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# New Classes of Preinvex Functions and Variational-Like Inequalities

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*Dedicated to our respected Parents and Teachers*

**Abstract.** In this paper, we define and introduce some new concepts of the higher order strongly generalized preinvex functions and higher order strongly monotone operators involving the arbitrary bifunction and function. Some new relationships among various concepts of higher order strongly general preinvex functions have been established. It is shown that the optimality conditions of the higher order strongly general preinvex functions are characterized by a class of variational inequalities, which is called the higher order strongly generalized variational-like inequality. Auxiliary principle technique is used to suggest an implicit method for solving higher order strongly generalized variational-like inequalities. Convergence analysis of the proposed method is investigated using the pseudo-monotonicity of the operator. It is shown that the new parallelogram laws for Banach spaces can be obtained as applications of higher order strongly affine generalized preinvex functions, which is itself a novel application. Some special cases also discussed. Results obtained in this paper can be viewed as refinement and improvement of previously known results.

## 1. Introduction

In recent years, several extensions and generalizations have been considered for classical convexity. Noor and Noor [27, 28] and Mohsen et al[16] introduced the concept of higher order strongly convex functions and studied their properties. These results can be viewed as significant refinement of the results of Lin and Fukushima [13] and Alabdali et al [1] for higher order strongly (uniformly) convex functions. Higher order strongly convex functions include the strongly convex functions as special case, which were introduced and studied by Polyak [33]. With appropriate choice of non-negative arbitrary functions, one can obtain various known and new classes of convex functions. For the properties of the strongly convex functions and their variant forms, see Adamek [2], Karamardian [12] Nikodem et al. [18] and Noor and Noor [25–28].

Hanson [11] introduced the concept of invex function for the differentiable functions in mathematical programming. Ben-Israel and Mond [4] introduced the concept of invex set and preinvex functions. It is known that the differentiable preinvex function are invex functions. The converse also holds under certain conditions, see [15]. Noor [20] proved that the minimum of the differentiable preinvex functions on the invex set can be characterized by a class of variational inequalities, which is known as the variational-like

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inequality. For the recent developments in variational-like inequalities and invex equilibrium problems, see [18, 19, 27] and the references therein. Noor at el. [20, 21, 26] investigated the properties of the strongly preinvex functions and their variant forms.
In a many problems, a set may not be a convex set. To overcome this drawback, the underlying set can be made a convex set with respect to an arbitrary function. This fact motivated Youness [41] to introduce the concept of general convex sets and general convex functions involving an arbitrary function. Cristescu at al[8, 9] have investigated algebraic and topological properties of the general convex sets defined by Noor [24] in order to deduce their shape. These general sets are a subclass of star-shaped sets, which have Youness [41] type convexity. A representation theorem based on extremal points is given for the class of bounded general convex sets. Results showing that this convexity is a frequent property in connection with a wild range of applications are given, see, for example,[8, 9]. Noor [24] has shown that the optimality conditions of the differentiable general convex functions can be characterized a class of variational inequalities called general variational inequality, the origin of which can be traced back to Stampacchia [35]. Noor and Noor [23, 25–28] introduced the higher order strongly general convex functions and studied their properties. For the formulation, applications, numerical methods, sensitivity analysis and other aspects of general variational inequalities, see [17–22, 26–28, 30–33, 35, 38, 43] and the references therein.

We would like to point out that preinvex functions and general convex functions are two distinctly different generalizations and extensions of convex functions in various directions. These type of functions have played a leading role in the developments of various branches of pure and applied sciences. It is natural to unify these classes and investigate their characterizations. Inspired by the research work going in this field, we introduce and consider another class of non-convex functions with respect to the arbitrary non-negative bifunction and a function This class of non-convex functions is called the higher order strongly generalized preinvex functions. Several new concepts of monotonicity are introduced. We establish the relationship between these classes and derive some new results under some mild conditions. As a novel and innovative application of these higher order strongly affine generalized preinvex functions, we obtain the parallelogram-like laws for uniformly Banach spaces. We have shown that the minimum of a differentiable higher order strongly generalized preinvex functions on the general invex set can be characterized by a class of variational-like inequality. This results inspired us to consider the higher order strongly generalized variational-like inequalities. Due to the inherent nonlinearity, the projection method and its variant form can not be used to suggest the iterative methods for solving these generalized variational-like inequalities. To overcome these drawbacks, we use the technique of the auxiliary principle [10, 14, 23, 30, 43] to suggest an implicit method for solving general variational-like inequalities. Convergence analysis of the proposed method is investigated under pseudo-monotonicity, which is a weaker condition than monotonicity. As special cases, on can obtain various new and refined versions of the known results. It is expected that the ideas and techniques of this paper may stimulate further research in this field.

## 2. Preliminary Results

Let $K$ be a nonempty closed set in a real Hilbert space $H$. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively. Let $F : K \to R$ be a continuous function and let $g : [0, \infty) \to R$ be a non-negative function.

**Definition 2.1.** [9, 17].*The set $K$ in $H$ is said to be a convex set, if*

$$u + t(v - u) \in K, \qquad \forall u, v \in K, t \in [0, 1].$$

**Definition 2.2.** [9, 17] *A function $F$ is said to be a convex function, if*

$$F((1 - t)u + tv) \leq (1 - t)F(u) + tF(v), \quad \forall u, v \in K, \quad t \in [0, 1].$$

If the convex function $F$ is differentiable, then $u \in K$ is the minimum of the $F$, if and only if, $u \in K$ satisfies the inequality

$$\langle F'(u), v - u\rangle \geq 0, \quad \forall v \in K,$$

which is called the variational inequality, introduced and studied by Stampacchia [35] in 1964. For the applications, formulation, sensitivity, dynamical systems, generalizations, and other aspects of the variational inequalities, see [10, 19–23, 26, 30, 35, 43] and the references therein.

It is known that in many problems the underlying set may not be a convex set. To overcome this drawback, Youness [24] introduced the general convex sets with respect to an arbitrary function.

**Definition 2.3.** [24].*The set $K_g$ in $H$ is said to be general convex set, if there exists an arbitrary function $g$, such that*

$$(1-t)g(u) + tg(v) \in K_g, \quad \forall u, v \in H : g(u), g(v) \in K_g, t \in [0,1].$$

If $g = I$, the identity operator, then general convex set reduces to the classical convex set. Clearly every convex set is a general convex set, but the converse is not true.

For the sake of simplicity, we always assume that $\forall u, v \in H : g(u), g(v) \in K_g$, unless otherwise.

**Definition 2.4.** *A function $F$ is said to be general convex($g$-convex) function, if there exists an arbitrary non-negative function $g$, such that*

$$F((1-t)g(u) + tg(v)) \leq (1-t)F(g(u)) + tF(g(v)), \quad \forall g(u), g(v) \in K_g, \quad t \in [0,1].$$

The general convex functions were introduced by Noor [24]. Noor [24] proved that the minimum $u \in H : g(u) \in K_g$ of the differentiable general convex functions $F$ can be characterized by the class of variational inequalities of the type:

$$\langle F'(g(u)), g(v) - g(u)\rangle \geq 0, \quad \forall v \in H : g(v) \in K_g,$$

which is known as general variational inequalities, introduced and studied by Noor [24].

Ben-Israel and Mond [4] introduced the concept of invex set and preinvex functions, which has inspired a great deal of interest of the applications of invex sets and preinvex functions in mathematical programming and optimization problems.

**Definition 2.5.** [4].*The set $K_\eta$ in $H$ is said to be an invex set with respect to an arbitrary bifunction $\eta(\cdot,\cdot)$, if*

$$u + t\eta(v, u) \in K_\eta, \qquad \forall u, v \in K_\eta, t \in [0,1].$$

The invex set $K_\eta$ is also called $\eta$-connected set. Note that the invex set with $\eta(v, u) = v - u$ is a convex set $K$, but the converse is not true. For example, the set $K_\eta = R - (-\frac{1}{2}, \frac{1}{2})$ is an invex set with respect to $\eta$, where

$$\eta(v,u) = \begin{cases} v - u, & \text{for} \quad v > 0, u > 0 \quad \text{or} \quad v < 0, u < 0 \\ u - v, & \text{for} \quad v < 0, u > 0 \quad \text{or} \quad v < 0, u < 0. \end{cases}$$

It is clear that $K_\eta$ is not a convex set.

From now onward $K_\eta$ is a nonempty closed invex set in $H$ with respect to the bifunction $\eta(\cdot,\cdot)$, unless otherwise specified.

Clearly the general convex sets and invex sets are two different generalizations of the convex sets and have important applications. It is natural to unifies these concepts. These facts and observations motivated us to introduce the following:

**Definition 2.6.** *A set $K_{g\eta} \subset H$ is said to be a generalized invex set with respect to an arbitrary function $g$ and bifunction $\eta(.,.)$, if and only if*

$$g(u) + t\eta(g(v), g(u)) \in K_{g\eta}, \quad \forall u, v \in H : g(u), g(v) \in K_{g\eta}, t \in [0.1],$$

which was introduced by Awan et al[3] in 2015.

**Definition 2.7.** *The function $F$ on the generalized invex set $K_{g\eta}$ is said to be higher order strongly generalized preinvex with respect to the bifunction $\eta(\cdot,\cdot)$ and an arbitrary function $g$, if there exists a constant $\mu \geq 0$, such that*

$$\begin{aligned} F(g(u) + t\eta(g(v), g(u))) \leq\ & (1-t)F(g(u)) + tF(g(v)) \\ & -\mu\{t^p(1-t) + t(1-t)^p\}\|\eta(g(v), g(u))\|^p, \\ & \forall g(u), g(v) \in K_{g\eta}, t \in [0,1], p \geq 1. \end{aligned}$$

The function $F$ is said to be higher order strongly generalized preconcave, if and only if, $-F$ is higher order strongly generalized preinvex function. Note that every higher order strongly generalized convex function is a higher order strongly generalized preinvex, but the converse is not true.

**I.** If $\eta(g(v), g(u)) = g(v) - g(u)$, then the higher order strongly generalized preinvex function becomes higher order strongly general convex functions, that is,

**Definition 2.8.** *The function $F$ on the general convex set $K_g$ is said to be higher order strongly general convex with respect to function $g$, if there exists a constant $\mu \geq 0$, such that*

$$\begin{aligned} F(g(u) + t(g(v) - g(u))) \leq\ & (1-t)F(g(u)) + tF(g(v)) \\ - \ & \mu\{t^p(1-t) + t(1-t)^p\}\|g(v) - g(u)\|^p, \\ & \forall g(u), g(v) \in K_g, t \in [0,1], p \geq 1, \end{aligned}$$

*which were investigated by Noor et al. [29, 32].*

For the properties of the higher order strongly general convex functions in variational inequalities and equilibrium problems, see Noor et al. [28, 29, 32].
**II.** If $\eta(g(v), g(u)) = v - u$, then the higher order strongly generalized preinvex function becomes higher order strongly convex functions, that is,

**Definition 2.9.** *The function $F$ on the convex set $K$ is said to be higher order strongly convex, if there exists a constant $\mu \geq 0$, such that*

$$F(u + t(v-u)) \leq (1-t)F(u) + tF(v) - \mu\{t^p(1-t) + t(1-t)^p\}\|v - u\|^p, \\ \forall u, v \in K, t \in [0,1], p \geq 1,$$

*which were introduced and studied by Mohsen et al [16].*

For the properties of the higher order strongly convex functions in variational inequalities and equilibrium problems, see Noor et al. [20–22, 25, 27–29, 32].

**III.** If $p = 2$, then Definition 2.7 becomes:

**Definition 2.10.** *A function $F$ on the generalized invex set $K_{g\eta}$ is said to be strongly generalized preinvex function with respect to the bifunction $\eta(.,.)$ and function $g$, if there exists a constant $\mu \geq 0$, such that*

$$F(g(u) + t\eta(g(v), g(u))) \leq (1-t)F(g(u)) + tF(g(v)) - \mu t(1-t)\|\eta(g(v), g(u))\|^2, \\ \forall g(u), g(v) \in K_{g\eta}, t \in [0,1],$$

*which appears to be a new one.*

**IV.** If $\mu = 0$, then Definition 2.7 becomes:

**Definition 2.11.** *A function F on the generalized invex set $K_{g\eta}$ is said to be general preinvex function with respect to the bifunction $\eta(.,.)$ and function $g$, if*

$$F(g(u) + t\eta(g(v), g(u))) \leq (1-t)F(g(u)) + tF(g(v)), \quad \forall g(u), g(v) \in K_{g\eta}, t \in [0,1],$$

which was introduced and studied by Awan et al[3].

**Definition 2.12.** *The function F on the generalized invex set $K_{g\eta}$ is said to be higher order strongly generalized quasi preinvex with respect to the bifunction $\eta(\cdot,\cdot)$ and an arbitrary function $g$, if there exists a constant $\mu \geq 0$, such that*

$$\begin{aligned} F(g(u) + t\eta(g(v), g(u))) \leq \max\{F(g(u)), F(g(v))\} - \mu\{t^p(1-t) + t(1-t)^p\}\|\eta(g(v), g(u))\|^p, \\ \forall g(u), g(v) \in K_{g\eta}, t \in [0,1], p \geq 1. \end{aligned}$$

**Definition 2.13.** *The function F on the generalized invex set $K_{g\eta}$ is said to be higher order strongly generalized log-preinvex with respect to the bifunction $\eta(\cdot,\cdot)$ and function $g$, if there exists a constant $\mu \geq 0$, such that*

$$\begin{aligned} F(g(u) + t\eta(g(v), g(u))) \leq (F(g(u)))^{1-t}(F(g(v)))^t - \mu\{t^p(1-t) + t(1-t)^p\}\|\eta(g(v), g(u))\|^p, \\ \forall g(u), g(v) \in K_{g\eta}, t \in [0,1], p \geq 1, \end{aligned}$$

*where $F(\cdot) > 0$.*

From the above definitions, we have

$$\begin{aligned} F(g(u) + t\eta(g(v), g(u))) &\leq (F(g(u)))^{1-t}(F(g(v)))^t \\ &\quad -\mu\{t^p(1-t) + t(1-t)^p\}\|\eta(g(v), g(u))\|^p \\ &\leq (1-t)F(g(u)) + tF(g(v))) \\ &\quad -\mu\{t^p(1-t) + t(1-t)^p\}\|\eta(g(v), g(u))\|^p \\ &\leq \max\{F(g(u)), F(g(v))\} \\ &\quad -\mu\{t^p(1-t) + t(1-t)^p\}\|\eta(g(v), g(u))\|^p. \end{aligned}$$

This shows that every higher order strongly generalized log-preinvex function is higher order strongly generalized preinvex function and every higher order strongly generalized preinvex function is a higher order strongly generalized quasi-preinvex function. However, the converse is not true.

**Definition 2.14.** *The function F on the generalized invex set $K_{g\eta}$ is said to be higher order strongly affine generalized preinvex with respect to the bifunction $\eta(\cdot,\cdot)$ and function $g$, if there exists a constant $\mu \geq 0$, such that*

$$\begin{aligned} F(g(u) + t\eta(g(v), g(u))) &= (1-t)F(g(u)) + tF(g(v)) \\ &- \mu\{t^p(1-t) + t(1-t)^p\}\|\eta(g(v), g(u))\|^p, \\ &\forall g(u), g(v) \in K_{g\eta}, t \in [0,1], p \geq 1. \end{aligned}$$

For $t = 1$, Definitions 2.7 and 2.14 reduce to the following condition:
**Condition A**.

$$F(g(u) + \eta(g(v), g(u))) \leq F(g(v)), \quad \forall g(u), g(v) \in K_{g\eta}.$$

**Definition 2.15.** *The differentiable function F on the generalized invex set $K_{g\eta}$ is said to be higher order strongly generalized invex function with respect to the bifunction $\eta(\cdot,\cdot)$ and function $g$, if there exists a constant $\mu \geq 0$, such that*

$$F(g(v)) - F(g(u)) \geq \langle F'(g(u)), \eta(g(v), g(u))\rangle + \mu\|\eta(g(v), g(u))\|^p, \quad \forall g(u), g(v) \in K_{g\eta}, p \geq 1,$$

*where $F'(g(u)))$ is the differential of F at $g(u)$.*

It is noted that, if $\mu = 0$ and $g = I$, then the Definition 2.10 reduces to the definition of the invex function as introduced by Hanson [11]. It is well known that the concepts of preinvex and invex functions play significant roles in the mathematical programming and optimization theory, see [4, 11, 15, 34, 37, 38, 42] and the references therein.

**Remark 2.16.** *Note that, if $\mu = 0$, then the Definitions 2.10-2.13 appear to be new ones.*

**Definition 2.17.** *An operator $T : K_{g\eta} \to H$ is said to be:*

1. *higher order strongly $g\eta$-monotone, iff, there exists a constant $\alpha > 0$ such that*
$$\begin{aligned}\langle Tu, \eta(g(v), g(u))\rangle &+ \langle Tv, \eta(g(u), g(v))\rangle \\ &\leq -\alpha\{\|\eta(g(v), g(u))\|^p + \|\eta(g(u), g(v))\|^p\}, \quad \forall g(u), g(v) \in K_{g\eta}, p > 0.\end{aligned}$$
2. *$\eta$-monotone, iff,*
$$\langle Tu, \eta(g(v), g(u))\rangle + \langle Tv, \eta(g(u), g(v))\rangle \leq 0, \quad \forall g(u), g(v) \in K_{g\eta}, p > 0.$$
3. *higher order strongly $g\eta$-pseudomonotone, iff, there exists a constant $\nu > 0$ such that*
$$\begin{aligned}\langle Tu, \eta(g(v), g(u))\rangle + \nu\|\eta(g(v), g(u))\|^p \geq 0 \Rightarrow -\langle Tv, \eta(g(u), g(v)))\rangle \geq 0,\\ \forall g(u), g(v) \in K_{g\eta}, p > 0.\end{aligned}$$
4. *higher order strongly relaxed $g\eta$-pseudomonotone, iff, there exists a constant $\mu > 0$ such that*
$$\begin{aligned}\langle Tu, \eta(g(v), g(u))\rangle \geq 0 \Rightarrow -\langle Tv, \eta(g(u), g(v)\rangle + \mu\|\eta(g(u), g(v))\|^p \geq 0,\\ \forall g(u), g(v) \in K_{g\eta}, p > 0.\end{aligned}$$
5. *strictly $g\eta$-monotone, iff,*
$$\langle Tu, \eta(g(v), g(u))\rangle + \langle Tv, \eta(g(u), g(v))\rangle < 0, \quad \forall g(u), g(v) \in K_{g\eta}, p > 0.$$
6. *$g\eta$-pseudomonotone, iff,*
$$\begin{aligned}\langle Tu, \eta(g(v), g(u))\rangle \geq 0 \Rightarrow \langle Tv, \eta(g(u), g(v))\rangle \leq 0,\\ \forall \quad g(u), g(v) \in K_{g\eta}, p > 0.\end{aligned}$$
7. *quasi $g\eta$-monotone, iff,*
$$\langle Tu, \eta(g(v), g(u))\rangle > 0 \Rightarrow \langle Tv, \eta(g(u), g(v))\rangle \leq 0, \quad \forall g(u), g(v) \in K_{g\eta}, p > 0.$$
8. *strictly $g\eta$-pseudomonotone, iff,*
$$\langle Tu, \eta(g(v), g(u))\rangle \geq 0 \Rightarrow \langle Tv, \eta(g(u), g(v))\rangle < 0, \quad \forall g(u), g(v) \in K_{g\eta}, p > 0.$$

**Definition 2.18.** *A differentiable function $F$ on the generalized invex set $K_{g\eta}$ is said to be higher order strongly pseudo $g\eta$-invex function, iff, there exists a constant $\mu \geq 0$ such that*
$$\begin{aligned}&\langle F'(u), \eta(g(v), g(u))\rangle + \mu\|\eta(g(v), g(u))\|^p \geq 0\\ &\Rightarrow F(g(v)) - F(g(u)) \geq 0, \quad \forall g(u), g(v) \in K_{g\eta}, p > 1.\end{aligned}$$

**Definition 2.19.** *A differentiable function $F$ on the generalized invex set $K_{g\eta}$ is said to be higher order strongly quasi-invex function, iff, if there exists a constant $\mu > 0$ such that*
$$\begin{aligned}&F(g(v)) \leq F(g(u))\\ &\Rightarrow\\ &\langle F'(g(u)), \eta(g(v), g(u))\rangle + \mu\|\eta(g(u), g(v))\|^p \leq 0,\\ &\quad \forall g(u), g(v) \in K_{g\eta}, p > 1.\end{aligned}$$

**Definition 2.20.** *The function F on the generalized invex set $K_{g\eta}$ is said to be pseudo-invex, if*

$$\langle F'(g(u)), \eta(g(v), g(u))\rangle \geq 0 \Rightarrow F(g(v)) \geq F(g(u)), \qquad \forall g(u), g(v) \in K_{g\eta}.$$

**Definition 2.21.** *The differentiable function F on the generalized invex $K_{g\eta}$ is said to be higher order strongly generalized quasi-invex function, if*

$$F(g(v)) \leq F(g(u)) \Rightarrow \langle F'(g(u)), \eta(g(v), g(u))\rangle \leq 0, \qquad \forall g(u), g(v) \in K_{g\eta}.$$

We also need the following assumption regarding the bifunction $\eta(.,.)$, which can be viewed as a generalization of the condition of Mohan and Neogy [15].

**Condition C.** Let $\eta(\cdot,\cdot) : K_{g\eta} \times K_{g\eta} \to H$ satisfy assumptions

$$\eta(g(u), g(u) + t\eta(g(v), g(u))) = -t\eta(g(v), g(u))$$
$$\eta(g(v), g(u) + t\eta(g(v), g(u))) = (1-t)\eta(g(v), g(u)), \quad \forall g(u), g(v) \in K_{g\eta}, t \in [0,1].$$

Clearly for $t = 0$, we have $\eta(g(u), g(v)) = 0$, if and only if $g(u) = g(v), \forall u, v \in K_{g\eta}$. One can easily show [11, 15] that $\eta(g(u) + t\eta(g(v), g(u)), g(u)) = t\eta(g(v), g(u)), \forall g(u), g(v) \in K_{g\eta}$.

## 3. Main Results

In this section, we discuss some basic properties of higher order strongly generalized preinvex functions on the generalized invex set $K_{g\eta}$.

**Theorem 3.1.** *Let F be a differentiable function on the generalized invex set $K_{g\eta}$ and let the condition C hold. Then a function F is higher order strongly generalized preinvex function, if and only if, F is a higher order strongly generalized invex function.*

*Proof.* Let $F$ be a higher order strongly generalized preinvex function on the generalized invex set $K_{g\eta}$. Then

$$F(g(u) + t\eta(g(v), g(u))) \leq (1-t)F(g(u)) + tF(g(v)) - \mu\{t^p(1-t) + t(1-t)^p\}\|\eta(g(v), g(u))\|^p,$$
$$\forall g(u), g(v) \in K_{g\eta}, t \in [0,1], p > 1.$$

which can be written as

$$\begin{aligned} F(g(v)) - F(g(u)) \quad &\geq \quad \{\frac{F(g(u) + t\eta(g(v), g(u))) - F(g(u))}{t}\} \\ &\quad + \mu\{t^{p-1}(1-t) + (1-t)^p\}\|\eta(g(v), gu))\|^p. \end{aligned}$$

Taking the limit in the above inequality as $t \to 0$ , we have

$$F(g(v)) - F(g(u)) \geq \langle F'(g(u)), \eta(g(v), g(u)))\rangle + \mu\|\eta(g(v), g(u))\|^p.$$

This shows that $F$ is a higher order strongly generalized invex function.

Conversely, let $F$ be a higher order strongly generalized invex function on the generalized invex set $K_{g\eta}$. Then, $\forall g(u), g(v) \in K_{g\eta}, t \in [0,1]$, $g(v_t) = g(u) + t\eta(g(v), g(u)) \in K_{g\eta}$ and using the condition C, we have

$$\begin{aligned} & F(g(v)) - F(g(u) + t\eta(g(v), g(u))) \\ \geq \quad & \langle F'(g(u) + t\eta(g(v), g(u))), \eta(g(v), g(u) + t\eta(g(v), g(u)))\rangle \\ & + \mu\|\eta(g(v), g(u) + t\eta(g(v), g(u)))\|^p \\ = \quad & (1-t)\langle F'(g(u) + t\eta(g(v), g(u))), \eta(g(v), g(u))\rangle + \mu(1-t)^p\|\eta(g(v), g(u))\|^p. \end{aligned} \tag{3.1}$$

In a similar way, we have

$$\begin{aligned}
& F(g(u)) - F(g(u) + t\eta(g(v), g(u))) \\
\geq\ & \langle F'(g(u) + t\eta(g(v), g(u))), \eta(g(u), g(u) + t\eta(g(v), g(u)))\rangle \\
& + \mu\|\eta(g(u), g(u) + t\eta(g(v), g(u)))\|^p \\
=\ & -t\langle F'(g(u) + t\eta(g(v), g(u))), \eta(g(v), g(u))\rangle + \mu t^p\|\eta(g(v), g(u))\|^p. \qquad (3.2)
\end{aligned}$$

Multiplying (3.1) by $t$ and (3.2) by $(1-t)$ and adding the resultant, we have

$$\begin{aligned}
F(g(u) + t\eta(g(v), g(u))) \leq\ & (1-t)F(g(u)) + tF(g(v)) \\
& -\mu\{t^p(1-t) + t(1-t)^p\}\|\eta(g(v), g(u))\|^p,
\end{aligned}$$

showing that $F$ is a higher order strongly generalized preinvex function. □

**Theorem 3.2.** *Let $F$ be a differentiable higher order strongly generalized preinvex function on the generalized invex set $K_{g\eta}$. If $F$ is a higher order strongly generalized invex function, then*

$$\begin{aligned}
& \langle F'(g(u)), \eta(g(v), g(u))\rangle + \langle F'(g(v)), \eta(g(u), g(v))\rangle \\
\leq\ & -\mu\{\|\eta(g(v), g(u))\|^p + \|\eta(g(u), g(v))\|^p\}, \forall g(u), g(v) \in K_{g\eta}. \qquad (3.3)
\end{aligned}$$

*Proof.* Let $F$ be a higher order strongly general invex function on the generalized invex set $K_{g\eta}$. Then

$$F(g(v)) - F(g(u)) \geq \langle F'(g(u)), \eta(g(v), g(u))\rangle + \mu\|\eta(g(v), g(u))\|^p, \quad \forall g(u), g(v) \in K_{g\eta}. \qquad (3.4)$$

Changing the role of $g(u)$ and $g(v)$ in (3.4), we have

$$F(g(u)) - F(g(v)) \geq \langle F'(g(v)), \eta(g(u), g(v))\rangle + \mu\|\eta(g(u), g(v))\|^p, \quad \forall g(u), g(v) \in K_{g\eta}. \qquad (3.5)$$

Adding (3.4) and (3.5), we have

$$\begin{aligned}
& \langle F'(g(u)), \eta(g(v), g(u))\rangle + \langle F'(g(v)), \eta(g(u), g(v))\rangle \\
\leq\ & -\mu\{\|\eta(g(v), g(u))\|^p + \|\eta(g(u), g(v))\|^p\}, \forall g(u), g(v) \in K_{g\eta}, \qquad (3.6)
\end{aligned}$$

which shows that $F'(.)$ is higher order strongly $g\eta$-monotone operator. □

We note that the converse of Theorem 3.2 is true only for $p = 2$. However, we have:

**Theorem 3.3.** *If the differential $F'(.)$ is a higher order strongly $g\eta$-monotone operator, then*

$$F(g(v)) - F(g(u)) \geq \langle F'(g(u)), \eta(g(v), g(u))\rangle + \frac{2}{p}\mu\|\eta(g(v), g(u))\|^p.$$

*Proof.* Let $F'(.)$ be a higher order strongly $g\eta$-monotone operator. From (3.6), we have

$$\begin{aligned}
\langle F'(g(v)), \eta(g(u), g(v))\rangle \leq\ & -\langle F'(g(u)), \eta(g(v), g(u))\rangle \\
& -\mu\{\|\eta(g(v), g(u))\|^p + \|\eta(g(u), g(v))\|^p\}. \qquad (3.7)
\end{aligned}$$

Since $K_{g\eta}$ is a generalized invex set, $\forall g(u), g(v) \in K_{g\eta}$, $t \in [0, 1]$

$$g(v_t) = g(u) + t\eta(g(v), g(u)) \in K_{g\eta}.$$

Taking $g(v) = g(v_t)$ in (3.7) and using Condition C, we have

$$\begin{aligned}
\langle F'(g(v_t)), \eta(g(u), g(u) + t\eta(g(v), g(u)))\rangle \leq\ & \langle F'(g(u)), \eta(g(u) + t\eta(g(v), g(u)), g(u))\rangle \\
& -\mu\{\|\eta(g(u) + t\eta(g(v), g(u)), g(u))\|^p \\
& +\|\eta(g(u), u + t\eta(g(v), g(u))\|^p\} \\
=\ & -t\langle F'(g(u)), \eta(g(v), g(u))\rangle \\
& -2t^p\mu\|\eta(g(v), g(u))\|^p,
\end{aligned}$$

which implies that

$$\langle F'(v_t), \eta(g(v), g(u))\rangle \geq \langle F'(g(u)), \eta(g(v), g(u))\rangle + 2\mu t^{p-1}\|\eta(g(v), g(u))\|^p. \tag{3.8}$$

Let $\xi(t) = F(g(u) + t\eta(g(v), g(u))$. Then, from (3.8), we have

$$\begin{aligned}\xi'(t) &= \langle F'(g(u) + t\eta(g(v), g(u))), \eta(g(v), g(u))\rangle \\ &\geq \langle F'(g(u)), \eta(g(v), g(u))\rangle + 2\mu t^{p-1}\|\eta(g(v), g(u))\|^p.\end{aligned} \tag{3.9}$$

Integrating (3.9) between 0 and 1, we have

$$\xi(1) - \xi(0) \geq \langle F'(g(u)), \eta(g(v), g(u))\rangle + \frac{2}{p}\mu\|\eta(g(v), g(u))\|^p.$$

that is,

$$F(g(u) + t\eta(g(v), g(u))) - F(g(u)) \geq \langle F'(g(u)), \eta(g(v), g(u))\rangle + \frac{2}{p}\mu\|\eta(g(v), g(u)\|^p).$$

By using Condition A, we have

$$F(g(v)) - F(g(u)) \geq \langle F'(g(u)), \eta(g(v), g(u))\rangle + \frac{2}{p}\mu\|\eta(g(v), g(u))\|^p.$$

the required result. □

We now give a necessary condition for higher order strongly $g\eta$-pseudo-invex function.

**Theorem 3.4.** *Let $F'(.)$ be a higher order strongly relaxed $g\eta$-pseudomonotone operator and Condition A and C hold. Then $F$ is a higher order strongly $\eta$-pseudo-invex function.*

*Proof.* Let $F'$ be higher order strongly relaxed $\eta$-pseudomonotone. Then,

$$\langle F'(g(u)), \eta(g(v), g(u))\rangle \geq 0, \forall g(u), g(v) \in K_{g\eta},$$

implies that

$$-\langle F'(g(v)), \eta(g(u), g(v))\rangle \geq \alpha\|\eta(g(u), g(v))\|^p. \tag{3.10}$$

Since $K_{g\eta}$ is a generalized invex set, $\forall g(u), g(v) \in K_{g\eta}$, $t \in [0, 1]$,
$g(v_t) = g(u) + t\eta(g(v), g(u)) \in K_{g\eta}$. Taking $g(v) = g(v_t)$ in (3.10) and using condition Condition C, we have

$$-\langle F'(g(u) + t\eta(g(v), g(u))), \eta(g(u), g(v))\rangle \geq t\alpha\|\eta(g(v), g(u))\|^p. \tag{3.11}$$

Let

$$\xi(t) = F(g(u) + t\eta(g(v), g(u))), \quad \forall g(u), g(v) \in K_{g\eta}, t \in [0, 1].$$

Then, using (3.11), we have

$$\xi'(t) = \langle F'(g(u) + t\eta(g(v), g(u))), \eta(g(u), g(v))\rangle \geq t\alpha\|\eta(g(v), g(u))\|^p.$$

Integrating the above relation between 0 to 1, we have

$$\xi(1) - \xi(0) \geq \frac{\alpha}{2}\|\eta(g(v), g(u))\|^p,$$

that is,

$$F(g(u) + t\eta(g(v), g(u))) - F(g(u)) \geq \frac{\alpha}{2}\|\eta(g(v), g(u))\|^p,$$

which implies, using Condition A,

$$F(g(v)) - F(g(u)) \geq \frac{\alpha}{2}\|\eta(g(v), g(u))\|^p,$$

showing that $F$ is a higher order strongly $g\eta$-pseudo-invex function. □

**Theorem 3.5.** *Let the differential $F'(u)$ of a differentiable higher order strongly generalized preinvex function $F(u)$ be Lipschitz continuous on the generalized invex set $K_{g\eta}$ with a constant $\beta > 0$. Then*

$$F(g(u) + \eta(g(v), g(u))) - F(g(u)) \leq \langle F'(g(u)), \eta(g(v), g(u))\rangle + \frac{\beta}{2}\|\eta(g(v), g(u))\|^2, \quad \forall g(u), g(v) \in K_{g\eta}.$$

*Proof.* Its proof follows from Noor and Noor [26]. □

**Definition 3.6.** *The function $F$ is said to be sharply higher order strongly generalized pseudo preinvex on the generalized invex set $K_{g\eta}$, if there exists a constant $\mu > 0$ such that*

$$\langle F'(g(u)), \eta(g(v), g(u))\rangle \geq 0$$
$$\Rightarrow$$
$$F(g(v)) \geq F(g(v) + t\eta(g(u), g(v))) + \mu\{t^p(1-t) + t(1-t)^p\}\|\eta(g(v), g(u))\|^p, \quad \forall g(u), g(v) \in K_{g\eta}, t \in [0,1].$$

**Theorem 3.7.** *Let $F$ be a sharply higher order strongly generalized pseudo preinvex function on the generalized invex set $K_{g\eta}$ with a constant $\mu > 0$. Then*

$$-\langle F'(g(v)), \eta(g(v), g(u))\rangle \geq \mu\|\eta(g(v), g(u))\|^p, \quad \forall g(u), g(v) \in K_{g\eta}.$$

*Proof.* Let $F$ be a sharply higher order strongly generalized pseudo preinvex function on the generalized invex set $K_{g\eta}$. Then

$$F(g(v)) \geq F(g(v) + t\eta(g(u), g(v))) + \mu\{t^p(1-t) + t(1-t)^p\}\|\eta(g(v), g(u))\|^p, \quad \forall g(u), g(v) \in K_{g\eta}, t \in [0,1],$$

from which we have

$$\frac{F(g(v) + t\eta(g(u), g(v))) - F(g(v))}{t} + \mu\{t^{p-1}(1-t) + (1-t)^p\}\|\eta(g(v), g(u)\|^p) \leq 0.$$

Taking limit in the above inequality, as $t \to 0$, we have

$$-\langle F'(g(v)), \eta(g(v), g(u))\rangle \geq \mu\|\eta(g(v), g(u))\|^p,$$

the required result. □

**Definition 3.8.** *A function $F$ is said to be a generalized pseudo preinvex function with respect to a strictly positive bifunction $B(.,.)$, if*

$$F(g(v)) < F(g(u))$$
$$\Rightarrow$$
$$F(g(u) + t\eta(g(v), g(u))) < F(g(u)) + t(t-1)B(g(v), g(u)), \forall g(u), g(v) \in K_{g\eta}, t \in [0,1].$$

**Theorem 3.9.** *If the function $F$ is higher order strongly generalized preinvex function such that $F(g(v)) < F(g(u))$, then the function $F$ is higher order strongly generalized pseudo preinvex.*

*Proof.* Since $F(g(v)) < F(g(u))$ and $F$ is higher order strongly preinvex function, then $\forall g(u), g(v) \in K_{g\eta}, \quad t \in [0,1]$, we have

$$\begin{aligned} F(g(u)+t\eta(g(v),g(u))) &\le F(g(u)) + t(F(g(v)) - F(g(u))) \\ &\quad -\mu\{t^p(1-t)+t(1-t)^p\}\|\eta(g(v),g(u))\|^p \\ &< F(g(u)) + t(1-t)(F(g(v)) - F(g(u))) \\ &\quad -\mu\{t^p(1-t)+t(1-t)^p\}\|\eta(g(v),g(u))\|^p \\ &= F(g(u)) + t(t-1)(F(g(u)) - F(g(v))) \\ &\quad -\mu\{t^p(1-t)+t(1-t)^p\}\|\eta(g(v),g(u))\|^p \\ &< F(u) + t(t-1)B(g(u),g(v)) \\ &\quad -\mu\{t^p(1-t)+t(1-t)^p\}\|\eta(g(v),g(u))\|^p, \forall g(u), g(v) \in K_{g\eta}, \end{aligned}$$

□

where $B(g(u), g(v)) = F(g(u)) - F(g(v)) > 0$. This shows that the function $F$ is higher order strongly generalized pseudo preinvex.

## 4. Applications

In this section, we show that the characterizations of uniformly Banach spaces involving the notion of higher order strongly generalized invexity are given.

Taking $F(u) = \|u\|^p$ in Definition2.14, we have

$$\begin{aligned} \|g(u)+t\eta(g(v),g(u))\|^p &= (1-t)\|g(u)\|^p + t\|g(v)\|^p \\ &- \mu\{t^p(1-t)+t(1-t)^p\}\|\eta(g(v),g(u))\|^p, \\ &\quad \forall g(u), g(v) \in K_{g\eta}, t \in [0,1], p > 1. \end{aligned} \tag{4.1}$$

Taking $t = \frac{1}{2}$ in (4.1), we have

$$\begin{aligned} \left\|\frac{2g(u)+\eta(g(v),g(u))}{2}\right\|^p + \mu\frac{1}{2^p}\|\eta(g(v),g(u))\|^p &= \frac{1}{2}\|g(u)\|^p + \frac{1}{2}\|g(v)\|^p, \\ &\quad \forall g(u), g(v) \in K_{g\eta}, \end{aligned} \tag{4.2}$$

which is known as the parallelogram-like laws for the Banach spaces involving bifunction $\eta(.,.)$ and the arbitrary function $g$.

If $\eta(g(v), g(u)) = g(v) - g(u)$, then (4.2) reduces to the parallelogram-like law as:

$$\|g(v)+g(u)\|^p + \mu\|g(v)-g(u)\|^p = 2^{p-1}\{\|g(u)\|^p + \|g(v)\|^p\}, \forall g(u), g(v) \in K_g, \tag{4.3}$$

which is called the parallelogram-like law and can be used to characterize the uniform Banach spaces involving the arbitrary function;.
If $g = I$,then (4.3) reduces to the parallelogram-like law as:

$$\|v+u\|^p + \mu\|v-u\|^p = 2^{p-1}\{\|u\|^p + \|v\|^p\}, \forall u, v \in K, \tag{4.4}$$

which are known as the parallelogram-like law for the uniform Banach spaces. Xi [36] obtained these characterizations of $p$-uniform convexity and $q$-uniform smoothness of a Banach space via the functionals $\|.\|^p$ and $\|.\|^q$, respectively. Bynum [5] and Chen et al [6, 7] have studied the properties and applications of the parallelogram laws for the Banach spaces in prediction theory and applied sciences.

## 5. Generalized variational-like inequalities

In this section, we introduce and consider a new class of generalized variational-like inequalities, which arises as a optimality condition of differentiable higher order strongly generalized preinvex functions. This is the main motivation of our next result.

**Theorem 5.1.** *Let F be a differentiable higher order strongly generalized preinvex function with modulus $\mu > 0$. If $u \in H : g(u) \in K_{g\eta}$ is the minimum of the function F, then*

$$F(g(v)) - F(g(u)) \geq \mu\|\eta(g(v), g(u))\|^p, \quad \forall g(u), g(v) \in K_{g\eta}. \tag{5.1}$$

*Proof.* Let $u \in H : g(u) \in K_{g\eta}$ be a minimum of the function F. Then

$$F(g(u)) \leq F(g(v)), \forall g(v) \in K_{g\eta}. \tag{5.2}$$

*Since $K_{g\eta}$ is a generalized invex set, so, $\forall g(u), g(v) \in K_{g\eta}, \quad t \in [0, 1]$,*

$$g(v_t) = g(u) + t\eta(g(v), g(u)) \in K_{g\eta}.$$

*Taking $g(v) = g(v_t)$ in (5.2), we have*

$$0 \leq \lim_{t\to 0}\{\frac{F(g(u) + t\eta((g(v), g(u))) - F(g(u))}{t}\} = \langle F'(g(u)), \eta(g(v), g(u))\rangle. \tag{5.3}$$

*Since F is differentiable higher order strongly generalized preinvex function, so*

$$\begin{aligned} F(g(u) + t\eta(g(v), g(u))) &\leq F(g(u)) + t(F(g(v)) - F(g(u))) \\ &\quad -\mu\{t^p(1-t) + t(1-t)^p\}\|\eta(g(v).g(u))\|^p, \forall g(u), g(v) \in K_{g\eta}, \end{aligned}$$

*from which, using (5.3), we have*

$$\begin{aligned} F(g(v)) - F(g(u)) &\geq \lim_{t\to 0}\{\frac{F(g(u) + t\eta(g(v), g(u))) - F(g(u))}{t}\} + \mu\|\eta(g(v), g(u))\|^p \\ &= \langle F'(g(u)), \eta(g(v), g(u))\rangle + \mu\|\eta(g(v), g(u))\|^p, \end{aligned}$$

*the required result (5.1).* □

**Remark:** We would like to mention that, if $u \in H : g(u) \in K_{g\eta}$ satisfies the inequality

$$\langle F'(u), \eta(g(v), u)\rangle + \mu\|\eta(g(v), u)\|^p \geq 0, \quad \forall u, g(v) \in K_{g\eta}, \tag{5.4}$$

then $u \in H : g(u) \in K_{g\eta}$ is the minimum of the function $F$.

The inequality of the type (5.4) is called the higher order strongly generalized variational-like inequality. It is well known that the inequalities of the type (5.4) occur, which do not arises as a minimum of the differentiable functions. These facts motivated us to consider a more generalized variational-like inequality of which (5.4) is a special case.

For given two operators $T, g$, we consider the problem of finding $u \in H : g(u) \in K_{g\eta}$ for a constant $\mu$ such that

$$\langle Tu, \eta(g(v), g(u))\rangle + \mu\|\eta(g(v), g(u))\|^p \geq 0, \quad \forall g(v) \in K_{g\eta}, p > 1, \tag{5.5}$$

which is called the higher order strongly generalized variational-like inequality.
We now discuss some special cases of the problem (5.5).

**(i).** If $Tu = F'(g(u))$, then problem (5.5) is exactly the problem (5.4).

**(ii).** If $\mu = 0$, then (5.5) is equivalent to finding $u \in H : g(u) \in K_{g\eta}$ such that

$$\langle Tu, \eta(g(v), g(u))\rangle \geq 0, \quad \forall g(v) \in K_{g\eta}, \tag{5.6}$$

which is known as the generalized variational-like inequality.
**(iii).** If $\eta(g(v), g(u)) = g(v) - g(u)$, then problem (5.5) reduces to the problem of finding $u \in H : g(u) \in K_g$ such that

$$\langle Tu, g(v) - g(u)\rangle + \mu\|g(v) - g(u)\|^p \geq 0, \quad \forall g(v) \in K_g, p > 1, \tag{5.7}$$

which is called the higher order general variational inequality, introduced and studied by Noor et al. [32].

For suitable and appropriate choice of the parameters $\mu, p$, operators $g$ and bifunction $\eta(.,.)$, one can obtain several new and known classes of variational-like inequalities, variational inequalities and related optimization problems.

We note that the projection method and its variant forms can be used to study the higher order strongly generalized variational-like inequalities (5.5)due to its inherent structure. This fact motivated us to consider the auxiliary principle technique, which is mainly due to Lions and Stampacchia [14] Glowinski et al[10] as developed by Noor [20, 21, 23] and Noor et al[28, 30, 31]. We use this technique to suggest some iterative methods for solving the higher order strongly generalized variational-like inequalities (5.5).

For a given $u \in H : g(u) \in K_{g\eta}$ satisfying (5.5), consider the problem of finding $w \in H : g(w) \in K_{g\eta}$, such that

$$\langle \rho Tw, \eta(g(v), g(w))\rangle + \langle w - u, v - w\rangle + \nu\|\eta(g(v), g(w))\|^p \geq 0, \quad \forall g(v) \in K_{g\eta}, p > 1, \tag{5.8}$$

where $\rho > 0$ is a parameter. The problem (5.8) is called the auxiliary higher order strongly generalized variational-like inequality. It is clear that the relation (5.8) defines a mapping connecting the problems (5.5) and (5.8). We not that, if $g(w(u)) = g(u)$, then $w$ is a solution of problem (5.5). This simple observation enables to suggest an iterative method for solving (5.5).

**Algorithm 5.2.** *For given $u_0$, find the approximate solution $u_{n+1}$ by the scheme*

$$\begin{aligned}\langle \rho Tu_{n+1}, \eta(g(v), g(u_{n+1}))\rangle &+ \langle u_{n+1} - u_n, v - u_{n+1}\rangle \\ &+ \nu\|\eta(g(v), g(u_{n+1}))\|^p \geq 0. \quad \forall g(v) \in K_{g\eta}, p > 1.\end{aligned} \tag{5.9}$$

The Algorithm 5.2 is known as the implicit method. Such type of methods have been studied extensively for various classes of variational-like inequalities. See [20, 21, 23, 28, 30] and the reference therein.
If $\nu = 0$, then Algorithm 5.2 reduces to:

**Algorithm 5.3.** *For given $u_0$, find the approximate solution $u_{n+1}$ by the scheme*

$$\langle \rho Tu_{n+1}, \eta(g(v), g(u_{n+1}))\rangle + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \forall g(v) \in K_{g\eta},$$

which appears to new ones even for solving the generalized variational-like inequalities(5.6).

In to study the convergence analysis of Algorithm 5.2, we need the following concept.

**Definition 5.4.** *The operator $T$ is said to be pseudo $g\eta$-monotone with respect to $\mu\|\eta(g(v), g(u))\|^p, p > 1$, if*

$$\begin{aligned}&\langle \rho Tu, \eta(g(v), g(u))\rangle + \mu\|\eta(g(v), g(u))\|^p \geq 0, \forall g(v) \in K_{g\eta}, p > 1, \\ &\Longrightarrow \\ &-\langle \rho Tv, \eta(g(v), g(u))\rangle - \mu\|\eta(g(v), g(u))\|^p \geq 0, \forall g(v) \in K_{g\eta}, p > 1\end{aligned}$$

If $\mu = 0$, then Definition 5.4 reduces to:

**Definition 5.5.** *The operator $T$ is said to be pseudo $g\eta$-monotone, if*

$$\langle \rho Tu, \eta(g(v), g(u)) \rangle \geq 0, \forall g(v) \in K_{g\eta}$$
$$\Longrightarrow$$
$$-\langle \rho Tv, \eta(g(v), g(u)) \rangle \geq 0, \forall g(v) \in K_{g\eta},$$

which appears to be a new one.
We now study the convergence analysis of Algorithm 5.2.

**Theorem 5.6.** *Let $u \in H : g(u) \in K_{g\eta}$ be a solution of (5.5) and $u_{n+1}$ be the approximate solution obtained from Algorithm 5.2. If $T$ is a pseudo $g\eta$- monotone operator, then*

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2. \tag{5.10}$$

*Proof.* Let $u \in H : g(u) \in\in K_{g\eta}$ be a solution of (5.5), then

$$\langle \rho Tu, \eta(g(v), g(u)) \rangle + \mu \|\eta(g(v), g(u))\|^p \geq 0, \forall g(v) \in K_{g\eta},$$

implies that

$$-\langle \rho Tv, \eta(g(v), g(u)) \rangle - \mu \|\eta(g(v), g(u))\|^p \geq 0, \forall g(v) \in K_{g\eta}, \tag{5.11}$$

Now taking $v = u_{n+1}$ in (5.11), we have

$$-\langle \rho Tu_{n+1}, \eta(g(u_{n+1}), g(u)) \rangle - \mu \|\eta(g(u_{n+1}), g(u))\|^p \geq 0. \tag{5.12}$$

Taking $v = u$ in (5.9), we have

$$\langle \rho Tu_{n+1}, \eta(g(u), g(u_{n+1})) \rangle + \langle u_{n+1} - u_n, v - u_{n+1} \rangle + \nu \|\eta(g(u), g(u_{n+1}))\|^p \geq 0. \quad \forall g(v) \in K, p > 1. \tag{5.13}$$

Combining (5.12) and (5.13), we have

$$\langle u_{n+1} - u_n, u_{n+1} - u \rangle \geq 0.$$

Using the inequality

$$2\langle a, b \rangle = \|a + b\|^2 - \|a\|^2 - \|b\|^2, \forall a, b \in H,$$

we obtain

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2,$$

the required result (5.10). □

**Theorem 5.7.** *Let the operator $T$ be a pseudo $g\eta$-monotone. If $u_{n+1}$ is the approximate solution obtained from Algorithm 5.2 and $u \in H : g(u) \in\in K_{g\eta}$ is the exact solution (5.5), then*

$$\lim_{n \to \infty} u_n = u.$$

*Proof.* Let $u \in H : g(u) \in K_{g\eta}$ be a solution of (5.5). Then, from (5.10), it follows that the sequence $\{\|u - u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. From (5.10), we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leq \|u_0 - u\|^2,$$

from which, it follows that

$$\lim_{n \to \infty} \|u_{n+1} - u_n\| = 0. \tag{5.14}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $u_n$ converge to $\hat{u} \in H$. Replacing $u_n$ by $u_{n_j}$ in (5.9), taking the limit $n_j \to 0$ and from (5.14), we have

$$\langle T\hat{u}, \eta(g(v), g(\hat{u}))\rangle + \mu\|\eta(g(v), g(\hat{u}))\|^p \geq 0, , \quad \forall g(v) \in K_{g\eta}, p > 1.$$

This implies that $\hat{u} \in H : g(\hat{u})K_{g\eta}$ satisfies (5.5) and

$$\|u_{n+1} - u_n\|^2 \leq \|u_n - \hat{u}\|^2.$$

Thus it follows from the above inequality that the sequence $u_n$ has exactly one cluster point $\hat{u}$ and

$$\lim_{n\to\infty} = \hat{u}.$$

□

In order to implement the implicit Algorithm 5.2, one uses the predictor-corrector technique. Consequently, Algorithm 5.2 is equivalent to the following iterative method for solving the higher order strongly generalized variational-like inequality (5.5).

**Algorithm 5.8.** *For a given $u_0 \in K_{g\eta}$, find the approximate solution $u_{n+1}$ by the schemes*

$$\begin{aligned} \langle \rho T u_n, \eta(g(v), g(y_n))\rangle &+ \langle y_n - u_n, v - y_n\rangle + \mu\|\eta(g(v), g(y_n))\|^p \geq 0, \forall g(v) \in K_{g\eta}, p > 1 \\ \langle \rho T y_n, \eta(g(v), g(u_n))\rangle &+ \langle u_n - y_n, v - y_n\rangle + \mu\|\eta(g(v), g(u_n))\|^p \geq 0, \forall g(v) \in K_{g\eta}, p > 1. \end{aligned}$$

Algorithm 5.8 is called the two-step method and appears to be a new one.

Using the auxiliary principle technique, we now suggest an other iterative method for solving the higher order strongly generalized variational-like inequalities and related optimization problems.

For a given $u \in H : g(u) \in K_{g\eta}$ satisfying (5.5), consider the problem of finding $w \in H : h(w) \in K_{g\eta}$, such that

$$\langle \rho T u, \eta(g(v), g(w))\rangle + \langle g(w) - g(u), g(v) - g(w)\rangle + \nu\|\eta(g(v), g(w))\|^p \geq 0, \quad \forall g(v) \in K_{g\eta}, p > 1, \tag{5.15}$$

where $\rho > 0$ is a parameter. The problem (5.15) is called the auxiliary higher order strongly generalized variational-like inequality. It is clear that the relation (5.15) defines a mapping connecting the problems (5.5) and (5.15). We note that, if $g(w(u)) = g(u)$, then $w$ is a solution of problem (5.5). This simple observation enables us to suggest the following iterative method for solving (5.5).

**Algorithm 5.9.** *For given $u_0$, find the approximate solution $u_{n+1}$ by the scheme*

$$\begin{aligned} &\langle \rho T u_n, \eta(g(v), g(u_{n+1}))\rangle + \langle g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle \\ &\qquad + \nu\|\eta(g(v), g(u_{n+1}))\|^p \geq 0, \forall g(v) \in K_{g\eta}, p > 1, \end{aligned} \tag{5.16}$$

which is an explicit algorithm.

Using the auxiliary principle technique, on can suggest several iterative methods for solving the higher order strongly generalized variational-like inequalities and related optimization problems. We have only given some glimpse of the higher order strongly generalized variational-like inequalities. It is an interesting problem to explore the applications of such type of generalized variational-like inequalities in various fields of pure and applied sciences.

## Conclusion

In this paper, we have introduced and studied a new class of convex functions, which is called higher order strongly generalized preinvex function. It is shown that several new classes of strongly convex functions can be obtained as special cases of these higher order strongly generalized preinvex functions. We have studied the basic properties of these functions. New parallelogram laws for uniformly Banach spaces have been derived as applications of the higher order strongly generalized preinvex functions. It is an open problem to study the applications of these parallelogram laws. We have also considered a new class of higher order strongly generalized variational-like inequalities. Using the auxiliary principle technique, an implicit iterative method is suggested for finding the approximate solution of generalized variational-like inequalities. Using the pseudo-monotonicity of the operator, convergence criteria is discussed. Some special cases are considered as application of the main results. The ideas and techniques of this paper may be starting point for further research.

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