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A Discussion on the Coincidence Quasi-Best Proximity Points

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Abstract. In this paper, we first introduce a new class of the pointwise cyclic-noncyclic proximal contraction pairs. Then we consider the coincidence quasi-best proximity point problem for this class. Finally, we study the coincidence quasi-best proximity points of weak cyclic-noncyclic Kannan contraction pairs. We consider an example to indicate the validity of the main result.

1. Introduction

Best proximity theory is one of the interesting research topics of nonlinear functional analysis. Roughly speaking, best proximity appears when a mapping is a fixed point free, that is, $Tx \neq x$ for any x in the domain of T. Here, the following natural question appears: Is there x such that the distance between x and its image, Tx, is as small as possible. The affirmative answer reveals the definition best proximity point. Regarding the connection with the fixed point theory and its potential application, this topic has attracted the attention of researchers, see e.g. [1–3, 6, 9–12, 14–23, 28, 29, 31]

A self-mapping T on a subset K of a Banach space X is called Lipschitz if there exists a positive constant L such that

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in K. \tag{1}$$

The constant L, in the inequality (1), is called Lipschitz constant. If $L \in [0, 1)$, then the self-mapping T is called contraction. Due to Banach, it is know that each contraction in a complete normed space possesses a unique fixed point. The analog of Banach result, in the context of a complete metric space, was proved by Caccиopoli [5]. Incase of $L = 1$, the mapping T is called nonexpansive. Despite contraction mapping, in the setting of a complete norm space, nonexpansive mappings can be fixed point free. We may consider $Tx = 1 + x$, defined on a real line, as a concrete example for this fact.

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For a nonempty, bounded and convex subset F of a Banach space X, we put:

$$\begin{aligned} r_x(F) &= \sup\{\|x-y\| : y \in F\}; \\ r(F) &= \inf\{r_x(F) : x \in F\}; \\ F_c &= \{x \in F : r_x(F) = r(F)\}. \end{aligned}$$

In this paper, when we say that a pair (A, B) of sets in a Banach space satisfies a property $(*)$ if each of the sets A and B has the property $(*)$. For example, the pair (A, B) is called convex if both A and B are convex. The symbol $\subseteq$ refer to a relation between two pairs in the following way:

$$(A, B) \subseteq (E, F) \Leftrightarrow A \subseteq E \text{ and } B \subseteq F.$$

In addition, we will use the following notations:

$$\begin{aligned} \delta(A, B) &= \sup\{\|x-y\| : x \in A, y \in B\}; \\ \delta(x, B) &= \sup\{\|x-y\| : y \in B\}. \end{aligned}$$

Let $T, S : A \cup B \to A \cup B$ be self-mapping, where A, B are nonempty subsets of Banach space X. We say that a self-mapping T (respectively, S) is called *cyclic* (respectively, *noncyclic*) provided that $T(A) \subseteq B$ and $T(B) \subseteq A$; (respectively, $S(A) \subseteq A$ and $S(B) \subseteq B$).

In particular, for a cyclic mapping $T : A \cup B \to A \cup B$, a pair $(x, y) \in A \cup B$ is said to be a best proximity pair provided that

$$\| x - Tx \| = \| y - Ty \| = \operatorname{dist}(A, B).$$

Note that if a mapping T has a fixed point then $\operatorname{dist}(A, B) = 0$, and, further, $A \cap B \neq \emptyset$.

Gabeleh *et al.* [11] considered a cyclic-noncyclic pair $(T; S)$ on $A \cup B$ that is, $T : A \cup B \to A \cup B$ is cyclic and $S : A \cup B \to A \cup B$ is noncyclic, where (A, B) is a nonempty pair of subsets of Banach space X. A point $p \in A \cup B$ a *coincidence best proximity point* [11] for $(T; S)$ provided that

$$d(Sp, Tp) = \operatorname{dist}(A, B).$$

Note that if $S = I$, identity map, then $p \in A \cup B$ is a best proximity point for T, where I is the identity map on $A \cup B$.

In year 2019, Abkar and Norouzian [28] introduced the problem of the coincidence quasi-best proximity point as follows and proved the existence of such points for quasi-cyclic-noncyclic contraction pairs. The problem of the coincidence quasi-best proximity point is important because it also includes the best proximity point and the coincidence best proximity point in the particular case.

Definition 1.1. *([28]) Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and $T, S : X \to X$ be a quasi-cyclic-noncyclic pair on $A \cup B$; that is, $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. A point $p \in A \cup B$ is said to be a coincidence quasi-best proximity point for $(T; S)$ provided that*

$$d(Sp, Tp) = \operatorname{dist}(S(A), S(B)).$$

Note that if $S = I$, identity map, then p reduces to a best proximity point for T.

Definition 1.2. *([24]) We say that a Banach space X has the property $(\mathbf{N})$ if every bounded decreasing sequence of nonempty, closed and convex subsets of X have a nonempty intersection.*

For $C \subseteq X$, we denote the diameter of C by $\delta(C)$. A point $x \in C$ is a diametral point of C provided that $\sup\{\|x - y\| : y \in C\} = \delta(C)$. A convex set $K \subseteq X$ is said to have normal structure if for each bounded convex subset H of K which contains at least two points, there is some point $x \in H$ which is not a diametral point of H.

Lemma 1.3. *([24]) Let F be a closed and convex subset of a Banach space X.*

(i) If X has the property ($\mathbf{N}$), then F_c is nonempty, closed and convex.

(ii) If F has normal structure and contains at least two points, then $\delta(F_c) < \delta(F)$.

Definition 1.4. *Let (A, B) be a nonempty pair of subsets of a Banach space X and $S : A \cup B \to A \cup B$ be a noncyclic mapping on $A \cup B$. Then, we say that a convex pair $(S(A), S(B))$ is a proximal pair if for each $(a_1, b_1) \in A \times B$, there exists $(a_2, b_2) \in A \times B$ such that for each $i, j \in \{1, 2\}$ which i, j are different from each other, we have*

$$\|Sa_i - Sb_j\| = \mathrm{dist}(S(A), S(B)).$$

Given (A, B) a pair of nonempty subsets of a Banach space X, then proximal pair $(S(A), S(B))$ is the pair $(S(A_0^s), S(B_0^s))$ given by

$$A_0^s := \{a \in A : \|Sa - Sb\| = \mathrm{dist}(S(A), S(B)) \text{ for some } b \in B\},$$

$$B_0^s := \{b \in B : \|Sa - Sb\| = \mathrm{dist}(S(A), S(B)) \text{ for some } a \in A\}.$$

In fact, if the pair $(S(A), S(B))$ be nonempty, weakly compact and convex, then the pair $(S(A_0^s), S(B_0^s))$ is nonempty, weakly compact and convex. Furthermore, we have

$$\mathrm{dist}(S(A_0^s), S(B_0^s)) = \mathrm{dist}(S(A), S(B)). \tag{2}$$

The proof of the above statement (2) can be derived in [7] and [3].

In [7], Eldred *et al.* introduced the notion of proximal normal structure and used it to study the existence of a best proximity point for relatively nonexpansive mapping.

Definition 1.5. *[7] Let (K_1, K_2) be a nonempty pair of subsets of a Banach space X and $S : K_1 \cup K_2 \to K_1 \cup K_2$ be a noncyclic mapping on $K_1 \cup K_2$. We say that a convex pair $(S(K_1), S(K_2))$ has proximal normal structure (**PNS**) if for any closed, bounded, convex and proximal pair $(S(H_1), S(H_2)) \subseteq (S(K_1), S(K_2))$ which*

$$\mathrm{dist}(S(H_1), S(H_2)) = \mathrm{dist}(S(K_1), S(K_2)), \qquad \delta(S(H_1), S(H_2)) > \mathrm{dist}(S(H_1), S(H_2)),$$

there exists $(x, y) \in H_1 \times H_2$ such that

$$\delta(Sx, S(H_2)) < \delta(S(H_1), S(H_2)), \qquad \delta(Sy, S(H_1)) < \delta(S(H_1), S(H_2)).$$

Note that the pair (K, K) has proximal normal structure if and only if K has normal structure in the sense of Brodskii and Milman, see e.g. [4] and [24]. We mention that every compact and convex subset of a Banach space X has normal structure, see e.g. [13]. Moreover, every bounded, closed and convex subset of a uniformly convex Banach space X has also normal structure, see e.g. [27].

Theorem 1.6. *([7]) Every bounded, closed and convex pair in a uniformly convex Banach space X has proximal normal structure.*

In what follows we recall the notion the pointwise contraction:

Definition 1.7. *[3] Let X be a Banach space. A mapping $T : X \longrightarrow X$ is called a pointwise contraction if for each $x \in X$ there exists $0 \leq \alpha(x) < 1$ such that*

$$\|Tx - Ty\| \leq \alpha(x) \, \| x - y \| \qquad \text{for all } y \in X.$$

Roughly speaking, we can say that the point-wise contraction mapping lies between the contraction mappings and non-expansive mappings. The initial result for pointwise contraction mapping is given below:

Theorem 1.8. *([25] Theorem 1.2) Let K be a weakly compact convex subset of a Banach space X and $T : K \longrightarrow K$ be a pointwise contraction. Then there exists a unique fixed point and $\{T^n(x)\}$ converges to the unique fixed point.*

J.Anuradha *et al.* [3] introduced the notion of proximal pointwise contraction to obtain the existence of a best proximity pair. Indeed, they generalized the result on pointwise contraction from a weakly compact subset to a weakly compact convex pair of a Banach space.

Definition 1.9. *[3] Assume that (A, B) is a nonempty pair of subsets of a Banach space X and $T : A \cup B \to A \cup B$ is a cyclic mapping. The mapping T is said to be a proximal pointwise contraction if for each $(x, y) \in A \times B$, there exist $0 \leq \alpha(x), \alpha(y) < 1$ such that*

$$\|Tx - Ty\| \leqslant \max\{\alpha(x)\|x - y\|, dist(A, B)\} \quad \text{for all } y \in B, \text{ and}$$

$$\|Tx - Ty\| \leqslant \max\{\alpha(y)\|x - y\|, dist(A, B)\} \quad \text{for all } x \in A.$$

It is clear that the proximal pointwise contraction map is a relatively nonexpansive map.

Next, we recall the definition of weak cyclic Kannan contraction theorem given by Petric [29].

Theorem 1.10. *[29] Let (A, B) be a nonempty pair in a Banach space X and $T : A \cup B \to A \cup B$ be a cyclic mapping.We say that T is a weak cyclic Kannan contraction mapping if*

$$\|Tx - Ty\| \leq \alpha\{\|x - Tx\| + \|y - Ty\|\} + (1 - 2\alpha)\mathrm{dist}(A, B), \tag{3}$$

for some$\alpha \in (0, \frac{1}{2})$ and for all $(x, y) \in A \times B$. Then T has a unique best proximity point $z \in A$.

Note that a mapping $T : A \cup B \to A \cup B$ that satisfies (3) is called a weak cyclic Kannan contraction mapping. In this paper, we aim to introduce and examine a new class of the pointwise cyclic-noncyclic proximal contraction pairs. More precisely, we shall investigate the coincidence quasi-best proximity point problem for this class. We also examine the coincidence quasi-best proximity points of weak cyclic-noncyclic Kannan contraction pairs.

2. Pointwise cyclic-noncyclic proximal contraction pair

In this section, first we introduce a pointwise cyclic-noncyclic proximal contraction pair.

Definition 2.1. *Assume that (A, B) is a nonempty pair of subsets of a Banach space X and $T, S : A \cup B \to A \cup B$ are two mappings. A pair $(T; S)$ is said to be a pointwise cyclic-noncyclic proximal contraction pair that $(T; S)$ is a cyclic-noncyclic pair and for any $(x, y) \in A \times B$,there exist $0 \leq \alpha(x, y) < 1$ such that*

$$\|Tx - Ty\| \leqslant \max\{\alpha(x, y)\|Sx - Sy\|, dist(S(A), S(B))\},$$

for any $(x, y) \in A \times B$

We note that the class of pointwise cyclic-noncyclic proximal contraction pairs contains the class of proximal pointwise contractions as a subclass. Indeed, it follows by letting $S = I$.

Theorem 2.2. *Let K be a weakly compact convex subset of a Banach space and $T, S : K \to K$. If $(T; S)$ is a pointwise cyclic-noncyclic proximal contraction pair, then there exists $p \in K$ such that $\|Tp - Sp\| = 0$.*

Proof. Suppose that Γ determine the collection of all nonempty,weakly compact and convex subsets of K such that $T, S : K \to K$. By Zorn's Lemma Γ has a minimal member which we deter mine by F. We perfect the proof by display that F consists of a single point.

Assume that $x \in F_c$. In this case, for any $y \in F_c$ we have

$$\begin{aligned}\|Sx - y\| &\leq \sup\{\|z - y\| : z \in F\} \\ &= r_y(F) = r(F).\end{aligned}$$

Consequently, we find that

$$\sup\{\|Sx - y\| : x \in F_c\} \leq r(F).$$

Thus, we get

$$\begin{aligned} r_{Sx}(F) &= \sup\{\|Sx - y\| : y \in F\} \\ &\leq \sup\{\|Sx - y\| : x \in F_c, y \in F\} \\ &\leq \sup\{r(F), y \in F\} \\ &= r(F). \end{aligned}$$

Then, for any $x \in F_c$ we have $r_{Sx}(F) = r(F)$; that is, $S : F_c \to F_c$. Moreover, for any $x, y \in F_c$ we have $\|Sx - Sy\| \leq r(F)$.

On other hand, for any $x, y \in F_c$,

$$\begin{aligned} \|Tx - Ty\| &\leqslant \alpha(x, y)\|Sx - Sy\| \\ &\leqslant \alpha(x, y)r(F) \\ &\leqslant r(F), \end{aligned}$$

that is, $r_{Tx}(F) = r(F)$. Then, $T : F_c \to F_c$.

By Lemma 1.3, we have $F_c \in \Gamma$. If $\delta(F) > 0$, then, again by Lemma 1.3, F_c is properly contained in F.

Since this contradicts the minimality of F, we conclude that $\delta(F) = 0$ and F consists of a single point, that is, there exists a $p \in K$ such that $Tp = p$ and $Sp = p$.

So, there exists a $p \in K$ such that

$$\|Tp - Sp\| = 0.$$

□

Theorem 2.3. *Let (A, B) be a nonempty pair of subsets in a Banach space X and $T, S : A \cup B \to A \cup B$. Suppose that $(T; S)$ is a pointwise cyclic-noncyclic proximal contraction pair such that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. Suppose also that $(S(A), S(B))$ be a weakly compact and convex pair of subsets in X. Then there exists $(x, y) \in A \times B$ such that for $p \in \{x, y\}$ we have*

$$\|Tp - Sp\| = \mathrm{dist}(S(A), S(B)).$$

Proof. The proof is trivial via the Theorem 2.2 if $\mathrm{dist}(S(A), S(B)) = 0$. Consequently, we suppose that $\mathrm{dist}(S(A), S(B)) > 0$. Let $(S(A_0^s), S(B_0^s))$ be the proximal pair related to $(S(A), S(B))$. As we have observed, $S(A_0^s)$ and $S(B_0^s)$ are nonempty, weakly compact and convex, and

$$\mathrm{dist}(S(A_0^s), S(B_0^s)) = \mathrm{dist}(S(A), S(B)).$$

Assume that $x \in A_0^s$, then there exists $y \in B_0^s$ such that $\|Sx - Sy\| = \mathrm{dist}(S(A), S(B))$. On other hand, $(T; S)$ is a pointwise cyclic-noncyclic proximal contraction pair. Thus,

$$\|T(Sx) - T(Sy)\| = \mathrm{dist}(S(A), S(B)), \qquad \|S(Sx) - S(Sy)\| = \mathrm{dist}(S(A), S(B)).$$

This yields that

$$\|S(Sx) - S(Sy)\| = \mathrm{dist}(S(A_0^s), S(B_0^s)),$$

and

$$\|T(Sx) - T(Sy)\| = \mathrm{dist}(S(A_0^s), S(B_0^s)).$$

Therefore, we have

$$T(Sx) \in S(B_0^s), \qquad T(Sy) \in S(A_0^s),$$

that is,

$$T(S(A_0^s)) \subseteq S(B_0^s), \qquad T(S(B_0^s)) \subseteq S(A_0^s).$$

Similarly, we find

$$S(S(A_0^s)) \subseteq S(A_0^s), \qquad S(S(B_0^s)) \subseteq S(B_0^s).$$

So, for each $x \in A_0^s$ and $y \in B_0^s$ we have

$$\|T(Sx) - T(Sy)\| = \mathrm{dist}(S(A_0^s), S(B_0^s))$$

and

$$\|S(Sx) - S(Sy)\| = \mathrm{dist}(S(A_0^s), S(B_0^s))$$

By Theorem 1.6 $(S(A), S(B))$ has a proximal normal structure. Clearly $(S(A_0^s), S(B_0^s))$ also has a proximal normal structure. Now, assume that Ω determine the collection of all nonempty subsets $S(F)$ of $S(A_0^s) \cup S(B_0^s)$ where $S(F) \cap S(A_0^s)$ and $S(F) \cap S(B_0^s)$ are nonempty, closed and convex such that

$$T(S(F) \cap S(A_0^s)) \subseteq S(F) \cap S(B_0^s), \quad T(S(F) \cap S(B_0^s)) \subseteq S(F) \cap S(A_0^s),$$

and

$$S(S(F) \cap S(A_0^s)) \subseteq S(F) \cap S(A_0^s), \quad S(S(F) \cap S(B_0^s)) \subseteq S(F) \cap S(B_0^s).$$

So, we find

$$\mathrm{dist}(S(F) \cap S(A_0^s), S(F) \cap S(B_0^s)) = \mathrm{dist}(S(A), S(B)).$$

Since, $S(A_0^s) \cup S(B_0^s) \in \Omega$ and Ω is nonempty, we assume that $\{S(F_\alpha)\}_{\alpha\in\Omega}$ is a decreasing chain in Ω such that $S(F_0) = \cap_{\alpha\in\Omega} S(F_\alpha)$. Then $S(F_0) \cap S(A_0^s) = \cap_{\alpha\in\Omega}(S(F_\alpha) \cap S(A_0^s))$, so $S(F_0) \cap S(A_0^s)$ is nonempty, closed and convex.

Similarly, $S(F_0) \cap S(B_0^s)$ is nonempty, closed and convex. Also,

$$T(S(F_0) \cap S(A_0^s)) \subseteq S(F_0) \cap S(B_0^s), \quad T(S(F_0) \cap S(B_0^s)) \subseteq S(F_0) \cap S(A_0^s)$$

and

$$S(S(F_0) \cap S(A_0^s)) \subseteq S(F_0) \cap S(A_0^s), \quad S(S(F_0) \cap S(B_0^s)) \subseteq S(F_0) \cap S(B_0^s).$$

To show that $S(F_0) \in \Omega$ we only need to show that

$$\mathrm{dist}(S(F_0) \cap S(A_0^s), S(F_0) \cap S(B_0^s)) = \mathrm{dist}(S(A), S(B)).$$

However, for each $\alpha \in J$ it is possible to select

$$Sx_\alpha \in S(F_\alpha) \cap S(A_0^s), \qquad Sy_\alpha \in S(F_\alpha) \cap S(B_0^s),$$

such that

$$\|Sx_\alpha - Sy_\alpha\| = \mathrm{dist}(S(A), S(B)).$$

It is also possible to choose convergent subnets $\{Sx_{\alpha'}\}$ and $\{Sy_{\alpha'}\}$ (with the same indices), say

$$\lim_{\alpha'} Sx_{\alpha'} = Sx, \quad \lim_{\alpha'} Sy_{\alpha'} = Sy.$$

Then clearly $Sx \in S(F_0) \cap S(A_0^s)$ and $Sy \in S(F_0) \cap S(B_0^s)$. By weak lower semicontinuity of the norm, we have $\|Sx - Sy\| \leq \text{dist}(S(A), S(B))$; hence,

$$\text{dist}(S(A), S(B)) \leq \text{dist}(S(F_0) \cap S(A_0^s), S(F_0) \cap S(B_0^s)) \leq \|Sx - Sy\| \leq \text{dist}(S(A), S(B)).$$

Therefore,

$$\text{dist}(S(F_0) \cap S(A_0^s), S(F_0) \cap S(B_0^s)) = \text{dist}(S(A), S(B)).$$

Since, every chain in Ω is bounded below by a member of Ω, Zorn's Lemma implies that Ω has a minimal member, say $S(K)$. Assume that $S(K_1) = S(K) \cap S(A_0^s)$ and $S(K_2) = S(K) \cap S(B_0^s)$.

Observe that if

$$\delta(S(K_1), S(K_2)) = \text{dist}(S(K_1), S(K_2)),$$

then for any $x \in S(K_1)$, we have

$$\|Tx - Sx\| = \text{dist}(S(K_1), S(K_2)) = \text{dist}(S(A), S(B)).$$

Similarly, for any $y \in S(K_2)$, we have

$$\|Ty - Sy\| = \text{dist}(S(K_1), S(K_2)) = \text{dist}(S(A), S(B)).$$

Thus proof is finished. So, we assume that

$$\delta(S(K_1), S(K_2)) > \text{dist}(S(K_1), S(K_2)).$$

We complete the proof by showing that this leads to a contradiction. Since, $S(K)$ is minimal it follows that $(S(K_1), S(K_2))$ is a proximal pair in $(S(A_0^s), S(B_0^s))$. By *PNS* there exists $(x_1, y_1) \in K_1 \times K_2$ and $\beta \in (0, 1)$ such that

$$\delta(Sx_1, S(K_2)) \leq \beta\delta(S(K_1), S(K_2)) \qquad \text{and} \qquad \delta(Sy_1, S(K_1)) \leq \beta\delta(S(K_1), S(K_2)).$$

Since, $(S(K_1), S(K_2))$ is a proximal pair, there exists $(x_2, y_2) \in K_1 \times K_2$ such that for each $i, j \in \{1, 2\}$ which are different from each other, we have

$$\|Sx_i - Sy_j\| = \text{dist}(S(K_1), S(K_2)).$$

So, for each $u \in S(K_2)$ we have

$$\begin{aligned}
\|\frac{Sx_1 + Sx_2}{2} - u\| &\leq \|\frac{Sx_1 - u}{2}\| + \|\frac{Sx_2 - u}{2}\| \\
&\leq \frac{\beta\delta(S(K_1), S(K_2))}{2} + \frac{\delta(S(K_1), S(K_2))}{2} \\
&= \alpha\delta(S(K_1), S(K_2)),
\end{aligned}$$

where $\alpha = \frac{1+\beta}{2} \in (0, 1)$.

Assume that $Sw_1 = \frac{(Sx_1+Sx_2)}{2}$ and $Sw_2 = \frac{(Sy_1+Sy_2)}{2}$. Then

$$\delta(Sw_1, S(K_2)) \leq \alpha\delta(S(K_1), S(K_2)) \qquad \text{and} \qquad \delta(Sw_2, S(K_1)) \leq \alpha\delta(S(K_1), S(K_2)).$$

Since

$$\begin{aligned}\operatorname{dist}(S(K_1), S(K_2)) &\le \|Sw_1 - Sw_2\|\\ &= \|\frac{(Sx_1 + Sx_2)}{2} - \frac{(Sy_1 + Sy_2)}{2}\|\\ &\le \frac{1}{2}[\|Sx_1 - Sy_2\| + \|Sx_2 - Sy_1\|]\\ &= \operatorname{dist}(S(K_1), S(K_2)),\end{aligned}$$

we have $\|Sw_1 - Sw_2\| = \operatorname{dist}(S(K_1), S(K_2))$.

Consider the following sets

$$S(L_1) = \{Sx \in S(K_1) : \delta(Sx, S(K_2)) \le \alpha\delta(S(K_1), S(K_2))\},$$
$$S(L_2) = \{Sy \in S(K_2) : \delta(Sy, S(K_1)) \le \alpha\delta(S(K_1), S(K_2))\}.$$

Then for each $i = 1, 2$, $S(L_i)$ is a nonempty, closed and convex subset of $S(K_i)$ and since $Sw_1 \in S(L_1)$ and $Sw_2 \in S(L_2)$, we have

$$\operatorname{dist}(S(K_1), S(K_2)) \le \operatorname{dist}(S(L_1), S(L_2)) \le \|Sw_1 - Sw_2\| = \operatorname{dist}(S(K_1), S(K_2)).$$

Therefore,

$$\operatorname{dist}(S(L_1), S(L_2)) = \operatorname{dist}(S(K_1), S(K_2)) = \operatorname{dist}(S(A), S(B)).$$

Now, assume that $Sx \in S(L_1)$ and $Sy \in S(K_2)$. Then $Sx \in S(A_0^s)$ and $Sy \in S(B_0^s)$; that is, $x \in A_0^s$ and $y \in B_0^s$. Thus,

$$\|T(Sx) - T(Sy)\| = \operatorname{dist}(S(A), S(B)) \le \delta(Sx, S(K_2)) \le \alpha\delta(S(K_1), S(K_2)).$$

So, $T(Sy) \in B(T(Sx); \alpha\delta(S(K_1), S(K_2))) \cap S(K_1)$; that is,

$$T(S(K_2)) \subseteq B(T(Sx); \alpha\delta(S(K_1), S(K_2))) \cap S(K_1) := S(K_1').$$

Clearly $S(K_1')$ is closed and convex. Also, if $Sy \in S(K_2)$ satisfies $\|Sx - Sy\| = \operatorname{dist}(S(A), S(B))$, then

$$\|T(Sx) - T(Sy)\| = \operatorname{dist}(S(K_1), S(K_2)).$$

Since, $T(Sy) \in S(K_1')$, we conclude that $\operatorname{dist}(S(K_1'), S(K_2)) = \operatorname{dist}(S(A), S(B))$. Therefore, $S(K_1') \cup S(K_2) \in \Omega$ and by minimality of $S(K)$ it must be the case that $S(K_1') = S(K_1)$. Hence, $S(K_1) \subseteq B(T(Sx); \alpha\delta(S(K_1), S(K_2)))$; that is, $\delta(T(Sx), S(K_1)) \le \alpha\delta(S(K_1), S(K_2))$ and since $Sx \in S(L_1)$ was arbitrary this proves $T(S(L_1)) \subseteq S(L_2)$. Similarly,

$T(S(L_2)) \subseteq S(L_1)$, $S(S(L_1)) \subseteq S(L_1)$

and $S(S(L_2)) \subseteq S(L_2)$. Thus, $S(L_1) \cup S(L_2) \in \Omega$, but $\delta(S(L_1), S(L_2)) \le \alpha\delta(S(K_1), S(K_2))$ and this contradicts the minimality of $S(K)$. □

Corollary 2.4. *let (A, B) be a nonempty pair in Banach space X and $(T; S)$ that $T, S : A \cup B \to A \cup B$ be a pointwise cyclic-noncyclic proximal contraction pair such that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. Suppose that $(S(A), S(B))$ be a nonempty closed bounded convex pair in a reflexive Banach space X. Then there exists $(x, y) \in A \times B$ such that for $p \in \{x, y\}$ we have*

$$\|Tp - Sp\| = \operatorname{dist}(S(A), S(B)).$$

Corollary 2.5. *let (A, B) be a nonempty weakly compact convex pair in Banach space X and suppose that T is a proximal pointwise contraction map. Then there exists $(x, y) \in A \times B$ such that*

$$\|x - Tx\| = \|y - Ty\| = \operatorname{dist}(A, B).$$

Example 2.6. *Assume that $A = [-4, 0]$ and $B = [0, 4]$ be subsets in a uniformly convex Banach space $(\mathbb{R}, |.|)$ and for any $x \in A \cup B$, we have*

$$Tx = -\frac{1}{4}x, \qquad Sx = \frac{1}{2}x.$$

Then,

$$T(A) = [0, 1] \subseteq [0, 2] = S(B), \qquad T(B) = [-1, 0] \subseteq [-2, 0] = S(A).$$

Moreover, we suppose that for any $(x, y) \in A \times B$,

$$\alpha(x, y) = \begin{cases} 0, & if \quad x = y \\ 1, & if \quad x \neq y. \end{cases}$$

If $(x, y) \in A \times B$ such that $\|x - y\| = \operatorname{dist}(S(A), S(B)) = 0$, then $x = y$ and

$$\|Tx - Ty\| = \operatorname{dist}(S(A), S(B)), \qquad \|Sx - Sy\| = \operatorname{dist}(S(A), S(B)).$$

In other wise,

$$\begin{aligned} \|Tx - Ty\| &= \|\tfrac{1}{4}y - \tfrac{1}{4}x\| = \tfrac{1}{2}\|\tfrac{1}{2}y - \tfrac{1}{2}x\| \\ &= \tfrac{1}{2}\|Sy - Sx\| = \tfrac{1}{2}\|Sx - Sy\| \\ &\leq \|Sx - Sy\|. \end{aligned}$$

Thus, $(T; S)$ is a pointwise cyclic-noncyclic proximal contraction pair and by Corollary 2.4, there exists $(x, y) \in A \times B$ such that

$$\|Tx - Sx\| = \operatorname{dist}(S(A), S(B)), \qquad \|Ty - Sy\| = \operatorname{dist}(S(A), S(B)).$$

3. Weak cyclic-noncyclic Kannan contraction

In this section, we introduce a weak cyclic-noncyclic Kannan contraction pair.

Definition 3.1. *Let (A, B) be a nonempty pair in a Banach space X and $T, S : A \cup B \to A \cup B$ be two self-mappings. A pair $(T; S)$ is said to be a weak cyclic-noncyclic Kannan contraction pair that $(T; S)$ is a cyclic-noncyclic pair and for any $(x, y) \in A \times B$,then*

$$\|Tx - Ty\| \leq \alpha\{\|Sx - Tx\| + \|Sy - Ty\|\} + (1 - 2\alpha)\operatorname{dist}(S(A), S(B))$$

for some $\alpha \in (0, \frac{1}{2})$.

In this section, we discuss the coincidence quasi-best proximity point problem for weak cyclic-noncyclic Kannan contraction pairs.

Lemma 3.2. *Assume that (A, B) is a nonempty pair of subsets in a Banach space X and $(T; S)$ that $T, S : A \cup B \to A \cup B$ is a weak cyclic-noncyclic Kannan contraction pair such that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. Suppose that $(S(A), S(B))$ be a weakly compact and convex pair of subsets in X. Then, there exists $(S(K_1), S(K_2)) \subseteq (S(A_0^s), S(B_0^s)) \subseteq (S(A), S(B))$ which is minimal with respect to being nonempty, closed, convex and T and S-invariant pair of subsets of $(S(A), S(B))$, such that*

$$\operatorname{dist}(S(K_1), S(K_2)) = \operatorname{dist}(S(A), S(B)).$$

Moreover, the pair $(S(K_1), S(K_2))$ is proximal.

Proof. *We skip the proof by regarding the analogue with Theorem 2.3.* □

Theorem 3.3. *Assume that (A, B) is a nonempty pair of subsets in a Banach space X and $(T; S)$ that $T, S : A \cup B \to A \cup B$ is a weak cyclic-noncyclic Kannan contraction pair such that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. Suppose that $(S(A), S(B))$ be a weakly compact and convex pair of subsets in X. Then $(T; S)$ has a coincidence quasi-best proximity point.*

Proof. By Lemma 3.2, assume that $(S(K_1), S(K_2))$ be a minimal, weakly compact, convex and proximal pair which T and S-invariant, such that $\mathrm{dist}(S(K_1), S(K_2)) = \mathrm{dist}(S(A), S(B))$. Notice that

$$\overline{con}(T(S(K_1))) \subseteq S(K_2)$$

and so,

$$T(\overline{con}(T(S(K_1)))) \subseteq T(S(K_2)) \subseteq \overline{con}(T(S(K_2))).$$

Similarly,

$$T(\overline{con}(T(S(K_2)))) \subseteq \overline{con}(T(S(K_1)));$$

that is, T is cyclic on $\overline{con}(T(S(K_1))) \cup \overline{con}(T(S(K_2)))$.

On other hand, S is noncyclic on $\overline{con}(S(S(K_1))) \cup \overline{con}(S(S(K_2)))$. Minimality of $(S(K_1), S(K_2))$ implies that

$$\overline{con}(T(S(K_1))) = S(K_2) \qquad \text{and} \qquad \overline{con}(T(S(K_2))) = S(K_1).$$

Besides,

$$\overline{con}(S(S(K_1))) = S(K_1) \qquad \text{and} \qquad \overline{con}(S(S(K_2))) = S(K_2).$$

Let $a \in K_1$ then,

$$S(K_2) \subseteq B(a, \delta(a, S(K_2)).$$

Now if $y \in K_2$ then,

$$\|Ta - Ty\| \le \alpha\{\|Sa - Ta\| + \|Sy - Ty\|\} + (1 - 2\alpha)\mathrm{dist}(S(A), S(B))$$

$$\le 2\alpha\delta(S(K_1), S(K_2)) + (1 - 2\alpha)\mathrm{dist}(S(A), S(B)))$$

Hence for all$y \in K_2$,we have

$$T(S(K_2)) \subseteq B(Ta, 2\alpha\delta(S(K_1), S(K_2)) + (1 - 2\alpha)\mathrm{dist}(S(A), S(B)))$$

Therefore

$$S(K_1) = \overline{con}(T(S(K_2))) \subseteq B(Ta, 2\alpha\delta(S(K_1), S(K_2)) + (1 - 2\alpha)\mathrm{dist}(S(A), S(B)))$$

This implies that

$$\|Sx - Ta\| \le 2\alpha\delta(S(K_1), S(K_2)) + (1 - 2\alpha)\mathrm{dist}(S(A), S(B))) \quad \forall S(x) \in S(K_1)$$

so that

$$\delta(Ta, S(K_1)) \le 2\alpha\delta(S(K_1), S(K_2)) + (1 - 2\alpha)\mathrm{dist}(S(A), S(B))). \quad (3.1)$$

Similarly, if $b \in K_2$

$$\delta(Tb, S(K_2)) \leq 2\alpha\delta(S(K_1), S(K_2)) + (1 - 2\alpha)\mathrm{dist}(S(A), S(B))). \quad (3.2)$$

Now we put

$$S(E_1) := \{Sx \in S(K_1)\,; \delta(Sx, S(K_2)) \leq 2\alpha\delta(S(K_1), S(K_2)) + (1 - 2\alpha)\mathrm{dist}(S(A), S(B))\},$$

and

$$S(E_2) := \{Sy \in S(K_2)\,; \delta(Sy, S(K_1)) \leq 2\alpha\delta(S(K_1), S(K_2)) + (1 - 2\alpha)\mathrm{dist}(S(A), S(B))\}$$

Then $(S(E_1), S(E_2))$ is nonempty and

$$S(E_1) = \cap_{Sy \in S(K_2)} B(Sy; 2\alpha\delta(S(K_1), S(K_2)) + (1 - 2\alpha)\mathrm{dist}(S(A), S(B))) \cap S(K_1),$$

and

$$S(E_2) = \cap_{Sx \in S(K_1)} B(Sx; 2\alpha\delta(S(K_1), S(K_2)) + (1 - 2\alpha)\mathrm{dist}(S(A), S(B))) \cap S(K_2).$$

Beside by (3.1) and (3.2) it is easy to see that

$$T(S(E_1)) \subseteq S(E_2) \qquad \text{and} \qquad T(S(E_2))) \subseteq S(E_1).$$

Besides,

$$S(S(E_1)) \subseteq S(E_1) \qquad \text{and} \qquad S(S(E_2)) = S(E_2).$$

Hence, T is cyclic and S is noncyclic on $S(E_1) \cup S(E_2)$. Minimality of $(S(K_1), S(K_2))$ implies that

$$S(E_1) = S(K_1) \qquad \text{and} \qquad S(E_2) = S(K_2).$$

Therefore we have

$$\delta(Sx, S(K_2)) \leq 2\alpha\delta(S(K_1), S(K_2)) + (1 - 2\alpha)\mathrm{dist}(S(A), S(B)). \quad \forall Sx \in S(K_1).$$

Now we obtain

$$\delta(S(K_1), S(K_2)) = \sup_{x \in K_1} \delta(Sx, S(K_2)) \leq 2\alpha\delta(S(K_1), S(K_2)) + (1 - 2\alpha)\mathrm{dist}(S(A), S(B)).$$

Therefore, we find

$$\delta(S(K_1), S(K_2)) = \mathrm{dist}(S(A), S(B)).$$

Now we have

$$\|Sx - Ty\| \leq \delta(Sx, S(K_2)) = \mathrm{dist}(S(A), S(B)),$$

$$\|Sy - Tx\| \leq \delta(Sy, S(K_1)) = \mathrm{dist}(S(A), S(B)).$$

This shows that each point in $S(K_1) \cup S(K_2)$ is a coincidence quasi-best proximity point of $(T; S)$. □

Corollary 3.4. *let (A, B) be a nonempty pair in Banach space X and $(T; S)$ that $T, S : A \cup B \to A \cup B$ be a Weak cyclic-noncyclic Kannan contraction pair such that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. Suppose that $(S(A), S(B))$ be a nonempty closed bounded convex pair in a reflexive Banach space X. Then there exists $(x, y) \in A \times B$ such that for $p \in \{x, y\}$ we have*

$$\|Tp - Sp\| = \mathrm{dist}(S(A), S(B)).$$

Corollary 3.5. *let (A, B) be a nonempty weakly compact convex pair in Banach space X and T that $T : A \cup B \to A \cup B$ is a Weak cyclic Kannan contraction mapping. Then there exists $(x, y) \in A \times B$ such that for $p \in \{x, y\}$ we have*

$$\|Tp - p\| = \mathrm{dist}(A, B).$$

Example 3.6. *Assume that $A = (-\infty, -1]$ and $B = [1, +\infty)$ are subsets in the uniformly convex Banach space $\mathbb{R}$ with the standard norm $|\cdot|$. For any $x \in A \cup B$, we define*

$$Tx = \begin{cases} -x, & if \quad x \in A \cup B \\ 0, & if \quad \text{otherwise}, \end{cases}$$

and

$$Sx = \begin{cases} x+1, & if \quad x \in A, \\ x-1, & if \quad x \in B, \\ 0, & if \quad \text{otherwise}. \end{cases}$$

Then,

$$T(A) = [1, +\infty) \subseteq [0, +\infty) = S(B), \qquad T(B) = (-\infty, -1] \subseteq (-\infty, 0] = S(A).$$

Moreover, for any $(x, y) \in A \times B$ we have

$$\mathrm{dist}(S(A), S(B)) = 0.$$

Furthermore, we observe that

$$\begin{aligned} \|Tx - Ty\| &= \|y - x\| \\ &\le \alpha\{\|x + 1 + x\| + \|y - 1 + y\|\} + (1 - 2\alpha\, \mathrm{dist}(S(A), S(B)) \\ &\le \alpha(\|2x + 1\| + \|2y - 1\|). \end{aligned}$$

It yields that for any $\alpha \in (0, \frac{1}{2})$ the above relationship holds. Thus, $(T; S)$ is a weak cyclic-noncyclic Kannan contraction pair, so that by Corollary 3.4, there exists $(x, y) \in A \times B$ such that

$$\|Tx - Sx\| = \mathrm{dist}(S(A), S(B)), \qquad \|Ty - Sy\| = \mathrm{dist}(S(A), S(B)).$$

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