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Perturbation Results in the Fredholm Theory and M-Essential Spectra of Some Matrix Operators

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Abstract. In this paper, we will use some new properties of non-compactness measure, in order to establish a description of the M-essential spectrum for some matrix operators on Banach spaces.

1. Introduction

In this paper we shall study the M-essential spectra of a general class of operators defined by a 2×2 block operator matrix acting in a product of Banach spaces $X \times X$

$$L_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where the operators occurring in the representation of L_0 are unbounded. A acts on the Banach space X and has the domain $\mathcal{D}(A)$, D is defined on $\mathcal{D}(D)$ and acts on X. The intertwining operators B, C are defined respectively on $\mathcal{D}(B)$, $\mathcal{D}(C)$ and act on X. Below, we shall assume that $\mathcal{D}(A) \subset \mathcal{D}(C)$ and $\mathcal{D}(B) \subset \mathcal{D}(D)$. Then the matrix L_0 defines a linear operator in X with domain $\mathcal{D}(A) \times \mathcal{D}(B)$.

Note that in general L_0 is not closed or closable, even if its entries are closed. But the authors in [4], give some sufficient conditions under which L_0 is closable and describe its closure which we shall denote L. Remark that in the work [7], M. Faierman, R. Mennicken and M. Möller give a method for dealing with the spectral theory for pencils of the form $L_0 - \mu M$, where M is a bounded operator.

To study the Wolf essential spectrum of the operator matrix L in Banach spaces, the authors in [4] (resp. in [12]) used the compactness condition for the operator $(\lambda - A)^{-1}$ (resp. $C(\lambda - A)^{-1}$ and $((\lambda - A)^{-1}B)^*$). Recently, in [1] the author describes the Fredholm essential spectra of L with the help of the measures of weak-noncompactness, where X is a Banach space which possess the Dunford-Pettis property. In this paper, we prove some localization results on the M-essential spectra of the matrix operator L via the concept of some quantities. The purpose of this work is to pursue the analysis started in [1, 4, 12].

Our paper is organized as follows : In Section 2, we recall some notations and definitions. In Section 3, we prove some results needed in the rest of the paper. In Section 4, we investigate the M-essential spectra of a general class of operators defined by a 2×2 block operator matrix by means of some quantities.

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2. Notations and definitions

Let X and Y be two infinite-dimensional Banach spaces. We denote by $\mathcal{C}(X,Y)$ (resp., $\mathcal{L}(X,Y)$) the set of all closed densely defined linear operators (resp., the space of all bounded linear operators) acting from X into Y. The subspace of all compact operators of $\mathcal{L}(X,Y)$ is denoted $\mathcal{K}(X,Y)$. If $X=Y$, the sets $\mathcal{C}(X,Y)$, $\mathcal{L}(X,Y)$, $\mathcal{K}(X,Y)$ are replaced respectively $\mathcal{C}(X)$, $\mathcal{L}(X)$, $\mathcal{K}(X)$. For $T\in\mathcal{C}(X)$ we use the following notations: $\mathcal{D}(T)$ is the domain, $\mathcal{N}(T)$ is the kernel and $\mathcal{R}(T)$ is the range of T. The nullity, $n(T)$, of T is defined as the dimension of $\mathcal{N}(T)$ and the deficiency, $d(T)$, of T is defined as the codimension of $\mathcal{R}(T)$ in X. We use $\sigma(T)$ and $\rho(T)$ to denote the spectrum and the resolvent set of T. We denote by $\Phi_+(X)$ and $\Phi_-(X)$ the classes of upper semi-Fredholm and lower semi-Fredholm. $\Phi(X):=\Phi_+(X)\cap\Phi_-(X)$ is the set of Fredholm operators in $\mathcal{C}(X)$. If $T\in\Phi_+(X)\cup\Phi_-(X)$, the number $i(T):=n(T)-d(T)$ is called the index of T.
Recall that, for $T\in\mathcal{C}(X)$, $X_T:=\mathcal{D}(T)$ endowed with the graph norm $\|.\|_T$ is a Banach space and we have $T\in\mathcal{L}(X_T,X)$. We denote by $\widehat{T}$ the restriction of T to $\mathcal{D}(T)$. Let J be a linear operator on X such that $X_T\subset\mathcal{D}(J)$. We say that J is T-bounded if its restriction to X_T, $\widehat{J}$ belongs to $\mathcal{L}(X_T,X)$.
Notice that $T\in\Phi(X)$ (resp., $\Phi_+(X)$) if and only if $\widehat{T}\in\Phi(X_T,X)$ (resp., $\Phi_+(X_T,X)$).

Definition 2.1. *Let X and Y be two Banach spaces.*

1. *Let $T\in\mathcal{L}(X,Y)$. T is said to have a left-Fredholm inverse (resp., a right-Fredholm inverse) if there exists $T_l\in\mathcal{L}(Y,X)$ (resp., $T_r\in\mathcal{L}(Y,X)$) and $K\in\mathcal{K}(X)$ such that $T_lT=I_X-K$ (resp., $I_Y-TT_r\in\mathcal{K}(Y)$). The operator T_l (resp., T_r) is called left-Fredholm (resp., right-Fredholm) inverse of T. The operator T is said to have a Fredholm inverse if there exists a map which is both a left and a right Fredholm inverse of T.*

2. *Let $T\in\mathcal{C}(X)$. T is said to have a left-Fredholm inverse (resp., right-Fredholm inverse, Fredholm inverse) if $\widehat{T}$ has a left-Fredholm inverse (resp., right-Fredholm inverse, Fredholm inverse).*

The sets of operators having left and right-Fredholm inverses are respectively defined by:

$$\Phi_l(X):=\{T\in\mathcal{C}(X)\text{ such that }T\text{ has a left Fredholm inverse}\},$$
$$\Phi_r(X):=\{T\in\mathcal{C}(X)\text{ such that }T\text{ has a right Fredholm inverse}\}.$$

Let $S\in\mathcal{L}(X)$ and $T\in\mathcal{C}(X)$. A complex number λ is in $\Phi_{lS}(T)$, $\Phi_{rS}(T)$ or $\Phi_S(T)$ if $\lambda S-T$ is in $\Phi_l(X)$, $\Phi_r(X)$ or $\Phi(X)$ respectively. We define the S-resolvent set (resp., the S-spectrum) of T by: $\rho_S(T):=\{\lambda\in\mathbb{C},\ \lambda S-T\text{ has a bounded inverse}\}$ (resp., $\sigma_S(T)=\mathbb{C}\setminus\rho_S(T)$).

In this paper, for $S\in\mathcal{L}(X)$, we are concerned with the following S-essential spectra:

$$\sigma_{eF,S}(T):=\{\lambda\in\mathbb{C}\ \text{ such that }\ \lambda S-T\notin\Phi(X)\},$$
$$\sigma_{eW,S}(T):=\mathbb{C}\setminus\{\lambda\in\Phi_S(T)\text{ such that }i(\lambda S-T)=0\},$$
$$\sigma_{eB,S}(T):=\mathbb{C}\setminus\{\lambda\in\mathbb{C}\text{ such that all scalars near }\lambda\text{ are in }\rho_S(T)\text{ and that }i(\lambda S-T)=0\}.$$
$$\sigma_{le,S}(T):=\{\lambda\in\mathbb{C}\text{ such that }\lambda S-T\notin\Phi_l(X)\},$$
$$\sigma_{re,S}(T):=\{\lambda\in\mathbb{C}\text{ such that }\lambda S-T\notin\Phi_r(X)\}.$$

$\sigma_{eF,S}(.)$ is the Fredholm S-essential spectrum. $\sigma_{eW,S}(.)$ is the Wolf S-essential spectrum. $\sigma_{eB,S}(.)$ is the Browder S-essential spectrum and $\sigma_{le}(.)$ (resp., $\sigma_{re}(.)$) is the left (resp., right) S-essential spectrum. Note that if $S=I$, we recover the usual definition of the essential spectra of $T\in\mathcal{C}(X)$.

We write $\overline{\mathbb{D}}(0,r)$ for the closure of the disc $\mathbb{D}(0,r)$. We use $\mathcal{C}[r_1,r_2]:=\overline{\mathbb{D}}(0,r_2)\backslash\mathbb{D}(0,r_1)$, for $r_1\le r_2$ and we denote by $\mathcal{C}(0,r)$ the circle with center 0 and radius r.

3. Some localization results on the S-essential spectra of a bounded operator

3.1. Perturbation results

Our purpose is to give some results concerning the class of Fredholm operators via the concept of some quantities. We write M_X for the family of all nonempty and bounded subset of X. Here, we deal with a specific measure: the Kuratowski measure of noncompactness defined as follows (see [6])

$$\gamma_X(A) = \inf\{\varepsilon > 0 \ : \ A \text{ may be covered by finitely many subsets of } X \text{ of diameter} \leq \varepsilon\}.$$

For $T \in \mathcal{L}(X, Y)$, we define the two non-negative quantities associated with T by:

$$\begin{cases} \alpha(T) = \sup\left\{\dfrac{\gamma_Y(T(A))}{\gamma_X(A)};\ A \in M_X,\ \gamma(A) > 0\right\} \\ \text{and} \\ \beta(T) = \inf\left\{\dfrac{\gamma_Y(T(A))}{\gamma_X(A)};\ A \in M_X,\ \gamma(A) > 0\right\}. \end{cases}$$

If no confusion can arise, then we write simply $\gamma(A)$ (resp., $\gamma(T(A))$) instead of $\gamma_X(A)$ (resp., $\gamma_Y(T(A))$).

We start this section by the following:

Theorem 3.1. *Let $A \in \mathcal{C}(X)$ and T be an A-bounded operator.*

(i) Let B be a bounded operator in $\Phi(X_A)$, $S \in \mathcal{L}(X_A)$ and assume that there exists A_l a left-Fredholm inverse of A.

If $\alpha(A_l\widehat{T}) < \beta(SB_r)$, then $\widehat{T}B + \widehat{A}S \in \Phi_+(X_A, X)$ and $i(\widehat{T}B + \widehat{A}S) = i(S) + i(A)$.
If moreover $S \in \Phi(X_A)$, then $\widehat{T}B + \widehat{A}S \in \Phi_l(X_A, X)$.

(ii) Let $B, S \in \Phi(X)$ and assume that there exists A_r a right-Fredholm inverse of A.

If $\alpha(\widehat{T}A_r) < \beta(B_lS)$, then $B\widehat{T} + S\widehat{A} \in \Phi_r(X_A, X)$ and $i(B\widehat{T} + S\widehat{A}) = i(S) + i(A)$.

Proof. (i) According to [3, Theorem 2.2], $A_l\widehat{T} + SB_r \in \Phi_+(X_A)$ and $i(A_l\widehat{T} + SB_r) = i(S) - i(B)$. Furthermore $A_l(\widehat{T}B + \widehat{A}S)B_r = A_l\widehat{T} + SB_r + K$, where K is compact in $\mathcal{L}(X_A)$. Since $A_l\widehat{T} + SB_r + K \in \Phi_+(X_A)$ and $B \in \Phi(X_A)$, then $\widehat{T}B + \widehat{A}S \in \Phi_+(X_A, X)$. Furthermore, we have $i(A_l(\widehat{T}B + \widehat{A}S)B_r) = i(A_l\widehat{T} + SB_r) = i(S) - i(B)$, which implies that $i(\widehat{T}B + \widehat{A}S) = i(S) + i(A)$. Suppose moreover that $S \in \Phi(X_A)$, then $i(A_l\widehat{T} + SB_r) < +\infty$. Thus $A_l\widehat{T} + SB_r + K \in \Phi(X_A)$ and therefore $\widehat{T}B + \widehat{A}S \in \Phi_l(X_A, X)$.

(ii) Arguing as in the proof of (i) and the fact that $B, S \in \Phi(X)$, yield $B_l(B\widehat{T} + S\widehat{A})A_r = \widehat{T}A_r + B_lS + K \in \Phi(X)$, where $K \in \mathcal{K}(X)$ and $i(\widehat{T}A_r + B_lS) = i(S) - i(B)$. Thus, $(B\widehat{T} + S\widehat{A})A_r \in \Phi(X)$ and therefore $B\widehat{T} + S\widehat{A} \in \Phi_r(X_A, X)$. Furthermore, we have $i(B\widehat{T} + S\widehat{A}) = i(S) + i(A)$.

In the following corollary we prove some localization results of the S-essential spectra of a bounded operator T. For this, define $\alpha_0(T)$ (resp, $\beta_0(T)$) to be the limit of the sequence $\alpha(T^n)^{\frac{1}{n}}$ (resp, $\beta(T^n)^{\frac{1}{n}}$). For the existence of these limits see [10, Lemma 2.1]. According to [3, Proposition 2.1], we remark that $\alpha_0(T) \leq \alpha(T)$ and $\beta(T) \leq \beta_0(T)$. We denote by:

$$\widetilde{\alpha}(T) = \begin{cases} \alpha_0(T) & \text{if } ST = TS, \\ \alpha(T) & \text{if } ST \neq TS. \end{cases} \quad \widetilde{\beta}(T) = \begin{cases} \beta_0(T) & \text{if } ST = TS, \\ \beta(T) & \text{if } ST \neq TS. \end{cases}$$

Corollary 3.2. *Let $S, T \in \mathcal{L}(X)$ such that $\beta_0(S) > 0$. Then one has the following.*

(i) Suppose that $S \in \Phi(X)$, then $\sigma_{eF,S}(T) \subset \overline{\mathbb{D}}(0, \dfrac{\widetilde{\alpha}(T)}{\beta_0(S)})$.

(ii) Suppose that $S \in \Phi(X)$ with $i(S) = 0$, then $\sigma_{eW,S}(T) \subset \overline{\mathbb{D}}(0, \dfrac{\widetilde{\alpha}(T)}{\beta_0(S)})$.

If moreover $i(T) = 0$, then $\sigma_{eW,S}(T) \subset \mathcal{C}([\dfrac{\widetilde{\beta}(T)}{\alpha_0(S)}, \dfrac{\widetilde{\alpha}(T)}{\beta_0(S)}])$.

(iii) Suppose that $T \notin \Phi_-(X)$, *then* $\mathbb{D}(0, \frac{\widetilde{\beta}(T)}{\alpha_0(S)}) \subset \sigma_{eF,S}(T)$.

(iv) Suppose that $T \in \Phi_-(X)$, *then* $\sigma_{eF,S}(T) \subset C([\frac{\widetilde{\beta}(T)}{\alpha_0(S)}, \frac{\widetilde{\alpha}(T)}{\beta_0(S)}])$.

Proof. Suppose that $TS = ST$. Let $n \in \mathbb{N}^*$ and assume that $\beta(\lambda^n S^n) > \alpha(T^n)$. Then according to Theorem [3, Theorem 2.2], $\lambda S - T \in \Phi(X)$ and $i(\lambda S - T) = i(S)$. Hence, if $|\lambda| > \frac{\widetilde{\alpha}(T)}{\beta_0(S)}$, then $\lambda \notin \sigma_{eF,S}(T)$ proving (*i*). If furthermore $i(S) = 0$, then $\lambda \notin \sigma_{eW,S}(T)$, which proves the first statement of (*ii*). Notice that if $\beta(T) = 0$, then $\widetilde{\beta}(T) = 0$ and the results are all trivial. Suppose that $\beta(T) > 0$. For $\alpha_0(\lambda S) < \beta_0(T)$, there exists $n \in \mathbb{N}^*$ such that $\alpha((\lambda S)^n) < \beta(T^n)$, then by Theorem [3, Theorem 2.2] we get $\lambda S - T \in \Phi_+(X)$ and $i(\lambda S - T) = i(T)$. Hence, we get easily (*ii*) – (*iv*).

3.2. Example: Unilateral backward weighted shift operators

Let $t = (t_n)_{n\in\mathbb{N}}$ and $s = (s_n)_{n\in\mathbb{N}}$ be two bounded complex sequences. Consider the unilateral backward weighted shift operator $T(t,p)$ defined on $X = l^r(\mathbb{N},\mathbb{C})$, $r \geq 1$, by: $T(t,p)(x_0, x_1, \ldots) = (t_p x_p, t_{p+1}x_{p+1}, \ldots)$. In [2, 9] the authors prove some localization results on the essential spectra of $T(t,p)$. In this example, we describe the S-essential spectra of $T(t,p)$ where S is defined on X by: $S(s,q)(x_0, x_1, ...) = (s_q x_q, s_{q+1}x_{q+1}, ...)$. Recall that (see [2, Proposition 3.8])

$$\alpha(T(t,p)) = t_+ := \limsup_{n\to\infty} |t_n| \text{ and } \beta(T(t,p)) = t_- := \liminf_{n\to\infty} |t_n|.$$

Hence, if 0 is a cluster point for the sequence $(|t_n|)_n$, then $\beta(T(t,p)) = 0$. If not, then $T(t,p)$ is a Fredholm operator with index p. More precisely $n(T(t,p)) = p + card(F_0)$ and $d(T(t,p)) = card(F_0)$, where $F_0 := \{n \geq p$ such that $t_n = 0\}$.

In what follows we investigate more precisely the S-essential spectra of $T(t,p)$, where $S, T \in \Phi(X)$ with $i(S) = q$ and $i(T) = p \neq 0$.

Proposition 3.3. *(i)* $\sigma_{eF,S}(T) \subset C([\frac{t_-}{s_+}, \frac{t_+}{s_-}])$.

(ii) If $i(S) = 0$, *then* $\sigma_{eW,S}(T) \subset \overline{\mathbb{D}}(0, \frac{t_+}{s_-})$.

(iii) Suppose that $\lim_{n\to+\infty} |t_n| = a$ *and* $\lim_{n\to+\infty} |s_n| = b$. *Then* $\sigma_{eF,S}(T) = C(0, \frac{a}{b})$.

Proof. (*i*) and (*ii*) are a direct consequence of Corollary 3.2.

(*iii*) We have $\alpha(T) = \beta(T) = a$ and $\alpha(S) = \beta(S) = b$. According to (*i*), $\sigma_{eF,S}(T) = C(0, \frac{a}{b})$.

4. The *M*-essential spectra of some matrix operator

Let L_0 be a matrix operator and M be a bounded matrix operator acting on the Banach space $X \times X$ and which are formally defined as follows :

$$L_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},$$

where M_2 and M_3 are compact operators. The operators A, B, C and D acts on X and has the domain $\mathcal{D}(A)$, $\mathcal{D}(B)$, $\mathcal{D}(C)$ and $\mathcal{D}(D)$ respectively. In this section we will study some properties of the M-essential spectra of L the closure of L_0. For this, we require the following assumptions verified:

(H_1) $A \in \mathcal{C}(X)$ with nonempty M_1-resolvent set $\rho_{M_1}(A)$.

(H_2) The operator $B \in \mathcal{C}(X)$ and for some (hence for all) $\mu \in \rho_{M_1}(A)$, the operator $(A-\mu M_1)^{-1}B$ is closable. We denote by $G(\mu) := \overline{(A-\mu M_1)^{-1}(B-\mu M_2)}$.

(H_3) The operator C satisfies $\mathcal{D}(A) \subset \mathcal{D}(C)$, and for some (hence for all) $\mu \in \rho_{M_1}(A)$, the operator $C(A-\mu M_1)^{-1}$ is bounded. We denote $F(\mu) = (C-\mu M_3)(A-\mu M_1)^{-1}$.

(H_4) The lineal $\mathcal{D}(B) \cap \mathcal{D}(D)$ is dense in X, and for some (hence for all) $\mu \in \rho_{M_1}(A)$, the operator $D - C(A-\mu M_1)^{-1}B$ is closable. We will denote by $S(\mu)$ the closure of the operator $D-(C-\mu M_3)(A-\mu M_1)^{-1}(B-\mu M_2)$.

The following theorem gives some sufficient conditions for the closeness of L_0.

Theorem 4.1. *[7] Suppose that the conditions (H_1)-(H_3) are satisfied and the lineal $\mathcal{D}(B) \cap \mathcal{D}(D)$ is dense in X. Then the operator L_0 is closable if and only if the operator $D - C(A-\mu M_1)^{-1}B$ is closable in X, for some $\mu \in \rho_{M_1}(A)$. Moreover, the closure L of L_0 is given by:*

$$L = \mu M + \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} A-\mu M_1 & 0 \\ 0 & S(\mu)-\mu M_4 \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}.$$

For $\lambda \in \mathbb{C}$ and $\mu \in \rho_{M_1}(A)$, we will denote $A_{\lambda M_1} = \lambda M_1 - A$ and $S_{\lambda M_4}(\mu) = \lambda M_4 - S(\mu)$. Then $L_{\lambda M}$ can be written as follows:

$$L_{\lambda M} := UV(\lambda)W - (\lambda-\mu)\mathcal{R}(\mu), \tag{1}$$

where $U = \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix}$, $W = \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}$, $V(\lambda) = \begin{pmatrix} A_{\lambda M_1} & 0 \\ 0 & S_{\lambda M_4}(\mu) \end{pmatrix}$
and $\mathcal{R}(\mu) = \begin{pmatrix} 0 & M_1G(\mu)-M_2 \\ F(\mu)M_1 - M_3 & F(\mu)M_1G(\mu) \end{pmatrix}$.

In [5, Theorem 3.3.2], the authors constrict the measures of noncompactness in cartesian product of a given finite collection of Banach spaces. More precisely, we have:

Lemma 4.2. *[5, Theorem 3.3.2] Let $E_1, \ldots, E_n$ be a finite collection of Banach spaces, let $\mu_1, \ldots, \mu_n$ the measures of noncompactness in $E_1, \ldots, E_n$ respectively. Assume the function $F : ([0,+\infty[)^n \to [0,+\infty[$ is convex and $F(x_1, \ldots, x_n) = 0$ if and only if $x_i = 0$ for $i = 1, \ldots, n$. Then*

$$\mu(x) = F(\mu_1(\pi_1(x)), \ldots, \mu_n(\pi_n(x)))$$

defines a measure of noncompactness in $E_1 \times E_2 \times \ldots \times E_n$.

Here $\pi_i(x)$ denotes the natural projection of x into E_i.

According to the previous lemma, for all $A \in M_{X^2}$, the quantity $\gamma(A) = \max(\gamma(\pi_1(A)), \gamma(\pi_2(A)))$ defines a measure of noncompactness in X^2. For $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, consider the measure of noncompactness of T

$$\alpha_{X\times X}(T) = \sup\left\{ \frac{\gamma(T(A))}{\gamma(A)};\ A \in M_{X^2} \text{ and } \gamma(A) > 0 \right\}.$$

Define a measure of noncompactness of T by:

$$\alpha(T) := \max\{\alpha(A) + \alpha(B); \alpha(C) + \alpha(D)\}.$$

The following proposition gives the relationship between $\alpha(T)$ and $\alpha_{X\times X}(T)$.

Proposition 4.3. $\alpha_{X\times X}(T) \leq \alpha(T)$.

Proof. For all $H \in M_{X^2}$, we have $H \subset \pi_1(H) \times \pi_2(H)$. Hence, $T(H) \subset T(\pi_1(H) \times \pi_2(H))$. If we denote by $H_i := \pi_i(H)$, $i = 1, 2$, then

$$\begin{aligned}
\alpha_{X\times X}(T) &= \sup\left\{\frac{\gamma(T(H))}{\gamma(H)};\ H \in M_{X^2} \text{ and } \gamma(H) > 0\right\}\\
&\leq \sup\left\{\frac{\max(\gamma(A(H_1)) + \gamma(B(H_2)),\ \gamma(C(H_1)) + \gamma(D(H_2)))}{\max(\gamma(H_1),\ \gamma(H_2))},\ \gamma(H_j) > 0, \forall j = 1,2\right\}\\
&\leq \max\begin{pmatrix} \sup\left\{\frac{\gamma(A(H_1))}{\gamma(H1)},\ H_1 \in M_X,\ \gamma(H_1) > 0\right\} + \sup\left\{\frac{\gamma(B(H_2))}{\gamma(H_2)},\ H_2 \in M_X,\ \gamma(H_2) > 0\right\};\\ \sup\left\{\frac{\gamma(C(H_1))}{\gamma(H1)},\ H_1 \in M_X,\ \gamma(H_1) > 0\right\} + \sup\left\{\frac{\gamma(D(H_2))}{\gamma(H_2)},\ H_2 \in M_X,\ \gamma(H_2) > 0\right\} \end{pmatrix}\\
&\leq \max\{\alpha(A) + \alpha(B); \alpha(C) + \alpha(D)\}.
\end{aligned}$$

Unless otherwise stated in all what follows, we suppose that, for some $\mu \in \rho_{M_1}(A)$, $F(\mu)$ and $G(\mu)$ satisfy the condition:

$$(H) : \max\left(\alpha(G(\mu)),\ \alpha(F(\mu))\right) < 1.$$

Furthermore we take λ in the disk with center μ and radius 1.

Theorem 4.4. (*i*) *Suppose that there exist $A^l_{\lambda M_1}$ a left-Fredholm inverse of $A_{\lambda M_1}$ and $S^l_{\lambda M_4}(\mu)$ a left-Fredholm inverse of $S_{\lambda M_4}(\mu)$. Suppose further that:*

$$\alpha(S^l_{\lambda M_4}(\mu)F(\mu)M_1) < \frac{1}{2} \text{ and } \alpha(A^l_{\lambda M_1}M_1G(\mu)) < \frac{1}{2}$$

Then $L_{\lambda M} \in \Phi_l(X \times X)$ and $i(L_{\lambda M}) = i(V(\lambda))$.

(*ii*) *Suppose that there exist $A^r_{\lambda M_1}$ a right-Fredholm inverse of $A_{\lambda M_1}$ and $S^r_{\lambda M_4}(\mu)$ a right-Fredholm inverse of $S_{\lambda M_4}(\mu)$. Suppose further that:*

$$\alpha(S^l_{\lambda M_4}(\mu)M_1G(\mu)) < \frac{1}{2} \text{ and } \alpha(A^l_{\lambda M_1}F(\mu)M_1) < 1$$

Then $L_{\lambda M} \in \Phi_r(X \times X)$ and $i(L_{\lambda M}) = i(V(\lambda))$.

(*iii*) *Suppose that the hypotheses of* (*i*) *and* (*ii*) *hold true. Then*

$$L_{\lambda M} \in \Phi(X \times X) \text{and } i(L_{\lambda M}) = i(V(\lambda)).$$

Proof.

(*i*) Let $T_\lambda = UV(\lambda)W$ and $V^l_\lambda = \begin{pmatrix} A^l_{\lambda M_1} & 0 \\ 0 & S^l_{\lambda M_4}(\mu)\end{pmatrix}$. It is easy to see that V^l_λ is a left-Fredholm inverse of $V(\lambda)$. Thus, $T^l_\lambda = W^{-1}V^l_\lambda U^{-1}$ is a left-Fredholm inverse of T_λ. On the other hand, we have:

$$T^l_\lambda\mathcal{R}(\mu) = \begin{pmatrix} -G(\mu)S^l_{\lambda M_4}(\mu)(F(\mu)M_1 - M_3) & A^l_{\lambda M_1}(M_1G(\mu) - M_2) - G(\mu)S^l_{\lambda M_4}(\mu)F(\mu)M_2 \\ S^l_{\lambda M_4}(\mu)(F(\mu)M_1 - M_3) & S^l_{\lambda M_4}(\mu)F(\mu)M_2 \end{pmatrix}.$$

Now, since M_2 and M_3 are compact operators, then

$$\begin{aligned}
\alpha(T^l_\lambda\mathcal{R}(\mu)) &\leq max\{\alpha(S^l_{\lambda M_4}(\mu)F(\mu)M_1) + \alpha(A^l_{\lambda M_1}M_1G(\mu)); \alpha(S^l_{\lambda M_4}(\mu)F(\mu)M_1)\}\\
&\leq \alpha(S^l_{\lambda M_4}(\mu)F(\mu)M_1) + \alpha(A^l_{\lambda M_1}M_1G(\mu)).
\end{aligned}$$

By hypotheses, we get $\alpha(T^l_\lambda\mathcal{R}(\mu)) < 1$. Hence, by the fact that $\mathcal{R}(\mu)$ is T_λ-bounded and $|\lambda - \mu| < 1$, we deduce that $\alpha((\lambda - \mu)T^l_\lambda\mathcal{R}(\mu)) < 1$. Finally, the results follow from Proposition 4.3 and Theorem 3.1(*i*).

(ii) Let $V_\lambda^r = \begin{pmatrix} A^r_{\lambda M_1} & K_1' \\ K_2' & S^r_{\lambda M_4}(\mu) \end{pmatrix}$ be such that K_1' and K_2' are compact operators. In the same way one checks that $T_\lambda^r = W^{-1}V_\lambda^r U^{-1}$ is a right-Fredholm inverse of T_λ. Arguing as in the proof of (i) and according to the hypotheses we obtain:

$$\alpha((\lambda-\mu)\mathcal{R}(\mu)T_\lambda^r) < 1.$$

Finally, the results follow from Proposition 4.3 and Theorem 3.1 (ii).

(iii) Is a deduction from (i) and (ii).

Now, the question is to find out under what conditions we have that $L_{\lambda M}$ has a Fredholm inverse. For this we consider $H(\mu) = S(\mu) - CG(\mu)$ and $H_{\lambda M_4}(\mu) = \lambda M_4 - H(\mu),\ \mu \in \rho_{M_1}(A)$.

Theorem 4.5. (i) *Suppose that $A_{\lambda M_1}$ (resp. $H_{\lambda M_4}(\mu)$) has a left-Fredholm inverse $A^l_{\lambda M_1}$ (resp. $H^l_{\lambda M_4}(\mu)$). Suppose further that:*

$H^l_{\lambda M_4}(\mu)C$ and $A^l_{\lambda M_1}A_{\mu M_1}G(\mu)$ are compact operators and $\alpha(A^l_{\lambda M_1}M_1G(\mu)) < 1$.

Then

$$V(\lambda) \in \Phi_l(X) \text{ and } i(L_{\lambda M}) = i(V(\lambda)).$$

(ii) *Suppose that $A_{\lambda M_1}$ (resp. $H_{\lambda M_4}(\mu)$) has a right-Fredholm inverse $A^r_{\lambda M_1}$ (resp. $H^r_{\lambda M_4}(\mu)$) satisfying:*

$$\begin{cases} \bullet\ A_{\mu M_1}G(\mu)H^r_{\lambda M_4}(\mu) \text{ and } CA^r_\lambda \text{ are compact operators.} \\ \bullet\ \alpha(H^l_{\lambda M_4}(\mu)M_1G(\mu)) < \frac{1}{2} \text{ and } \alpha(A^l_{\lambda M_1}F(\mu)M_1) < \frac{1}{2}. \end{cases}$$

Then

$$V(\lambda) \in \Phi_r(X) \text{ and } i(L_{\lambda M}) = i(V(\lambda)).$$

(iii) *Suppose that B and C are compact operators. Then*

$$V(\lambda) \in \Phi(X) \text{ and } i(L_{\lambda M}) = i(V(\lambda)).$$

To prove Theorem 4.5 we shall need the following lemma.

Lemma 4.6. (i) *Suppose that there exists $H^l_{\lambda M_4}(\mu)$ (resp. $A^l_{\lambda M_1}$) a left-Fredholm inverse of $H_{\lambda M_4}(\mu)$ (resp. $A_{\lambda M_1}$) satisfying:*

$$H^l_{\lambda M_4}(\mu)C \text{ and } A^l_{\lambda M_1}A_{\mu M_1}G(\mu) \text{ are compact operators.}$$

Then $L_{\lambda M}$ has a left-Fredholm inverse defined by: $L^l_{\lambda M} = \begin{pmatrix} A^l_{\lambda M_1} & K_1 \\ K_2 & H^l_{\lambda M_4}(\mu) \end{pmatrix}$, where K_1, K_2 are compact operators.

(ii) *Suppose that there exists $H^r_{\lambda M_4}(\mu)$ (resp. $A^r_{\lambda M_1}$) a right-Fredholm inverse of $H_{\lambda M_4}(\mu)$ (resp. $A_{\lambda M_1}$) satisfying:*

$$A_{\mu M_1}G(\mu)H^r_{\lambda M_4}(\mu) \text{ and } CA^r_{\lambda M_1} \text{ are compact operators.}$$

Then $L_{\lambda M}$ has a right-Fredholm inverse defined by: $L^r_{\lambda M} = \begin{pmatrix} A^r_\lambda & K_1' \\ K_2' & H^r_{\lambda M_4}(\mu) \end{pmatrix}$, where K_1', K_2' are compact operators.

Proof of Theorem 4.5

(i) According to the hypotheses and using Lemma 4.6, there exist tow compact operators K_1 and K_2 such that $L^l_{\lambda M} = \begin{pmatrix} A^l_{\lambda M_1} & K_1 \\ K_2 & H^l_{\lambda M_4}(\mu) \end{pmatrix}$ is a left-Fredholm inverse of $L_{\lambda M}$. Then we have:

$$L^l_{\lambda M}\mathcal{R}(\mu) = \begin{pmatrix} K_1(F(\mu)M_1 - M_3) & A^l_{\lambda M_1}(M_1G(\mu) - M_2) + K_1F(\mu)M_1G(\mu) \\ H^l_{\lambda M_4}(\mu)(F(\mu)M_1 - M_3) & K_2(M_1G(\mu) - M_2)) + H^l_{\lambda M_4}(\mu)F(\mu)M_1G(\mu) \end{pmatrix}.$$

Given that M_2 and M_3 are compacts, the hypotheses yield $\alpha(L^l_{\lambda M}\mathcal{R}(\mu)) < 1$. Finally, the results follow from Proposition 4.3 and Theorem 3.1.

(*ii*) From the hypotheses there exist tow compact operators K'_1 and K'_2 such that $L^r_{\lambda M} = \begin{pmatrix} A^r_{\lambda M_1} & K'_1 \\ K'_2 & H^r_{\lambda M_4}(\mu) \end{pmatrix}$ is a right-Fredholm inverse of $L_{\lambda M}$. Thus, according to the hypotheses, we obtain: $\alpha(\mathcal{R}(\mu)L^r_{\lambda M}) < 1$. Finally, the results follow from Proposition 4.3 and Theorem 3.1.

(*iii*) Is a deduction from (*i*) and (*ii*). Q.E.D.

Theorem 4.7. *The following assertions hold.*

(*i*) *Suppose that, for each $\lambda \in \Phi_{lM_1}(A) \cap \Phi_{lM_4}(S(\mu)) \cap \Phi_{lM_4}(H(\mu))$, we have $H^l_{\lambda M_4}(\mu)C$ and $A^l_{\lambda M_1}A_{\mu M_1}G(\mu)$ are compact operators and*

$$\alpha(S^l_{\lambda M_4}(\mu)F(\mu)M_1) < \frac{1}{2} \text{ and } \alpha(A^l_{\lambda M_1}M_1G(\mu)) < \frac{1}{2}$$

Then $\sigma_{le,M}(L) = \sigma_{le,M_1}(A) \cup \sigma_{le,M_4}(S(\mu))$.
(*ii*) *Suppose that, for each $\lambda \in \Phi_{rM_1}(A) \cap \Phi_{rM_4}(S(\mu)) \cap \Phi_r(H(\mu))$, we have $A_{\mu M_1}G(\mu)H^r_{\lambda M_4}(\mu)$, $CA^r_{\lambda M_1}$ are compact operators and*

$$\alpha(S^l_{\lambda M_4}(\mu)M_1G(\mu)) < \frac{1}{2}, \alpha(A^l_{\lambda M_1}F(\mu)M_1) < 1, \alpha(H^l_{\lambda M_4}(\mu)M_1G(\mu)) < \frac{1}{2}.$$

Then

$$\sigma_{re,M}(L) = \sigma_{re,M_1}(A) \cup \sigma_{re,M_4}(S(\mu)).$$

(*iii*) *Suppose that B and C are compact operators, then*

$$\sigma_{eF,M}(L) = \sigma_{eF,M_1}(A) \cup \sigma_{eF,M_4}(S(\mu)) \text{ and } \sigma_{eW,M}(L) = \sigma_{eW,M_1}(A) \cup \sigma_{eW,M_4}(S(\mu)).$$

If in addition, $\mathbb{C}\backslash\sigma_{eF,M}(L)$, $\mathbb{C}\backslash\sigma_{eF,M_1}(A)$ and $\mathbb{C}\backslash\sigma_{eF,M_4}(S(\mu))$ are connected, $\rho(L) \neq \emptyset$ and $\rho(S(\mu)) \neq \emptyset$, then

$$\sigma_{eB,M}(L) = \sigma_{eB,M_1}(A) \cup \sigma_{eB,M_4}(S(\mu)).$$

Proof (*i*) Suppose that, for each $\lambda \in \Phi_{lM_1}(A) \cap \Phi_{lM_4}(S(\mu))$, $\alpha(S^l_{\lambda M_4}(\mu)F(\mu)M_1) < \frac{1}{2}$ and $\alpha(A^l_{\lambda M_1}M_1G(\mu)) < \frac{1}{2}$, then by apply Theorem 4.4(i), we get

$$\sigma_{le,M}(L) \subset \sigma_{le,M_1}(A) \cup \sigma_{le,M_4}(S(\mu)).$$

Suppose that, for each $\lambda \in \Phi_{lM_1}(A) \cap \Phi_{lM_4}(H(\mu))$, $H^l_{\lambda M_4}(\mu)C$ and $A^l_{\lambda M_1}A_{\mu M_1}G(\mu)$ are compact operators and $\alpha(A^l_{\lambda M_1}M_1G(\mu)) < \frac{1}{2}$, then according to Theorem 4.5(i), we obtain

$$\sigma_{le,M_1}(A) \cup \sigma_{le,M_4}(S(\mu)) \subset \sigma_{le,M}(L).$$

The same reasoning as (*i*) and by apply Theorem 4.4(*ii*) and Theorem 4.5(*ii*), we prove the assertion (*ii*).

The first part of assertion (*iii*) is a deduction from (*i*) and (*ii*). To describe the Browder essential spectrum of L, we have $\sigma_{eF,M}(L) \subset \sigma_{eB,M}(L)$. Thus, since $n(L_{\lambda M})$ and $d(L_{\lambda M})$ are constant on any component of $\Phi_M(L)$ except possibly on a discrete set of points at which they have large values (see for example, [8, 11]), then $\sigma_{eB,M}(L) \subset \sigma_{eF,M}(L)$ and therefore $\sigma_{eB,M}(L) = \sigma_{eF,M}(L)$. Using the same reasoning as before, we show that $\sigma_{eB,M_1}(A) = \sigma_{eF,M_1}(A)$ and $\sigma_{eB,M_4}(S(\mu)) = \sigma_{eF,M_4}(S(\mu))$.

The following corollary provides an extension of Theorem 2 in [12].

Corollary 4.8. *The following assertions hold.*

(*i*) *Suppose that $G(\mu)$ is compact and, for each $\lambda \in \Phi_{lM_1}(A) \cap \Phi_{lM_4}(S(\mu))$, we have $\overline{D^l_{\lambda M_4}}C$ is a compact operator, then $\sigma_{le,M}(L) = \sigma_{le,M_1}(A) \cup \sigma_{le,M_4}S(\mu))$.*

(*ii*) *Suppose that, for each* $\lambda \in \Phi_{rM_1}(A) \cap \Phi_{rM_4}(S(\mu)) \cap \Phi_r(H_\lambda(\mu))$*, we have* $CA^r_{\lambda M_1}$ *and* $F(\mu)M_1A^r_{\lambda M_1}$ *are compact operators, then*

$$\sigma_{re,M}(L) = \sigma_{re,M_1}(A) \cup \sigma_{re,M_4}(S(\mu)).$$

(*iii*) *Suppose that the hypotheses of* (*i*) *and* (*ii*) *hold, then*

$$\sigma_{eF,M}(L) = \sigma_{eF,M_1}(A) \cup \sigma_{eF,M_4}(S(\mu)) \text{ and } \sigma_{eW,M}(L) = \sigma_{eW,M_1}(A) \cup \sigma_{eW,M_4}(S(\mu)).$$

If in addition, $\mathbb{C}\backslash\sigma_{eF,M}(L)$*,* $\mathbb{C}\backslash\sigma_{eF,M_1}(A)$ *and* $\mathbb{C}\backslash\sigma_{eF,M_4}(S(\mu))$ *are connected,* $\rho(L) \neq \emptyset$ *and* $\rho(S(\mu)) \neq \emptyset$*, then*

$$\sigma_{eB,M}(L) = \sigma_{eB,M_1}(A) \cup \sigma_{eB,M_4}(S(\mu)).$$

Remark 4.9. *Suppose that* $G(\mu)$ *and* $F(\mu)M_1$ *are compact operators, then* $\mathcal{R}(\mu)$ *is compact. Thus according to the equation (1)*

$$\sigma_{le,M}(L) = \sigma_{le,M_1}(A) \cup \sigma_{le,M_4}S(\mu)),$$
$$\sigma_{re,M}(L) = \sigma_{re,M_1}(A) \cup \sigma_{re,M_4}(S(\mu)),$$
$$\sigma_{eI,M}(L) = \sigma_{eI,M_1}(A) \cup \sigma_{eI,M_4}(S(\mu)), \ \forall I \in \{F, W\}.$$

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