \cap Y_n \subseteq U$. Hence $\mathbb{B}$ is a network for X. □

Theorem 3.6. *For a Hausdorff space X, the following are equivalent:*

(a) *The space $P_{ph}(X)$ is separable.*
(b) *The space $P_h(X)$ is separable.*
(c) *X is the continuous image of a countable topological sum of Peano spaces.*
(d) *X is a σ-Peano space.*

Proof. (a) $\Rightarrow$ (b) It follows from the fact that $P_h(X) \leq P_{ph}(X)$.

(b) $\Rightarrow$ (c) Let $P_h(X)$ be separable and $\mathcal{F} = \{f_n : n \in N\}$ be a countable dense subset of $P_h(X)$. Then by the above discussion for each $n \in \mathbb{N}$, $Y_n = f_n(I)$ is a Peano space. Now define $Y = \oplus_{n\in\mathbb{N}} Y_n$ and define a map $\phi : Y \to X$ by $\phi(y) = y$ for each $y \in Y_n$. It is easy to see that ϕ is a continuous map. To show ϕ is surjective, take $x \in X$. Since $\mathcal{F}$ is dense in $P_h(X)$, there exists $n \in \mathbb{N}$ such that $f_n \in [I, x]^-$. Thus $x \in Y_n \subseteq Y$.

(c) $\Rightarrow$ (d) Let X be the continuous image of a countable topological sum of Peano spaces. So there exists a sequence $\{Z_n\}$ of Peano spaces and a map $\phi : \oplus_{n\in\mathbb{N}} Z_n \to X$ such that ϕ is a continuous surjection. Since continuous image of a Peano space is Peano, for each $n \in \mathbb{N}$, $\phi(Z_n)$ is a Peano subspace in X. Now we show that $\cup\phi(Z_n) = X$. Let $x \in X$. Since ϕ is surjective, there exists $n \in \mathbb{N}$ such that $x \in \phi(Z_n)$. Thus $X = \cup\phi(Z_n)$. Hence X is a σ-Peano space.

(d) $\Rightarrow$ (a) Let $X = \cup_{n=1}^{\infty} Y_n$, where Y_n is a Peano subspace of X, and let $\mathcal{Y} = \{Y_n : n \in \mathbb{N}\}$. By using Lemma 3.4, we have the countable family $\mathcal{G}$ of all the path components of X. By Lemma 3.5, there exists a countable network $\mathcal{K} = \{K_n : n \in \mathbb{N}\}$ of pathwise connected subsets of X. For each $K_n \in \mathcal{K}$, fix two members k_n^1 and k_n^2. Let $C = \{[p, q] : p, q \in \mathbb{Q} \cap [0, 1], p < q\}$ and let $\hat{C}^n$ be the set of $([p_1, q_1], \ldots, [p_n, q_n]) \in C^n$ such that $p_1 < q_1 < p_2 < q_2 < \ldots < p_n < q_n$. For $m, n \in \mathbb{N}$, let $\mathcal{J}_n^m$ denote the collection of all subsets of the set $\{1, 2, \ldots, m + n\}$ of cardinality m.

Let $\mathcal{S}_{n,m} = \{(([p_1, q_1], \ldots, [p_{n+m}, q_{n+m}]), (K_{t_1}, \ldots, K_{t_m}), (Y_{s_1}, \ldots, Y_{s_n}), J_n^m, G) \in \hat{C}^{n+m} \times \mathcal{K}^m \times \mathcal{Y}^n \times \mathcal{J}_n^m \times \mathcal{G} : K_{t_r} \subseteq G, Y_{s_l} \subseteq G, \text{ for } 1 \leq l \leq n, 1 \leq r \leq m\}$.

Since every Peano space is pathwise connected and every pathwise connected subset of a space is contained in some path component of the space, for each $n, m \in \mathbb{N}$, $\mathcal{S}_{n,m} \neq \emptyset$ and countable. Therefore $\mathcal{S} = \cup_{n,m\in\mathbb{N}} \mathcal{S}_{n,m}$ is countable.

Pick $S \in \mathcal{S}$, then there exists $n, m \in \mathbb{N}$ such that $S \in \mathcal{S}_{n,m}$ and

$$S = (([p_1, q_1], \ldots, [p_{n+m}, q_{n+m}]), (K_{t_1}, \ldots, K_{t_m}), (Y_{s_1}, \ldots, Y_{s_n}), J_n^m, G).$$

Now we construct a continuous function f_S in $P(X)$. We have K_{t_i} is pathwise connected for $1 \leq i \leq m$, therefore for each $j \in J_n^m$, there exists a continuous function

$$f_S^j : [p_j, q_j] \to K_{t_{\phi(j)}}$$

such that $f_S^j(p_j) = k^1_{t_{\phi(j)}}$ and $f_S^j(q_j) = k^2_{t_{\phi(j)}}$, where ϕ is a function from J_n^m to $\{1, \ldots, m\}$ defined by $\phi(j) =$ the position of j in the set J_n^m after taking the members of J_n^m in an increasing order.

Since every Peano space is a continuous image of a closed and bounded interval, for each $j \in I_n^m = \{1, 2, \ldots, m + n\} \setminus J_n^m$, there exists a continuous function

$$f_S^j : [p_j, q_j] \to Y_{s_{\psi(j)}}$$

such that $f_S^j([p_j, q_j]) = Ys_{\psi(j)}$, where ψ is a function from I_n^m to $\{1, \ldots, n\}$ defined by $\psi(j) =$ the position of j in the set I_n^m after taking the members of I_n^m in an increasing order.

For $1 \leq r \leq m, 1 \leq l \leq n$, we have $K_{t_r} \subseteq G$ and $Y_{s_l} \subseteq G$, and G is pathwise connected. Therefore, for each $1 \leq j \leq m + n - 1$ there exists a continuous function

$$g_S^j : [q_j, p_{j+1}] \to G$$

such that

$$g_S^j(q_j) = f_S^j(q_j)$$

and

$$g_S^j(p_{j+1}) = f_S^{j+1}(p_{j+1}).$$

Define $f_S : [0,1] \to X$ by

$$f_S(y) = \begin{cases} f_S^j(y) & y \in [p_j, q_j], 1 \le j \le n+m \\ g_S^j(y) & y \in [q_j, p_{j+1}], 1 \le j \le n+m-1 \\ f_S^1(p_1) & y \in [0, p_1] \\ f_S^{n+m}(q_{n+m}) & y \in [q_{n+m}, 1]. \end{cases}$$

Clearly $f_S \in P(X)$. Now we show that the countable collection $\mathcal{F} = \{f_S : S \in \mathcal{S}\}$ is a dense subset of $P_{ph}(X)$.

Let W be any open set in $P_{ph}(X)$ and $g \in W$. By Proposition 2.2, there exists $\tilde{W} = [r_{i_1}, V_{i_1}]^+ \cap \ldots \cap [r_{i_m}, V_{i_m}]^+ \cap [U_{l_1}, x_{l_1}]^- \cap \ldots \cap [U_{l_n}, x_{l_n}]^-$ a nonempty basic open set in $P_{ph}(X)$ such that $g \in \tilde{W} \subseteq W$, where for $1 \le j < k \le m$ and $1 \le t < s \le n$, $r_{i_j} \in [0,1]$, V_{i_j} is open in X, $r_{i_j} < r_{i_k}$, U_{l_t} is open in $[0,1]$, $x_{l_t} \in X$ and $\sup U_{l_t} < \inf U_{l_s}$. We can choose $n+m$ intervals $[p_1, q_1], \ldots, [p_{n+m}, q_{n+m}]$ with rational end points in $[0,1]$ such that $p_1 < q_1 < \ldots < p_{n+m} < q_{n+m}$ and each U_{l_t} contains exactly one interval and each interval in the remaining m intervals contains exactly one r_{i_j}. Take $J_n^m = \{j \in \{1, \ldots, m+n\} : r_{i_s} \in [p_j, q_j]$ for some $s \in \{1, \ldots, m\}\}$ and $C^{m+n} = ([p_1, q_1], \ldots, [p_{n+m}, q_{n+m}]) \in \hat{C}^{m+n}$.

Since $\mathcal{K}$ is a network for X, for each $1 \le j \le m$, there exists $K_{i_j} \in \mathcal{K}$ such that $g(r_{i_j}) \in K_{i_j} \subseteq V_{i_j}$. As $X = \cup_{k \in \mathbb{N}} Y_k$, for each $1 \le t \le n$, there exists $Y_{l_t} \in \mathcal{Y}$ such that $x_{l_t} \in Y_{l_t}$. Also $\mathcal{G}$ is the countable family of all path components of X, so there exists $G \in \mathcal{G}$ such that $g([0,1]) \subseteq G$. Therefore, for each $1 \le j \le m$, $1 \le t \le n$, we have $K_{i_j} \subseteq G$ and $Y_{l_t} \subseteq G$.

Take $K^m = (K_{i_1}, \ldots, K_{i_m})$ and $Y^n = (Y_{l_1}, \ldots, Y_{l_n})$ then $S = (C^{m+n}, K^m, Y^n, J_n^m, G) \in \mathcal{S}$. Therefore, we have $f_S \in \mathcal{F}$. Now for each $j \in J_n^m$, there is exactly one $u \in \{1, \ldots, m\}$ such that $r_{i_u} \in [p_j, q_j]$. Note that this u denotes the position of j in the set J_n^m after taking the members of J_n^m in an increasing order. Thus $f_S([p_j, q_j]) \subseteq K_{i_u} \subseteq V_{i_u}$. And for each $j \in I_n^m = \{1, 2, \ldots, m+n\} \setminus J_n^m$, there is exactly one $t \in \{1, \ldots, n\}$ such that $[p_j, q_j] \subseteq U_{l_t}$. Note that this t denotes the position of j in the set I_n^m after taking the members of I_n^m in an increasing order. Therefore, $x_{l_t} \in f_S([p_j, q_j]) = Y_{l_t}$. Then $f_S \in \tilde{W} \cap \mathcal{F}$ and hence $\mathcal{F}$ is dense in $P_{ph}(X)$. □

Corollary 3.7. *If X is a Peano space, then the spaces $P_{ph}(X)$ and $P_h(X)$ are separable.*

Example 3.8. (Example 38, page 65 in [8]) Denote the Hilbert Cube by $X = [0,1]^{\mathbb{N}}$, then X is a Peano space. Hence the spaces $P_h(X)$ and $P_{ph}(X)$ are separable.

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