Filomat 35:7 (2021), 2129–2140
https://doi.org/10.2298/FIL2107129S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Second Hankel Determinant for Certain Subclass of Bi-Univalent Functions

Safa Salehian[a], Ahmad Motamednezhad[b]

[a] *Department of Mathematics, Gorgan Branch, Islamic Azad University, Gorgan, Iran*
[b] *Faculty of Mathematical Sciences, Shahrood University of Technology, P.O.Box 316-36155, Shahrood, Iran*

Abstract. The main purpose of this paper is to obtain an upper bound for the second Hankel determinant for functions belonging to a subclass of bi-univalent functions in the open unit disk in the complex plane. Furthermore, the presented results in this work improve or generalize the recent works of other authors.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f defined on the unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let $\mathcal{S}$ be the subclass of functions in $\mathcal{A}$ that are univalent in $\mathbb{U}$. For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$ and we write it as $f(z) \prec g(z)$ if there exists a Schwartz function w, which is analytic in $\mathbb{U}$ with $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{U})$ such that

$$f(z) = g(w(z)), \quad z \in \mathbb{U}.$$

Indeed, it is known that $f(z) \prec g(z) \Longrightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.
In particular, if the function g is univalent in $\mathbb{U}$, then $f(z) \prec g(z) \Longleftrightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$, [7].

The well-known *Koebe One-Quarter Theorem* established that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}) \quad \text{and} \quad f(f^{-1}(w)) = w \quad (|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}),$$

2010 *Mathematics Subject Classification*. Primary 30C45; Secondary 30C50
Keywords. Bi-univalent functions, Coefficient estimates, Second Hankel determinant, Subordination
Received: 25 May 2020; Revised: 13 September 2020; Accepted: 20 September 2020
Communicated by Miodrag Mateljević
Corresponding author: Safa Salehian
Email addresses: s.salehian84@gmail.com (Safa Salehian), a.motamedne@gmail.com (Ahmad Motamednezhad)

where

$$\begin{aligned} g(w) = f^{-1}(w) &= w + \sum_{n=2}^{\infty} A_n w^n \\ &= w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \end{aligned} \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$.

Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). For a brief history and interesting examples of functions in the class $\mathcal{S}$, see [24]. In 1967, Lewin [14] introduced the concept of bi-univalent analytic functions and proved that the second coefficient satisfies $|a_2| < 1.51$. In the following, various subclasses of the bi-univalent functions were introduced and the first two coefficients $|a_2|$, $|a_3|$ in the Taylor-Maclaurin series expansion [1, 4, 9, 16, 21–25] were estimated. But the coefficient problem for each of the following Taylor-Maclaurin coefficients

$$|a_n| \quad (n \in \mathbb{N} - \{1, 2\};\ \mathbb{N} = \{1, 2, 3, \cdots\})$$

is still an open problem (see [15, 20]).

The q^{th} determinant for $q \geq 1$ and $n \geq 0$ is stated by Noonan and Thomas [18] as below.

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (a_1 = 1).$$

This determinant has also been considered by several authors. For example, Ehrenborg [8] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman [13].

Note that

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} \qquad \textit{and} \qquad H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix},$$

where the Hankel determinants $H_2(1) = a_3 - a_2^2$ and $H_2(2) = a_2 a_4 - a_2^3$ are well-known as Fekete-Szegö and second Hankel determinant functionals, respectively.

Recently, Çağlar et al. [2, 3], Qadeem et al. [5], Deniz et al. [6], Kanas et al. [11] and Orhan et al. [19] obtained the upper bound for the functional $H_2(2) = a_2 a_4 - a_2^3$ for the subclasses of bi-univalent functions.

In this work, we assume that the function φ is an analytic function with positive real part in the unit disk $\mathbb{U}$, satisfying $\varphi(0) = 1, \varphi'(0) > 0$, such that $\varphi(\mathbb{U})$ is symmetric with respect to the real axis. Such a function has the power series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots , \quad (B_1 > 0).$$

Recently, El-Qadeem and Mamon [5] defined the subclass $\mathcal{H}_\Sigma(\tau, \lambda, \delta; \varphi)$ of bi-univalent functions and obtained upper bound of the second Hankel determinant for functions in this class.

In this paper, we improve the estimates of second Hankel determinant which obtained by El-Qadeem and Mamon [5], Çağlar et al. [2], Murugusundaramoorthy and Vijaya [17] and Khani et al. [12].

1.1. Preliminaries

Lemma 1.1. *[11] If $w(z) = \sum_{n=1}^{\infty} w_n z^n$, $z \in \mathbb{U}$, is a Schwarz function, then*

$$w_2 = h(1 - w_1^2)$$

and

$$w_3 = (1 - w_1^2)(1 - |h|^2)s - w_1(1 - w_1^2)h^2,$$

for some h, s, with $|h| \le 1$ and $|s| \le 1$.

Lemma 1.2. *[7] Let v be analytic function in the unit disk $\mathbb{U}$, with $v(0) = 0$, and $|v(z)| < 1$ for all $z \in \mathbb{U}$, with the power series expansion*

$$v(z) = \sum_{n=1}^{\infty} c_n z^n.$$

Then, $|c_n| \le 1$ for all $n \in \mathbb{N}$. Furthermore, $|c_n| = 1$ for some $n \in \mathbb{N}$ if and only if $v(z) = e^{i\theta} z^n$, $\theta \in \mathbb{R}$.

2. Coefficient bounds

Definition 2.1. *[5] A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{H}_\Sigma(\tau, \lambda, \delta; \varphi)$ $(\lambda \ge 1, \delta \ge 0, \tau \in \mathbb{C} - \{0\})$, if the following conditions are satisfied:*

$$1 + \frac{1}{\tau}\left((1 - \lambda)\frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) - 1\right) \prec \varphi(z), \qquad (z \in \mathbb{U})$$

and

$$1 + \frac{1}{\tau}\left((1 - \lambda)\frac{g(w)}{w} + \lambda g'(w) + \delta w g''(w) - 1\right) \prec \varphi(w), \qquad (w \in \mathbb{U}),$$

where the function g is the inverse of f given by (2).

Remark 2.2. *For special choices of the parameters λ, τ, δ and the function φ, we can obtain the following classes as below.*

(I) By putting $\tau = 1$, $\delta = 0$ and $\varphi(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \le \beta < 1)$, we have

$$\mathcal{H}_\Sigma(1, \lambda, 0; \frac{1 + (1 - 2\beta)z}{1 - z}) = \mathcal{F}_\Sigma(\beta, \lambda),$$

where the bi-univalent function class consists of functions f satisfying the following conditions:

$$\Re\Big((1 - \lambda)\frac{f(z)}{z} + \lambda f'(z)\Big) > \beta \qquad \text{and} \qquad \Re\Big((1 - \lambda)\frac{g(w)}{w} + \lambda g'(w)\Big) > \beta.$$

The bi-univalent function class $\mathcal{F}_\Sigma(\beta, \lambda)$ was studied by Frasin and Aouf [10].

(II) By putting $\tau = \lambda = 1$ and $\varphi(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \le \beta < 1)$, we have

$$\mathcal{H}_\Sigma(1, 1, \delta; \frac{1 + (1 - 2\beta)z}{1 - z}) = \mathcal{H}_\Sigma(\delta, \beta),$$

where the bi-univalent function class consists of functions f satisfying the following conditions:

$$\Re\,(f'(z) + \delta z f''(z)) > \beta \qquad \text{and} \qquad \Re\,(g'(w) + \delta w g''(w)) > \beta.$$

The bi-univalent function class $\mathcal{H}_\Sigma(\delta, \beta)$ was studied by Frasin [9].

(III) *By putting* $\tau = \lambda = 1$, $\delta = 0$ *and* $\varphi(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \le \beta < 1)$, *we have*

$$\mathcal{H}_\Sigma(1, 1, 0; \frac{1 + (1 - 2\beta)z}{1 - z}) = \mathcal{N}_\sigma(\beta),$$

where the bi-univalent function class consists of functions f *satisfying the following conditions:*

$$\Re\left(f'(z)\right) > \beta \qquad \text{and} \qquad \Re\left(g'(w)\right) > \beta.$$

The bi-univalent function class $\mathcal{N}_\sigma(\beta)$ *was studied by Srivastava et al. [24].*

(IV) *By putting* $\tau = \lambda = 1$, $\delta = 0$ *and* $\varphi(z) = \left(\frac{1+z}{1-z}\right)^\alpha$ $(0 < \alpha \le 1)$, *we have*

$$\mathcal{H}_\Sigma(1, 1, 0; \left(\frac{1 + z}{1 - z}\right)^\alpha) = \mathcal{N}^\alpha_\sigma,$$

where the bi-univalent function class consists of functions f *satisfying the following conditions:*

$$\left|\arg\left(f'(z)\right)\right| < \frac{\alpha\pi}{2} \qquad \text{and} \qquad \left|\arg\left(g'(w)\right)\right| \frac{\alpha\pi}{2}.$$

The bi-univalent function class $\mathcal{N}^\alpha_\sigma$ *was studied by Srivastava et al. [24].*

Theorem 2.3. *If* $f \in \Sigma$ *of the form* (1) *belongs to the class* $\mathcal{H}_\Sigma(\tau, \lambda, \delta; \varphi)$, *then*

$$|a_2a_4 - a_3^2| \le B_1|\tau|^2 \begin{cases} \frac{B_1}{(1+2\lambda+6\delta)^2}; & T \le 0, S \le -T \\ \left|\frac{B_3}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} - \frac{\tau^2 B_1^3}{(1+\lambda+2\delta)^4}\right|; & (T \ge 0, S \ge -\frac{T}{2}) \ \text{or} \ (T \le 0, S \ge -T) \\ \frac{4SU-T^2}{4S}; & T > 0, S \le -\frac{T}{2} \end{cases}$$

where

$$S = \left|\frac{B_3}{(1 + \lambda + 2\delta)(1 + 3\lambda + 12\delta)} - \frac{\tau^2 B_1^3}{(1 + \lambda + 2\delta)^4}\right| - \frac{|\tau|B_1^2}{2(1 + \lambda + 2\delta)^2(1 + 2\lambda + 6\delta)} - \frac{2|B_2| + B_1}{(1 + \lambda + 2\delta)(1 + 3\lambda + 12\delta)} + \frac{B_1}{(1 + 2\lambda + 6\delta)^2},$$

$$T = \frac{|\tau|B_1^2}{2(1 + \lambda + 2\delta)^2(1 + 2\lambda + 6\delta)} + \frac{2|B_2| + B_1}{(1 + \lambda + 2\delta)(1 + 3\lambda + 12\delta)} - \frac{2B_1}{(1 + 2\lambda + 6\delta)^2}$$

and

$$U = \frac{B_1}{(1 + 2\lambda + 6\delta)^2}.$$

Proof. Since $f \in \mathcal{H}_\Sigma(\tau, \lambda, \delta; \varphi)$, there exist two Schwartz functions u, v in $\mathbb{U}$, of the form $u(z) = \sum_{n=1}^{\infty} c_n z^n$ and $v(z) = \sum_{n=1}^{\infty} d_n w^n$, with $u(0) = 0$, $v(0) = 0$ and $|u(z)| < 1$, $|v(w)| < 1$ such that

$$1 + \frac{1}{\tau}\left((1 - \lambda)\frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) - 1\right) = \varphi(u(z)) \tag{3}$$

and

$$1+\frac{1}{\tau}\left((1-\lambda)\frac{g(w)}{w}+\lambda g'(w)+\delta w g''(w)-1\right)=\varphi(v(w)), \tag{4}$$

where

$$\varphi(u(z))=1+B_1c_1z+(B_1c_2+B_2c_1^2)z^2+(B_1c_3+2B_2c_1c_2+B_3c_1^3)z^3+\cdots \tag{5}$$

and

$$\varphi(v(w))=1+B_1d_1w+(B_1d_2+B_2d_1^2)w^2+(B_1d_3+2B_2d_1d_2+B_3d_1^3)w^3+\cdots. \tag{6}$$

Since $f\in\Sigma$ has the Taylor series expansion (1) and $g=f^{-1}$ the series (2), we have

$$1+\frac{1}{\tau}\left((1-\lambda)\frac{f(z)}{z}+\lambda f'(z)+\delta z f''(z)-1\right)=1+\sum_{n=2}^{\infty}\left(\frac{1+(n-1)(\lambda+n\delta)}{\tau}\right)a_nz^n \tag{7}$$

and

$$1+\frac{1}{\tau}\left((1-\lambda)\frac{g(w)}{w}+\lambda g'(w)+\delta w g''(w)-1\right)=1+\sum_{n=2}^{\infty}\left(\frac{1+(n-1)(\lambda+n\delta)}{\tau}\right)A_nw^n. \tag{8}$$

Now, from (3), (5) and (7), we get

$$\frac{(1+\lambda+2\delta)}{\tau}a_2=B_1c_1, \tag{9}$$

$$\frac{(1+2\lambda+6\delta)}{\tau}a_3=B_1c_2+B_2c_1^2, \tag{10}$$

$$\frac{(1+3\lambda+12\delta)}{\tau}a_4=B_1c_3+2B_2c_1c_2+B_3c_1^3. \tag{11}$$

Similarly, from (4), (6) and (8), we have

$$-\frac{(1+\lambda+2\delta)}{\tau}a_2=B_1d_1, \tag{12}$$

$$\frac{(1+2\lambda+6\delta)}{\tau}(2a_2^2-a_3)=B_1d_2+B_2d_1^2, \tag{13}$$

$$-\frac{(1+3\lambda+12\delta)}{\tau}(5a_2^3-5aa_3+a_4)=B_1d_3+2B_2d_1d_2+B_3d_1^3. \tag{14}$$

From (9) and (12), we have

$$c_1=-d_1 \tag{15}$$

and

$$a_2=\frac{B_1c_1\tau}{1+\lambda+2\delta}. \tag{16}$$

Now from (10) and (13), we get that

$$a_3=\frac{B_1^2c_1^2\tau^2}{(1+\lambda+2\delta)^2}+\frac{B_1\tau(c_2-d_2)}{2(1+2\lambda+6\delta)}. \tag{17}$$

Also from (11) and (14), we get that

$$a_4=\frac{5B_1^2c_1\tau^2(c_2-d_2)}{4(1+\lambda+2\delta)(1+2\lambda+6\delta)}+\frac{B_1\tau(c_3-d_3)}{2(1+3\lambda+12\delta)}+\frac{B_3c_1^3\tau}{1+3\lambda+12\delta}+\frac{B_2c_1\tau(c_2+d_2)}{1+3\lambda+12\delta}. \tag{18}$$

Thus we can easily obtain that

$$a_2a_4-a_3^2=\left(\frac{B_1B_3\tau^2}{(1+\lambda+2\delta)(1+3\lambda+12\delta)}-\frac{B_1^4\tau^4}{(1+\lambda+2\delta)^4}\right)c_1^4 + \frac{B_1^3c_1^2\tau^3(c_2-d_2)}{4(1+\lambda+2\delta)^2(1+2\lambda+6\delta)}+\frac{B_1B_2c_1^2\tau^2(c_2+d_2)}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} + \frac{B_1^2c_1\tau^2(c_3-d_3)}{2(1+\lambda+2\delta)(1+3\lambda+12\delta)}-\frac{B_1^2\tau^2(c_2-d_2)^2}{4(1+2\lambda+6\delta)^2}. \tag{19}$$

According to Lemma 1.1, we have

$$c_2=h(1-c_1^2) \quad , \quad d_2=j(1-d_1^2), \tag{20}$$

$$c_3=(1-c_1^2)(1-|h|^2)s-c_1(1-c_1^2)h^2, \tag{21}$$

$$d_3=(1-d_1^2)(1-|j|^2)w-d_1(1-d_1^2)j^2, \tag{22}$$

for some h, j, s, w with $|h|\le 1$, $|j|\le 1$, $|s|\le 1$ and $|w|\le 1$.

Hence by (20), (21) and (22), we have

$$c_2+d_2=(1-c_1^2)(h+j) \quad , \quad c_2-d_2=(1-c_1^2)(h-j), \tag{23}$$

$$c_3-d_3=(1-c_1^2)\Big((1-|h|^2)s-(1-|j|^2)w\Big)-c_1(1-c_1^2)(h^2+j^2). \tag{24}$$

By substituting the relations (23), (24) in (19), we obtain

$$a_2a_4-a_3^2=\left(\frac{B_1B_3\tau^2}{(1+\lambda+2\delta)(1+3\lambda+12\delta)}-\frac{B_1^4\tau^4}{(1+\lambda+2\delta)^4}\right)c_1^4 + \frac{B_1^3c_1^2\tau^3(1-c_1^2)(h-j)}{4(1+\lambda+2\delta)^2(1+2\lambda+6\delta)}+\frac{B_1B_2c_1^2\tau^2(1-c_1^2)(h+j)}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} + \frac{B_1^2c_1\tau^2(1-c_1^2)\Big((1-|h|^2)s-(1-|j|^2)w-c_1(h^2+j^2)\Big)}{2(1+\lambda+2\delta)(1+3\lambda+12\delta)} - \frac{B_1^2\tau^2(1-c_1^2)^2(h-j)^2}{4(1+2\lambda+6\delta)^2}. \tag{25}$$

It follows that

$$|a_2a_4-a_3^2|=\Bigg|\left(\frac{B_1B_3\tau^2}{(1+\lambda+2\delta)(1+3\lambda+12\delta)}-\frac{B_1^4\tau^4}{(1+\lambda+2\delta)^4}\right)c_1^4 + \frac{B_1^3c_1^2\tau^3(1-c_1^2)(h-j)}{4(1+\lambda+2\delta)^2(1+2\lambda+6\delta)}+\frac{B_1B_2c_1^2\tau^2(1-c_1^2)(h+j)}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} + \frac{B_1^2c_1\tau^2(1-c_1^2)\Big((1-|h|^2)s-(1-|j|^2)w\Big)}{2(1+\lambda+2\delta)(1+3\lambda+12\delta)} - \frac{B_1^2c_1^2\tau^2(1-c_1^2)(h^2+j^2)}{2(1+\lambda+2\delta)(1+3\lambda+12\delta)}-\frac{B_1^2\tau^2(1-c_1^2)^2(h-j)^2}{4(1+2\lambda+6\delta)^2}\Bigg|. \tag{26}$$

As $|c_1| \leq 1$, we may assume without restriction that $c = c_1 \in [0,1]$, so

$$\begin{aligned}|a_2a_4 - a_3^2| \leq & B_1|\tau|^2\Bigg\{\left|\frac{B_3}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} - \frac{B_1^3\tau^2}{(1+\lambda+2\delta)^4}\right|c^4 \\ & + \frac{|\tau|B_1^2c^2(1-c^2)(|h|+|j|)}{4(1+\lambda+2\delta)^2(1+2\lambda+6\delta)} + \frac{|B_2|c^2(1-c^2)(|h|+|j|)}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} \\ & + \frac{B_1c(1-c^2)\left((1-|h|^2)|s| + (1-|j|^2)|w|\right)}{2(1+\lambda+2\delta)(1+3\lambda+12\delta)} \\ & + \frac{B_1c^2(1-c^2)(|h|^2+|j|^2)}{2(1+\lambda+2\delta)(1+3\lambda+12\delta)} + \frac{B_1(1-c^2)^2(|h|+|j|)^2}{4(1+2\lambda+6\delta)^2}\Bigg\}. \end{aligned} \tag{27}$$

Since $|s| \leq 1$ and $|w| \leq 1$, we get

$$\begin{aligned}|a_2a_4 - a_3^2| \leq & B_1|\tau|^2\Bigg\{\left|\frac{B_3}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} - \frac{B_1^3\tau^2}{(1+\lambda+2\delta)^4}\right|c^4 \\ & + \frac{|\tau|B_1^2c^2(1-c^2)(|h|+|j|)}{4(1+\lambda+2\delta)^2(1+2\lambda+6\delta)} + \frac{|B_2|c^2(1-c^2)(|h|+|j|)}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} \\ & + \frac{B_1(c^2-c)(1-c^2)(|h|^2+|j|^2)}{2(1+\lambda+2\delta)(1+3\lambda+12\delta)} + \frac{B_1c(1-c^2)}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} \\ & + \frac{B_1(1-c^2)^2(|h|+|j|)^2}{4(1+2\lambda+6\delta)^2}\Bigg\}. \end{aligned} \tag{28}$$

Now, for $\mu = |h| \leq 1$ and $\gamma = |j| \leq 1$, we get

$$|a_2a_4 - a_3^2| \leq B_1|\tau|^2\left[T_1 + (\mu+\gamma)T_2 + (\mu^2+\gamma^2)T_3 + (\mu+\gamma)^2T_4\right] = B_1|\tau|^2F(\mu,\gamma), \tag{29}$$

where

$$\begin{aligned} T_1 = T_1(c) &= \left|\frac{B_3}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} - \frac{B_1^3\tau^2}{(1+\lambda+2\delta)^4}\right|c^4 + \frac{B_1c(1-c^2)}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} \geq 0 \\ T_2 = T_2(c) &= \frac{c^2(1-c^2)}{1+\lambda+2\delta}\left(\frac{|\tau|B_1^2}{4(1+\lambda+2\delta)1+2\lambda+6\delta} + \frac{|B_2|}{(1+3\lambda+12\delta)}\right) \geq 0 \\ T_3 = T_3(c) &= \frac{B_1(c^2-c)(1-c^2)}{2(1+\lambda+2\delta)(1+3\lambda+12\delta)} \leq 0 \\ T_4 = T_4(c) &= \frac{B_1(1-c^2)^2}{4(1+2\lambda+6\delta)^2} \geq 0. \end{aligned}$$

Now, we need to maximize $F(\mu,\gamma)$ in the closed square $S = [0,1] \times [0,1]$ for $c \in [0,1]$. We must investigate the maximum of $F(\mu,\gamma)$ according to $c = 0$, $c = 1$, and $c \in (0,1)$ taking into account the sign of $F_{\mu\mu}F_{\gamma\gamma} - (F_{\mu\gamma})^2$.

First, for $c = 0$, we have

$$F(\mu,\gamma) = \frac{B_1(\mu+\gamma)^2}{4(1+2\lambda+6\delta)^2}. \tag{30}$$

Thus, we have

$$\max\{F(\mu,\gamma) : (\mu,\gamma) \in [0,1]\times[0,1]\} = F(1,1) = \frac{B_1}{(1+2\lambda+6\delta)^2}. \tag{31}$$

Second, for $c = 1$, we have

$$F(\mu,\gamma) = \left|\frac{B_3}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} - \frac{B_1^3\tau^2}{(1+\lambda+2\delta)^4}\right|. \tag{32}$$

Thus, we get

$$\max\{F(\mu,\gamma):\ (\mu,\gamma)\in[0,1]\times[0,1]\} = \left|\frac{B_3}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} - \frac{B_1^3\tau^2}{(1+\lambda+2\delta)^4}\right|. \tag{33}$$

At the end, let $c \in (0,1)$. Since $T_3 < 0$ and $T_3 + 2T_4 > 0$, we conclude that

$$F_{\mu\mu}F_{\gamma\gamma} - (F_{\mu\gamma})^2 < 0.$$

Thus, the function $F(\mu,\gamma)$ can't have a local maximum in the interior of the square S. So, we investigate the maximum of $F(\mu,\gamma)$ on the boundary of the square S.

For $\mu = 0$ and $0 \le \gamma \le 1$ (similarly $\gamma = 0$ and $0 \le \mu \le 1$), we obtain

$$F(0,\gamma) = H(\gamma) = T_1 + \gamma T_2 + \gamma^2(T_3 + T_4). \tag{34}$$

In order to find the maximum of $H(\gamma)$, we consider the situation of the function $H(\gamma)$ as increasing or decreasing as follows:

$$H'(\gamma) = T_2 + 2\gamma(T_3 + T_4).$$

(*i*) Suppose that $T_3 + T_4 \ge 0$. In this case $H'(\gamma) > 0$; that is, $H(\gamma)$ is an increasing function. Hence the maximum of $H(\gamma)$ occurs at $\gamma = 1$ and

$$\max\{H(\gamma):\ \gamma\in[0,1]\} = H(1) = T_1 + T_2 + T_3 + T_4.$$

(*ii*) Suppose that $T_3 + T_4 < 0$. Then we consider for critical point $\gamma = \frac{T_2}{-2(T_3+T_4)} = \frac{T_2}{2\theta}$ where $\theta = -(T_3+T_4) > 0$, the following two cases:

Case 2.4. *Let* $\gamma = \frac{T_2}{2\theta} > 1$. *Then* $\theta < \frac{T_2}{2} \le T_2$, *and* $T_2 + T_3 + T_4 \ge 0$. *Therefore*

$$H(0) = T_1 \le T_1 + T_2 + T_3 + T_4 = H(1).$$

Case 2.5. *Let* $\gamma = \frac{T_2}{2\theta} \le 1$. *Since* $\frac{T_2}{2} \ge 0$, *we get* $\frac{T_2^2}{4\theta} \le \frac{T_2}{2} \le T_2$. *Also, we have* $H(1) = T_1 + T_2 + T_3 + T_4 \le T_1 + T_2$. *Therefore,*

$$H(0) = T_1 \le T_1 + \frac{T_2^2}{4\theta} = H\left(\frac{T_2}{2\theta}\right) \le T_1 + T_2.$$

Thus, we observe that the maximum of $H(\gamma)$ occurs when $T_3 + T_4 \ge 0$, it means

$$\max\{H(\gamma):\ \gamma\in[0,1]\} = H(1) = T_1 + T_2 + \underbrace{(T_3 + T_4)}_{\ge 0} \tag{35}$$

for any fixed $c \in (0,1)$.

For $\mu = 1$ and $0 \le \gamma \le 1$ (similarly $\gamma = 1$ and $0 \le \mu \le 1$), we obtain

$$F(1,\gamma) = G(\gamma) = T_1 + T_2 + T_3 + T_4 + \gamma(T_2 + 2T_4) + \gamma^2(T_3 + T_4). \tag{36}$$

In order to obtain the maximum of $G(\gamma)$, we consider the situation of the function $G(\gamma)$ as increasing or decreasing as follows:

$$G'(\gamma) = T_2 + 2T_4 + 2(T_3 + T_4)\gamma.$$

(*iii*) Suppose that $T_3 + T_4 \geq 0$. In this case $G'(\gamma) > 0$; that is, $G(\gamma)$ is an increasing function. Hence the maximum of $G(\gamma)$ occurs at $\gamma = 1$ and

$$\max\{G(\gamma) : \gamma \in [0,1]\} = G(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

(*iv*) Suppose that $T_3 + T_4 < 0$. Then we consider for critical point $\gamma = \frac{T_2+2T_4}{-2(T_3+T_4)} = \frac{T_2+2T_4}{2\theta}$ where $\theta = -(T_3 + T_4) > 0$, the following two cases:

Case 2.6. *Let* $\gamma = \frac{T_2+2T_4}{2\theta} > 1$. *Then* $\theta < \frac{T_2+2T_4}{2} \leq T_2 + 2T_4$, *and* $T_2 + T_3 + 3T_4 \geq 0$. *Therefore*

$$\begin{aligned} G(0) = T_1 + T_2 + T_3 + T_4 &\leq T_1 + T_2 + T_3 + T_4 + (T_2 + T_3 + 3T_4) \\ &= G(1) = T_1 + 2T_2 + 2T_3 + 4T_4. \end{aligned}$$

Case 2.7. *Let* $\gamma = \frac{T_2+2T_4}{2\theta} \leq 1$. *Since* $\frac{T_2+2T_4}{2} \geq 0$, *we get that*

$$\frac{(T_2 + 2T_4)^2}{4\theta} \leq \frac{T_2 + 2T_4}{2} \leq T_2 + 2T_4.$$

Therefore,

$$\begin{aligned} G(0) =& T_1 + T_2 + T_3 + T_4 \leq T_1 + T_2 + T_3 + T_4 + \frac{(T_2 + 2T_4)^2}{4\theta} \\ =& G\left(\frac{T_2 + 2T_4}{2\theta}\right) \leq T_1 + T_2 + T_3 + T_4 + T_2 + 2T_4 = T_1 + 2T_2 + T_3 + 3T_4 \\ =& T_1 + 2T_2 + \underbrace{(T_3 + T_4)}_{\leq 0} + 2T_4. \end{aligned}$$

Thus, the function $G(\gamma)$ gets its maximum when $T_3 + T_4 \geq 0$, it means

$$\max\{G(\gamma) : \gamma \in [0,1]\} = G(1) = T_1 + 2T_2 + 2\underbrace{(T_3 + T_4)}_{\geq 0} + 2T_4$$

for any fixed $c \in (0, 1)$.

Since $H(1) \leq G(1)$ for $c \in [0, 1]$, then

$$\max\{F(\mu,\gamma) : (\mu,\gamma) \in [0,1] \times [0,1]\} = F(1,1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

Let $K : [0, 1] \to \mathbb{R}$,

$$\begin{aligned} K(c) = B_1|\tau|^2 \max\{F(\mu,\gamma) : (\mu,\gamma) \in [0,1] \times [0,1]\} &= B_1|\tau|^2 F(1,1) \\ &= B_1|\tau|^2(T_1 + 2T_2 + 2T_3 + 4T_4). \end{aligned} \tag{37}$$

Now puting T_1, T_2, T_3 and T_4 in the function K, we have

$$\begin{aligned} K(c) = B_1|\tau|^2 \Bigg\{ \Bigg(& \left| \frac{B_3}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} - \frac{\tau^2 B_1^3}{(1+\lambda+2\delta)^4} \right| - \frac{|\tau| B_1^2}{2(1+\lambda+2\delta)^2(1+2\lambda+6\delta)} \\ & - \frac{2|B_2| + B_1}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} + \frac{B_1}{(1+2\lambda+6\delta)^2} \Bigg) c^4 \\ & + \left(\frac{|\tau| B_1^2}{2(1+\lambda+2\delta)^2(1+2\lambda+6\delta)} + \frac{2|B_2| + B_1}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} - \frac{2B_1}{(1+2\lambda+6\delta)^2} \right) c^2 \\ & + \frac{B_1}{(1+2\lambda+6\delta)^2} \Bigg\}. \end{aligned} \tag{38}$$

By putting $c^2 = t$ in equation (38), it may have the form

$$K(t) = B_1|\tau|^2(St^2 + Tt + U), \quad t \in [0,1]$$

where

$$S = \left|\frac{B_3}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} - \frac{\tau^2 B_1^3}{(1+\lambda+2\delta)^4}\right| - \frac{|\tau|B_1^2}{2(1+\lambda+2\delta)^2(1+2\lambda+6\delta)}$$
$$- \frac{2|B_2| + B_1}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} + \frac{B_1}{(1+2\lambda+6\delta)^2},$$
$$T = \frac{|\tau|B_1^2}{2(1+\lambda+2\delta)^2(1+2\lambda+6\delta)} + \frac{2|B_2| + B_1}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} - \frac{2B_1}{(1+2\lambda+6\delta)^2}$$
$$U = \frac{B_1}{(1+2\lambda+6\delta)^2}.$$

Since

$$\max_{0\le t\le 1}(St^2 + Tt + U) = \begin{cases} U; & T \le 0, S \le -T \\ S + T + U; & (T \ge 0, S \ge -\frac{T}{2}) \text{ or } (T \le 0, S \ge -T) \\ \frac{4SU - T^2}{4S}; & T > 0, S \le -\frac{T}{2}, \end{cases}$$

it gives,

$$|a_2a_4 - a_3^2| \le B_1|\tau|^2 \begin{cases} U; & T \le 0, S \le -T \\ S + T + U; & (T \ge 0, S \ge -\frac{T}{2}) \text{ or } (T \le 0, S \ge -T) \\ \frac{4SU - T^2}{4S}; & T > 0, S \le -\frac{T}{2}. \end{cases}$$

This completes the proof. □

3. Corollaries and Consequences

By taking

$$\tau = 1,\ \delta = 0 \text{ and } \varphi(z) = \frac{1 + (1-2\beta)z}{1-z} \quad (0 \le \beta < 1)$$

in Theorem 2.3, we conclude the following corollary.

Corollary 3.1. *Let f given by* (1) *be in the class $\mathcal{F}_\Sigma(\beta, \lambda)$. Then*

$$|a_2a_4 - a_3^2| \le \frac{(1-\beta)^2}{(1+2\lambda)^2}\left(4 - \frac{1}{(1+3\lambda)}\frac{\xi}{\zeta}\right)$$

where

$$\xi = \Big[(1+2\lambda)(1+3\lambda)(1-\beta) + (1+\lambda)(1+4\lambda+6\lambda^2)\Big]^2,$$
$$\zeta = (1+2\lambda)^2\left|(1+\lambda)^3 - 4(1+3\lambda)(1-\beta)^2\right| + (1+\lambda)^2\Big[(1+2\lambda)(1+3\lambda)\beta$$
$$-\left(9\lambda^3 + 23\lambda^2 + 15\lambda + 3\right)\Big].$$

Remark 3.2. *The bound on* $|a_2a_4 - a_3^2|$ *given in Corollary 3.1 is better than that given in [17, Concluding Remarks]. Because*

$$\frac{(1-\beta)^2}{(1+2\lambda)^2}\left(4 - \frac{1}{(1+3\lambda)}\frac{\xi}{\zeta}\right) \le \frac{9(1+\lambda)^2(1-\beta)^2}{2(1+3\lambda)\left[(1+\lambda)^3 - 2(1-\beta)^2(1+3\lambda)\right]};$$

$$(1-\beta)^2 \ge \frac{(1+\lambda)^3}{2(1+3\lambda)} - \frac{9(1+2\lambda)^2(1+\lambda)^2}{16(1+3\lambda)^2}\left(\frac{\vartheta}{\vartheta + [1+5\lambda+8\lambda^2+3\lambda^3]^2}\right),$$

where

$$\vartheta = (1+3\lambda)(1+\lambda)^2(1+5\lambda+9\lambda^2+5\lambda^3).$$

By taking

$$\tau = \lambda = 1 \text{ and } \varphi(z) = \frac{1+(1-2\beta)z}{1-z} \quad (0 \le \beta < 1)$$

in Theorem 2.3, we conclude the following corollary.

Corollary 3.3. *Let* f *given by* (1) *be in the class* $\mathcal{H}_\Sigma(\delta, \beta)$. *Then*

$$|a_2a_4 - a_3^2| \le \frac{(1-\beta)^2}{(1+2\delta)^2}\left(\frac{4}{9} - \frac{1}{72(1+3\delta)}\frac{\rho}{\nu}\right),$$

where

$$\rho = \left[6(1+2\delta)(1+3\delta)(1-\beta) + (1+\delta)(11+44\delta+60\delta^2)\right]^2,$$
$$\nu = 9(1+2\delta)^2\left|(1+\delta)^3 - 2(1+3\delta)(1-\beta)^2\right| + (1+\delta)^2\Big[6(1+2\delta)(1+3\delta)\beta - (25+125\delta+196\delta^2+84\delta^3)\Big].$$

By taking $\delta = 0$ in Corollary 3.3, we conclude the following corollary.

Corollary 3.4. *Let* f *given by* (1) *be in the class* $\mathcal{N}_\sigma(\beta)$. *Then*

$$|a_2a_4 - a_3^2| \le (1-\beta)^2\left(\frac{4}{9} - \frac{(17-6\beta)^2}{36^2\left[|\frac{1}{2} - (1-\beta)^2| + \frac{\beta}{3} - \frac{25}{18}\right]}\right).$$

Remark 3.5. *The bound on* $|a_2a_4 - a_3^2|$ *given in Corollary 3.4 is better than that given in [2, Theorem 1]. Because*

$$(1-\beta)^2\left(\frac{4}{9} - \frac{(17-6\beta)^2}{36^2\left[|\frac{1}{2} - (1-\beta)^2| + \frac{\beta}{3} - \frac{25}{18}\right]}\right) \le \frac{(1-\beta)^2}{2}\left(2(1-\beta)^2+1\right); \;\; \beta \le 1 - \sqrt{\frac{83}{240}}.$$

By taking

$$\tau = \lambda = 1, \; \delta = 0 \text{ and } \varphi(z) = \left(\frac{1+z}{1-z}\right)^\alpha \quad (0 < \alpha \le 1)$$

in Theorem 2.3, we conclude the following corollary.

Corollary 3.6. *Let* f *given by* (1) *be in the class* $\mathcal{N}_\sigma^\alpha$. *Then*

$$|a_2a_4 - a_3^2| \le \begin{cases} \frac{4\alpha^2}{9}, & 0 < \alpha \le \frac{7}{24} \\ \alpha^2\left(\frac{4}{9} - \frac{\left(\frac{2}{3}\alpha - \frac{7}{36}\right)^2}{2\left[|\frac{1}{12} - \frac{\alpha^2}{3}| - \frac{2}{3}\alpha - \frac{1}{36}\right]}\right), & \frac{7}{24} < \alpha \le 1. \end{cases}$$

Remark 3.7. *The bound on* $|a_2a_4 - a_3^2|$ *given in Corollary 3.6 is better than that given in [2, Theorem 2] and [12, Corollary 2.6]. Because*

$$\alpha^2\left(\frac{4}{9} - \frac{\left(\frac{2}{3}\alpha - \frac{7}{36}\right)^2}{2\left[\left|\frac{1}{12} - \frac{\alpha^2}{3}\right| - \frac{2}{3}\alpha - \frac{1}{36}\right]}\right) \leq \frac{\alpha^2(8\alpha^2+1)}{6}; \quad \alpha \geq \frac{\sqrt{406}}{32}.$$

Remark 3.8. *The bound on* $|a_2a_4 - a_3^2|$ *given in Theorem 2.3 is better than that given in [5, Theorem 2.1].*

References

[1] D. Bansal, H M. Srivastava, Coefficient estimates for a subclass of analytic and bi-univalent functions, J. Egyptian Math. Soc. 23(2) (2015) 242–246.

[2] M. Çağlar, E. Deniz, H. M. Srivastava, Second Hankel determinant for certain subclasses of bi-univalent functions, Turk. J. Math. 41 (2017) 694–706.

[3] M. Çağlar, E. Y. Erdağı, E. Deniz, Upper bound of second Hankel determinant for a subclass of bi-univalent functions, AIP Conference Proceedings 1833, 020009 (2017); https://doi.org/10.1063/1.4981657

[4] M. Çağlar, H. Orhan, N. Yağmur, Coefficient bounds for new subclasses of bi-univalent functions, Filomat 27 (2013), 1165–1171.

[5] A. H. El-Qadeem, M. A. Mamon, Upper bound of second Hankel determinant of subclass of bi-univalent functions defined by subordination, arXiv:1908.07350v1.

[6] E. Deniz E, M. Çağlar, H. Orhan, Second Hankel determinant for bi-starlike and bi-convex functions of order β, Appl. Math. Comput. 271 (2015) 301–307.

[7] P. L. Duren, Univalent Functions, Grundlehren der mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.

[8] R. Ehrenborg, The Hankel determinant of exponantial polynomials, Amer. Math. Monthly 107 (2000) 557–560.

[9] B. A. Frasin, Coefficient bounds for certain classes of bi-univalent functions, Hacet. J. Math. Stat. 43(3) (2014) 383–389.

[10] B. A. Frasin, M. K. Aouf, New subclasses of bi-univalent functions, Appl. Math. Lett. **24** (2011), no. 9, 1569–1573.

[11] S. Kanas, E. A. Adegani, A. Zireh, An unified approach to second Hankel determinant of bi-subordinate functions, Mediterr. J. Math. 14(6) (2017) 12 pp.

[12] M. H. Khani, A. Zireh, E. A. Adegani, The second Hankel determinant problem for a class of bi-univalent functions, J. Math. Fund. Sci. 51(2) (2019) 191–203.

[13] J. W. Layman, The Hankel transform and some of its properties, J. Integer Seq. 4 (2001) 1–11.

[14] M. Lewin, On a coefficient problem for bi-univalent functions, Proc Amer Math Soc 18 (1967) 63–68.

[15] A. Motamednezhad, S. Salehian, Faber polynomial coefficient estimates for a certain subclass of meromorphic bi-univalent functions, Asian-Eur. J. Math. 13(1) (2020) 2050076 (12 pages).

[16] A. Motamednezhad, S. Salehian, New subclass of bi-univalent functions by (p, q)-derivative operator, Honam Math. J. 41(2) (2019) 381–390.

[17] G. Murugusundaramoorthy, K. Vijaya, Second Hankel determinant for bi-univalent analytic functions associated with Hohlov operator, *International Journal of Analysis and Applications* 8(1) (2015) 22–29.

[18] J. W. Noonan, D. K. Thomas, On the second Hankel determinant of a real ly mean p-valent functions, Trans. Amer. Math. Soc. 223(2) (1976) 337–346.

[19] H. Orhan, N. Magesh, J. Yamini, Bounds for the second Hankel determinant of certain bi-univalent functions, Turk. J. Math. 40 (2016) 679–687.

[20] S. Salehian, A. Zireh, Coefficient estimats for certain subclass of meromorphic and bi-univalent functions, Commun. Korean Math. Soc. 32(2) (2017) 389–397.

[21] H. M. Srivastava, S. Bulut, M. Çağlar, N. Yağmur, Coefficient estimates for a general subclass of analytic and bi-univalent functions, Filomat 27 (2013), 831–842.

[22] H. M. Srivastava, S. Altinkaya, S. Yalcin, Certain subclasses of bi-univalent functions associated with the Horadam polynomials, Iran. J. Sci. Technol. Trans. A Sci. 43(4) (2019) 1873–1879.

[23] H. M. Srivastava, S. Gaboury, F. Ghanim, Coefficient estimates for a general subclass of analytic and bi-univalent functions of the Ma-Minda type, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 112(4) (2018) 1157–1168

[24] H. M. Srivastava, A. K. Mishra, P. Gochhayat, Certain sublcasses of analytic and bi-univalent functions, Appl. Math. Lett. 23 (2010) 1188–1192.

[25] A. Zireh, S. Salehian, On the certain subclass of analytic and bi-univalent functions defined by convolution, Acta Univ. Apulensis Math. Inform. 44 (2015) 9–19.