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Maximal Summability Operators On the Dyadic Hardy Spaces

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Abstract. It is proved that the maximal operators of subsequences of Nörlund logarithmic means and Cesáro means with varying parameters of Walsh-Fourier series is bounded from the dyadic Hardy spaces H_p to L_p. This implies an almost everywhere convergence for the subsequences of the summability means.

1. Walsh System

We shall denote the set of all non-negative integers by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$ and the set of dyadic rational numbers in the unit interval $\mathbb{I} := [0,1)$ by $\mathbb{Q}$. In particular, each element of $\mathbb{Q}$ has the form $\frac{p}{2^n}$ for some $p, n \in \mathbb{N}$, $0 \leq p < 2^n$. By a dyadic interval in $\mathbb{I}$ we mean one of the form $\left[\frac{l}{2^k}, \frac{l+1}{2^k}\right)$ for some $k \in \mathbb{N}$, $0 \leq l < 2^k$. Denote $I_n := [0, 2^{-n}), I_n(x) := x \dotplus I_n$. For $0 < n \in \mathbb{N}$ denote by $|n| := \max\left\{j \in \mathbb{N} : n_j \neq 0\right\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$.. The σ-algebra generatered by the dyadic intervals $\{I_n(x) : x \in \mathbb{I}\}$ will be denoted by $\mathcal{A}_n$ $(n \in \mathbb{N})$. Let

$$x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)}$$

be the dyadic expansion of $x \in \mathbb{I}$, where $x_n = 0$ or 1 and if x is a dyadic rational number we choose the expansion which terminate in 0's.

Denote the dyadic expension of $n \in \mathbb{N}$ by

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n) 2^j, \varepsilon_j(n) = 0, 1.$$

Denote by $\dotplus$ the logical addition on $\mathbb{I}$. That is, for any $x, y \in \mathbb{I}$

$$x \dotplus y := \sum_{n=0}^{\infty} \left|x_n - y_n\right| 2^{-(n+1)}.$$

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Define the binary operator $\oplus : \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ by

$$k \oplus n = \sum_{i=0}^{\infty} |\varepsilon_i(k) - \varepsilon_i(n)| 2^i. \tag{1}$$

It is well-known (see, e.g. [18], p. 5) that

$$w_{m\oplus n}(x) = w_m(x) w_n(x), x \in [0,1), n, m \in \mathbb{N}. \tag{2}$$

The Rademacher system is defined by

$$\rho_n(x) := (-1)^{x_n} \quad (x \in \mathbb{I}, n \in \mathbb{N}).$$

The Walsh-Paley system is defined as the sequence of the Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (\rho_k(x))^{n_k} = (-1)^{\sum\limits_{k=0}^{|n|} n_k x_k} (x \in \mathbb{I}, n \in \mathbb{N}).$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x) (n \in \mathbb{N}).$$

Recall that (see [18])

$$D_{2^n}(x) = \begin{cases} 2^n, \text{if } x \in I_n(0) \\ 0, \ \text{if } x \in \mathbb{I}\backslash I_n(0) \end{cases}. \tag{3}$$

Let $f \in L_1(\mathbb{I})$. The partial sums of the Walsh-Fourier series are defined as follows:

$$S_M(x,f) := \sum_{i=0}^{M-1} \widehat{f}(i) w_i(x),$$

where the number

$$\widehat{f}(i) = \int_{\mathbb{I}} f(t) w_i(t) dt$$

is said to be the ith Walsh-Fourier coefficient of the function f. Set $E_n(x,f) = S_{2^n}(x,f)$ .The maximal function is defined by

$$E^*(x,f) = \sup_{n\in\mathbb{N}} \left|E_n(x,f)\right|.$$

2. Dyadic Hardy Spaces

The norm (or quasinorm) of the space $L_p(\mathbb{I})$ is defined by

$$\|f\|_p := \left(\int_{\mathbb{I}} |f(x)|^p dx \right)^{1/p} \quad (0 < p < +\infty).$$

In case $p = \infty$, by $L^p(\mathbb{I})$ we mean $L^\infty(\mathbb{I})$, endoved with the supremum norm.
The space weak-$L_1(\mathbb{I})$ consists of all measurable functions f for which

$$\|f\|_{\text{weak}-L_1(\mathbb{I})} := \sup_{\lambda>0} \lambda \left|\left(|f| > \lambda\right)\right| < +\infty.$$

The notiation $a \lesssim b$ in the proofs stands for $a < c \cdot b$, where c is an absolute constant.
Let $f \in L_1(\mathbb{I})$. For $0 < p < \infty$ the Hardy space $H_p(\mathbb{I})$ consists all functions for which

$$\|f\|_{H_p} := \|E^*(f)\|_p < \infty.$$

A bounded measurable function a is a p-atom, if there exists a dyadic interval I, such that
a) $\int_I a = 0$;
b) $\|a\|_\infty \le |I|^{-1/p}$;
c) supp $a \subset I$.
An operator T be called p-quasi-local if there exist a constant $c_p > 0$ such that for every p-atom a

$$\int_{\mathbb{I}\setminus I} |Ta|^p \le c_p < \infty,$$

where I is the support of the atom. We shall need the following

Theorem W1 1 (Weisz [23]). *Suppose that the operator T is σ-sublinear and p-quasi-local for each $0 < p \le 1$. If T is bounded from $L_\infty(\mathbb{I})$ to $L_\infty(\mathbb{I})$, then*

$$\|Tf\|_p \le c_p \|f\|_p \qquad (f \in H_p(\mathbb{I}))$$

for every $0 < p < \infty$. In particular for $f \in L_1(\mathbb{I})$, it holds

$$\|Tf\|_{weak\_L_1(\mathbb{I})} \le C \|f\|_1 .$$

3. Nörlund Logarithmic means

In the literature, there is the notion of Riesz's logarithmic means of a Fourier series. The n-th Riesz's logarithmic means of the Fourier series of an integrable function f is defined by

$$R_n(x, f) := \frac{1}{l_n} \sum_{k=1}^{n} \frac{S_k(x, f)}{k},$$

where $l_n := \sum_{k=1}^{n} (1/k)$.

Riesz's logarithmic means with respect to the trigonometric system was studied by a lot of authors. This means with respect to the Walsh and Vilenkin systems was discussed by Simon [19], Blahota, Gát [4], Gát [7], Gát, Goginava [9], Tephnadze [20], Person, Tephnadze and Wall [17].

Let $\{q_k : k \ge 0\}$ be a sequence of nonnegative numbers. The nth Nörlund means for the Fourier series of f is defined by

$$\frac{1}{Q_n} \sum_{k=0}^{n-1} q_{n-k} S_k(f),$$

where

$$Q_n := \sum_{k=1}^{n} q_k.$$

If $q_k = k$, then we get the Nörlund logarithmic means

$$t_n(x, f) := \frac{1}{l_n} \sum_{k=0}^{n-1} \frac{S_k(x, f)}{n-k}.$$

In this paper we call it logarithmic mean altough, it is a kind of "reverse" Riesz's logarithmic mean.

It is easy to see that

$$t_n(x, f) = \int_{\mathbb{I}} f(t) F_n(x \dotplus t)\, dt,$$

where by $F_n(t)$ we denote nth logarithmic kernel, i. e.

$$F_n(t) := \frac{1}{l_n} \sum_{k=0}^{n-1} \frac{D_k(t)}{n-k}, l_n = \sum_{k=1}^{n} \frac{1}{k}.$$

The Fejér kernel is defined by

$$K_n(t) := \frac{1}{n} \sum_{k=1}^{n} D_k(t).$$

For $n = \sum_{j=0}^{\infty} \varepsilon_j(n) 2^j, \varepsilon_j(n) = 0, 1$ we define

$$n(k) := \sum_{j=0}^{k} \varepsilon_j(n) 2^j.$$

It is easy to see that $n(|n|) = n$.

For a non-negative integer n let us denote the dyadic variation

$$V(n) := \sum_{i=0}^{\infty} |\varepsilon_i(n) - \varepsilon_{i+1}(n)| + \varepsilon_0(n).$$

We define the weighted version of variation of an $n \in \mathbb{N}$ with binary coefficients $(\varepsilon_k(n) : k \in \mathbb{N})$ by

$$L(n) := \sum_{k=1}^{|n|} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| \, l_{n(k)}.$$

Set for positive reals K the subset of natural numbers

$$L_K := \left\{ n \in \mathbb{N} : \frac{L(n)}{|n|} \leq K \right\}.$$

It is known [17] that if $n_j < n_{j+1}$ and

$$\sup_j V\left(n_j\right) < \infty, \tag{4}$$

then a. e. $S_{n_j}(f) \to f$. On the other hand, Konyagin [14] proved that the condition (4) is not necessary for a. e. convergence of subsequence of partial sums. Moreover, he gave negative answer to the question of Balashov and proved the validity of the following theorem.

Theorem K 1 (Konyagin [14]). *Suppose $\{n_A\}$ is an increasing sequence of natural numbers, $k_A := \left[\log_2 n_A\right] + 1$, and 2^{k_A} is a divider of n_{A+1} for all A. Then $S_{n_A}(f) \to f$ a. e. for any function $f \in L_1(\mathbb{I})$.*

For instance, for the sequence $\{n_A\}, n_A := 2^{A^2} \sum_{i=0}^{A} 4^i$, such that $\sup_{n_A} V(n_A) = \infty$, satisfies the hypotheses of the theorem.

Almost ewerywhere convergence of $\{t_{2^A}(f) : A \geq 1\}$ with respect to Walsh-Paley system was studied by first author [11]. In particular, the following is proved

Theorem G1 1. *Let $f \in L_1(\mathbb{I})$. Then $t_{2^A}(x, f) \to f(x)$ as $A \to \infty$ a. e. $x \in \mathbb{I}$.*

In [16], Nagy established a similar result for the Walsh-Kaczmarz system. Memić [15] improved Theorem G1. However, a divergence on the set with positive measure for the whole sequence $\{t_n(f) : n \geq 1\}$ was proved by Gát and Goginava [8].

In [12] the following is proved.

Theorem G2 1. *Let $f \in L_1(\mathbb{I})$ and $K > 0$. Then $\lim_{L_K \ni n \to \infty} t_n(x, f) = f(x)$ for a. e. $x \in \mathbb{I}$.*

We define the maximal operator

$$t_*(x; f) := \sup_{n \in L_K} \left(\left|f * F_n\right|\right)(x).$$

In this section it is proved that the maximal operator of subsequences of Nörlund logarithmic means of Walsh-Fourier series is bounded from the dyadic Hardy spaces H_p to L_p. This implies an almost everywhere convergence for the subsequences of the summability means.

Theorem 3.1. *Let $p > 0$. Then there exists a positive constant c_p such that*

$$\left\|t_*(f)\right\|_p \leq c_p \left\|\left|f\right|\right\|_{H_p} \quad \left(\left|f\right| \in H_p, p > 0\right)$$

and

$$\left\|t_*(f)\right\|_{weak-L_1(\mathbb{I})} \lesssim \left\|f\right\|_1.$$

Corollary 3.2 (see [12]). *Let $f \in L_1(\mathbb{I})$. Then*

$$\lim_{L_K \ni n \to \infty} t_n(x, f) = f(x) \quad \text{for a. e. } x \in \mathbb{I}.$$

Theorem 3.3. *Let $\{m_A : A \in \mathbb{N}\}$ be a subsequence for which there does not exist K such that $\{m_A : A \in \mathbb{N}\} \notin L_K$ for all $K \in \mathbb{N}$, i. e. the condition*

$$\sup_A \frac{1}{|m_A|} \sum_{k=1}^{|m_A|} \left|\varepsilon_k(m_A) - \varepsilon_{k+1}(m_A)\right| l_{m_A(k)} = \infty$$

holds. The operator $t_{m_A}(f)$ is not bounded from the dyadic Hardy spaces $H_1(\mathbb{I})$ to the space $L_1(\mathbb{I})$.

Proof. [Proof of Theorem 3.1] The following representation is known (see [12])

$$l_n F_n(t) = H_n^{(1)}(t) + H_n^{(2)}(t),$$

where

$$H_n^{(1)}(t) =: w_n(t)\left(\sum_{j=1}^{|n|}\varepsilon_j(n)\,D_{2^j}(t)\,\rho_j(t)\,l_{n(j)}\right),$$

$$H_n^{(2)}(t) =: \left(\sum_{j=1}^{|n|}\varepsilon_j(n)\sum_{k=1}^{2^j}\frac{D_k(t)}{k+n(j-1)}\right)\prod_{s=j+1}^{|n|}(\rho_s(t))^{\varepsilon_s(n)}.$$

Hence, we have

$$f * F_n(x) = \left(f * \frac{H_n^{(1)}}{l_n}\right)(x) + \left(f * \frac{H_n^{(2)}}{l_n}\right)(x). \tag{5}$$

It is easy to see that

$$\begin{aligned}
w_n(t)\,H_n^{(1)}(t) &= \sum_{j=1}^{|n|}\varepsilon_j(n)\left(D_{2^{j+1}}(t)-D_{2^j}(t)\right)l_{n(j)} \\
&= \sum_{j=1}^{|n|-1}\left(\varepsilon_j(n)\,l_{n(j)}-\varepsilon_{j+1}(n)\,l_{n(j+1)}\right)D_{2^{j+1}}(t) \\
&\quad +\varepsilon_{|n|}(n)\,l_{n(|n|)}D_{2^{|n|+1}}(t)-\varepsilon_1(n)\,l_{n(1)}D_2(t) \\
&= \sum_{j=1}^{|n|-1}\left(\varepsilon_j(n)-\varepsilon_{j+1}(n)\right)l_{n(j)}D_{2^{j+1}}(t) \\
&\quad +\sum_{j=1}^{|n|-1}\varepsilon_{j+1}(n)\left(l_{n(j)}-l_{n(j+1)}\right)D_{2^{j+1}}(t) \\
&\quad +\varepsilon_{|n|}(n)(n)\,l_{n(|n|)}D_{2^{|n|+1}}(t)-\varepsilon_1(n)\,l_{n(1)}D_2(t).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\frac{\left|H_n^{(1)}(t)\right|}{l_n} \\
\le\ & \frac{1}{l_n}\sum_{j=1}^{|n|-1}\left|\varepsilon_j(n)-\varepsilon_{j+1}(n)\right|l_{n(j)}D_{2^{j+1}}(t) \\
& +\sum_{j=1}^{|n|-1}\varepsilon_{j+1}(n)\left(l_{n(j+1)}-l_{n(j)}\right)D_{2^{j+1}}(t) \\
& +\varepsilon_{|n|}(n)(n)\,l_{n(|n|)}D_{2^{|n|+1}}(t)+\varepsilon_1(n)\,l_{n(1)}D_2(t). \\
=\ & :P_n(t)
\end{aligned} \tag{6}$$

Let $n \in L_K$. Then we can write

$$\begin{aligned}
&\left|\left(f * P_n\right)(x)\right| \\
&\leq \frac{1}{l_n} \sum_{j=1}^{|n|-1} \left|\varepsilon_j(n) - \varepsilon_{j+1}(n)\right| l_{n(j)} \left(f * D_{2^{j+1}}\right)(x) \\
&+ \frac{1}{l_n} \sum_{j=1}^{|n|} \varepsilon_j(n) \left(l_{n(j+1)} - l_{n(j)}\right) \left(f * D_{2^{j+1}}\right)(x) \\
&+ \frac{\varepsilon_{|n|}(n)(n)\, l_{n(|n|)}}{l_n} \left(f * D_{2^{|n|+1}}\right)(x) \\
&+ \frac{\varepsilon_1(n)\, l_{n(1)}}{l_n} \left(f * D_2\right)(x) \\
&\leq E^*(x, f) \left\{ \frac{1}{l_n} \sum_{j=1}^{|n|-1} \left|\varepsilon_j(n) - \varepsilon_{j+1}(n)\right| l_{n(j)} \right. \\
&\left. + \frac{1}{l_n} \sum_{j=1}^{|n|} \varepsilon_j(n) \left(l_{n(j+1)} - l_{n(j)}\right) + 2 \right\} \\
&\lesssim L_K E^*(x, f).
\end{aligned}$$

Since (see [18, 23])

$$\left\|E^*(f)\right\|_p \leq c_p \left\|f\right\|_{H_p} \quad (p > 0), \tag{7}$$

and

$$\left\|E^*(f)\right\|_{\text{weak}-L_1(\mathbb{I})} \leq c \left\|f\right\|_1, \tag{8}$$

we have

$$\left\|\sup_{n \in L_K} \left|\left(f * P_n\right)(x)\right|\right\|_p \leq c_p \left\|f\right\|_{H_p} \quad \left(f \in H_p, p > 0\right) \tag{9}$$

and

$$\left\|\sup_{n \in L_K} \left|\left(f * P_n\right)\right|\right\|_{\text{weak}-L_1(\mathbb{I})} \lesssim \left\|f\right\|_1. \tag{10}$$

Now, we can write

$$\frac{\left|H_n^{(2)}(t)\right|}{l_n} \leq \frac{1}{l_n} \sum_{j=1}^{|n|} \varepsilon_j(n) \left| \sum_{k=1}^{2^j} \frac{D_k(t)}{k + n(j-1)} \right|.$$

Using Abel's transformation we obtain

$$\begin{aligned}
&\sum_{k=1}^{2^j} \frac{D_k(t)}{k + n(j-1)} \\
&= \sum_{k=1}^{2^j-1} \left(\frac{1}{k + n(j-1)} - \frac{1}{k + 1 + n(j-1)} \right) k K_k(t) \\
&+ \frac{2^j}{2^j + n(j-1)} K_{2^j}(t).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\left|H_n^{(2)}(x)\right| \\
\leq\ & \frac{1}{l_n}\sum_{j=1}^{|n|}\varepsilon_j(n)\sum_{k=1}^{2^j-1}\left(\frac{1}{k+n(j-1)}-\frac{1}{k+1+n(j-1)}\right)k\left|K_k(x)\right| \\
&+\frac{1}{l_n}\sum_{j=1}^{|n|}\varepsilon_j(n)\frac{2^j}{2^j+n(j-1)}K_{2^j}(x) \\
=\ & : H_n^{(21)}(x)+H_n^{(22)}(x).
\end{aligned} \tag{11}$$

Since (see [18], p. 46)

$$|K_l(x)| \leq 3\cdot 2^{-s}\sum_{i=0}^{s-1}\sum_{j=0}^{i}2^jD_{2^i}\left(x\dotplus 2^{-j-1}\right) \tag{12}$$

when $2^{s-1}\leq l<2^s$. We have

$$\begin{aligned}
&\left|H_n^{(21)}(x)\right| \\
\leq\ & \frac{3}{l_n}\sum_{j=1}^{|n|}\varepsilon_j(n)\sum_{s=1}^{j}\sum_{l=2^{s-1}}^{2^s-1} \\
&\left(\frac{1}{l+n(j-1)}-\frac{1}{l+1+n(j-1)}\right) \\
&\times\sum_{k=0}^{s-1}\sum_{r=0}^{k}2^rD_{2^k}\left(x\dotplus 2^{-r-1}\right) \\
=\ & \frac{3}{l_n}\sum_{j=1}^{|n|}\varepsilon_j(n)\sum_{s=1}^{j}\left(\frac{1}{2^{s-1}+n(j-1)}-\frac{1}{2^s+n(j-1)}\right) \\
&\times\sum_{k=0}^{s-1}\sum_{r=0}^{k}2^rD_{2^k}\left(x\dotplus 2^{-r-1}\right).
\end{aligned} \tag{13}$$

It is well known (see [18], p. 47) that if $j\in\mathbb{N}$ then

$$K_{2^j}(x)=\frac{1}{2}\left(2^{-j}D_{2^j}(x)+\sum_{l=0}^{j}2^{l-j}D_{2^j}\left(x\dotplus\frac{1}{2^{l+1}}\right)\right). \tag{14}$$

In particular, $K_{2^n}\geq 0$ everywhere on $\mathbb{I}$. Then we have

$$\begin{aligned}
H_n^{(22)}(x) \leq\ & \frac{1}{2l_n}\sum_{j=1}^{|n|}\varepsilon_j(n)2^{-j}D_{2^j}(x) \\
&+\frac{1}{2l_n}\sum_{j=1}^{|n|}\varepsilon_j(n)\sum_{l=0}^{j}2^{l-j}D_{2^j}\left(x\dotplus\frac{1}{2^{l+1}}\right).
\end{aligned} \tag{15}$$

Combining (11), (13) and (15) we have

$$
\begin{aligned}
&\left|H_n^{(2)}(x)\right| &(16)\\
\lesssim\ & \frac{1}{l_n}\sum_{j=1}^{|n|}\varepsilon_j(n)\sum_{s=1}^{j}\left(\frac{1}{2^{s-1}+n(j-1)}-\frac{1}{2^s+n(j-1)}\right)\\
&\times\sum_{k=0}^{s}\sum_{r=0}^{k}2^rD_{2^k}\left(x\dotplus 2^{-r-1}\right)\\
=\ & :Q_n(x).
\end{aligned}
$$

We can write

$$
\begin{aligned}
&(f*Q_n)(x)\\
=\ & f*\left(\frac{c}{l_n}\sum_{j=1}^{|n|}\varepsilon_j(n)\sum_{s=1}^{j}\left(\frac{1}{2^{s-1}+n(j-1)}-\frac{1}{2^s+n(j-1)}\right)\right.\\
&\left.\times\sum_{k=0}^{s}\sum_{r=0}^{k}2^rD_{2^k}\left(\cdot\dotplus 2^{-r-1}\right)\right)(x).
\end{aligned}
$$

First, we prove that the operator $f*Q_n$ is bounded from $L_\infty(\mathbb{I})$ to $L_\infty(\mathbb{I})$. Indeed, since

$$
\begin{aligned}
&\sup_{n\in\mathbb{N}}\|Q_n\|_1\\
\lesssim\ & \sup_{n\in\mathbb{N}}\frac{1}{l_n}\sum_{j=1}^{|n|}\varepsilon_j(n)\sum_{s=1}^{j}2^s\sum_{k=2^{s-1}}^{2^s-1}\left(\frac{1}{k+n(j-1)}-\frac{1}{k+1+n(j-1)}\right)\\
\lesssim\ & \sup_{n\in\mathbb{N}}\frac{1}{l_n}\sum_{j=1}^{|n|}\varepsilon_j(n)\sum_{s=1}^{j}\sum_{k=2^{s-1}}^{2^s-1}k\left(\frac{1}{k+n(j-1)}-\frac{1}{k+1+n(j-1)}\right)\\
\lesssim\ & \sup_{n\in\mathbb{N}}\frac{1}{l_n}\sum_{j=1}^{|n|}\varepsilon_j(n)\sum_{k=1}^{2^j-1}\frac{k}{(k+n(j-1))^2}\\
\lesssim\ & \sup_{n\in\mathbb{N}}\frac{1}{l_n}\sum_{j=1}^{|n|}\varepsilon_j(n)\sum_{k=1}^{2^j-1}\left(\frac{1}{k+n(j-1)}+\frac{n(j-1)}{(k+n(j-1))^2}\right)\\
\lesssim\ & \sup_{n\in\mathbb{N}}\frac{1}{l_n}\sum_{j=2}^{|n|}\varepsilon_j(n)\left(l_{n(j)}-l_{n(j-1)}+1\right)\\
\le\ & c<\infty.
\end{aligned}
$$

we obtain that

$$
\sup_{n\in\mathbb{N}}\left\|f*Q_n\right\|_\infty\le c\left\|f\right\|_\infty.
$$

Hence, the operator $f*Q_n$ is bounded from $L_\infty(\mathbb{I})$ to $L_\infty(\mathbb{I})$.

We suppose that $f\in H_p(\mathbb{I})$. Let function a be an H_p atom. It means that either a is constant or there is an interval $I_N(u)$ such that $\operatorname{supp}(a)\subset I_N(u)$, $\|a\|_\infty\le 2^{N/p}$ and $\int a=0$. Without lost of generality we can suppose that $u=0$. Consequently, for any function g which is $\mathcal{A}_N$-measurable we have that $\int ag=0$. We prove that the operator $\sup_{n>N} (f * Q_n)(x)$ is H_p-quasi local. That is,

$$\int_{\bar{I}_N} \left(\sup_{n>N} |a * Q_n| \right)^p \leq c_p. \tag{17}$$

Let $x \in \bar{I}_N$. Then we can write

$$\begin{aligned}
& |(a * Q_n)(x)| \\
= & \left| \frac{1}{l_n} \int_{I_N} a(t) \left(\sum_{j=N+1}^{|n|} \varepsilon_j(n) \sum_{s=N+1}^{j} \right. \right. \\
& \left(\frac{1}{2^{s-1} + n(j-1)} - \frac{1}{2^s + n(j-1)} \right) \\
& \left. \left. \times \sum_{k=N+1}^{s} \sum_{r=0}^{k} 2^r D_{2^k} \left(x \dotplus t \dotplus 2^{-r-1} \right) \right) dt \right| \\
\leq & \frac{2^{N/p}}{l_n} \sum_{j=N+1}^{|n|} \varepsilon_j(n) \sum_{s=N+1}^{j} \\
& \left(\frac{1}{2^{s-1} + n(j-1)} - \frac{1}{2^s + n(j-1)} \right) \\
\times & \sum_{k=N+1}^{s} \sum_{r=0}^{k} 2^r \int_{I_N} D_{2^k} \left(x \dotplus t \dotplus 2^{-r-1} \right) dt \\
= & \frac{2^{N/p}}{l_n} \sum_{j=N+1}^{|n|} \varepsilon_j(n) \sum_{k=N+1}^{j} \sum_{s=k+1}^{j} \\
& \left(\frac{1}{2^{s-1} + n(j-1)} - \frac{1}{2^s + n(j-1)} \right) \\
& \times \sum_{r=0}^{k} 2^r \int_{I_N} D_{2^k} \left(x \dotplus t \dotplus 2^{-r-1} \right) dt \\
\leq & \frac{2^{N/p}}{l_n} \sum_{j=N+1}^{|n|} \varepsilon_j(n) \left(\frac{1}{2^N + n(j-1)} - \frac{1}{n(j)} \right) \\
& \times \sum_{k=N+1}^{j} \sum_{r=0}^{k} 2^r \int_{I_N} D_{2^k} \left(x \dotplus t \dotplus 2^{-r-1} \right) dt \\
= & \frac{2^{N/p}}{l_n} \sum_{j=N+1}^{|n|} \varepsilon_j(n) \left(\frac{1}{2^N + n(j-1)} - \frac{1}{n(j)} \right) \\
& \times \left(\sum_{r=N+1}^{j} \sum_{k=r}^{j} + \sum_{r=0}^{N} \sum_{k=N+1}^{j} \right) 2^r \int_{I_N} D_{2^k} \left(x \dotplus t \dotplus 2^{-r-1} \right) dt.
\end{aligned}$$

Since

$$\sum_{r=N+1}^{j}\sum_{k=r}^{j}2^{r}\int_{I_N}D_{2^k}\left(x\dotplus t\dotplus 2^{-r-1}\right)dt=0\quad\left(x\in\bar{I}_N\right),$$

we have

$$\begin{aligned}&|a*Q_n|\\ \leq\ & \frac{2^{N/p}}{l_n}\sum_{j=N+1}^{|n|}\varepsilon_j(n)(j-N)\left(\frac{1}{2^N+n(j-1)}-\frac{1}{n(j)}\right)\\ &\times\sum_{r=0}^{N}2^{r}\mathbf{1}_{I_N\left(2^{-r-1}\right)}(x).\end{aligned}$$

Since

$$\begin{aligned}&\sum_{j=N+1}^{|n|}\varepsilon_j(n)(j-N)\left(\frac{1}{2^N+n(j-1)}-\frac{1}{n(j)}\right)\\ \leq\ & |n|\sum_{j=N+1}^{|n|}\left(\frac{1}{2^N+n(j-1)}-\frac{1}{n(j)}\right)\\ \leq\ & \frac{|n|}{2^N},\end{aligned}$$

we have

$$|a*Q_n|\leq\frac{2^{N/p}}{2^N}\sum_{r=0}^{N}2^{r}\mathbf{1}_{I_N\left(2^{-r-1}\right)}(x),$$

where $\mathbf{1}_E$ is characteristic function of the set E and consequently,

$$\int_{\bar{I}_N}\sup_{n\geq N}|a*Q_n|^p\leq\frac{2^N}{2^{Np}}\sum_{r=0}^{N}2^{rp}\int_{\bar{I}_N}\mathbf{1}_{I_N\left(2^{-r-1}\right)}\leq c_p.$$

Hence,

$$\left\|\sup_{n\in\mathbb{N}}\left|f*Q_n\right|\right\|_p\leq c_p\left\|f\right\|_{H_p}\quad\left(f\in H_p,p>0\right)\tag{18}$$

and

$$\left\|\sup_{n\in\mathbb{N}}\left|f*Q_n\right|\right\|_{\text{weak}-L_1(\mathbb{I})}\lesssim\left\|f\right\|_1.\tag{19}$$

Since

$$\left|f*F_n\right|(x)\leq\left|f\right|*P_n+\left|f\right|*Q_n,$$

from (9), (10), (18) and (19) we have

$$\left\|\sup_{n\in L_K}\left|f*F_n\right|\right\|_p\leq c_p\left\|\left|f\right|\right\|_{H_p}\quad\left(\left|f\right|\in H_p,p>0\right)$$

and

$$\left\|\sup_{n\in L_K}\left|f*F_n\right|\right\|_{\text{weak}-L_1(\mathbb{I})}\lesssim\left\|f\right\|_1.$$

Which complete the proof of Theorem 3.1. □

Proof. [Proof of Theorem 3.3] Set

$$f_A:=D_{2^{|m_A|+1}}-D_{2^{|m_A|}}.$$

Then it is easy to see that

$$\sup_{n\in\mathbb{N}}\left|S_{2^n}\left(f_A\right)\right|=D_{2^{|m_A|}}$$

and consequently,

$$\left\|f_A\right\|_{H_1}=\left\|\sup_{n\in\mathbb{N}}\left|S_{2^n}\left(f_A\right)\right|\right\|_1=\left\|D_{2^{|m_A|}}\right\|_1=1.$$

Set

$$m_A=2^{|m_A|}+q_A,$$

where

$$q_A:=\sum_{j=0}^{|m_A|-1}\varepsilon_j\left(m_A\right)2^j.$$

Then we can write

$$t_{m_A}\left(f_A\right)=\frac{1}{l_{m_A}}\sum_{k=2^{|m_A|}+1}^{2^{|m_A|}+q_A-1}\frac{S_k\left(f_A\right)}{m_A-k}.$$

It is easy to see that

$$\begin{aligned}S_k\left(f_A\right)&=S_k\left(D_{2^{|m_A|+1}}-D_{2^{|m_A|}}\right)\\&=D_k-D_{2^{|m_A|}}.\end{aligned}$$

Hence, we have

$$\begin{aligned}t_{m_A}\left(f_A\right)&=\frac{1}{l_{m_A}}\sum_{k=2^{|m_A|}+1}^{2^{|m_A|}+q_A-1}\frac{D_k-D_{2^{|m_A|}}}{m_A-k}\\&=\frac{1}{l_{m_A}}\sum_{k=1}^{q_A-1}\frac{D_{k+2^{|m_A|}}-D_{2^{|m_A|}}}{m_A-k}\\&=\frac{w_{2^{|m_A|}}}{l_{m_A}}\sum_{k=1}^{q_A-1}\frac{D_k}{q_A-k}.\end{aligned}$$

From the condition of Theorem 3.3 we conclude that

$$\sup_{A\in\mathbb{N}}\left\|t_{m_A}\left(f_A\right)\right\|_1=\sup_{A\in\mathbb{N}}\left\|F_{m_A}\right\|_1=\infty.$$

Theorem 3.3 is proved. □

4. Cesàro Means with Varying Parameters

The (C,α_n) means of the Walsh-Fourier series of the function f is given by

$$\sigma_n^{\alpha_n}(f,x)=\frac{1}{A_{n-1}^{\alpha_n}}\sum_{j=1}^{n}A_{n-j}^{\alpha_n-1}S_j(f,x)=\frac{1}{A_{n-1}^{\alpha_n}}\sum_{j=0}^{n-1}A_{n-1-j}^{\alpha_n}\widehat{f}(j)\,w_j(x),$$

where

$$A_n^{\alpha_n}:=\frac{(1+\alpha_n)\dots(n+\alpha_n)}{n!}$$

for any $n\in\mathbb{N},\alpha_n\neq-1,-2,\dots$.

It is known that [26]

$$A_n^{\alpha_n}=\sum_{k=0}^{n}A_k^{\alpha_n-1},A_n^{\alpha_n-1}=\frac{\alpha_n}{\alpha_n+n}A_n^{\alpha_n}. \tag{20}$$

The (C,α_n) kernel is defined by

$$K_n^{\alpha_n}=\frac{1}{A_{n-1}^{\alpha}}\sum_{j=1}^{n}A_{n-j}^{\alpha_n-1}D_j=\frac{1}{A_{n-1}^{\alpha_n}}\sum_{j=0}^{n-1}A_{n-j-1}^{\alpha_n}w_j.$$

The following estimations was proved by Akhobadze [2, 3] : Let $k,n\in\mathbb{N}$. Then

$$c_1(1+\alpha_n)(2+\alpha_n)k^{\alpha_n} < A_k^{\alpha_n}<c_2(1+\alpha_n)(2+\alpha_n)k^{\alpha_n}, \quad \text{when } -2<\alpha_n<-1; \tag{21}$$

$$c_1(1+\alpha_n)k^{\alpha_n}<A_k^{\alpha_n}<c_2(1+\alpha_n)k^{\alpha_n},\ \text{when } -1<\alpha_n<0; \tag{22}$$

$$c_1(d)k^{\alpha_n}<A_k^{\alpha_n}<c_2(d)k^{\alpha_n},\ \text{when } 0<\alpha_n\le d. \tag{23}$$

The idea of Cesàro means with variable parameters of numerical sequences is due to Kaplan [13] and the introduction of these (C,α_n) means of Fourier series is due to Akhobadze (see [3] or [2]) who investigated the behavior of the L_1-norm convergence of $\sigma_n^{\alpha_n}(f)\to f$ for the trigonometric system.

The first result with respect to the a.e. convergence of the Walsh-Fejér means $\sigma_n^{\alpha_n}(f)$ for all integrable function f with constant sequence $\alpha_n=\alpha>0$ is due to Fine [5] (see also Weisz [22]). On the rate of convergence of Cesàro means in this constant case see the paper of Yano [25], Fridli [**?**].

For $n:=\sum_{i=0}^{\infty}\varepsilon_i(n)2^i\ (\varepsilon_i(n)=0,1,i\in\mathbb{N})$ set two variable function

$$P(n,\alpha):=\sum_{i=0}^{\infty}\varepsilon_i(n)2^{i\alpha_n}\quad(n\in\mathbb{N}),\alpha:=\{\alpha_n:n\in\mathbb{N}\}.$$

The function $P(n,\alpha)$ was introduced by Abu Joudeh and Gát in [1]. Also set for sequence $\alpha:=\{\alpha_n:n\in\mathbb{N}\}$ and positive reals K the subset of natural numbers

$$P_K(\alpha):=\left\{n\in\mathbb{N}:\frac{P(n,\alpha)}{n^{\alpha_n}}\le K\right\}.$$

The a.e. divergence of Cesàro means with varying parameters of Walsh-Fourier series was investigated by Tetunashvili [21]. Abu Joudeh and Gát in [1] proved the almost everywhere convergence (with some restrictions) of the Cesàro (C,α_n) means of integrable functions. In particular, the following is proved

Theorem JG 1. *Suppose that $\alpha_n \in (0,1)$. Let $f \in L_1(\mathbb{I})$. Then we have the almost everywhere convergence $\sigma_n^{\alpha_n}(f) \to f$ provided that $P_K(\alpha) \ni n \to \infty$.*

In this section we define the weighted version of variation of an $n \in \mathbb{N}$ with binary coefficients $(\varepsilon_k(n) : k \in \mathbb{N})$ by

$$V(n,\alpha) := \sum_{i=0}^{\infty} |\varepsilon_i(n) - \varepsilon_{i+1}(n)| \, 2^{i\alpha_n} \quad (n \in \mathbb{N}).$$

Set for sequence $\alpha := \{\alpha_n : n \in \mathbb{N}\}$ and positive reals K the subset of natural numbers

$$V_K(\alpha) := \left\{ n \in \mathbb{N} : \frac{V(n,\alpha)}{n^{\alpha_n}} \leq K < \infty \right\}.$$

It is easy to see that $P_K(\alpha) \subsetneq V_{2K}(\alpha)$. On the other hand, if $\alpha_n \to 0$, then there exists K such that $2^n - 1 \in V_K(\alpha)$ for all n, but there does not exists K, such that $2^n - 1 \in P_K(\alpha)$ for all n.

The boundedness of maximal operators of subsequences of $(C, \alpha_n)-$ means of partial sums of Walsh-Fourier series from the Hardy space H_p into the space L_p is studied in [10]. In particular, the following is proved.

Theorem GG2 1. *Let $p > 0$. Then there exists a positive constant c_p such that*

$$\left\| \sup_{N \in \mathbb{N}} \left| f * \left| K_{2^N}^{\alpha_N} \right| \right| \right\|_p \leq c_p \|f\|_{H_p} \quad \left(f \in H_p\right).$$

Weisz [24] generalized Theorem GG2 for both the Cesàro and Riesz means by taking the supremum over all indicies $n \in \mathbb{N}_v$. Here $\mathbb{N}_v$ denotes the set of all $n = 2^{n_1} + \cdots + 2^{n_v}$ with a fixed parameter v. In particular, the following is proved.

Theorem W2 1. *Let $p > 0$. Then there exists a positive constant c_p such that*

$$\left\| \sup_{n \in P_K(\alpha)} \left| f * K_n^{\alpha_n} \right| \right\|_p \leq c_p \left\| |f| \right\|_{H_p} \quad \left(|f| \in H_p\right).$$

In this section we are going to improve Theorem W2. We prove that the maximal operator of subsequences of Cesàro means with varying parameters of Walsh-Fourier series is bounded from the dyadic Hardy spaces H_p to L_p. This implies an almost everywhere convergence for the subsequences of the summability means.

Theorem 4.1. *Let $p > 0$. Then there exists a positive constant c_p such that*

$$\left\| \sup_{n \in V_K(\alpha)} \left| f * K_n^{\alpha_n} \right| \right\|_p \leq c_p \left\| |f| \right\|_{H_p} \quad \left(|f| \in H_p\right)$$

and

$$\left\| \sup_{n \in V_K(\alpha)} \left| f * K_n^{\alpha_n} \right| \right\|_{weak\_L_1(\mathbb{I})} \leq c \|f\|_1 \quad (f \in L_1).$$

Corollary 4.2. *Let $f \in L_1(\mathbb{I})$. Then*

$$\lim_{V_K(\alpha) \ni n \to \infty} \sigma_n^{\alpha_n}(x,f) = f(x) \text{ for a. e. } x \in \mathbb{I}.$$

Remark 4.3. *We suspect that Theorem 3.1 and Theorem 4.1 will be valid in the case when $f \in H_p(\mathbb{I})$ $(p>0)$, but we could not proved these.*

Now, we prove Theorem 4.1.

Proof. We can write

$$\begin{aligned} K_n^{\alpha_n} &= \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) w_{n^{(s)}-1} \sum_{j=1}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-2} j K_j \\ &\quad - \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) w_{n^{(s)}-1} A_{n_{(s)}-1}^{\alpha_n-1} 2^s K_{2^s} \\ &\quad + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) w_{n^{(s)}-1} A_{n_{(s)}-1}^{\alpha_n} D_{2^s} \\ &=: T_n^{(1)} + T_n^{(2)} + T_n^{(3)}, \end{aligned} \tag{24}$$

and

$$\sup_{n\in\mathbb{N}} \left(f * \left| T_n^{(3)} \right| \right) \le c_K E^*(x,f)$$

Then from (7) and (8) we have

$$\left\| \sup_{n\in\mathbb{N}} \left(f * \left| T_n^{(3)} \right| \right) \right\|_p \le c_p \|f\|_{H_p} \quad \left(f \in H_p,\ p>0 \right) \tag{25}$$

and

$$\left\| \sup_{n\in\mathbb{N}} \left(f * \left| T_n^{(3)} \right| \right) \right\|_{\text{weak\_}L_1(\mathbb{I})} \le c \|f\|_1 \quad (f \in L_1(\mathbb{I})). \tag{26}$$

It is easy to see that (see (20))

$$\left| T_n^{(1)} \right| \le \frac{2}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{j=1}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \left| K_j \right|,$$

then, from (12) we have

$$\begin{aligned} &\left| T_n^{(1)} \right| + \left| T_n^{(2)} \right| \\ \le\ & \frac{6}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{l=1}^{s} \frac{1}{2^l} \\ &\times \sum_{j=2^{l-1}}^{2^l-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \sum_{k=0}^{l-1} \sum_{r=0}^{k} 2^r D_{2^k} \left(x \dotplus 2^{-r-1} \right) \\ &+ \frac{1}{2A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) A_{n_{(s)}-1}^{\alpha_n-1} D_{2^s}(x) \\ &+ \frac{1}{2A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) A_{n_{(s)}-1}^{\alpha_n-1} \sum_{l=0}^{s} 2^l D_{2^s} \left(x \dotplus \frac{1}{2^{l+1}} \right) \\ =\ &: \widetilde{T}_n(x). \end{aligned}$$

Now, we discuss the operator $\sup_{n\in\mathbb{N}}\left(f*\widetilde{T}_n\right)$. First, we show that the operator is bounded from $L_\infty(\mathbb{I})$ to $L_\infty(\mathbb{I})$. Indeed, since (see [18]) $\sup_n \|K_n\|_1 < 2$ and from (22), (23) we have

$$
\begin{aligned}
&\sup_{n\in\mathbb{N}}\left\|\widetilde{T}_n\right\|_1 \\
\lesssim\ & \sup_{n\in\mathbb{N}}\frac{1}{A_{n-1}^{\alpha_n}}\sum_{s=0}^{|n|}\varepsilon_s(n)\sum_{l=1}^{s}\frac{1}{2^l} \\
&\times\sum_{j=2^{l-1}}^{2^l-1}A_{n_{(s-1)}+j}^{\alpha_n-1}\sum_{k=0}^{l-1}\sum_{r=0}^{k}2^r\int_{\mathbb{I}}D_{2^k}\left(x\dotplus 2^{-r-1}\right)dx \\
&+\sup_{n\in\mathbb{N}}\frac{1}{A_{n-1}^{\alpha_n}}\sum_{s=0}^{|n|}\varepsilon_s(n)A_{n_{(s)}-1}^{\alpha_n-1}\int_{\mathbb{I}}D_{2^s}(x)\,dx \\
&+\sup_{n\in\mathbb{N}}\frac{1}{A_{n-1}^{\alpha_n}}\sum_{s=0}^{|n|}\varepsilon_s(n)A_{n_{(s)}-1}^{\alpha_n-1}\sum_{l=0}^{s}2^l\int_{\mathbb{I}}D_{2^s}\left(x\dotplus\frac{1}{2^{l+1}}\right)dx \\
\lesssim\ & \sup_{n\in\mathbb{N}}\frac{1}{A_{n-1}^{\alpha_n}}\sum_{s=0}^{|n|}\varepsilon_s(n)\sum_{j=1}^{2^s-1}A_{n_{(s-1)}+j}^{\alpha_n-1} \\
&+\sup_{n\in\mathbb{N}}\frac{1}{A_{n-1}^{\alpha_n}}\sum_{s=0}^{|n|}\varepsilon_s(n)2^sA_{n_{(s)}-1}^{\alpha_n-1} \\
\lesssim\ & \sup_{n\in\mathbb{N}}\frac{1}{A_{n-1}^{\alpha_n}}\sum_{s=0}^{|n|}\varepsilon_s(n)\left(A_{n_{(s)}}^{\alpha_n}-A_{n_{(s-1)}}^{\alpha_n}\right) \\
&+\sup_{n\in\mathbb{N}}\frac{\alpha_n}{n^{\alpha_n}}\sum_{s=0}^{|n|}\varepsilon_s(n)2^{s\alpha_n} \\
\leq\ & c<\infty,
\end{aligned}
$$

which implies the boundedness of operator $\sup_{n\in\mathbb{N}}\left(f*\widetilde{T}_n\right)$ from the space $L_\infty(\mathbb{I})$ to the space $L_\infty(\mathbb{I})$.

We suppose that $f\in H_p(\mathbb{I})$. Let function a be an H_p atom. It means that either a is constant or there is an interval $I_N(u)$ such that $\mathrm{supp}(a)\subset I_N(u)$, $\|a\|_\infty\leq 2^{N/p}$ and $\int a=0$. Without lost of generality we can suppose that $u=0$. Consequently, for any function g which is $\mathcal{A}_N$-measurable we have that $\int ag=0$. We prove that operator $\sup_{n\in\mathbb{N}}\left(f*\widetilde{T}_n\right)$ is H_p-quasi local. That is,

$$
\int_{\overline{I}_N}\left(\sup_{n>N}\left|a*\widetilde{T}_n\right|\right)^p\leq c_p.
$$

Let $x \in \bar{I}_N$. Then from (3) we can write

$$
\begin{aligned}
& \left|a * \widetilde{T}_n\right| \\
= & \left| \int_{I_N} a(t) \left(\frac{6}{A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \varepsilon_s(n) \sum_{l=N+1}^{s} \frac{1}{2^l} \right.\right. \\
\times & \sum_{j=2^{l-1}}^{2^l-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \sum_{k=N+1}^{l-1} \sum_{r=0}^{k} 2^r D_{2^k}\left(x \dotplus t \dotplus 2^{-r-1}\right) \\
& + \frac{1}{2A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \varepsilon_s(n) A_{n_{(s)}-1}^{\alpha_n-1} D_{2^s}(x \dotplus t) \\
& \left.\left. + \frac{1}{2A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \varepsilon_s(n) A_{n_{(s)}-1}^{\alpha_n-1} \sum_{l=0}^{s} 2^l D_{2^s}\left(x \dotplus t \dotplus \frac{1}{2^{l+1}}\right) dt \right) \right| \\
\lesssim & \frac{2^{N/p}}{A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \varepsilon_s(n) \sum_{l=N+1}^{s} \frac{1}{2^l} \\
& \times \sum_{j=2^{l-1}}^{2^l-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \sum_{k=N+1}^{l} \sum_{r=0}^{k} 2^r \int_{I_N} D_{2^k}\left(x \dotplus t \dotplus 2^{-r-1}\right) dt \\
& + \frac{2^{N/p}}{A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \varepsilon_s(n) A_{n_{(s)}-1}^{\alpha_n-1} \sum_{l=0}^{N-1} 2^l \int_{I_N} D_{2^s}\left(x \dotplus t \dotplus \frac{1}{2^{l+1}}\right) dt.
\end{aligned}
$$

Since

$$
\begin{aligned}
\sum_{j=2^{l-1}}^{2^l-1} A_{n_{(s-1)}+j}^{\alpha_n-1} &= \sum_{j=2^{l-1}}^{2^l-1} \left(A_{n_{(s-1)}+j}^{\alpha_n} - A_{n_{(s-1)}+j-1}^{\alpha_n} \right) \\
&= A_{n_{(s-1)}+2^l-1}^{\alpha_n} - A_{n_{(s-1)}+2^{l-1}-1}^{\alpha_n}
\end{aligned}
$$

we have

$$
\begin{aligned}
& \left|a * \widetilde{T}_n^{(1)}\right| \\
\leq & \frac{2^{N/p+1}}{A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \varepsilon_s(n) \sum_{l=N+1}^{s} \frac{1}{2^l} \left(A_{n_{(s-1)}+2^l-1}^{\alpha_n} - A_{n_{(s-1)}+2^{l-1}-1}^{\alpha_n} \right) \\
& \times \sum_{k=N+1}^{l} \sum_{r=0}^{k} 2^r \int_{I_N} D_{2^k}\left(x \dotplus t \dotplus 2^{-r-1}\right) dt \\
& + \frac{2^{N/p}}{A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \varepsilon_s(n) A_{n_{(s)}-1}^{\alpha_n-1} \sum_{l=0}^{N-1} 2^l \mathbf{1}_{I_N(2^{-l-1})}(x)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{2^{N/p+1}}{A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \varepsilon_s(n) \sum_{l=N+1}^{s} \frac{1}{2^l} \left(A_{n_{(s-1)}+2^l-1}^{\alpha_n} - A_{n_{(s-1)}+2^{l-1}-1}^{\alpha_n} \right) \\
& \times \left(\sum_{r=0}^{N} 2^r \sum_{k=N+1}^{l-1} \int_{I_N} D_{2^k}\left(x \dotplus t \dotplus 2^{-r-1} \right) dt \right. \\
& \left. + \sum_{r=N+1}^{l-1} 2^r \sum_{k=r}^{l-1} \int_{I_N} D_{2^k}\left(x \dotplus t \dotplus 2^{-r-1} \right) dt \right) \\
& + \frac{\alpha_n 2^{N/p}}{n^{\alpha_n}} \sum_{s=N+1}^{|n|} 2^{s(\alpha_n-1)} \sum_{l=0}^{N-1} 2^l \mathbf{1}_{I_N\left(2^{-l-1}\right)}(x).
\end{aligned}
$$

Since

$$
\sum_{r=N+1}^{l-1} 2^r \sum_{k=r}^{l-1} \int_{I_N} D_{2^k}\left(x \dotplus t \dotplus 2^{-r-1} \right) dt = 0 \quad \left(x \in \overline{I}_N \right)
$$

we get

$$
\begin{aligned}
& \left| a * \widetilde{T}_n^{(1)} \right| \\
\lesssim \quad & \frac{2^{N/p}}{A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \varepsilon_s(n) \sum_{l=N+1}^{s} \frac{1}{2^l} \left(A_{n_{(s-1)}+2^l-1}^{\alpha_n} - A_{n_{(s-1)}+2^{l-1}-1}^{\alpha_n} \right) \\
& \times \sum_{r=0}^{N} 2^r \sum_{k=N+1}^{l} \int_{I_N} D_{2^k}\left(x \dotplus t \dotplus 2^{-r-1} \right) dt \\
& + \frac{\alpha_n 2^{N/p}}{2^N} \sum_{l=0}^{N-1} 2^l \mathbf{1}_{I_N\left(2^{-l-1}\right)}(x) \\
\lesssim \quad & \frac{2^{N/p}}{A_{n-1}^{\alpha_n}} \sum_{s=N+1}^{|n|} \varepsilon_s(n) \sum_{l=N+1}^{s} \frac{(l-N)}{2^l} \\
& \times \left(A_{n_{(s-1)}+2^l-1}^{\alpha_n} - A_{n_{(s-1)}+2^{l-1}-1}^{\alpha_n} \right) \sum_{r=0}^{N} 2^r \mathbf{1}_{I_N\left(2^{-r-1}\right)}(x) \\
& + \frac{\alpha_n 2^{N/p}}{2^N} \sum_{l=0}^{N-1} 2^l 1_{I_N\left(2^{-l-1}\right)}(x) \\
\lesssim \quad & \frac{2^{N/p}}{A_{n-1}^{\alpha_n} 2^N} \sum_{s=N+1}^{|n|} \varepsilon_s(n) \sum_{l=N+1}^{s} \left(A_{n_{(s-1)}+2^l-1}^{\alpha_n} - A_{n_{(s-1)}+2^{l-1}-1}^{\alpha_n} \right) \\
& \times \sum_{r=0}^{N} 2^r \mathbf{1}_{I_N\left(2^{-r-1}\right)}(x) \\
& + \frac{\alpha_n 2^{N/p}}{2^N} \sum_{l=0}^{N-1} 2^l 1_{I_N\left(2^{-l-1}\right)}(x)
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \frac{2^{N/p}}{A_{n-1}^{\alpha_n}2^N}\sum_{s=N+1}^{|n|}\varepsilon_s(n)\left(A_{n_{(s)}-1}^{\alpha_n}-A_{n_{(s-1)}-1}^{\alpha_n}\right)\\
&\quad\times\sum_{r=0}^{N}2^r\mathbf{1}_{I_N\left(2^{-r-1}\right)}(x)\\
&\quad+\frac{\alpha_n 2^{N/p}}{2^N}\sum_{l=0}^{N-1}2^l 1_{I_N\left(2^{-l-1}\right)}(x)\\
&\lesssim \frac{2^{N/p}}{2^N}\sum_{r=0}^{N}2^r\mathbf{1}_{I_N\left(2^{-r-1}\right)}(x).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_{\bar{I}_N}\left(\sup_{n>N}\left|a*\widetilde{T}_n^{(1)}(x)\right|\right)^p dx\\
&\leq \frac{2^N}{2^{Np}}\sum_{r=0}^{N}2^{rp}\int_{\bar{I}_N}\mathbf{1}_{I_N\left(2^{-r-1}\right)}(x)\,dx\\
&= \frac{1}{2^{Np}}\sum_{r=0}^{N}2^{rp}\leq c_p<\infty.
\end{aligned}
$$

and consequently,

$$
\left\|\sup_{n\in\mathbb{N}}\left(f*\widetilde{T}_n\right)\right\|_p\leq c_p\left\|f\right\|_{H_p}\quad\left(f\in H_p,\ \ p>0\right) \tag{27}
$$

and

$$
\left\|\sup_{n\in\mathbb{N}}\left(f*\widetilde{T}_n\right)\right\|_{\text{weak\_}L_1(\mathbb{I})}\leq c\left\|f\right\|_1\left(f\in L_1(\mathbb{I})\right). \tag{28}
$$

Since

$$
\sup_{n\in V_K(\alpha)}\left|f*K_n^{\alpha_n}\right|\leq\sup_{n\in\mathbb{N}}\left(f*\widetilde{T}_n\right)+\sup_{n\in\mathbb{N}}\left(f*\left|T_n^{(3)}\right|\right)
$$

Combining (24), (25), (26), (27) and (28) we complete the proof of Theorem 4.1. □

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References

[1] A. A. Abu Joudeh, G. Gát, Miskolc Math. Notes 19 (2018), no. 1, 303–317.
[2] T. Akhobadze, On the convergence of generalized Cesàro means of trigonometric Fourier series. I. Acta Math. Hungar. 115 (2007), no. 1-2, 59–78.
[3] T. Akhobadze, On the convergence of generalized Cesàro means of trigonometric Fourier series. II. Acta Math. Hungar. 115 (2007), no. 1-2, 79–100.
[4] I. Blahota, G. Gát, Norm summability of Nörlund logarithmic means on unbounded Vilenkin groups. Anal. Theory Appl. 24 (2008), no. 1, 1–17.

[5] N. Fine, On the Walsh functions. Trans. Amer. Math. Soc. 65 (1949), 372–414.
[6] S. Fridli, Approximation by Vilenkin-Fourier sums. Acta Math. Hungar. 47 (1986), no. 1-2, 33–44.
[7] G. Gát, Investigations of certain operators with respect to the Vilenkin system. Acta Math. Hungar. 61 (1993), no. 1-2, 131–149.
[8] G. Gát, U. Goginava, On the divergence of Nörlund logarithmic means of Walsh-Fourier series. Acta Math. Sin. (Engl. Ser.) 25 (2009), no. 6, 903–916.
[9] G. Gát, U. Goginava, Norm convergence of logarithmic means on unbounded Vilenkin groups. Banach J. Math. Anal. 12 (2018), no. 2, 422–438.
[10] G. Gát, U. Goginava, Maximal operators of Cesàro means with varying parameters of Walsh-Fourier series. Acta Math. Hungar. 159 (2019), no. 2, 653–668.
[11] U. Goginava, Almost everywhere convergence of subsequence of logarithmic means of Walsh-Fourier series. Acta Math. Acad. Paedagog. Nyházi. (N.S.) 21 (2005), no. 2, 169–175.
[12] U. Goginava, Logarithmic means of Walsh-Fourier series. Miskolc Math. Notes 20 (2019), no. 1, 255–270.
[13] I. B. Kaplan, Cesàro means of variable order. (Russian) Izv. Vysš. Učebn. Zaved. Matematika 1960 1960 no. 5 (18) 62–73.
[14] S. V. Konyagin, On a subsequence of Fourier-Walsh partial sums. (Russian) ; translated from Mat. Zametki 54 (1993), no. 4, 69–75, 158 Math. Notes 54 (1993), no. 3-4, 1026–1030 (1994).
[15] N. Memić, Almost everywhere convergence of some subsequences of the Nörlund logarithmic means of Walsh-Fourier series. Anal. Math. 41 (2015), no. 1-2, 45–54.
[16] K. Nagy, Almost everywhere convergence of a subsequence of the Nörlund logarithmic means of Walsh-Kaczmarz-Fourier series. J. Math. Inequal. 3 (2009), no. 4, 499–510.
[17] L.-E. Persson, G. Tephnadze, P. Wall, On the Nörlund logarithmic means with respect to Vilenkin system in the martingale Hardy space H_1. Acta Math. Hungar. 154 (2018), no. 2, 289–301.
[18] F. Schipp, W. R. Wade, P. Simon, Walsh series. An introduction to dyadic harmonic analysis. With the collaboration of J. Pál. Adam Hilger, Ltd., Bristol, 1990.
[19] P. Simon, Strong convergence of certain means with respect to the Walsh-Fourier series. Acta Math. Hungar. 49 (1987), no. 3-4, 425–431.
[20] G. Tephnadze, On the maximal operators of Riesz logarithmic means of Vilenkin-Fourier series. Studia Sci. Math. Hungar. 51 (2014), no. 1, 105–120.
[21] Sh. Tetunashvili, On divergence of Fourier series by some methods of summability. J. Funct. Spaces Appl. 2012, Art. ID 542607, 9 pp.
[22] F. Weisz, Cesàro summability of one- and two-dimensional Walsh-Fourier series. Anal. Math. 22 (1996), no. 3, 229–242.
[23] F. Weisz, Summability of multi-dimensional Fourier series and Hardy spaces. Mathematics and its Applications, 541. Kluwer Academic Publishers, Dordrecht, 2002.
[24] F. Weisz, Cesàro and Riesz summability with varying parameters of multi-dimensional Walsh-Fourier series. Acta Math. Hungar. 161 (2020), no. 1, 292–312.
[25] Sh. Yano, Cesàro summability of Fourier series. J. Math. Tokyo 1, (1951). 32–34.
[26] A. Zygmund, Trigonometric series. Vol. I, II. Third edition. With a foreword by Robert A. Fefferman. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2002.