



Laguerre Hypersurfaces With Definite Laguerre Tensor

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Abstract. $x : M^{n-1} \rightarrow \mathbb{R}^n$ be an umbilical free hypersurface with non-zero principal curvatures. Then x is associated with a Laguerre metric \mathbf{g} , a Laguerre tensor \mathbf{L} , a Laguerre form \mathbf{C} , and a Laguerre second fundamental form \mathbf{B} , which are invariants of x under Laguerre transformation group. In this paper, we will study the Laguerre hypersurface with parallel Laguerre form and nonnegative or non-positive Laguerre tensor.

1. Introduction

Let UIR^n be the unit tangent bundle over \mathbb{R}^n . An oriented sphere in \mathbb{R}^n centered at p with radius r can be regarded as the "oriented sphere" $\{(x, \xi) | x - p = r\xi\}$ in UIR^n , where x is the position vector and ξ is the unit normal of the sphere. An oriented hyperplane in \mathbb{R}^n with constant unit normal ξ and constant real number c can be regarded as the "oriented hyperplane" $\{(x, \xi) | x \cdot \xi = c\}$ in \mathbb{R}^n . A diffeomorphism $\phi : UIR^n \rightarrow UIR^n$ which takes oriented spheres to oriented spheres, oriented hyperplanes to oriented hyperplanes, preserving the tangential distance of any two spheres, is called a Laguerre transformation. All Laguerre transformations in UIR^n form a group of dimension $(n+1)(n+2)/2$, called the Laguerre transformation group.

An oriented hypersurface $x : M \rightarrow \mathbb{R}^n$ can be identified as the submanifold $(x, \xi) : M \rightarrow UIR^n$, where ξ is the unit normal of x . Two hypersurfaces $x, x^* : M \rightarrow \mathbb{R}^n$ are called Laguerre equivalent, if there is a Laguerre transformation $\phi : UIR^n \rightarrow UIR^n$ such that $(x^*, \xi^*) = \phi \circ (x, \xi)$. In Laguerre geometry one studies properties and invariants of hypersurfaces in \mathbb{R}^n under the Laguerre transformation group.

Laguerre geometry of surfaces in \mathbb{R}^3 has been developed by Blaschke and his school (see [1]). Recently, there has been some renewed interest in Laguerre geometry (see [2–4, 6, 8–11]).

In [7], Li and Wang studied Laguerre differential geometry of oriented hypersurfaces in \mathbb{R}^n . For any umbilical-free hypersurface $x : M \rightarrow \mathbb{R}^n$ with non-zero principal curvatures, Li and Wang defined the Laguerre invariant metric \mathbf{g} , the Laguerre second fundamental form \mathbf{B} , the Laguerre form \mathbf{C} and the

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Laguerre tensor \mathbf{L} on M , and showed that $\{\mathbf{g}, \mathbf{B}\}$ is a complete Laguerre invariant system for hypersurfaces in \mathbb{R}^n with $n \geq 4$. In the case $n = 3$, a complete Laguerre invariant system for surfaces in \mathbb{R}^3 is given by $\{\mathbf{g}, \mathbf{B}, \mathbf{L}\}$. Based on this foundation, in [5], Li et al. gave the classification of hypersurfaces with constant Laguerre eigenvalues and vanishing Laguerre form in \mathbb{R}^n .

The Laguerre tensor is said to be nonnegative or non-positive, if all the eigenvalues of Laguerre tensor are nonnegative or non-positive. As we know, under the condition of constant Laguerre eigenvalues, vanishing Laguerre form is equivalent to parallel Laguerre form. Thus, it is very interesting and important to study hypersurfaces with parallel Laguerre form and nonnegative or non-positive Laguerre tensor.

In this paper, we will prove the following result:

Theorem 1.1. *Let $x : M^{n-1} \rightarrow \mathbb{R}^n$ be an umbilical free hypersurface with non-zero principal curvatures. If the Laguerre form \mathbf{C} is parallel and the Laguerre tensor \mathbf{L} is nonnegative or non-positive, then the Laguerre tensor \mathbf{L} is must identically zero, which means that M^{n-1} is laguerre equivalent to an open subset of the hypersurface: the image of τ of the oriented hypersurface $x : \mathbb{R}^{n-1} \rightarrow \mathbb{R}_0^n$ given by the following example 3.1 in Section 3.*

We organize the paper as follows. In Section 2 we give Laguerre invariants for hypersurfaces in \mathbb{R}^n . In Section 3, an interesting example is presented. Finally, in Section 4, we prove the Main Theorem.

2. Laguerre geometry of hypersurfaces in \mathbb{R}^n

In this section we review the Laguerre invariants and structure equations for hypersurfaces in \mathbb{R}^n . For the detail we refer to [7].

Let \mathbb{R}_2^{n+3} be the space \mathbb{R}^{n+3} equipped with the inner product

$$\langle X, Y \rangle = -x_1y_1 + x_2y_2 + \cdots + x_{n+2}y_{n+2} - x_{n+3}y_{n+3}.$$

Let C^{n+2} be the light-cone in \mathbb{R}_2^{n+3} given by $C^{n+2} = \{X \in \mathbb{R}_2^{n+3} | \langle X, X \rangle = 0\}$. Let LG be the subgroup of orthogonal group $O(n+1, 2)$ on \mathbb{R}_2^{n+3} given by

$$LG = \{T \in O(n+1, 2) | \zeta T = \zeta\},$$

where $\zeta = (1, -1, \mathbf{0}, 0)$, where $\mathbf{0} \in \mathbb{R}^n$, is a light-like vector in \mathbb{R}_2^{n+3} .

Let $x : M \rightarrow \mathbb{R}^n$ be an umbilical free hypersurface with non-zero principal curvatures. Let $\xi : M \rightarrow \mathbb{S}^{n-1}$ be its unit normal. Let $\{e_1, e_2, \dots, e_{n-1}\}$ be the orthonormal basis for TM with respect to $dx \cdot dx$, consisting of unit principal vectors. We write the structure equations of $x : M \rightarrow \mathbb{R}^n$ by

$$e_j(e_i(x)) = \sum_k \Gamma_{ij}^k e_k(x) + k_i \delta_{ij} \xi; e_i(\xi) = -k_i e_i(x), 1 \leq i, k \leq n-1,$$

where $k_i \neq 0$ is the principal curvature corresponding to e_i . Let

$$r_i = \frac{1}{k_i}, r = \frac{r_1 + r_2 + \cdots + r_{n-1}}{n-1},$$

be the curvature radius and mean curvature radius of x , respectively. We define Laguerre position vector of x by

$$Y = \rho(x \cdot \xi, -x \cdot \xi, \xi, 1) : M \rightarrow C^{n+2} \subset \mathbb{R}_2^{n+3},$$

where $\rho = \sqrt{\sum_i (r_i - r)^2} > 0$.

Theorem 2.1. *Let $x, \tilde{x} : M \rightarrow \mathbb{R}^n$ be two umbilical oriented hypersurfaces with non-zero principal curvatures. Then x and \tilde{x} are Laguerre equivalent if and only if there exists $T \in LG$ such that $\tilde{Y} = YT$.*

From the theorem we know that

$$\mathbf{g} = \langle dY, dY \rangle = \rho^2 d\xi \cdot d\xi = \rho^2 III$$

is a Laguerre invariant metric, where III is the third fundamental form of x . we call \mathbf{g} the Laguerre metric of x . Let Δ be the Laplacian operator of \mathbf{g} , then we have

$$N = \frac{1}{n-1} \Delta Y + \frac{1}{2(n-1)^2} \langle \Delta Y, \Delta Y \rangle Y \tag{1}$$

and

$$\eta = \left(\frac{1}{2}(1 + |x|^2), \frac{1}{2}(1 - |x|^2), x, 0 \right) + r(x \cdot \xi, -x \cdot \xi, \xi, 1).$$

From (1) we get

$$\langle Y, Y \rangle = \langle N, N \rangle = 0, \langle N, Y \rangle = -1, \langle \eta, \eta \rangle = 0, \langle \eta, \zeta \rangle = -1.$$

Let $\{E_1, E_2, \dots, E_{n-1}\}$ be an orthonormal basis for $g = \langle dY, dY \rangle$ with dual basis $\{\omega_1, \omega_2, \dots, \omega_{n-1}\}$ and write $Y_i = E_i(Y), 1 \leq i \leq n-1$. Then we have the following orthogonal decomposition,

$$R_2^{n+3} = \text{Span}\{Y, N\} \oplus \text{Span}\{Y_1, Y_2, \dots, Y_{n-1}\} \oplus \text{span}\{\eta, \zeta\}.$$

We call $\{Y, N, Y_1, \dots, Y_{n-1}, \eta, \zeta\}$ a Laguerre moving frame in R_2^{n+3} of x . By taking derivatives of this frame we obtain the following structure equations:

$$E_i(N) = \sum_j L_{ij} Y_j + C_i \zeta, \tag{2}$$

$$E_j(Y_i) = L_{ij} Y + \delta_{ij} N + \sum_k \Gamma_{ij}^k Y_k + B_{ij} \zeta, \tag{3}$$

$$E_i(\eta) = -C_i Y + \sum_j B_{ij} Y_j. \tag{4}$$

From these equations we obtain the following basic Laguerre invariants: the Laguerre metric $\mathbf{g} = \langle dY, dY \rangle$, the Laguerre second fundamental form $\mathbf{B} = \sum_{ij} B_{ij} \omega_i \otimes \omega_j$, the Laguerre tensor $\mathbf{L} = \sum_{ij} L_{ij} \omega_i \otimes \omega_j$ and the Laguerre form $\mathbf{C} = \sum_i C_i \omega_i$, where $L_{ij} = L_{ji}, B_{ij} = B_{ji}$.

Define the covariant derivatives of \mathbf{L}, \mathbf{B} and \mathbf{C} by

$$\sum_k L_{ij,k} \omega_k = dL_{ij} + \sum_k L_{ik} \omega_{kj} + \sum_k L_{kj} \omega_{ki}, \tag{5}$$

$$\sum_k B_{ij,k} \omega_k = dB_{ij} + \sum_k B_{ik} \omega_{kj} + \sum_k B_{kj} \omega_{ki}, \tag{6}$$

$$\sum_j C_{i,j} \omega_j = dC_i + \sum_j C_j \omega_{ji}, \tag{7}$$

$$d\omega_{ij} - \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2} R_{ijkl} \omega_k \wedge \omega_l, \quad R_{ijkl} = -R_{ijlk}. \tag{8}$$

Exterior differentiation of the structure equations gives their integrability conditions:

$$L_{ij,k} = L_{ik,j}, \tag{9}$$

$$C_{i,j} - C_{j,i} = \sum_k (B_{ik}L_{kj} - B_{kj}L_{ki}), \tag{10}$$

$$B_{ij,k} - B_{ik,j} = C_j\delta_{ik} - C_k\delta_{ij}, \tag{11}$$

$$R_{ijkl} = L_{jk}\delta_{il} + L_{il}\delta_{jk} - L_{ik}\delta_{jl} - L_{jl}\delta_{ik}. \tag{12}$$

Moreover, we have the following identities(see[7]):

$$\sum_{i,j} (B_{ij})^2 = 1, \quad \sum_i B_{ii} = 0, \quad \sum_i B_{ij,i} = (n - 2)C_j, \tag{13}$$

$$\sum_i L_{ii} = -\frac{1}{2(n-1)} \langle \Delta Y, \Delta Y \rangle Y, \tag{14}$$

$$R_{ik} = -(n-3)L_{ik} - \left(\sum_i L_{ii} \right) \delta_{ik} \tag{15}$$

and

$$R = -2(n-2) \sum_i L_{ii} = \frac{n-2}{(n-1)} \langle \Delta Y, \Delta Y \rangle Y \tag{16}$$

is the normalized scalar curvature.

In the case $n \geq 4$, we know from (13) and (16) that C_i and L_{ij} are completely determined by the Laguerre invariants $\{\mathbf{g}, \mathbf{B}\}$, thus we get

Theorem 2.2. *Two umbilical free oriented hypersurfaces in R^n ($n > 3$) with non-zero principal curvatures are Laguerre equivalent if and only if they have the same Laguerre metric \mathbf{g} and Laguerre second fundamental form \mathbf{B} .*

In the case $n = 3$, a complete Laguerre invariant system for surfaces in R^3 is given by $\{\mathbf{g}, \mathbf{B}, \mathbf{L}\}$.

3. example

The following example is taken from [4], we will show that the Laguerre second fundamental form \mathbf{B} is parallel, the Laguerre form \mathbf{C} and Laguerre tensor \mathbf{L} vanish.

Example 3.1 ([4]). *For any positive integers m_1, \dots, m_s with $m_1 + m_2 + \dots + m_s = n - 1$ and any non-zero constants $\lambda_1, \dots, \lambda_s$, we define $x : \mathbb{R}^{n-1} \rightarrow \mathbb{R}_0^n$ to be a space-like oriented hypersurface in \mathbb{R}_0^n given by*

$$x = \left(\frac{\lambda_1|u_1|^2 + \lambda_2|u_2|^2 + \dots + \lambda_s|u_s|^2}{2}, u_1, u_2, \dots, u_s, \frac{\lambda_1|u_1|^2 + \lambda_2|u_2|^2 + \dots + \lambda_s|u_s|^2}{2} \right),$$

where $(u_1, u_2, \dots, u_s) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \dots \times \mathbb{R}^{m_s} = \mathbb{R}^{n-1}$ and $|u_i|^2 = u_i \cdot u_i, i = 1, \dots, s$. Then $\tau \circ (x, \xi) = (x', \xi') : \mathbb{R}^{n-1} \rightarrow U\mathbb{R}^n$, and we get the hypersurfaces $x' : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$.

Let \mathbb{R}_1^{n+1} be the Minkowski space with the inner product

$$\langle X, Y \rangle = x_1y_1 + x_2y_2 + \dots + x_{n+2}y_{n+2} - x_{n+3}y_{n+3}. \tag{17}$$

Let $\nu = (1, \mathbf{0}, 1)$ be the light-like vector in \mathbb{R}_1^{n+1} with $\mathbf{0} \in \mathbb{R}^{n-1}$. Let \mathbb{R}_0^n be the degenerate hyperplane in \mathbb{R}_1^{n+1} defined by

$$\mathbb{R}_0^n = \{X \in \mathbb{R}_1^{n+1} : \langle X, \nu \rangle = 0\}. \tag{18}$$

We define

$$U\mathbb{R}_0^n = \{(x, \xi) \in \mathbb{R}_1^{n+1} \times \mathbb{R}_1^{n+1} : \langle x, \nu \rangle = 0, \langle \xi, \xi \rangle = 0, \langle \xi, \nu \rangle = 1\} \tag{19}$$

and $\tau : UR_0^n \rightarrow UR^n$ given by

$$\tau(x, \xi) = (x', \xi') \in UR^n \tag{20}$$

where $x = (x_1, x_0, x_1) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$, $\xi = (\xi_1 + 1, \xi_0, \xi_1) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$ and

$$x' = \left(-\frac{x_1}{\xi_1}, x_0 - \frac{x_1}{\xi_1} \xi_0\right), \xi' = \left(1 + \frac{1}{\xi_1}, \frac{x_0}{\xi_1}\right). \tag{21}$$

Let $x : M \rightarrow \mathbb{R}_0^n$ be a space-like oriented hypersurface in the degenerate hyperplane \mathbb{R}_0^n . Let ξ be the unique vector in \mathbb{R}_1^{n+1} satisfying

$$\langle \xi, dx \rangle = 0, \quad \langle \xi, \xi \rangle = 0, \quad \langle \xi, \nu \rangle = 1.$$

From $\tau(x, \xi) = (x', \xi') \in UR^n$, we get a hypersurface $x : M \rightarrow \mathbb{R}^n$.

Let $x : M \rightarrow \mathbb{R}_0^n$ be a space-like oriented hypersurface in the degenerate hyperplane \mathbb{R}_0^n . Let ξ be the unique vector in \mathbb{R}_1^{n+1} satisfying

$$\langle \xi, dx \rangle = 0, \quad \langle \xi, \xi \rangle = 0, \quad \langle \xi, \nu \rangle = 1.$$

We define the shape operator $\mathbf{S} : TM \rightarrow TM$ by $d\xi = -dx \circ \mathbf{S}$. Since \mathbf{S} is self-adjoint, all eigenvalues $\{k_i\}$ of \mathbf{S} are real. Let $\{e_1, e_2, \dots, e_{n-1}\}$ be the orthonormal basis for TM with respect to $dx \cdot dx$, consisting of eigenvectors of \mathbf{S} . We write the structure equation of $x : M \rightarrow \mathbb{R}_0^n$ by

$$e_j(e_i(x)) = \sum_k \Gamma_{ij}^k e_k(x) + k_i \delta_{ij} \nu, \quad e_i(\xi) = -k_i e_i(x), \quad 1 \leq i, j, k \leq n-1.$$

We assume that $k_i \neq 0$, and we define $r_i = 1/k_i$ as the curvature radius of x and $r = (r_1 + r_2 + \dots + r_{n-1})/(n-1)$ as the mean curvature radius of x . Let $\tau : UR_0^n \rightarrow UR^n$ be the Laguerre embedding defined by (20) and $(x', \xi') = \tau \circ (x, \xi)$. We get a hypersurface $x' : M \rightarrow \mathbb{R}^n$. By a direct calculation we can show that

$$Y = \rho(\langle x, \xi \rangle - \langle x, \xi \rangle, \xi) = \rho'(x' \cdot \xi', -x' \cdot \xi', \xi', 1) = Y'.$$

$$\begin{aligned} \eta &= \left(\frac{1}{2}(1 + \langle x, x \rangle), \frac{1}{2}(1 - \langle x, x \rangle), x\right) + r(\langle x, \xi \rangle, -\langle x, x \xi \rangle, \xi) \\ &= \eta' = \left(\frac{1}{2}(1 + |x'|^2), \frac{1}{2}(1 - |x'|^2), x', 0\right) + r'(x' \cdot \xi', -x' \cdot \xi', \xi', 1), \end{aligned}$$

where $\rho'^2 = \sum_i (r'_i - r')^2$ and $\rho^2 = \sum_i (r_i - r)^2$. Thus the Laguerre metric is given by

$$g = \rho^2 III = \rho'^2 III' = g',$$

where III and III' are the third fundamental forms of x and x' , respectively.

We define $\tilde{E}_i = r_i e_i, 1 \leq i \leq n-1$, then $\{\tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_{n-1}\}$ is an orthonormal basis for $III = d\xi \cdot d\xi$. Then $\{E_i = \rho^{-1} \tilde{E}_i | 1 \leq i \leq n-1\}$ is an orthonormal basis for the Laguerre metric $g = \langle dY, dY \rangle$, and we write $\{\omega_1, \omega_2, \dots, \omega_{n-1}\}$ for its dual basis. By direct calculation, the basis Laguerre invariants \mathbf{B} , \mathbf{L} and \mathbf{C} of $x : M \rightarrow \mathbb{R}_0^n$ have the following local expressions:

$$\mathbf{B} = \sum_{ij} B_{ij} \omega_i \otimes \omega_j, \quad \mathbf{L} = \sum_{ij} L_{ij} \omega_i \otimes \omega_j, \quad \mathbf{C} = C_i \omega_i,$$

where

$$B_{ij} = \rho^{-1}(r_i - r) \delta_{ij}, \quad C_i = -\rho^{-2} \{ \tilde{E}_i(r) - \tilde{E}_i(\log \rho)(r_i - r) \}, \tag{22}$$

$$L_{ij} = \rho^{-2} \{ Hess_{ij}(\log \rho) - \tilde{E}_i(\log \rho) \tilde{E}_j(\log \rho) + \frac{1}{2} |\nabla \log \rho|^2 \delta_{ij} \}, \tag{23}$$

where $Hess_{ij}$ and ∇ are Hessian matrix and the gradient with respect to $III = d\xi \cdot d\xi$.

Now we prove that $\nabla \mathbf{B} = 0$, $\mathbf{L} = 0$ and $\mathbf{C} = 0$. We define ξ as follows

$$\xi = \left(-\frac{|\lambda_1 u_1|^2 + \dots + |\lambda_s u_s|^2 - 1}{2}, \lambda_1 u_1, \lambda_2 u_2, \dots, \lambda_s u_s, -\frac{|\lambda_1 u_1|^2 + \dots + |\lambda_s u_s|^2 + 1}{2} \right).$$

Clearly ξ satisfies the following conditions

$$\langle \xi, dx \rangle = 0, \quad \langle \xi, \xi \rangle = 0, \quad \langle \xi, \nu \rangle = 1.$$

By a direct calculation we know that the eigenvalues of S are $\lambda_1, \lambda_2, \dots, \lambda_s$ with multiplicity m_1, m_2, \dots, m_s , respectively.

Set

$$r_i = \frac{1}{\lambda_i}, \quad r = \frac{1}{n-1}(r_1 + r_2 + \dots + r_{n-1}).$$

From (22), (23) we get the Laguerre invariants of x given by

$$\rho^2 = \sum_i (r_i - r)^2 = \text{constant}, \quad C_i = 0, \quad L_{ij} = 0, \quad 1 \leq i, j \leq n-1,$$

$$B_{ij} = b_i \delta_{ij}, \quad 1 + m_1 + \dots + m_{i-1} \leq i, j \leq 1 + m_1 + \dots + m_{i-1} + m_i.$$

where $b_i = \frac{r_i - r}{\sqrt{\sum_i (r_i - r)^2}}$.

Thus from (12) we get that x is flat with respect to g , i.e.,

$$R_{ijkl} = 0, \quad 1 \leq i, j, k, l \leq n-1.$$

and the Laguerre second fundamental form $\mathbf{B} = B_{ij} \omega_i \otimes \omega_j$ of x is parallel.

4. The proof of mian theorem

Firstly, we state the following result which is needed in the proof of main Theorem.

Lemma 4.1. *Let $x : M \rightarrow R^n$ be an umbilical free hypersurface with non-zero principal curvatures. Then we have*

$$|\nabla \mathbf{B}|^2 + (n-1)tr(\mathbf{B}(\nabla \mathbf{C})) - (n-1)tr(\mathbf{L}\mathbf{B}^2) - tr\mathbf{L} = 0 \tag{24}$$

where we denote

$$|\nabla \mathbf{B}| = \sum_{i,j,k} B_{ij,k}^2, \quad tr(\mathbf{B}(\nabla \mathbf{C})) = \sum_{i,j} B_{ij} C_{j,i}, \quad tr(\mathbf{L}\mathbf{B}^2) = \sum_{i,j,k} L_{ij} B_{jk} B_{ki}.$$

Proof. By definition, we have

$$\frac{1}{2} \Delta \left(\sum_{i,j} B_{ij}^2 \right) = \sum_{i,j,k} B_{ij,k}^2 + \sum_{i,j,k} B_{ij} B_{ij,kk},$$

which is equivalent to

$$\sum_{i,j,k} B_{ij,k}^2 + \sum_{i,j,k} B_{ij} B_{ij,kk} = 0. \tag{25}$$

The second covariant derivative of B_{ij} is defined by

$$\sum_l B_{ij,kl} \omega_l = dB_{ij,k} + \sum_l B_{ljk} \omega_{li} + \sum_l B_{il,k} \omega_{lj} + \sum_l B_{ij,l} \omega_{lk}. \tag{26}$$

We have the following Ricci identities,

$$B_{ij,kl} - B_{ij,lk} = \sum_m B_{mj}R_{mikl} + \sum_m B_{im}R_{mjkl}. \tag{27}$$

On the other hand, using (10), (26) and ((27), a standard calculation gives

$$\begin{aligned} B_{ij,kk} &= (B_{ik,j} + C_j\delta_{ik} - C_k\delta_{ij})_k = (B_{ik,jk} + C_{j,k}\delta_{ik} - C_{k,k}\delta_{ij}) \\ &= B_{ik,kj} + \sum_l B_{lk}R_{lij} + \sum_l B_{il}R_{lkj} + C_{j,k}\delta_{ik} - C_{k,k}\delta_{ij} \\ &= B_{kk,ij} + \sum_l B_{lk}R_{lij} + \sum_l B_{il}R_{lkj} + C_{i,j}\delta_{kk} - C_{k,j}\delta_{ki} + C_{j,k}\delta_{ik} - C_{k,k}\delta_{ij}. \end{aligned}$$

From (11), (12) and the above equation, we easily obtain

$$\begin{aligned} \sum_{i,j,k} B_{ij}B_{ij,kk} &= \sum_{i,j,k,l} B_{ij}B_{lk}R_{lij} + \sum_{i,j,k,l} B_{ij}B_{il}R_{lkj} + (n-1) \sum_{i,j} B_{ij}C_{j,i} \\ &= \sum_{i,j,k,l} B_{ij}B_{lk}(L_{ij}\delta_{lk} + L_{lk}\delta_{ij} - L_{lj}\delta_{ik} - L_{ik}\delta_{lj}) \\ &\quad + \sum_{i,j,k,l} B_{ij}B_{il}(L_{kj}\delta_{lk} + L_{lk}\delta_{kj} - L_{lj}\delta_{kk} - L_{kk}\delta_{lj}) + (n-1) \sum_{i,j} B_{ij}C_{j,i} \\ &= -(n-1)tr(\mathbf{LB}^2) - tr\mathbf{L} + (n-1)tr(\mathbf{B}(\nabla\mathbf{C})). \end{aligned}$$

Inserting the above into (25), we get (24). \square

Lemma 4.2. *Let $x : M \rightarrow R^n$ be an umbilical free hypersurface with non-zero principal curvatures. If the Laguerre second fundamental form is parallel and the Laguerre tensor is isotropic ($\mathbf{L} = \lambda\mathbf{g}$, where λ is a function), then $\lambda = 0$ and M^{n-1} is laguerre equivalent to the images of τ of the hypersurface \tilde{x} in R_0^n with mean curvature radius $r = 0$ and $\rho = \text{constant}$.*

Proof. From (12) and $\nabla\mathbf{B} = 0$, we obtain $\mathbf{C} = 0$, and hence $\nabla\mathbf{C} = 0$. From the formula (24), we obtain

$$-(n-1)\lambda tr(\mathbf{B}^2) - (n-1)\lambda = 0. \tag{28}$$

Noting that $tr(\mathbf{B}^2) = \sum_{i,j}(B_{ij})^2 = 1$, we obtain $\lambda = 0$. Hence

$$|\nabla\mathbf{B}| = 0, \quad \mathbf{L} = 0, \quad \mathbf{C} = 0. \tag{29}$$

By the classification theorem in [4], we know that M^{n-1} is laguerre equivalent to the images of τ of the hypersurface \tilde{x} in R_0^n with mean curvature radius $r = 0$ and $\rho = \text{constant}$, given by the example 3.1. \square

The proof of the main theorem. Let P be an arbitrary point in M . Since the $(n \times n)$ -matrix (L_{ij}) is a real symmetric matrix, we can choose orthonormal basis such that $(L_{ij}) = \lambda_i\delta_{ij}$ at P , where λ_i are the eigenvalues of the $(n \times n)$ -matrix (L_{ij}) at P . From the formula (24), $\nabla\mathbf{C} = 0$, $tr(\mathbf{LB}^2) = \sum_{i,j} \lambda_i(B_{ij})^2$, $tr(\mathbf{B}^2) = \sum_{i,j}(B_{ij})^2 = 1$. If the Laguerre tensor \mathbf{L} is non-positive, then $\lambda_i \leq 0$, hence and

$$0 = |\nabla\mathbf{B}|^2 - (n-1) \sum_{i,j} \lambda_i(B_{ij})^2 - \sum_i \lambda_i \geq |\nabla\mathbf{B}|^2 - 2(n-1) \max\{\lambda_i\} \geq |\nabla\mathbf{B}|^2 \geq 0, \tag{30}$$

and so $|\nabla\mathbf{B}| = 0$. Hence $\mathbf{C} = 0$, and the equality must hold every where in (30) by the arbitrariness of point P . Then $\lambda_i = 0$, i.e., $\mathbf{L} = 0$. Similarly, when the Laguerre tensor \mathbf{L} is nonnative, we can get the same results. By the lemma 3.2, M^{n-1} is laguerre equivalent to the images of τ of the hypersurface \tilde{x} in R_0^n with mean curvature radius $r = 0$ and $\rho = \text{constant}$, given by the example 3.1. This completes the proof.

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