Meir-Keeler Condensing Operators and Applications

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Abstract. Motivated by the open question posed by H. K. XU in [39] (Question 2.8), Belhadj, Ben Amar and Boumaiza introduced in [5] the concept of Meir-Keeler condensing operator for self-mappings in a Banach space via an arbitrary measure of weak noncompactness. In this paper, we introduce the concept of Meir-Keeler condensing operator for nonself-mappings in a Banach space via a measure of weak noncompactness and we establish fixed point results under the condition of Leray-Schauder type. Some basic hybrid fixed point theorems involving the sum as well as the product of two operators are also presented. These results generalize the results on the lines of Krasnoselskii and Dhage. An application is given to nonlinear hybrid linearly perturbed integral equations and an example is also presented.

1. Introduction

In some applications it is extremely difficult to find self mappings. To overcome such difficulty, we can refer to the famous Leray-Schauder principle ([32]) which is one of the most important theorems in nonlinear analysis and other variations of this principle ([8], [36]). These theorems are based on the compactness results and they are useful for giving solutions of nonlinear differential and integral equations in Banach spaces. In [21], the authors used the concept of Meir-Keeler condensing operator which is introduced in [1] and they proved fixed point theorems for nonself Meir-Keeler condensing mappings under the conditions of Leray-Schauder, Rothe and Altman types. They used a measure of noncompactness which can describe the degree of noncompactness for bounded sets greatly. Because the weak topology is the convenient and natural setting to investigate the existence problems of fixed points and eigenvectors for operators and solutions of various kinds of nonlinear differential equations and nonlinear integral equations in Banach spaces, the above mentioned result cannot be applied and this approach fails. These equations can be transformed into fixed point problems and nonlinear operator equations involving a broader class of nonlinear operators, in which the operators have the property that the image of any set in a certain sense more weakly compact than the original set itself. The major problem to face is that an infinite dimensional Banach space equipped with its weak topology does not admit open bounded sets. As a result, new theory was needed to complete the picture. The main scope of this paper is to give new existence results for weakly sequentially continuous nonself-mappings which satisfied a Meir-Keeler codensing property with respect to a measure of weak noncompactness.

For nonlinear integral equations of mixed type, the study of hybrid fixed point theorems initiated by

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Krasnoselskii [31] and Dhage [16] in a Banach space and a Banach algebra involve the arguments from geometry and topology. Naturally these results combine two basic fixed point theorems of analysis and topology namely. We prove in this work a Krasnosel'skii type fixed point theorem for weakly sequentially continuous mappings which cover and unify several earlier results from the literature and in particular the work of [35]. The hybrid fixed point theorem of Dhage, which contains a generalization of nonlinear $\mathcal{D}$−contraction, concerns the product of two operators and it is applied to the quadratically perturbed nonlinear integral equations for proving the existence theorems under some standard assumptions and since its appearance, it is used to study nonlinear hybrid differential and integral equations with quadratic perturbations (see $[11 − −15]$ and references therein). In [17], Dhage extented the geometrical condition of nonlinear $\mathcal{D}$−contraction to Meir-Keeler contraction. Meir-Keeler contractive maps are also source of investigations in metric fixed point theory. For more details, we refer the reader to [2], [20], [23], [24], [25], [26], [27], [28], [29], [30], and the references therein. In our work, we extend the result of Dhage to the weak topology setting and since the original $\mathcal{D}$−contraction condition is not applicable to nonlinear differential and integral equations we use an equivalent condition proved by Lim ([33]). Finally we apply the abstract hybrid fixed point theorem to a simple nonlinear hybrid integral equation in order to prove existence result under some geometrical and toplogical conditions. However, the study may be extended to other very complex and involved nonlinear integral equations with obvious modifications. We give also a numerical example to illustrate the abstract idea contained in the existence theorem.

2. Preliminaries

Let E be a Banach space endowed with the norm $\|.\|$. We denote by B_r the closed ball centered at 0 with radius r. For a subset C of E, we write $\overline{C}, \overline{C}^\omega$ and $co(C)$, to denote the closure, the weak closure and the convex hull of the subset C, respectively. Moreover, we write $x_n \to x$ and $x_n \rightharpoonup x$ to denote the strong convergence (with respect to the norm of E) and the weak convergence (with respect to the weak topology of E) of a sequence $(x_n)_n$ to x. Further denote by $\Omega_{(E)}$ the family of all nonempty and bounded subsets of a Banach space E and $\mathcal{W}_{(E)}$ is the subset of $\Omega_{(E)}$ consisting of all weakly compact subsets of E. In the sequel we need the following definition of a measure of weak noncompactness [10].

Definition 2.1. *Let E be a Banach space and $X_1, X_2 \in \Omega_{(E)}$. A mapping $\omega : \Omega_{(E)} \to [0, \infty)$ is said to be a measure of weak noncompactness if it satisfies the following conditions:*

1. *Regularity: $\omega(X_1) = 0$ if and only if X_1 is relatively weakly compact.*
2. *Monotonicity : If $X_1 \subseteq X_2$, then $\omega(X_1) \leqslant \omega(X_2)$.*
3. *Invariant under closure: $\omega(\overline{X_1}^\omega) = \omega(X_1)$.*
4. *Invariance under passage to the convex hull : $\omega(co(X_1)) = \omega(X_1)$.*
5. *$\omega(\lambda X_1 + (1 − \lambda)X_2) \leq \lambda\omega(X_1) + (1 − \lambda)\omega(X_2)$ for $\lambda \in [0, 1]$.*
6. *Generalized Cantor's intersection theorem: If $(X_n)_{n \geqslant 1}$ is a decreasing sequence of nonempty, bounded and weakly closed subsets of E with $\lim\limits_{n \to +\infty} \omega(X_n) = 0$, then $\bigcap\limits_{n=1}^{\infty} X_n \neq \emptyset$ and $\omega(\bigcap\limits_{n=1}^{\infty} X_n) = 0$ i.e. $\bigcap\limits_{n=1}^{\infty} X_n$ is relatively weakly compact. We say that a measure of weak noncompactness is regular if it satisfies additionally the following conditions :*
7. *The maximum property $\omega(X_1 \cup X_2) = \max\{\omega(X_1), \omega(X_2)\}$.*
8. *Algebraic semi-additivity : $\omega(X_1 + X_2) \leqslant \omega(X_1) + \omega(X_2)$.*
9. *$Ker(\omega) = \mathcal{W}_{(E)}$.*

In [10] De Blasi introduced the following example of a measure of weak noncompactness:

$$\beta(M) = \inf\{r > 0 : \text{ there exists a set } N \in \mathcal{W}_{(E)} \text{ such that } M \subseteq N + B_r\}$$

for $M \in \Omega_{(E)}$. Note that the De Blasi measure of weak noncompactness β is regular ([10]).

Definition 2.2. *[7] Let C be a nonempty subset of Banach space E. We say that $T : C \rightarrow E$ is condensing with respect to the measure of weak noncompactness ω if $T(X)$ is bounded, and*

$$\omega(T(X)) < \omega(X),$$

for all bounded subset X of C with $\omega(X) > 0$.

Definition 2.3. *[7] Let E be a Banach space. An operator $T : E \rightarrow E$ is said to be weakly compact if $T(C)$ is relatively weakly compact for every bounded subset $C \subset E$.*

Definition 2.4. *Let E be a Banach space. An operator $T : E \rightarrow E$ is said to be weakly sequentially continuous on E, if for every $(x_n)_n$ with $x_n \rightharpoonup x$, we have $Tx_n \rightharpoonup Tx$.*

We recall the weak version of the Schauder-Tikhonov fixed point principle which was obtained by Arino, Gautier and Penot:

Theorem 2.5. *[4] Let C be a nonempty, convex and weakly compact subset of a Banach space E and $T : C \rightarrow C$ a weakly sequentially continuous operator. Then T has at least one fixed point in the set C.*

Definition 2.6. *Let C be a nonempty subset of a Banach space E and ω an arbitrary measure of weak noncompactness on E. We say that an operator $T : C \rightarrow E$ is a Meir-Keeler condensing operator if for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leqslant \omega(X) < \varepsilon + \delta \;\Rightarrow\; \omega(T(X)) < \varepsilon, \tag{1}$$

for all bounded subset X of C.

The concept of Meir–Keeler condensing operator was introduced recently in [5] for self-mapping $T : C \rightarrow C$ and the following fixed point theorem was proved.

Theorem 2.7. *([5])Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and ω be an arbitrary measure of weak noncompactness on E. If $T : C \rightarrow C$ is a weakly sequentially continuous and Meir-keeler condensing, then T has at least one fixed point and the set of all fixed points of T in C is weakly compact.*

In [33], Lim introduced the notions of L-function and strictly L-function which are important to study Meir–Keeler condensing operator and in [5], Belhadj et al. gave a sufficient and necessary condition for Meir–Keeler condensing operator by virtue of *L*-function.

Definition 2.8. *$\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called an L-function if $\varphi(0) = 0, \varphi(s) > 0$ for $s \in (0, +\infty)$, and for every $s \in (0, +\infty)$ there exists $\delta > 0$ such that $\varphi(t) \leq s$ for $t \in [s, s + \delta]$. If $\varphi(t) \leq s$ is replaced with $\varphi(t) < s$ for $t \in [s, s + \delta]$, we say that φ is a strictly L-function*

Proposition 2.9. *([5]). Let C be a nonempty and bounded subset of a Banach space E, ω an arbitrary measure of weak noncompactness and $T : C \rightarrow C$ a mapping. Then T is a Meir-Keeler condensing operator if and only if there exists an L-function φ such that*

$$\omega(T(X)) < \varphi(\omega(X)),$$

for all $X \in \Omega_{(E)}$ with $X \subset C$ and $\omega(X) \neq 0$.
Moreover, if there exists a strictly L−function θ such that $\omega(T(X)) \leq \theta(\omega(X))$ for all $X \in \Omega_{(E)}$ with $X \subset C$ and $\omega(X) \neq 0$, then $T : C \rightarrow C$ is Meir-Keeler condensing.

Using Theorem 2.7 and Proposition 2.9, Belhadj, Ben Amar and Boumaiza state in [5] the following fixed point result.

Corollary 2.10. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E, ω an arbitrary measure of weak noncompactness and $T : C \rightarrow C$ a mapping. Assume that T is weakly sequentially continuous such that*

$$\omega(T(X)) < \varphi(\omega(X)) \;\; or \;\; \omega(T(X)) \leq \theta(\omega(X))$$

for $X \subseteq C$, where φ is an L-function and θ is a strictly L−function. Then, T has at least one fixed point and the set of all fixed points of T in C is weakly compact.

Definition 2.11. *[12] An upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$−function if $\psi(0) = 0$. The class of all $\mathcal{D}$−functions on $\mathbb{R}_+$ is denoted by $\mathcal{D}$.*

Definition 2.12. *[16] An operator $T : X \to X$ is called $\mathcal{D}$−Lipschitz if there exists a $\mathcal{D}$−function $\psi_T \in \mathcal{D}$ such that*

$$\|Tx - Ty\| \le \psi_T(\|x - y\|)$$

for all elements $x, y \in X$. Furthermore, if $\psi_T(r) < r$ for $r > 0$, then T is called a nonlinear $\mathcal{D}$−contraction on X. The class of all $\mathcal{D}$−functions satisfying the condition of nonlinear $\mathcal{D}$−contraction is denoted by $\mathcal{DR}$.

Definition 2.13. *[17] A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{DL}$−function if it is $\mathcal{D}$−function as well as strictly L−function. The class of $\mathcal{DL}$−functions is denoted by $\mathcal{DL}$.*

Remark 2.14. *It is clear that if $\psi \in \mathcal{DR}$, then $\psi \in \mathcal{DL}$, but the converse may not be true.*

3. Fixed Point Results for Meir-Keeler condensing operators

In this section we prove our main result for nonself Meir-Keeler condensing operator which is a generalization of the notion of Meir-Keeler contraction introduced by Meir and Keeler in 1969([34]).

Theorem 3.1. *Let K be a nonempty closed convex set in a Banach space E. In addition, let U be a weakly open subset of K and $x_0 \in U$, $T : \overline{U^\omega} \to E$ be a weakly sequentially continuous mapping such that $T(\overline{U^\omega})$ is bounded. If T is a Meir-Keeler condensing operator and satisfies Leray–Schauder condition*

$$(1 - \lambda)x_0 + \lambda Tx \ne x, \text{ for all } x \in \partial_K U \text{ and } \lambda \in (0, 1), \tag{2}$$

where $\partial_K U$ is the weak boundary of U relative to K, then T has at least one fixed point in $\overline{U^\omega}$, and the set of all fixed points of T is weakly compact.

Proof
Step 1: We have $x \ne (1 - \lambda)x_0 + \lambda Tx$, for all $x \in \partial_K U$ and $\lambda \in (0, 1)$. We observe that this supposition is satisfied also for $\lambda = 0$ (since $x_0 \in U$). If it is satisfied for $\lambda = 1$, then in this case we have a fixed point in $\partial_K U$ and there is nothing to prove. In conclusion, we can consider $x \ne (1 - \lambda)x_0 + \lambda Tx$ for all $x \in \partial_K U$ and $\lambda \in [0, 1]$. Let Σ be the set defined by

$$\Sigma = \{x \in \overline{U^\omega} : (1 - \lambda)x_0 + \lambda T(x) = x, \lambda \in [0, 1]\},$$

The set Σ is non-empty since $x_0 \in \Sigma$. The weak sequentially continuity of T implies that Σ is weakly sequentially closed. For that, let $(x_n)_n$ be a sequence of Σ such that $x_n \rightharpoonup x \in \overline{U^\omega}$. For all $n \in \mathbb{N}$, there exists a $\lambda_n \in [0, 1]$ such that $x_n = (1 - \lambda_n)x_0 + \lambda_n T(x_n)$. Since $(\lambda_n)_n \subset [0, 1]$, we can extract a subsequence $(\lambda_{n_j})_j$ such that $\lambda_{n_j} \longrightarrow \lambda \in [0, 1]$. Since T is weakly sequentially continuous, then $T(x_{n_j}) \rightharpoonup T(x)$. Consequently,

$$(1 - \lambda_{n_j})x_0 + \lambda_{n_j} T(x_{n_j}) \rightharpoonup (1 - \lambda)x_0 + \lambda T(x).$$

Hence $x = (1 - \lambda)x_0 + \lambda T(x)$ and $x \in \Sigma$. Thus, Σ is weakly sequentially closed.
We now claim that Σ is relatively weakly compact. Clearly,

$$\Sigma \subseteq co(T(\Sigma) \cup \{x_0\}). \tag{3}$$

Thus,

$$\omega(\Sigma) \le \omega(co(T(\Sigma) \cup \{x_0\})) \le \omega(T(\Sigma)). \tag{4}$$

Suppose $\omega(\Sigma) = \varepsilon_0 > 0$ and let $\delta = \delta(\varepsilon) > 0$ be chosen according to (1). Since T is a Meir-Keeler condensing operator, then

$$\omega(T(\Sigma)) < \varepsilon_0 = \omega(\Sigma).$$

which is a contradiction. Hence, $\omega(\Sigma) = 0$, and therefore $\overline{\Sigma^\omega}$ is weakly compact. This proves our claim. Now let $x \in \overline{\Sigma^\omega}$. Since $\overline{\Sigma^\omega}$ is weakly compact by the Eberlein-Smulian theorem [18], there exists a sequence $(x_n)_n$ in Σ which converges weakly to x. Since Σ is weakly sequentially closed, we have $x \in \Sigma$. Thus, $\overline{\Sigma^\omega} = \Sigma$. Hence, Σ is weakly closed and therefore weakly compact. From our assumptions we have $\Sigma \cap (K \setminus U) = \emptyset$. Since E endowed with its weak topology is a Hausdorff locally convex space then there exists a weakly continuous mapping $\rho : K \to [0,1]$ with $\rho(x) = 1$ for $x \in \Sigma$ and $\rho(x) = 0$ for $x \in K \setminus U$ (see [22] p. 146). Put $D = \overline{co}^\omega\{T(\overline{U}^\omega) \cup \{x_0\}\}$ which is a bounded convex closed set, and define $\tilde{T}$ as

$$\tilde{T}(x) = \begin{cases} (1 - \rho(x))x_0 + \rho(x)T(x) & \text{if } x \in D \cap \overline{U^\omega}, \\ 0 & \text{if } x \in D \setminus \overline{U^\omega}. \end{cases}$$

Because $\partial_K U = \partial_K \overline{U}^\omega$, ρ is weakly continuous and T is weakly sequentially continuous, we have that $\tilde{T} : D \to D$ is weakly sequentially continuous.

Step 3: By the definition of Meir-Keeler condensing operator we can define a function $\alpha : (0, \infty) \to (0, \infty)$, such that

$$\varepsilon \le \omega(X) < \varepsilon + 2\alpha(\varepsilon) \Rightarrow \omega(T(X)) < \varepsilon, \quad \text{for } \varepsilon \in (0, \infty). \tag{5}$$

Using such α, we define a nondecreasing function $\beta : (0, \infty) \to [0, \infty)$ as

$$\beta(t) = \inf\{\xi : t \le \xi + \alpha(\xi)\}$$

As $t \le t + \alpha(t)$, we have

$$\beta(t) \le t, \quad \text{for } t \in (0, \infty). \tag{6}$$

Now define a function φ from $[0, \infty)$ into itself as

$$\varphi(t) = \begin{cases} 0 & \text{if } t = 0, \\ \beta(t) & \text{if } t > 0 \text{ and } \min\{\xi > 0 : t \le \xi + \alpha(\xi)\} \text{ exists}, \\ \dfrac{\beta(t) + t}{2} & \text{otherwise}. \end{cases}$$

similar to the proofs of Theorem 2.6 in [1], we can prove that φ is an L-function (i.e. there exists $\delta_1(\varepsilon) > 0$ such that $\varepsilon \le t \le \varepsilon + \delta_1(\varepsilon) \Rightarrow \varphi(t) \le \varepsilon$) and

$$\omega(T(X)) < \varphi(\omega(X)), \tag{7}$$

for nonrelatively weakly compact set $X \subset \overline{U^\omega}$.

Step 4: We show that for $\varepsilon > 0$ there exists $\delta > 0$ such that when $\varepsilon \le \omega(S) < \varepsilon + \delta$, we have $\omega(T(S \cap \overline{U^\omega})) < \varepsilon$ for nonrelatively weakly compact set $S \subset D$ with $\omega(S \cap \overline{U^\omega}) > 0$. Let

$$A = \{\xi > 0 : \omega(S \cap \overline{U^\omega}) \le \xi + \alpha(\xi)\}, \quad B = \{\xi > 0 : \omega(S) \le \xi + \alpha(\xi)\}.$$

It follows from $\omega(S), \omega(S \cap \overline{U^\omega}) \in (0, +\infty)$ and

$$\omega(S \cap \overline{U^\omega}) \le \omega(S \cap \overline{U^\omega}) + \alpha(\omega(S \cap \overline{U^\omega})), \quad \omega(S) \le \omega(S) + \alpha(\omega(S)),$$

that $\omega(S \cap \overline{U^\omega}) \in A$ and $\omega(S) \in B$, hence both A and B are nonempty. Here we discuss the different cases.

- If both $\min A$ and $\min B$ exist, by the definition of φ and by β is nondecreasing we have

$$\varphi(\omega(S \cap \overline{U^\omega})) = \beta(\omega(S \cap \overline{U^\omega})) \le \beta(\omega(S)) = \varphi(\omega(S)),$$

- If min A exists but min B does not exist we have $\varphi(\omega(S \cap \overline{U^\omega})) = \beta(\omega(S \cap \overline{U^\omega}))$ from (6) if follows that $\beta(\omega(S \cap \overline{U^\omega})) \le \beta(\omega(S)) \le \omega(S)$, and hence

$$\varphi(\omega(S \cap \overline{U^\omega})) \le \frac{\beta(\omega(S)) + \omega(S)}{2} = \varphi(\omega(S)).$$

- If both min A and min B do not exist we have

$$\varphi(\omega(S \cap \overline{U^\omega})) = \frac{\beta(\omega(S \cap \overline{U^\omega}) + \omega(S \cap \overline{U^\omega})}{2} \le \frac{\beta(\omega(S)) + \omega(S)}{2} = \varphi(\omega(S)).$$

- If min A does not exist but min B exists we have

$$\varphi(\omega(S \cap \overline{U^\omega})) = \frac{\beta(\omega(S \cap \overline{U^\omega})) + \omega(S \cap \overline{U^\omega})}{2} \le \beta(\omega(S)) = \varphi(\omega(S))$$

In all above cases, we have

$$\varphi(\omega(S \cap \overline{U^\omega})) \le \varphi(\omega(S)). \tag{8}$$

On account of (7), (8) and since φ is L-function, we have that when $\varepsilon \le \omega(S) < \varepsilon + \delta_1(\varepsilon)$,

$$\omega(T(S \cup \overline{U^\omega})) < \varphi(\omega(S \cap \overline{U^\omega})) \le \varphi(\omega(S)) \le \varepsilon.$$

Now we deal with the last case
- If min A does not exist but min B exists, however

$$\omega(S \cap \overline{U^\omega}) > \beta(\omega(S)). \tag{9}$$

Since $\beta(\omega(S)) \in B$ we have

$$\omega(S) \le \beta(\omega(S)) + \alpha(\beta(\omega(S))). \tag{10}$$

So

$$\beta(\omega(S)) < \omega(S \cap \overline{U^\omega}) \le \omega(S) \le \beta(\omega(S)) + \alpha(\beta(\omega(S))) < \beta(\omega(S)) + 2\alpha(\beta(\omega(S))) \tag{11}$$

From (5) and (11) follows

$$\omega(T(S \cap \overline{U^\omega})) < \beta(\omega(S)) < \omega(S \cap \overline{U^\omega}) < \omega(S), \tag{12}$$

therefore according to (12), $\varepsilon \le \omega(S) < \varepsilon + 2\alpha(\varepsilon)$ implies

$$\omega(T(S \cap \overline{U^\omega})) < \omega(S \cap \overline{U^\omega}) < \varepsilon + 2\alpha(\varepsilon). \tag{13}$$

If $\omega(T(S \cap \overline{U^\omega})) \ge \varepsilon$, by (13) we have $\varepsilon \le \omega(S \cap \overline{U^\omega})) < \varepsilon + 2\alpha(\varepsilon)$, hence according to (5) we have $\omega(T(S \cap \overline{U^\omega})) < \varepsilon$ which is a contradiction. So $\omega(T(S \cap \overline{U^\omega})) < \varepsilon$. In a word, we can take $\delta = \min\{\delta_1(\varepsilon), 2\alpha(\varepsilon)\} > 0$, where $\delta_1(\varepsilon)$ and $\alpha(\varepsilon)$ appeared in step 3.

Step 5: Now we prove that $\tilde{T} : D \to D$ is a Meir-Keeler condensing operator. For $\varepsilon > 0$ and a nonrelatively weakly compact set $S \subset D$, we treat it in the following two situations.

(i) If $\omega(S \cap \overline{U^\omega}) = 0$, i.e., $S \cap \overline{U^\omega}$ is a relatively weakly compact set, the weakly sequentially continuity of $\tilde{T}$ implies that $\tilde{T}(S \cap \overline{U^\omega})$ is relatively weakly compact and $\omega(\tilde{T}(S \cap \overline{U^\omega}) = 0 < \varepsilon$. Then by the definition of $\tilde{T}$ we have

$$\begin{aligned} \omega(\tilde{T}(S)) &= \omega(\tilde{T}((S \cap \overline{U^\omega}) \cup (S \setminus \overline{U^\omega}))) \\ &= \omega(\{\tilde{T}(S \cap \overline{U^\omega}) \cup \{x_0\}\}) \\ &\le \omega(\tilde{T}(S \cap \overline{U^\omega})) < \varepsilon. \end{aligned} \tag{14}$$

(ii) If $\omega(S \cap \overline{U^\omega}) > 0$, for $x \in S \cap \overline{U^\omega}$ we have

$$(1 - \rho(x))x_0 + \rho(x)T(x) \in \overline{co}^\omega\{T(S \cap \overline{U^\omega}) \cup \{x_0\}\},$$

since $\rho(x) \in [0, 1]$. Therefore

$$\begin{aligned}
\overline{co}^\omega\{\tilde{T}(S) \cup \{x_0\}\} &= \overline{co}^\omega\{\tilde{T}(S \cap \overline{U^\omega}) \cup \{x_0\}\} \\
&= \overline{co}^\omega\{\{(1 - \rho(x))x_0 + \rho(x)T(x) : x \in S \cap \overline{U^\omega} \cup \{x_0\}\}\} \\
&\subset \overline{co}^\omega\{\overline{co}^\omega\{T(S \cap \overline{U^\omega}) \cup \{x_0\}\} \cup \{x_0\}\} \\
&= \overline{co}^\omega\{T(S \cap \overline{U^\omega}) \cup \{x_0\}\}
\end{aligned}$$

and

$$\begin{aligned}
\omega(\tilde{T}(S)) &= \omega(\{\tilde{T}(S) \cup \{x_0\}) \\
&= \omega(\overline{co}^\omega(\tilde{T}(S) \cup \{x_0\}) \\
&\leqslant \omega(\overline{co}^\omega\{T(S \cap \overline{U^\omega}) \cup \{x_0\}\}) \\
&= \omega(\{T(S \cap \overline{U^\omega}) \cup \{x_0\}\}) \\
&= \omega(T(S \cap \overline{U^\omega})).
\end{aligned} \tag{15}$$

By (15) and step 4, when $\varepsilon \leqslant \omega(S) < \varepsilon + \delta$ we have

$$\omega(\tilde{T}(S) \leqslant \omega(T(S \cap \overline{U^\omega})) < \varepsilon.$$

Hence $\tilde{T}$ is a Meir-Keeler condensing operator.

Step 6: An application of a Theorem 2.7 yields that $\tilde{T}$ has a fixed point in D. Let x^* be one of these fixed points, then $x^* \in \overline{U^\omega}$ and $(1 - \rho(x^*))x_0 + \rho(x^*)T(x^*) = x^*$ which implies $x^* \in \Sigma$ and $\rho(x^*) = 1$. Therefore $T(x^*) = x^*$ and T has fixed points in $\overline{U^\omega}$.

Step 7: We prove that the set of all fixed points of T is weakly compact.
Let $F = \{x \in \overline{U^\omega} : Tx = x\}$ and $\varepsilon_0 = \omega(F)$. If $\varepsilon_0 > 0$, there exists $\delta' > 0$ such that

$$\varepsilon_0 \leqslant \omega(F) < \varepsilon_0 + \delta' \Rightarrow \omega(T(F)) < \varepsilon_0,$$

since T is Meir–Keeler condensing. However $\varepsilon_0 = \omega(F) = \omega(T(F)) < \varepsilon_0$ is a contradiction, hence $\varepsilon_0 = 0$ and F is relatively weakly compact. Now taking into account any weakly convergent sequence $(x_n)_n \subset F$ and $x_n \rightharpoonup x$, we have $x \in \overline{U^\omega}$ because $\overline{U^\omega}$ is closed. The weakly sequentially continuity of T implies that $x_n = T(x_n) \rightharpoonup T(x)$ and $T(x) = x$ which means that $x \in F$, then F is weakly sequentially closed. Since $\overline{F^\omega}$ is weakly compact, by the Eberlein-Smulian theorem ([18], Theorem 8.12.4, p. 549), there exists a sequence $(x_n)_n \subset F$ such that $x_n \rightharpoonup x$, so $x \in F$. Hence $\overline{F^\omega} = F$ and F is weakly closed. Therefore, F is weakly compact. This completes the proof.

Corollary 3.2. *Let K be a closed convex set in Banach space E, U be a weakly open set in K and $x_0 \in U$. If $T : \overline{U^\omega} \to E$ is bounded weakly sequentially continuous and satisfies $\omega(T(X)) \leq \theta(\omega(X))$ for each bounded set $X \subset \overline{U^\omega}$, θ is a strictly L-function, moreover Leray–Schauder condition (2) holds, then T has at least one fixed point in $\overline{U^\omega}$, and the set of all fixed points of T is weakly compact.*

Proof. An application of Proposition 2.9 yields that T is a Meir-Keeler condensing operator. The result follows from Theorem 3.1. $\square$

A priori estimate theorem is also obtained.

Theorem 3.3. *Let $T : E \to E$ be a weakly sequentially continuous mapping and Meir-Keeler condensing operator. If the set*

$$D = \{x \in E : x = \lambda Tx, \ 0 \le \lambda \le 1\}$$

is bounded, then T has fixed point in $\overline{B}_R = \{x \in E : \|x\| \le R\}$, where $R = \sup\{\|x\| : x \in D\}$, especially, R is an arbitrary positive number when $D = \emptyset$.

Proof. For any positive integer k let $B_k = \{x \in E : \|x\| < R + \frac{1}{k}\}$. Let U_k be a weakly open set in B_k and $0 \in U_k$. Obviously $x \ne \lambda Tx$ for all $x \in \partial U_k$ (where ∂U_k is the weak boundary of U_k in B_k) and $\lambda \in [0, 1]$. Hence by Theorem 3.1, T has a fixed point x_k in U_k, that is, $x_k = Tx_k, k \in \mathbb{N}$. Denote $S = \{x_1, x_2, \ldots, x_k, \ldots\}$, then S is relatively weakly compact set by Theorem 3.1. and thus, by the Eberlein Smulian theorem, there exists a weakly convergent subsequence $x_{k_i} \rightharpoonup x^*$. By virtue of the weakly sequentially continuity of T and $\|x_k\| \le R + \frac{1}{k}, Tx_{k_i} = x_{k_i}$ implies that $x^* = Tx^*$ and $\|x^*\| \le \liminf \|x_k\| \le R$. $\quad\square$

4. Hybrid fixed point theorems

4.1. Krasnosel'skii type

In this section we prove the Krasnoselskii hybrid fixed point theorem involving the sum of two operators in a Banach space.

Theorem 4.1. *Let M be a nonempty bounded convex closed subset of a Banach space X and assume that $A, B : M \to X$ are two weakly sequentially continuous mappings. Suppose that $(I - B)^{-1}$ is well defined on $(I - B)(M)$ and the following conditions hold:*

1. $A(M) \subseteq (I - B)(M)$,
2. *$\forall \varepsilon \ge 0, \exists \delta > 0$ such that $\omega(S_{n+1}) < \varepsilon$ when $\varepsilon \le \omega(S_n) < \varepsilon + \delta$ for $n = 1, 2, \cdots$; here $S_1 = M$ and $S_{n+1} = \overline{co}((I - B)^{-1}AS_n)$, for $n = 1, 2, \cdots$; and ω is an arbitrary measure of weak noncompactness.*

Then there exists $x \in M$ with $x = Ax + Bx$.

Proof. Notice that $A(M) \subseteq (I - B)M$, so $(I - B)^{-1}AM \subseteq M$. This implies $S_2 \subseteq S_1$. Proceeding by induction we obtain $S_{n+1} \subseteq S_n$. If there exists an integer $N \ge 0$ such that $\omega(S_N) = 0$ and then $\lim_{n \to +\infty} \omega(S_n) = 0$. If not, then $\omega(S_n) \ne 0$ for all $n \ge 0$. Define $\varepsilon_n = \omega(S_n)$ and let $\delta_n = \delta_n(\varepsilon_n) > 0$ be chosen according to assumption (2). By the definition of ε_n, we have

$$\varepsilon_{n+1} = \omega(S_{n+1}) < \varepsilon_n.$$

Since $\{\varepsilon_n\}_{n \ge 0}$ is a positive decreasing sequence of real numbers, there exists $r \ge 0$ such that $\varepsilon_n \to r$ as $n \to \infty$. We show that $r = 0$. Suppose the contrary, then there there exists N_0 such that

$$n > N_0 \implies r \le \varepsilon_n < r + \delta(r),$$

then, we get $\varepsilon_{n+1} < r$. This is absurd, so $r = 0$. Consequently, by condition (6) in the definition of the measure of weak noncompactness, we deduce that the set $S_\infty = \cap_{n=1}^{\infty} S_n$ is nonempty, weakly closed convex. Further, since $\omega(S_\infty) \le \omega(S_n)$ for all $n \ge 1$, then $S_\infty \in \ker\omega$ and it follows that it is weakly compact. Also, since

$$(I - B)^{-1}AS_n \subseteq (I - B)^{-1}AS_{n-1} \subseteq \overline{co}(I - B)^{-1}AS_{n-1} = S_n \quad \forall n,$$

we have $(I - B)^{-1}AS_\infty \subset S_\infty$. Next, let us show that $(I - B)^{-1}A : S_\infty \to S_\infty$ is weakly sequentially continuous. To do so, let $(x_n)_n$ be a sequence in S_∞ which converges weakly to x. Since $(I - B)^{-1}AS_\infty$ is relatively weakly compact, it follows by the Eberlein Smulian's theorem that there exists a subsequence (x_{n_k}) of $(x_n)_n$ such that $(I - B)^{-1}A(x_{n_k}) \rightharpoonup y$. The weakly sequentially continuity of B leads to $B(I - B)^{-1}A(x_{n_k}) \rightharpoonup By$. Also from the equality $B(I - B)^{-1}A = -A + (I - B)^{-1}A$, it results that

$$-A(x_{n_k}) + (I - B)^{-1}A(x_{n_k}) \rightharpoonup -A(x) + y$$

So, $y = (I - B)^{-1}Ax$. We claim that $(I - B)^{-1}A(x_n) \rightharpoonup (I - B)^{-1}A(x)$. Suppose that this is not the case, then there exists a subsequence $(x_{\varphi_1(n)})_n$ and a week neighborhood V^ω of $(I - B)^{-1}A(x)$ such that $(I - B)^{-1}A(x_{\varphi_1(n)}) \notin V^\omega$, for all $n \in \mathbb{N}$. On the other hand, we have $x_{\varphi_1(n)} \rightharpoonup x$, than arguing as before, we find a subsequence $(x_{\varphi_1(\varphi_2(n))})_n$ such that $(I - B)^{-1}A(x_{\varphi_1(\varphi_2(n))})$ converges weakly to $(I - B)^{-1}Ax$, which is a contradiction and hence $(I - B)^{-1}A : S_\infty \to S_\infty$ is weakly sequentially continuous. Now, a use of the standard Arino-Gautier-Penot fixed point theorem gives us the desired result. □

Remark 4.2.   1. *The measure of weak noncompactness in Theorem 4.1 is arbitrary.*

2. *Theorem 4.1 is a generalization of Theorem 3.1 in [35].*

4.2. Dhage type

We study now the second important type of hybrid fixed point results. Since the product of two sequentially weakly continuous functions is not necessarily sequentially weakly continuous, we will introduce:

Definition 4.3. *We will say that the Banach algebra X satisfies condition (P) if*

$$(P) \begin{cases} \text{For any sequences } \{x_n\} \text{ and } \{y_n\} \text{ in } X \text{ such that } x_n \rightharpoonup x \text{ and } y_n \rightharpoonup y, \\ \text{then } x_n y_n \rightharpoonup xy; \end{cases}$$

Note that, every finite dimensional Banach algebra satisfies condition (P). Even, if X satisfies condition (P) then C(K, X) is also Banach algebra satisfying condition (P), where K is a compact Hausdorff space.

Theorem 4.4. *Let S be a nonempty, bounded, closed, and convex subset of a Banach algebra X satisfying the condition (P) and let $A : X \to X$ and $B : S \to X$ be two operators satisfying the following conditions*

1. *A and B are sequentially weakly continuous,*
2. *there exist $\mathcal{D}$−functions φ_A and φ_B such that $\omega(D(\Omega) \leq \varphi_D(\omega(\Omega))$, for $D = A$ and B for all non weakly relatively compact set $\Omega \subset X$,*
3. *for $\varepsilon > 0$ there exists a number $\delta > 0$ such that $M_A\varphi_B(r) + M_B\varphi_A(r) + \varphi_A(r)\varphi_B(r) < \varepsilon$ for all $r \in [\varepsilon, \varepsilon + \delta[$, where $M_B = \sup\{\|Bx\|; x \in S\}$ and $M_A = \sup\{\|Ax\|; x \in S\}$*
4. *A(S) and B(S) are bounded,*
5. *for all $x \in S, A(x)B(x) \in S$.*

Then the equation $x = A(x)B(x)$ has at least one solution in S and the set of all fixed points of AB in S is weakly compact.

Proof. The mapping $AB : S \to S$ is well defined. In view of assumption (1), condition (P) guarantees that AB is weakly sequentially continuous. Let now $\varepsilon > 0$ and let a non weakly relatively compact set $\Omega \subset S$. When $\varepsilon \leq \omega(\Omega) < \varepsilon + \delta$ we have (since $A(\Omega)B(\Omega)$ is bounded)

$$\omega(A(\Omega)B(\Omega)) \leq \|A(\Omega)\|\omega(B(\Omega)) + \omega(A(\Omega))\|B(\Omega)\| + \omega(A(\Omega))\omega(B(\Omega))$$
$$\leq \|A(\Omega)\|\varphi_B(\omega(\Omega)) + \|B(\Omega)\|\varphi_A(\omega(\Omega)) + \varphi_A(\omega(\Omega))\varphi_B(\omega(\Omega))$$
$$< \varepsilon. \text{ by assumption (3)}$$

Thus, the mapping AB is Meir-Keller condensing. From Theorem 2.7, AB has a fixed point and the set of all fixed points of this mapping in S is weakly compact. □

Theorem 4.5. *Let S be a nonempty, bounded, closed, and convex subset of a Banach algebra X. Let $A, C : X \to X$ and $B : S \to X$ be three operators such that*

1. *A and C are $\mathcal{D}$−Lipschitzians with the $\mathcal{D}$−functions φ_A and φ_C respectively,*
2. *A is regular on X, i.e., A maps X into the set of all invertible elements of X,*
3. *B is sequentially weakly continuous and B(S) is relatively weakly compact,*
4. *$(\frac{I-C}{A})^{-1}$ is sequentially weakly continuous on B(S),*

5. *for $\varepsilon > 0$ there exists a number $\delta > 0$ such that $M_B \varphi_A(r) + \varphi_C(r) < \varepsilon$ for all $r \in [\varepsilon, \varepsilon + \delta[$, where $M_B = \sup\{\|Bx\|; x \in S\}$.*

6. *$x = AxBy + Cx \implies x \in S$, for all $y \in S$.*

Then the equation $x = A(x)B(x) + C(x)$ has at least one solution in S.

Remark 4.6. *Recently, some fixed point theorems involving three operators in Banach algebras were established for completely continuous maps. Because every totally bounded subset of X is relatively weakly compact, Theorem 2.8 in [11] follows as a sequence of Theorem 4.5. Further, Theorem 4.5 is a generalization of many knows results of Dhage ([12–15]) in the weak topology setting and under weaker contraction condition. In Theorem 4.5, the continuity is not required.*

Proof. Let y be fixed in S and define the mapping

$$N_y : X \to X,$$
$$x \mapsto N_y(x) = AxBy + Cx.$$

Let $x_1, x_2 \in X$, by assumption (1), we have

$$\begin{aligned}
\|N_y(x_1) - N_y(x_2)\| &\le \|Ax_1 By - Ax_2 By\| + \|Cx_1 - Cx_2\| \\
&\le \|Ax_1 - Ax_2\|\|By\| + \|Cx_1 - Cx_2\| \\
&\le M_B \varphi_A(\|x_1 - x_2\|) + \varphi_C(\|x_1 - x_2\|) \\
&\le \varphi(\|x_1 - x_2\|)
\end{aligned}$$

where, $\varphi(r) = M_B \varphi_A(r) + \varphi_C(r)$ is a $\mathcal{DL}$−function on $\mathbb{R}_+$. Hence N_y is a Meir-Keeler contraction on X and by Meir-Keeler fixed point theorem, N_y has a unique fixed point, say $x_y \in X$. Then, we have

$$N_y(x_y) = Ax_y By + Cx_y = x_y.$$

By virtue of the hypothesis (6), $x_y \in S$. Therefore, the mapping $(\frac{I-C}{A})^{-1}$ is well defined on $B(S)$ and $(\frac{I-C}{A})^{-1}B(S) \subset S$. Since $(\frac{I-C}{A})^{-1}$ and B are sequentially weakly continuous, so, by composition we have $(\frac{I-C}{A})^{-1}B$ is sequentially weakly continuous. Finally, we claim that $(\frac{I-C}{A})^{-1}B(S)$ is relatively weakly compact. To see this, let $\{u_n\}$ be any sequence in S and let

$$v_n = (\frac{I-C}{A})^{-1}Bu_n.$$

Since $B(S)$ is relatively weakly compact, there is a renamed subsequence $\{Bu_n\}$ weakly converging to an element w. This fact, together with hypothesis (4) gives that

$$v_n = (\frac{I-C}{A})^{-1}Bu_n \rightharpoonup (\frac{I-C}{A})^{-1}w.$$

We infer that $(\frac{I-C}{A})^{-1}B$ is sequentially relatively weakly compact. An application of the Eberlein–Šmulian theorem [9] yields that $(\frac{I-C}{A})^{-1}B(S)$ is relatively weakly compact, which gives the result by Theorem 2.5. □

5. Application

Let $(X, \|.\|)$ be a Banach algebra. Let $J = [0,1]$ the closed and bounded interval in $\mathbb{R}$, the set of all real numbers. Let $E = C(J, X)$ the Banach algebra of all continuous functions from $[0,1]$ to X, endowed with the sup-norm $\|.\|_\infty$, defined by $\|f\|_\infty = \sup\{\|f(t)\|; t \in [0,1]\}$, for each $f \in C(J, X)$. We consider the nonlinear mixed both quadratic and linearly perturbed functional integral equation:

$$x(t) = (L_1 x)(t)\left[\left(q(t) + \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s))ds\right).u\right] + (L_2 x)(t), \quad 0 < \lambda < 1, \tag{16}$$

for all $t \in J$, where $u \neq 0$ is a fixed vector of X and the functions L_1, q, σ, p, L_2, are given, while $x = x(t)$ is an unknown function. We shall obtain the solution of (16) under some suitable conditions on the functions involved in (16). Suppose that the functions q, σ, p and the operators L_1 and L_2 verify the following conditions:

($\mathcal{H}_1$) $L_2 : C(J, X) \rightarrow C(J, X)$ is $D-$Lipschitzian with a $D-$function φ_{L_2} with $\|L_2 x\|_\infty < 1$.

($\mathcal{H}_2$) $\sigma : J \rightarrow J$ is a continuous and nondecreasing function.

($\mathcal{H}_3$) $q : J \rightarrow \mathbb{R}$ is a continuous function.

($\mathcal{H}_4$) The operator $L_1 : C(J, X) \rightarrow C(J, X)$ is such that
  (a) L_1 is $D-$Lipschitzian with a $D-$function φ_{L_1},
  (b) L_1 is regular on $C(J, X)$,
  (c) $(\frac{I}{L_1})^{-1}$ is well defined on $C(J, X)$,
  (d) $(\frac{I}{L_1})^{-1}$ is sequentially weakly continuous on $C(J, X)$.

($\mathcal{H}_5$) The function $p : J \times J \times X \times X \rightarrow \mathbb{R}$ is continuous such that for arbitrary fixed $s \in J$ and $x, y \in X$, the partial function $t \rightarrow p(t, s, x, y)$ is continuous uniformly for $(s, x, y) \in J \times X \times X$.

($\mathcal{H}_6$) There exists $r_0 > 0$ such that
  (a) $|p(t, s, x, y)| \leq r_0 - \|q\|_\infty$ for each $t, s \in J; x, y \in X$ such that $\|x\| \leq r_0$ and $\|y\| \leq r_0$,
  (b) $\|L_1 x\|_\infty \leq (1 - \frac{\|L_2 x\|_\infty}{r_0}) \frac{1}{\|u\|}$ for each $x \in C(J, X)$,
  (c) for all $\varepsilon > 0$, there exists $\delta > 0$ such that $r_0 \|u\| \varphi_{L_1}(r) + \varphi_{L_2}(r) < \varepsilon$ for all $r \in [\varepsilon, \varepsilon + \delta[$.

Theorem 5.1. *Under assumptions $(H_1) - (H_6)$, Eq. (16) has at least one solution $x = x(t)$ which belongs to the space $C(J, X)$.*

Proof. Let us define the subset S of $C(J, X)$ by

$$S = \{x \in C(J, X); \|x\|_\infty \leq r_0\}.$$

Obviously S is nonempty, convex and closed. Let us consider three operators A, B and C defined on $C(J, X)$ by

$$(Ax)(t) = (L_1 x)(t),$$

$$(Bx)(t) = \left(q(t) + \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s))\right).u, \quad 0 < \lambda < 1,$$

$$(Cx)(t) = (L_2 x)(t).$$

We shall prove that the operators A, B and C satisfy all the conditions of Theorem 4.5.

(i) From assumption $(\mathcal{H}_1)$ and $(\mathcal{H}_4)(a)$, A and C are $D-$Lipschitzian with $D-$functions φ_A and φ_C.

(ii) From assumption $(\mathcal{H}_4)(b)$, it follows that A is regular on $C(J, X)$.

(iii) Now, we show that B is sequentially weakly continuous on S. Firstly, we verify that if $x \in S$, then $Bx \in C(J, X)$. Let $\{t_n\}$ be any sequence in J converging to a point t in J. Then

$$\|(Bx)(t_n) - (Bx)(t)\| \leq \left\| \int_0^{\sigma(t_n)} p(t_n, s, x(s), x(\lambda s))ds - \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s))ds \right\|.\|u\|$$

$$\leq \left[\int_0^{\sigma(t_n)} |p(t_n, s, x(s), x(\lambda s)) - p(t, s, x(s), x(\lambda s))|ds \right] \|u\|$$

$$+ \left[\int_{\sigma(t_n)}^{\sigma(t)} |p(t, s, x(s), x(\lambda s))|ds \right] \|u\|$$

$$\leq \left[\int_0^1 |p(t_n, s, x(s), x(\lambda s)) - p(t, s, x(s), x(\lambda s))|ds \right] \|u\|$$

$$+ (r_0 - \|q\|_\infty)|\sigma(t_n) - \sigma(t)|\|u\|.$$

Since $t_n \to t$, so, $(t_n, s, x(s), x(\lambda s)) \to (t, s, x(s), x(\lambda s))$, for all $s \in J$. Taking into account the hypothesis $(\mathcal{H}_5)$, we obtain

$$p(t_n, s, x(s), x(\lambda s)) \to p(t, s, x(s), x(\lambda s))$$

in $\mathbb{R}$. Moreover, the use of assumption $(\mathcal{H}_6)$ leads to

$$|p(t_n, s, x(s), x(\lambda s)) - p(t, s, x(s), x(\lambda s))| \leq 2(r_0 - \|q\|_\infty)$$

for all $t, s \in J, \lambda \in (0, 1)$. Consider

$$\begin{cases} \varphi : J \to \mathbb{R} \\ \quad s \to \varphi(s) = 2(r_0 - \|q\|_\infty) \end{cases}$$

Clearly $\varphi \in L^1(J)$. Therefore, from the dominated convergence theorem and assumption $(\mathcal{H}_2)$, we obtain $(Bx)(t_n) \to (Bx)(t)$ in X. It follows that $Bx \in C(J, X)$. Next, we prove B is sequentially weakly continuous on S. Let $\{x_n\}$ be any sequence in S such that $x_n \rightharpoonup x \in S$. So, from assumptions $(\mathcal{H}_5) - (\mathcal{H}_6)$ and the dominated convergence theorem, we get

$$\lim_{n \to \infty} \int_0^1 p(t_n, s, x(s), x(\lambda s)) = \int_0^1 p(t, s, x(s), x(\lambda s)).$$

Which implies that $\lim_{n \to \infty}(Bx_n)(t) = (Bx)(t)$ in X. Since $(Bx_n)_n$ is bounded by $r_0\|u\|$, then by Theorem , we obtain that $Bx_n \rightharpoonup Bx$. We conclude that B is sequentially weakly continuous on S. We show that $B(S)$ is relatively weakly compact. By definition,

$$B(S) = \{B(x), \|x\|_\infty \leq r_0\}$$

For all $t \in J$, we have $B(S)(t) = \{B(x)(t), \|x\|_\infty \leq r_0\}$. We need now to show that $B(S)(t)$ is sequentially weakly relatively compact in X. To see this, let $\{x_n\}$ be any sequence in S, we have $(Bx_n)(t) = r_n(t).u$, where $r_n(t) = q(t) + \int_0^1 p(t, s, x_n(s), x_n(\lambda s))ds$. Since $\{r_n(t)\}$ is a real and bounded sequence, then there is a renamed subsequence such that $r_n(t) \to r(t)$ in $\mathbb{R}$, and, consequently $(Bx_n)(t) \to (q(t) + r(t)).u$ in X. We conclude that $B(S)(t)$ is sequentially relatively compact in X, then $B(S)(t)$ is sequentially relatively weakly compact in X. We prove now that $B(S)$ is weakly equicontinuous on J. If we take $\varepsilon > 0, x \in S, x^* \in X^*, t, t' \in J$ such that $t \leq t'$ and $t' - t \leq \varepsilon$. Then

$$|x^*((Bx)(t) - (Bx)(t'))| \leq [\omega(p, \varepsilon) + (r_0 - \|q\|_\infty)\omega(\sigma, \varepsilon)]\|x^*(u)\|,$$

where

$$\omega(p, \varepsilon) = \sup\{|p(t, s, x, y) - p(t', s, x, y)|; t, t', s \in J; |t - t'| \leq \varepsilon; x, y \in \mathcal{B}_{r_0}\}$$

$$\omega(\sigma, \varepsilon) = \sup\{|\sigma(t) - \sigma(t')|; t, t' \in J; |t - t'| \leq \varepsilon\}.$$

Taking into account the hypothesis $(\mathcal{H}_5)$ and in view of the uniform continuity of the function σ on the set J, it follows that $\omega(p, \varepsilon) \to 0$ and $\omega(\sigma, \varepsilon) \to 0$ as $\varepsilon \to 0$. An application of the Arzelà–Ascoli theorem [17], we conclude that $B(S)$ is sequentially weakly relatively compact in X. Again an application of the result of Eberlein–Šmulian theorem [10] yields that $B(S)$ is relatively weakly compact.

(*iv*) It is clear that hypothesis $\mathcal{H}_1, \mathcal{H}_4(c)$ and $\mathcal{H}_4(d)$ imply that $(\frac{I-C}{A})^{-1}$ is sequentially weakly continuous on $B(S)$.

(*v*) Finally, we need to prove the hypotheses (6) of Theorem 4.5. To see this, let $x \in C(J, X)$ and $y \in S$ such that $x = AxBy + Cx$, or, equivalently for all $t \in J$,

$$x(t) = L_2 x(t) + (L_1 x)(t)(By)(t).$$

But, for all $t \in J$, we have

$$\begin{aligned} \|x(t)\| &\leq \|x(t) - L_2 x(t)\| + \|L_2 x(t)\| \\ &\leq \|(L_1 x)(t)(By)(t)\| + \|L_2 x(t)\| \\ &\leq \|(L_1 x)\|_\infty r_0\|u\| + \|L_2 x\|_\infty \\ &\leq r_0. \end{aligned}$$

From the last inequality and taking the supremum over t, we obtain $\|x\|_\infty \leq r_0$, and, consequently $x \in S$. We conclude that the operators A, B and C satisfy all the requirements of Theorem 4.5. Thus, an application of it yields that equation (16) has a solution in the space $C(J, X)$. $\square$

Example 5.2. *Consider the Banach algebra $E = C([0, 1], \mathbb{R})$ of all continuous real-valued on $J = [0, 1]$ with norm $\|x\|_\infty = \sup_{t \in [0,1]} |x(t)|$. In this case $X = \mathbb{R}$. We consider the following nonlinear integral equation*

$$x(t) = f(t, x(t)) + \left(q(t) + \int_0^t p(t, s, x(s), x(\lambda s)) ds \right), \quad t \in J, \tag{17}$$

with $f(t, x)$ is given by

$$f(t, x) = \begin{cases} \ln(1 + \frac{x}{4}), & \text{if } -3 \leq x \leq 3 \\ \ln \frac{7}{4}, & \text{otherwise} \end{cases}$$

To show that equation (17) has a solution in E, we will verify that all conditions of Theorem 5.1 are satisfied. If we compare equation (16) with equation (17), we obtain $a = 1, u = 1, L_1 = Id, L_2(t, x(t)) = f(t, x(t))$ and $\sigma(t) = t$. Let $x, y \in \mathbb{R}_+$, then by definition of the function f, we obtain

$$|f(t, x) - f(t, y)| = |ln(1 + \frac{x}{4}) - ln(1 + \frac{y}{4})|$$

$$= ln\left(\frac{1 + \frac{x}{4}}{1 + \frac{y}{4}} \right)$$

$$= ln\left(1 + \frac{x - y}{4 + y} \right)$$

$$\leq ln\left(1 + \frac{|x - y|}{4 + y} \right)$$

$$\leq \varphi(|x - y|)$$

for all $t \in [0, 1]$, where $\varphi(r) = ln(1 + r)$ is a $\mathcal{DL}$−function on $\mathbb{R}_+$. If now $x, y \in \mathbb{R} \setminus (-3, 3)$, then also we have

$$|f(t, x) - f(t, y)| = 0 \leq \varphi(|x - y|)$$

for all $t \in [0, 1]$, where $\varphi(r) = ln(1 + r)$ and that $\varphi \in \mathcal{DL}$. So, all the conditions of Theorem 5.1 are satisfied. Consequently, the equation (17) has a solution defined on $[0, 1]$.

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