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Fixed Point Results for Hybrid Contractions in Menger Metric Spaces With Application to Integral Equations

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Abstract. Here the notion of $\alpha - H\theta-$ contraction has been proposed to construct some fixed point results of single-valued and multivalued mappings in Menger PM spaces. In addition, an existence result to an integral equation is concerned to justify the obtained results.

1. Introduction

In 1942, to qualify the space between two points, the idea of probabilistic metric space was proposed by Menger [8]. By applying this notion, it turns out in a metric space distribution functions can be seen in lieu of positive real numbers. Indeed, the idea of probabilistic metric space is applied in states which the distance between two points is not specified, but the probabel distance between two points is determined. After introducing the notion of probabilistic metric space, to study Menger's line of research, Sehgal and Bharucha [14] studied the probabel version of the classical Banach contraction principle. Since then, the theory of fixed point theory has been studied in probabilistic metric spaces by many authors to obtain theoridical results for different types of contraction mappings (see for example [2, 4, 5, 10, 11, 13–18]). Parvaneh et.al.[9] studied the notion of $\alpha - \eta - H\Theta-$ contraction to get some fixed point results. They assumed that the symbole Δ_Θ be the collection of all functions $h : (0, \infty) \longrightarrow [1, \infty)$ in which the following conditions hold true:

1° h is strictly increasing;

2° For all sequence $\{\alpha_n\} \subseteq (0, \infty)$,

$$\lim_{n\longrightarrow\infty} \gamma_n = 0 \Longleftrightarrow \lim_{n\longrightarrow\infty} h(\gamma_n) = 1;$$

3° There exist $0 < r < 1$ and $l \in (0, \infty)$ so that

$$\lim_{n\longrightarrow\infty} \frac{h(t) - 1}{t^r} = l;$$

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Also, the collection of all functions $D : {\mathbb{R}_+}^4 \longrightarrow \mathbb{R}_+$ which satisfies condition G(see [9]) was indicated by the symbol Δ_D. Next, they introduced the notion of $\alpha - \eta - H\Theta-$ contraction as follows:

Definition 1.1. *Assume that a self-mapping T has been defined on the metric space (X, d). Given two functions $\alpha, \eta : X \times X \longrightarrow [0, \infty)$, The mapping T is called an $\alpha - \eta - H\Theta-$ contraction if*

$$h(d(Tx, Ty)) \leq [h(x, y)]^{D\left(d(x,Tx), d(y,Ty),, d(x,Ty), d(y,Tx)\right)}$$

for all $x, y \in X$ with $\eta(x, Tx) \leq \alpha(x, y)$ and $d(Tx, Ty) > 0$ where $h \in \Delta_\Theta$ and $D \in \Delta_D$.

Now, we recall the following theorem from [9].

Theorem 1.2. *Assume that a self-mapping $T : X \longrightarrow X$ has been defined on the complete metric space X. Then a fixed point of the mapping T is obtained, if the following conditions hold true :*

1° *T is an $\alpha-$admissible mapping with respect to η;*

2° *T is an $\alpha - \eta - H\Theta-$contraction;*

3° *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;*

4° *T is $\alpha - \eta-$continuous.*

Moreover, if Given $x, y \in Fix(T)$ we have $\alpha(x, y) \geq \eta(x, x)$, then a unique fixed point of the mapping T is obtained .

Now, the following definition of [9] is given which will be applied in our consideration.

Definition 1.3. *A self mapping T on a metric space X is called orbitally continuous at $z \in X$ if*

$$\lim_{n \longrightarrow \infty} T^n x = z \Longrightarrow \lim_{n \longrightarrow \infty} TT^n x = Tz,$$

where $\{x_n\} \subseteq O(w)$ for some $w \in X$. Besides, if T be orbitally continuous for all $z \in X$, then T will be orbitally continuous on X.

Remark 1.4. *[7] Assume that a self-mapping $T : X \longrightarrow X$ has been defined on an orbitally $T-$complete metric space X. Moreover, suppose $\alpha : X \times X \longrightarrow [0, \infty)$ is defined by*

$$\alpha(x, y, t) = \begin{cases} 3 & \text{if } \quad x, y \in O(w), \\ 0 & \text{otherwise} \end{cases}$$

where $O(w)$ indicates an orbit of a point $w \in X$. Besides, T is an $\alpha - \eta-$continuous, when T is an orbitally continuous map on (X, d).

In this paper, as a primary goal, probabilistic versions of Theorem 1.2 will be considered. For this aim, first some new types of probabilistic versions are given in Menger PM spaces. Section 3 is devoted to consider some fixed point results of single-valued and orbitally continuous mappings in Menger PM spaces and the results are constructed in cases where the mentioned spaces are complete and partially orderd. Besides, in section 4, we study an existence result of an integral equation on a Banach space to illustrate the theoridical results.
Now some subsidiary facts which are concerned with discussion are presented.
In this paper the intervals $(-\infty, +\infty)$ and $[0, \infty)$ will be indicated by $\mathbb{R}$ and $\mathbb{R}_+$ respectively.
A mapping $G : \mathbb{R} \cup \{-\infty, +\infty\} \longrightarrow [0, 1]$ is named distribution function if G satisfies the following conditions:

1° G is nondecreasing;

2° Left continuous on $\mathbb{R}$;

3° $G(0) = 0$ and $G(+\infty) = 1$.

Let the collection of all probability distribution functions are denoted by the symbole Δ_+. Besides, the subset $E_+ \subseteq \Delta_+$ is defined by $E_+ = \{G \in \Delta_+; l^- G(+\infty) = 1\}$ where $l^- G(x) = \lim_{t \longrightarrow x_-} G(t)$. A maximal element for E_+ is given by

$$L(t) = \begin{cases} 0 & \text{if} \quad t \leq 0, \\ 1 & t > 0 \end{cases}$$

Definition 1.5. *[6] If $\Lambda : [0,1] \times [0,1] \longrightarrow [0,1]$ is the function with following conditions:*

1° *Λ is commutative and associative;*

2° *Λ is continuous;*

3° *$\Lambda(a, 1) = a$, for all $a \in [0,1]$;*

4° *$\Lambda(a, b) \leq \Lambda(c, d)$, whenever $a \leq c$ and $b \leq d$,*

Then Λ is called $t-$norm.
For example $\Lambda(a, b) = ab$ and $\Lambda_m(a, b) = \min\{a, b\}$ are product $t-$norm and min $t-$norm, respectively.

Definition 1.6. *[6, 13] Assume that X is a nonempty set, T is a continuous $t-$norm, and $D : X \times X \longrightarrow E_+$ be a mapping with the following conditions:*

1° $D_{x,y}(t) = L(t) \Longleftrightarrow x = y \quad t > 0;$

2° $D_{x,y}(t) = D_{y,x}(t)$ *for all* $x, y \in X, t > 0;$

3° $D_{x,y}(t + s) \geq T(D_{x,z}(t), D_{z,y}(s)), \qquad x, y, z \in X, \qquad s, t \geq 0,.$

Then, the triple (X, D, T) is named Menger PM space.

Definition 1.7. *[6, 13] Let (X, D, Λ) be a Menger PM space,*

1° *If $\lim_{n \longrightarrow \infty} D_{x_n,x}(t) = 1, \quad t > 0$, then $\{x_n\}$ will be called convergent to x in X.*

2° *If $\lim_{n,m \longrightarrow \infty} D_{x_n,x_m}(t) = 1, \quad t > 0$, then $\{x_n\}$ will be called cauchy sequence in X.*

3° *If every cauchy sequence of points in X has a limit that is also in X, then a Menger PM space (X, D, T) is called complete.*

From [13] we know that in the space (X, D, Λ) the collection of neighborhoods

$$\left\{ w_p(\varepsilon, \lambda); \quad p \in X, \lambda, \varepsilon > 0 \right\},$$

where

$$w_p(\varepsilon, \lambda) = \left\{ x \in X : D_{x,p}(\varepsilon) > 1 - \lambda \right\},$$

induces the topology τ on X. Assume that in the space (S, D, Λ) the symbol $CB(S)$ indicates the collection of all subsets of S which are closed in the topology τ. Moreover, the functions $F_{x,A}$ and $F_{A,B}$ are defined by:

$$D_{x,A}(t) = \sup_{y \in A} D_{x,y}(t), \quad t \geq 0,$$

and

$$\overline{D}_{A,B}(t) = \sup_{s<t} \Lambda(\inf_{x \in A} \sup_{y \in B} D_{x,y}(s), \inf_{y \in B} \sup_{x \in A} D_{x,y}(s)), \quad t \geq 0,$$

where $x \in S$ and $A, B \in CB(S)$.

Lemma 1.8. *[3] If in the space (S, D, Λ) , $A \in CB(S)$ and Λ be continuous t-norm, then the following conditions hold true:*

1° $D_{x,A}(t) = 1,\ t > 0$ *if and only if* $x \in A$;

2° $D_{x,A}(t_1 + t_2) \geq \Lambda(D_{x,y}(t_1), D_{y,A}(t_2)),\quad t_1, t_2 \geq 0$;

3° *For any* $A, B \in CB(S)$ *and* $x \in A$, $D_{x,B}(t) \geq \overline{D}_{A,B}(t)$, *for all* $t \geq 0$.

Definition 1.9. *[12] Given two functions* $\gamma, \beta : S \times S \times (0, \infty) \longrightarrow \mathbb{R}_+$, *the mulivalued mapping* $T : S \longrightarrow 2^S$ *is called an* $\gamma-$*admissible mapping with respect to* β *if*

$$\forall x \in S,\ \ \forall y \in Tx \quad \gamma(x, y, t) \leq \beta(x, y, t) \longrightarrow \gamma(y, z, t) \leq \beta(y, z, t)\ \ \forall z \in Ty,\quad t > 0.$$

2. On some Fixed point ideas in Menger PM- spaces.

In this section, first applying ideas and definitions presented in [12], we propose some notions in Menger PM spaces. Then, we inaugurate our main results in the mentioned spaces where the mentioned spaces are complete and partially ordered.
Now let Δ_h be the set of all functions $h : [0, 1] \longrightarrow (0, 1]$ provided that:

1° h is strictly increasing;

2° For all sequence $\{\alpha_n\} \subset [0, 1]$,

$$\lim_{n \longrightarrow \infty} \alpha_n = 1 \Longleftrightarrow \lim_{n \longrightarrow \infty} h(\alpha_n) = 1.$$

Besides, inspired from [9], the symbol $\overline{\Delta}_M$ is applied to present the set of all functions $M : [0, 1]^4 \longrightarrow \mathbb{R}_+$ so that if $\max\{t_1, t_2, t_3, t_4\} = 1$, then $M(t_1, t_2, t_3, t_4) = k$ where $k \in [0, 1)$.

Example 2.1. $M(l_1, l_2, l_3, l_4) = L \ln(\max\{l_1, l_2, l_3, l_4\}) + k$ *in which* $L \in \mathbb{R}_+$ *and* $k \in [0, 1)$.

Example 2.2. $M(l_1, l_2, l_3, l_4) = L(1 - \max\{l_1, l_2, l_3, l_4\}) + k$ *in which* $L \in \mathbb{R}_+$ *and* $k \in [0, 1)$.

Definition 2.3. *In a Menger PM space* (S, D, Λ), *a mapping* $T : S \longrightarrow 2^S$ *has the generalized approximate valued property, if for all* $a \in S$ *and* $x \in Ta$ *there exists* $y \in Tx$ *satisfying*

$$\overline{D}_{Ta,Tx}(t) - D_{x,y}(t) \leq \varphi(D_{x,y}(t), D_{x,Tx}(t)),\quad t > 0,$$

where $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ *with* $\varphi(s, t) = \frac{s}{2} - t$.

Definition 2.4. *In a Menger PM spac* (S, D, Λ), *a mapping* $T : S \longrightarrow 2^S$ *has the* $w-$ *generalized approximate valued property, if for all* $a \in S$ *and* $x \in Ta$ *there exists* $y \in Tx$ *satisfying*

$$\overline{D}_{Ta,T^2a}(t) - D_{x,y}(t) \leq \varphi(D_{x,y}(t), D_{x,Tx}(t)),\quad t > 0,$$

where $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ *with* $\varphi(s, t) = \frac{s}{2} - t$.

Definition 2.5. *Given the space* (S, D, Λ) *and the function* $\gamma : S \times S \times (0, \infty) \longrightarrow \mathbb{R}_+$, *a mulivalued mapping* $T : S \longrightarrow 2^S$ *is called an* $\gamma - Mh-$*contraction, if*

$$h(\overline{D}_{Tx,Ty}(t)) \geq \Big[h(D_{x,y}(t))\Big]^{M\left(D_{x,Tx}(t), D_{y,Ty}(t), D_{x,Ty}(t), D_{y,Tx}(t)\right)},\quad t > 0.$$

for all $x, y \in S$ *with* $\gamma(x, y, t) \leq 1$ *where* $M \in \overline{\Delta}_M$ *and* $h \in \Delta_h$.

Definition 2.6. *Let (S, D, Λ) be a Menger PM space and also $T : S \longrightarrow 2^S$ be a multivalued mapping. Given the function $\alpha : S \times S \times (0,\infty) \longrightarrow \mathbb{R}_+$, if for each sequence $\{x_n\}$ in S with $\alpha(x_n, x_{n+1}, t) \leq 1$ we have*

$$\lim_{n\longrightarrow\infty} \overline{F}_{Tx_n,Tx}(t) = 1, \quad t > 0,$$

then T is called continuous.

Now, due to [1], the following definition is presented which is more general compared to the Definition 1.9.

Definition 2.7. *Assume that the multivalued mapping $T : S \longrightarrow 2^S$ has been defined on the set S provided that the space (S, D, Λ) is a Menger PM space. Given the function $\gamma : S \times S \times (0,\infty) \longrightarrow \mathbb{R}_+$, if*

$$\forall x \in S, \ \ \forall y \in Tx \quad \gamma(x, y, t) \leq 1 \longrightarrow \gamma(y, z, t) \leq 1 \ \ \forall z \in Ty, \quad t > 0,$$

then, T is called $\gamma-$admissible.

Now, we state the first result as follows.

Theorem 2.8. *Suppose that the mapping $T : S \longrightarrow CB(S)$ has been defined on the set S provided that the space (S, D, Λ) is a Menger PM space and the mapping T has the generalized approximate valued property. Moreover, assume that the following conditions hold true:*

1° *T is an γ-admissible mapping;*

2° *T is an $\gamma - Mh$-contraction;*

3° *For some $x_0 \in S$ there exists $x_1 \in Tx_0$ so that $\gamma(x_0, x_1, t) \leq 1$ for all $t > 0$;*

4° *T is continuous.*

Then, a fixed point of the mapping T is obtained , in other words, there exists $x \in S$ so that $x \in Tx$. Moreover, if for all $x, y \in Fix(T)$ the condition $\gamma(x, y, t) \leq 1, \ \ t > 0$ is satisfied, then unique fixed point is obtained of the mapping T.

Proof. Assume that $x_0 \neq x_1$, otherwise we obtain the conclusion. Now, the generalized approximate valued property of T implies that there exists $x_2 \in Tx_1$ so that $\overline{D}_{Tx_0,Tx_1}(t) - D_{x_1,x_2}(t) \leq \varphi(D_{x_1,x_2}(t), D_{x_1,Tx_1}(t))$. Hence,

$$\begin{aligned}\overline{D}_{Tx_0,Tx_1}(t) - D_{x_1,x_2}(t) &\leq \frac{1}{2}D_{x_1,x_2}(t) - D_{x_1,Tx_1}(t)\\ &< D_{x_1,x_2}(t) - D_{x_1,x_2}(t) = 0, \quad t > 0.\end{aligned}$$

For $x_2 \in Tx_1$, applying γ-admissible property of T, we deduce that $\gamma(x_1, x_2, t) \leq 1$ for all $t > 0$. If $x_1 \in Tx_1$, then the proof is completed. Let $x_2 \neq x_1$. Again, according to the assumptions, there exists $x_3 \in Tx_2$ so that

$$\begin{aligned}&\overline{D}_{Tx_1,Tx_2}(t) - F_{x_2,x_3}(t)\\ &\leq \varphi(D_{x_2,x_3}(t), D_{x_2,Tx_2}(t))\\ &\leq \frac{1}{2}D_{x_2,x_3}(t) - D_{x_2,Tx_2}(t) < D_{x_2,x_3}(t) - D_{x_2,x_3}(t) = 0, \quad t > 0.\end{aligned}$$

Accordingly, $\overline{D}_{Tx_1,Tx_2}(t) \leq D_{x_2,x_3}(t)$. Besides, $\gamma(x_2, x_3, t) \leq 1 \quad t > 0$. By continuing this method, we construct the sequence $\{x_n\}$ in S so that for all $t > 0$ we get $\overline{D}_{Tx_{n-1},Tx_n}(t) \leq D_{x_n,x_{n+1}}(t)$ and $\gamma(x_n, x_{n+1}, t) \leq 1$. Now, since T

is an $\gamma - Mh$-cnotraction, for all $t > 0$ we derive that

$$\begin{aligned} & h(D_{x_n,x_{n+1}}(t)) \\ & \geq h(\overline{D}_{Tx_{n-1},Tx_n}(t)) \\ & \geq \Big[h(D_{x_{n-1},x_n}(t))\Big]^{M\Big(D_{x_n,Tx_n}(t),D_{x_{n-1},Tx_{n-1}}(t),D_{x_n,Tx_{n-1}}(t),D_{x_{n-1},Tx_n}(t)\Big)} \\ & = \Big[h(D_{x_{n-1},x_n}(t))\Big]^{M\Big(D_{x_n,Tx_n}(t),D_{x_{n-1},Tx_{n-1}}(t),D_{x_{n-1},Tx_n}(t),1\Big)}. \end{aligned} \tag{1}$$

Given that $\max\Big\{D_{x_n,Tx_n}(t), D_{x_{n-1},Tx_{n-1}}(t), D_{x_{n-1},Tx_n}(t), 1\Big\} = 1$ and $M \in \overline{\Delta}_M$, there exists $k \in [0,1)$ such that

$$M\Big(D_{x_n,Tx_n}(t), D_{x_{n-1},Tx_{n-1}}(t), D_{x_{n-1},Tx_n}(t), 1\Big) = k, \quad t > 0.$$

From (1), we have

$$\Big[h(D_{x_{n-1},x_n}(t))\Big]^k \leq h(D_{x_n,x_{n+1}}(t)) \leq 1, \quad t > 0,$$

that is,

$$\Big[h(D_{x_0,x_1}(t))\Big]^{k^n} \leq h(D_{x_n,x_{n+1}}(t)) \leq 1, \quad t > 0.$$

If n tends to infinitely in the obtained inequality, we deduce that $h((D_{x_n,x_{n+1}}(t)) \longrightarrow 1, \quad t > 0$. Inconsequence, by the definition of h, $\lim_{n\longrightarrow\infty} D_{x_n,x_{n+1}}(t) = 1$. Now, it is shown that $\{x_n\}$ is a cauchy sequence in S. For any $t > 0$ we have

$$\begin{aligned} D_{x_n,x_{n+p}}(t) &\geq \Delta\Big(D_{x_n,x_{n+1}}(\frac{t}{p}), D_{x_{n+1},x_{n+p}}(\frac{(p-1)t}{p})\Big) \\ &\geq \Delta\Big(D_{x_n,x_{n+1}}(\frac{t}{p}), \Delta(D_{x_{n+1},x_{n+2}}(\frac{t}{p}), ..., D_{x_{n+p-1},x_{n+p}}(\frac{t}{p})), ...\Big). \end{aligned}$$

If n tends to infinitely, we get $\lim_{n\longrightarrow\infty} D_{x_n,x_{n+p}}(t) = 1, \quad t > 0$. So the cauchy feature of the sequence $\{x_n\}$ is ontained. Since (S, D, Λ) is complete, for a point $x \in X$ we have $x_n \longrightarrow x$. Now, due to the continuity property of T, for any $t > 0$ we conclude that $\lim_{n\longrightarrow\infty} \overline{D}_{Tx_n,Tx}(t) = 1$. Consequently,

$$\begin{aligned} D_{x,Tx}(t) &\geq \Lambda(D_{x,x_{n+1}}(\frac{t}{2}), D_{x_{n+1},Tx}(\frac{t}{2})) \\ &\geq \Lambda(D_{x,x_{n+1}}(\frac{t}{2}), \overline{D}_{Tx_n,Tx}(\frac{t}{2})) \longrightarrow \Lambda(1,1) = 1. \end{aligned}$$

Hence, $D_{x,Tx}(t) = 1, \ t > 0$ which implies that $x \in Tx$. Assume $x \neq y \in Fix(T)$ in which $\gamma(x, y, t) \leq 1, \ t > 0$.

Hence using 2° we get,

$$h(\overline{D}_{Tx,Ty}(t)) \geq \Big[h(D_{x,y}(t))\Big]^{M\big(D_{x,Tx}(t),D_{y,Ty}(t),D_{x,Ty}(t),D_{y,Tx}(t)\big)}$$
$$= \Big[h(D_{x,y}(t))\Big]^{M\big(1,1,D_{x,Ty}(t),D_{y,Tx}(t)\big)}, \quad t > 0.$$

So, $\Big[h(D_{x,y}(t))\Big]^{k^n} \leq h(D_{x,y}(t)) \leq 1$ for all $t > 0$. If $n \longrightarrow \infty$ in the obtained estimate, we get, $h(D_{x,y}(t)) = 1$. Thus because of condition 2° of h, $x = y$ and the conclusion is followed. □

Now, in view of Example 2.1 and Theorem 2.8 we earn the following result.

Corollary 2.9. *Suppose that $T : S \longrightarrow CB(S)$ be a multivalued mapping on the set S provided that (S, D, Λ) is a complete Menger PM space in which Λ is a continuous t-norm. Besides, assume that T has the generalized approximate valued property and the following conditions hold true:*

1° *T is an γ-admissible mapping;*

2° *Given $x, y \in S$ and $t > 0$ with $\gamma(x, y, t) \leq 1$, we have*

$$h(\overline{D}_{Tx,Ty}(t)) \geq \Big[h(D_{x,y}(t))\Big]^{L\ln\Big(\max\Big\{D_{x,Tx}(t),D_{y,Ty}(t),D_{x,Ty}(t),D_{y,Tx}(t)\Big\}\Big)+k};$$

where $k \in [0, 1)$.

3° *For some $x_0 \in S$ we get $x_1 \in Tx_0$ so that $\gamma(x_0, x_1, t) \leq 1 \quad t > 0$;*

4° *T is continuous.*

Then, a fixed point is obtained for the mapping T, in other words, there exists $x \in S$ so that $x \in Tx$. Moreover, if Given $x, y \in Fix(T)$ the condition $\gamma(x, y, t) \leq 1, \quad t > 0$ is satisfied, then unique fixed point is obtained for the mapping T.

Theorem 2.10. *Suppose that the mapping $T : S \longrightarrow CB(S)$ has been defined on the set S provided that (S, D, Λ) is a complete Menger PM space in which Λ is a continuous t-norm. Moreover, assume that the mapping T has the generalized approximate valued property and the following conditions hold true:*

1° *T is an γ-admissible mapping;*

2° *T is an $\gamma - Mh$-contraction;*

3° *For some $x_0 \in S$ there exists $x_1 \in Tx_0$ such that $\gamma(x_0, x_1, t) \leq 1$ for all $t > 0$;*

4° *If $\{x_n\}$ be a sequence in S so that $\gamma(x_n, x_{n+1}, t) \leq 1, \quad n \in N$ and $x_n \longrightarrow x$, then a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is obtained provided that $\gamma(x_{n_k}, x, t) \leq 1, \ k \in N$.*

Then, a fixed point is obtained for the mapping T, in other words, for a point $x \in S$ we get $x \in Tx$. Moreover, if for all $x, y \in Fix(T)$ the condition $\gamma(x, y, t) \leq 1, \quad t > 0$ is satisfied, then unique fixed point is obtained for the mapping T.

Proof. Like proving Theorem 2.8, we construct the sequence $\{x_n\}$ in S so that $\gamma(x_n, x_{n+1}, t) \leq 1$ and $x_n \longrightarrow x$ for $x \in S$. Now due to 4° and given that T is $\gamma - Mh$-contraction, for all $t > 0$ we deduce that

$$h(\overline{D}_{Tx_{n_k},Tx}(t)) \geq (h(D_{x_{n_k},x}(t)))^{M\left(D_{x_{n_k},Tx_{n_k}}(t),D_{x,Tx}(t),D_{x_{n_k},Tx}(t),D_{x,Tx_{n_k}}(t)\right)}. \tag{2}$$

Further, in view of the proof of Theorem 2.8 for all $t > 0$ we have $D_{x_{n_k},Tx_{n_k}}(t) \geq D_{x_{n_k},x_{n_k+1}}(t)$ and $\lim_{k\longrightarrow\infty} D_{x_{n_k},x_{n_{k+1}}}(t) = 1$. Hence, $\lim_{k\longrightarrow\infty} D_{x_{n_k},Tx_{n_k}}(t) = 1, \ \ t > 0$. As M is continuous, we deduce that

$$\lim_{k\longrightarrow\infty} M(D_{x_{n_k},Tx_{n_k}}(t), D_{x,Tx}(t), D_{x_{n_k},Tx}(t), D_{x,Tx_{n_k}}(t)) = k < 1. \tag{3}$$

Due to (2) and (3), we conclude that $\lim_{k\longrightarrow\infty} h(D_{x_{n_k+1}}, T_x(t)) = 1, \ \ t > 0$. Accordingly,

$$\lim_{k\longrightarrow\infty} D_{x_{n_k+1}}, T_x(t) = 1, \ \ t > 0.$$

On the other hand,

$$D_{x,Tx}(t) \geq \Lambda\Big(D_{x,x_{n_{k+1}}}(t), D_{x_{n_{k+1}},Tx}(t)\Big) \longrightarrow \Lambda(1,1) = 1, \ \ t > 0.$$

Consequently, $D_{x,Tx}(t) = 1$ which implies $x \in Tx$. □

Due to Example 2.1 and Theorem 2.10, we get a result as follows.

Corollary 2.11. *Suppose that the mapping $T : S \longrightarrow CB(S)$ has been defined on the set S provided that (S, D, Λ) is a complete Menger PM space in which Λ is a continuous t-norm. Moreover, assume that the mapping T has the generalized approximate valued property and the following conditions hold true:*

1° *T is an γ-admissible mapping;*

2° *Given $x, y \in S$ with $\gamma(x, y, t) \leq 1$, the following inequality holds true:*

$$h(\overline{D}_{Tx,Ty}(t)) \geq \Big[h(D_{x,y}(t))\Big]^{L\ln\left(\max\left\{D_{x,Tx}(t),D_{y,Ty}(t),D_{x,Ty}(t),D_{y,Tx}(t)\right\}\right)+k}, \ \ t > 0.$$

3° *For some $x_0 \in S$ a point $x_1 \in Tx_0$ exists so that $\gamma(x_0, x_1, t) \leq 1, \ t > 0$;*

4° *If $\{x_n\}$ be a sequence in S so that $\gamma(x_n, x_{n+1}, t) \leq 1, \ \ n \in N$ and $x_n \longrightarrow x$, then a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is obtained provided that $\gamma(x_{n_k}, x, t) \leq 1, \ k \in N$.*

Then, a fixed point is obtained for the mapping T, in other words, for a point $x \in S$ we get $x \in Tx$. Moreover, if given $x, y \in Fix(T)$ the condition $\gamma(x, y, t) \leq 1, \ \ t > 0$ is satisfied, then unique fixed point is obtained for the mapping T.

Now, taking $L = 0$ in Corollary 2.11 the following corollary is followed.

Corollary 2.12. *Suppose that the mapping $T : S \longrightarrow CB(S)$ has been defined on the set S provided that (S, D, Λ) is a complete Menger PM space in which Λ is a continuous t-norm. Moreover, assume that the mapping T has the generalized approximate valued property and the following conditions hold true:*

1° *T is an γ-admissible mapping;*

2° *Given $x, y \in S$ with $\gamma(x, y, t) \leq 1$, the following inequality is satisfied:*

$$h(\overline{D}_{Tx,Ty}(t)) \geq \Big[h(D_{x,y}(t))\Big]^k; \ \ t > 0.$$

3° *For some $x_0 \in X$ there exists $x_1 \in Tx_0$ such that $\gamma(x_0, x_1, t) \leq 1,\ \ t > 0$;*

4° *If $\{x_n\}$ be a sequence in S so that $\gamma(x_n, x_{n+1}, t) \leq 1,\quad n \in N$ and $x_n \longrightarrow x$, then a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is obtained provided that $\gamma(x_{n_k}, x, t) \leq 1,\ \ k \in N$.*

Then, a fixed point is obtained of the mapping T, in other words, there exists $x \in S$ so that $x \in Tx$. Moreover, if given $x, y \in Fix(T)$ the condition $\gamma(x, y, t) \leq 1,\quad t > 0$ is satisfied, then unique fixed point is obtained of the mapping T.

Now, we present the following definition of [12] which will be applied in the next results.

Definition 2.13. *Suppose that the order relation $\preceq$ is defined on the set S. We say that $A \preceq B$ if given $a \in A$ and $b \in B$ we have $a \preceq b$.*

Theorem 2.14. *Suppose that the space $(S, D, \Lambda, \preceq)$ is a complete partially ordered Menger PM space in which Λ is a continuous t-norm. Moreover, assume that $T : S \longrightarrow CB(S)$ is a multivalued mapping which has the generalized approximate valued property and the following conditions hold true:*

1° *Given $x, y \in S$ with $x \preceq y$,*

$$h(\overline{D}_{Tx,Ty}(t)) \geq \left[h(D_{x,y}(t))\right]^{M\left(D_{x,Tx}(t), D_{y,Ty}(t), D_{x,Ty}(t), D_{y,Tx}(t)\right)},\quad t > 0;$$

2° *If $x \in S$ and $y \in Tx$ so that $x \preceq y$, then $\{y\} \preceq Ty$;*

3° *There exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$;*

4° *T is continuous or if $\{x_n\}$ is a sequence in S so that $x_n \preceq x_{n+1},\ \ n \in N$ and $x_n \longrightarrow x$, then we earn a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ so that $x_{n_k} \preceq x,\quad k \in N$.*

Then, a fixed point is obtained for the mapping T, in other words, there exists $x \in S$ such that $x \in Tx$. Moreover, T has a unique fixed point if given $x, y \in Fix(T)$ the relation $x \preceq y$ is satisfied.

Proof. Taking

$$\gamma(x, y, t) = \begin{cases} 1 & \text{if} \quad x \preceq y, \\ 0 & \text{otherwise} \end{cases}$$

in Theorems [2.8-2.10], we see that all conditions of Theorems [2.8-2.10] are satisfied. So from Theorems [2.8 -2.10] we conclude a fixed point of the mapping T. □

Theorem 2.15. *Suppose that the space (S, D, Λ) is a complete Menger PM space in which Λ is a continuous t-norm. Moreover, assume that $T : S \longrightarrow CB(S)$ is a multivalued mapping which has the w-generalized approximate valued property and the following conditions hold true:*

1° *T has the $\gamma-$admissible property;*

2° *Given $x, y \in X$ and $t > 0$ with $\gamma \leq 1$, the following inequality holds:*

$$h(\overline{D}_{T(x),T^2(x)}(t)) \geq \left[h(D_{x,Tx}(t))\right]^k; \tag{4}$$

where $k \in [0, 1)$ and $h \in \Delta_h$.

3° *For some $x_0 \in S$ a point $x_1 \in Tx_0$ exits so that $\gamma(x_0, x_1, t) \leq 1,\ \ t > 0$;*

4° *T is continuous;*

5° *Given* $t > 0$ *and* $x \in Fix(T^n)$ *we have* $\gamma(x, Tx, t) \leq 1$.

Then, $Fix(T) = Fix(T^n)$.

Proof. Like proving Theorem 2.8, we make a sequence $\{x_n\}$ in S so that $\gamma(x_n, x_{n+1}, t) \leq 1$ and $\overline{D}_{Tx_n,T^2x_n(t)} \leq D_{x_n,x_{n+1}}(t),\ \ t > 0$. Hence, from (4) we earn

$$1 \geq h(D_{x_n,x_{n+1}}(t)) \geq h(\overline{D}_{Tx_n,T^2_{x_n}}(t)) \geq \Big[h(D_{x_n,Tx_n}(t))\Big]^k$$
$$\geq \Big[h(D_{x_n,x_{n+1}}(t))\Big]^k \geq \cdots \geq \Big[h(D_{x_0,x_1}(t))\Big]^{k^n},\ \ t > 0.$$

If n tends to infinitely in the above estimate, for any $t > 0$ we obtain that

$$\lim_{k\longrightarrow\infty} h(D_{x_n,x_{n+1}}(t)) = 1.$$

Accordingly, $\lim_{n\longrightarrow\infty} F_{x_n,x_{n+1}}(t) = 1,\ \ t > 0$. Now, like proving Theorem 2.8, a fixed point is obtained for the mapping T. To prove $Fix(T) = Fix(T^n)$, we may assume that $Fix(T^n) \neq Fix(T)$. Hence, a point $w \in Fix(T^n)$ is obtained so that w dose not belong to $Fix(T)$. Inconsequence, from (4) and 5° , we have

$$1 \geq h(D_{w,Tw}(t)) \geq h(\overline{D}_{T(T^{n-1}(w)),T^2(T^{n-1}(w)}(t))) \geq \Big[h(\overline{D}_{T^{n-1}(w),T^n(w)}(t))\Big]^k$$
$$\geq \Big[h(\overline{D}_{T^{n-2}(w),T^{n-1}(w)}(t))\Big]^{k^2} \geq \cdots \geq \Big[h(D_{w,Tw}(t))\Big]^{k^n},\ \ t > 0.$$

Due to the above estimate , we earn $h(D_{w,Tw}(t)) = 1,\ \ t > 0$. Hence $w \in Tw$. So, $Fix(T) = Fix(T^n)$. □

3. Some new types of mappings in PM spaces

Here, applying Theorem 2.8, some new results will be constructed.

Theorem 3.1. *Let* (S, D, Λ) *be a complete Menger PM space in which* Λ *is continuous t-norms. Moreover, assume that the self mapping* f *has been defined on the set* S *and following conditions hold true:*

1° *Given* $x, y \in S$ *with* $\gamma(x, y, t) \leq 1$, *we earn*

$$h(D_{fx,fy}(t)) \geq \Big[h(D_{x,y}(t))\Big]^{M\left(D_{x,Tx}(t),D_{y,Ty}(t),D_{x,Ty}(t),D_{y,Tx}(t)\right)},\ \ t > 0;$$

2° *If* $x \in S$ *and* $\gamma(x, fx, t) \leq 1$, *then* $\gamma(fx, f^2x, t) \leq 1,\ \ t > 0$;

3° *A point* $x_0 \in S$ *exists so that* $\gamma(x_0, f(x_0), t) \leq 1,\ \ t > 0$;

4° T *is continuous or if* $\{x_n\}$ *is a sequence in* S *provided that* $x_n \leq x_{n+1},\ \ n \in N$ *and* $x_n \longrightarrow x$, *we obtain a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *provided that* $x_{n_k} \leq x,\ \ k \in N$.

Then, a fixed point of the mapping T *is obtained.*

Proof. Let us the mapping $T : S \longrightarrow 2^S$ is defined by $Tx = \{fx\}$. Hence, it can be easily seen that all conditions of Theorems [2.8-2.10] are satisfied and Theorems [2.8 -2.10] imply that T has a fixed point. □

Due to Theorem 3.1, the following corollaries are obtained.

Corollary 3.2. *Let (S, D, Λ) be a complete Menger PM space in which Λ is continuous t-norm and the relation $\preceq$ has been defined on the set S. Moreover, assume that the self-mapping T has been defined on the set S and the following conditions hold true:*

1° *Given $u, v \in S$ that $u \preceq v$, we get*

$$h(D_{Tx,Ty}(t)) \geq \Big[h(D_{x,y}(t))\Big]^{M\Big(D_{x,Tx}(t), D_{y,Ty}(t), D_{x,Ty}(t), D_{y,Tx}(t)\Big)} \quad t > 0;$$

2° *If $x \in S$ and $x \preceq Tx$ then $Tx \preceq T^2x$;*

3° *There exists $x_0 \in S$ so that $x_0 \preceq Tx_0$;*

4° *T is continuous or if $\{x_n\}$ is a sequence in S provided that $\gamma(x_n, x_{n+1}, t) \leq 1$, $n \in N$ and $x_n \longrightarrow x$, we get a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ so that $\gamma(x_{n_k}, x, t) \leq 1$ for all $k \in N$.*

Then, a fixed point is earned of the mapping T . Moreover, if given $x, y \in Fix(T)$ the relation $x \preceq y$ is satisfied, then T has a unique fixed point.

Corollary 3.3. *Let (S, D, Λ) be a complete Menger PM space in which Λ is continuous t-norm and the relation $\preceq$ has been defined on the set S. Moreover, assume that the self-mapping T has been defined on the set S which is nondecreasing with respect to $\preceq$ and the following conditions hold true:*

1° *Given $x, y \in S$ with $x \preceq y$, we have*

$$h(D_{Tx,Ty}(t)) \geq \Big[h(T_{x,y}(t))\Big]^{M\Big(D_{x,Tx}(t), D_{y,Ty}(t), D_{x,Tx}(t), D_{y,Tx}(t)\Big)} \quad t > 0;$$

2° *A point $x_0 \in S$ exists so that $x_0 \preceq Tx_0$;*

3° *T is continuous or if $\{x_n\}$ is a sequence in S with $x_n \preceq x_{n+1}$ and $x_n \longrightarrow x$, we get a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ provided that $x_{n_k} \preceq x$ for all $k \in N$.*

Then, a fixed point of the mapping T is obtained. Moreover, if for all $x, y \in Fix(T)$ the relation $x \preceq y$ is satisfied, then unique fixed point is obtained of the mapping T.

Theorem 3.4. *Assume that (S, D, Λ) is a complete Menger PM space in which Λ is a continuous t-norm. Moreover, suppose that $T : S \longrightarrow S$ is a self-mapping which satisfies the following conditions:*

1° *Given $x, y \in O(w)$ and $t > 0$ we earn*

$$h(D_{x,y}(t)) = h(D_{Tx,Ty}(t)) \geq \Big[h(D_{x,y}(t))\Big]^{M\Big(D_{x,Tx}(t), D_{y,Ty}(t), D_{x,Ty}(t), D_{y,Tx}(t)\Big)}; \tag{5}$$

2° *T is an orbitally continuous function.*

Then, a fixed point of the mapping T is obtained . Moreover, if $Fix(T) \subseteq O(m)$, then we get unique fixed point of T.

Proof. Define

$$\gamma(x,y,t) = \begin{cases} 1 & \text{if} \quad x,y \in O(m), \\ 0 & \text{o.w} \end{cases}$$

in which $O(m)$ indicates an orbit of a point $m \in X$. Following the proof of Remark 1.4, the continuity property of T is obtained. Now, given $x \in S$ provided that $\gamma(x, Tx, t) \leq 1, \quad t > 0$, we get $Tx, T^2x \in O(m)$ and $\gamma(Tx, T^2x, t) \leq 1$. Hence, the conditions of Theorem 3.1 are satisfied. Thus, we earn a fixed point of T. To prove uniqueness, let $x \neq y \in Fix(T) \subseteq O(w)$. Now, applying (5) we have

$$h(D_{x,y}(t)) \geq \left[h(D_{x,y}(t)) \right]^{k^n}, \quad t > 0.$$

By taking the limit of the above inequality as n approaches infinity , we get $h(D_{x,y}(t)) = 1$. Therefore due to condition 2° of the definition h, $x = y$. □

The following corollary immediately follows from the above theorem.

Corollary 3.5. *Assume that (S, D, Λ) is a complete Menger PM space in which Λ is a continuous t-norm. Moreover, suppose that $T : S \longrightarrow S$ is a self-mapping which satifies the following conditions:*

1° *Given $x, y \in O(w)$ and $t > 0$ we earn*

$$h(D_{Tx,Ty}(t)) \geq \left[h(D_{x,y}(t)) \right]^{k}, \quad t > 0;$$

2° *T is an orbitally continuous function.*

Then, we obtain a fixed point of the mapping T. Moreover, if $Fix(T) \subseteq O(w)$, then we get unique fixed point of T.

4. Solvability of an integral equation

Here an existence result of an integral equation is constructed on a Banach space.
Let $X = C([0, L], \mathbb{R})$ indicates the collection of all continuous real-valued fuctions on $[0, L]$. The space $Y = C([0, L], \mathbb{R})$ with the norm

$$\|y\|_\infty = \max_{t \in [0,L]} |y(t)|, \quad y \in C([0, L], \mathbb{R}),$$

is a Banach space. Define $d : Y \times Y \longrightarrow \mathbb{R}_+$ by

$$d(y_1, y_2) = \|y_1 - y_2\|_\infty, \quad y_1, y_2 \in Y.$$

Now assume that $H : Y \times Y \longrightarrow E^+$ has been defined as follows:

$$H_{y_1,y_2}(t) = \frac{t}{t + d(y_1, y_2)}.$$

The space (X, F, Δ_m) has been proposed as a complete Menger PM space which $C([0, L], \mathbb{R})$ induces it. Now, we deal with considering the following integral equation:

$$y(t) = \int_0^L b(t,s)M(s, y(s))ds + g(t), \quad t \in [0, L]. \tag{6}$$

Define $f : C([0, L], \mathbb{R}) \longrightarrow C([0, L], \mathbb{R})$ as follows:

$$fy(t) = \int_0^L b(t,s)M(s, y(s))ds + g(t), \quad t \in [0, L]. \tag{7}$$

Consider the following conditions:

1° The functions $M : [0, L] \times Y \longrightarrow \mathbb{R}$, $b : [0, L] \times [0, L] \longrightarrow \mathbb{R}$ and $g : [0, L] \longrightarrow \mathbb{R}$ are continuous;

2° There exists $\gamma \in C(C([0, L], \mathbb{R}) \times C([0, L], \mathbb{R}) \times (0, \infty), \mathbb{R})$ such that for all $t > 0$ and $x, y \in C(I, \mathbb{R})$ that $\alpha(x, y, t) \leq 1$, we have

$$|M(s, x(s)) - M(s, y(s))| \leq |x(s) - y(s)|;$$

3° A point $x_0 \in C(I, \mathbb{R})$ there exists so that $\gamma(x_0, fx_0, t) \leq 1, \quad t > 0$;

4° Given $x \in C(I, \mathbb{R})$ and $t > 0$, if $\gamma(x, fx, t) \leq 1$, then $\gamma(fx, f^2x, t) \leq 1$;

5° For a sequence $\{x_n\}$ in $C(I, \mathbb{R})$ provided that $\gamma(x_n, x_{n+1}, t) \leq 1 \quad n \in N$ and $x_n \longrightarrow x$, we earn a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ so that $\gamma(x_{n_k}, x, t) \leq 1 \quad k \in N$.

6° For all $t \in [0, L]$,

$$\int_0^L b(t, s) ds \leq 1.$$

Theorem 4.1. *Suppose that the assumptions* $(1° - 6°)$ *are satisfied. Then we earn at least one solution of the integral equation (6) in the space* $C(I, \mathbb{R})$.

.

Proof. Let $f : Y \longrightarrow Y$ be defined as follows:

$$fy(t) = \int_0^L b(t, s) M(s, y(s)) ds + g(t), \quad t \in [0, L],$$

Given $y_1, y_2 \in Y$ and $t > 0$ with $\gamma(y_1, y_2, t) \leq 1$ we conclude that

$$\begin{aligned} |f(y_1)(t) - f(y_2)(t)| &= |\int_0^L b(t, s)[M(s, y_1(s)) - M(s, y_2(s))] ds| \\ &\leq \int_0^L b(t, s)|y_1(s) - y_2(s)| ds \leq \|y_1(s) - y_2(s)\|_\infty, \end{aligned}$$

that is,

$$\|fy_1 - fy_2\|_\infty \leq \|y_1 - y_2\|_\infty.$$

Hence,

$$\begin{aligned} h(D_{fy_1, fy_2}(t)) &= h(\frac{t}{t + d(fy_1, fy_2)}) = h(\frac{t}{t + \|fy_1 - fy_2\|_\infty}) \\ &\geq h(\frac{t}{t + \|y_1 - y_2\|_\infty}) = h(D_{y_1, y_2}(t)) \geq h(D_{y_1, y_2}(t))^{\frac{1}{2}}. \end{aligned}$$

We see that the conditions of Theorem 3.1 are indefeasible with $M(t_1, t_2, t_3, t_4) = \frac{1}{2}$. Hence, from Theorem 3.1 a fixed point of f is obtained in $C(I, \mathbb{R})$ which is a solution of the integral equation (6). □

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