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Transmission Problem For The Sturm–Liouville Equation Involving a Retarded Argument

Erdoğan Şen[a]

[a]*Tekirdag Namik Kemal University, Kampus str. 1, TR-59030, Tekirdag, Turkey*

Abstract. In this work, spectral properties of a discontinuous boundary-value problem with retarded argument which contains a spectral parameter in the boundary conditions and in the transmission conditions at the point of discontinuity are investigated. To this aim, asymptotic formulas for the eigenvalues and eigenfunctions are obtained.

1. Introduction

Delay differential equations, especially differential equations with retarded argument arise in many areas of mathematical modelling: for example, population dynamics (taking into account the gestation times), infectious diseases (accounting for the incubation periods), physiological and pharmaceutical kinetics (modelling, for example, the body's reaction to CO_2, etc. in circulating blood) and chemical kinetics (such as mixing reactants), the navigational control of ships and aircraft and more general control problems (see [1] and the references therein).

Boundary value problems for ordinary differential equations with a parameter in the equation and/or in the boundary conditions were studied by many authors in various statements (e.g., see [2-10]).

Boundary value problems for differential equations of the second order with retarded argument were studied in [11-20].

The present article is devoted to studying the spectral properties of the eigenvalues and eigenfunctions of a boundary value problem with retarded argument. In the present considered problem's boundary conditions and transmission conditions involves a spectral parameter. The main result of the present paper is Theorem 3.3 and Theorem 3.4 on asymptotic formulas for eigenvalues and eigenfunctions.

The operator under consideration is defined by the differential expression

$$u''(t) + q(t)u(t - \Delta(t)) + \rho u(t) = 0 \tag{1}$$

on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, with boundary conditions

$$u(0)\sqrt{\rho}\cos\alpha + u'(0)\sin\alpha = 0, \tag{2}$$

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Email address: esen@nku.edu.tr (Erdoğan Şen)

$$u(\pi)\sqrt{\rho}\cos\beta + u'(\pi)\sin\beta = 0, \tag{3}$$

and transmission conditions

$$\frac{1}{\delta}u(\frac{\pi}{2}-0) = \sqrt[3]{\rho}u(\frac{\pi}{2}+0), \tag{4}$$

$$\frac{1}{\delta}u'(\frac{\pi}{2}-0) = \sqrt[3]{\rho}u'(\frac{\pi}{2}+0), \tag{5}$$

where the real-valued function $q(t)$ is continuous in $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ and has a finite limit $q(\frac{\pi}{2}\pm 0) = \lim_{t\to\frac{\pi}{2}\pm 0} q(t)$, the real valued function $\Delta(t)\geq 0$ is continuous in $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ and has a finite limit $\Delta(\frac{\pi}{2}\pm 0) = \lim_{t\to\frac{\pi}{2}\pm 0}\Delta(t)$, $t-\Delta(t)\geq 0$, if $t\in\left[0,\frac{\pi}{2}\right)$; $t-\Delta(t)\geq\frac{\pi}{2}$, if $t\in\left(\frac{\pi}{2},\pi\right]$; ρ is a real spectral parameter and $\delta\neq 0$ is an arbitrary real number.

It must be also noted that some applications of second-order boundary value problems which contains a spectral parameter in transmission conditions can be found in [21].

Let $w_1(t,\rho)$ be a solution of Eq. (1) on $\left[0,\frac{\pi}{2}\right]$, satisfying the initial conditions

$$w_1(0,\rho) = \sin\alpha, w_1'(0,\rho) = -\sqrt{\rho}\cos\alpha \tag{6}$$

The conditions (6) define a unique solution of Eq. (1) on $\left[0,\frac{\pi}{2}\right]$([12], p. 12).

After defining above solution, we shall define the solution $w_2(t,\rho)$ of Eq. (1) on $\left[\frac{\pi}{2},\pi\right]$ by means of the solution $w_1(t,\rho)$ by the initial conditions

$$w_2\left(\frac{\pi}{2},\rho\right) = \rho^{-1/3}\delta^{-1}w_1\left(\frac{\pi}{2},\rho\right), \quad w_2'(\frac{\pi}{2},\rho) = \rho^{-1/3}\delta^{-1}w_1'(\frac{\pi}{2},\rho). \tag{7}$$

The conditions (7) are defined as a unique solution of Eq. (1) on $\left[\frac{\pi}{2},\pi\right]$.

Consequently, the function $w(t,\rho)$ is defined on $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ by the equality

$$w(t,\rho) = \begin{cases} w_1(t,\rho), & t\in[0,\frac{\pi}{2}), \\ w_2(t,\rho), & t\in(\frac{\pi}{2},\pi] \end{cases}$$

is a such solution of the Eq. (1) on $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$; which satisfies one of the boundary conditions and both transmission conditions.

Lemma 1.1. *Let $w(t,\rho)$ be a solution of Eq.(1) and $\rho>0$. Then the following integral equations hold:*

$$w_1(t,\rho) = \sin(\alpha-st) - \frac{1}{s}\int_0^t q(\tau)\sin s(t-\tau)\,w_1(\tau-\Delta(\tau),\rho)\,d\tau \quad \left(s=\sqrt{\rho},\rho>0\right), \tag{8}$$

$$\begin{aligned} w_2(t,\rho) = {} & \frac{1}{s^{2/3}\delta}w_1\left(\frac{\pi}{2},\rho\right)\cos s\left(t-\frac{\pi}{2}\right) + \frac{w_1'\left(\frac{\pi}{2},\rho\right)}{s^{5/3}\delta}\sin s\left(t-\frac{\pi}{2}\right) \\ & - \frac{1}{s}\int_{\pi/2}^t q(\tau)\sin s(t-\tau)\,w_2(\tau-\Delta(\tau),\rho)\,d\tau \quad \left(s=\sqrt{\rho},\rho>0\right). \end{aligned} \tag{9}$$

Proof. To prove this, it is enough to substitute $-s^2w_1(\tau,\rho) - w_1''(\tau,\rho)$ and $-s^2w_2(\tau,\rho) - w_2''(\tau,\rho)$ instead of $-q(\tau)w_1(\tau-\Delta(\tau),\rho)$ and $-q(\tau)w_2(\tau-\Delta(\tau),\rho)$ in the integrals in (8) and (9) respectively and integrate by parts twice. □

2. An existence theorem

In this chapter, we show that the characteristic function of the problem (1)-(5) has an infinite set of roots.

Theorem 2.1. *The problem (1)-(5) can have only simple eigenvalues.*

Proof. Let $\widetilde{\rho}$ be an eigenvalue of the problem (1)-(5) and

$$\widetilde{u}(t,\widetilde{\rho}) = \begin{cases} \widetilde{u}_1(t,\widetilde{\rho}), & t \in [0, \frac{\pi}{2}), \\ \widetilde{u}_2(t,\widetilde{\rho}), & t \in (\frac{\pi}{2}, \pi] \end{cases}$$

be a corresponding eigenfunction. Then from (2) and (6) it follows that the determinant

$$W\left[\widetilde{u}_1(0,\widetilde{\rho}), w_1(0,\widetilde{\rho})\right] = \begin{vmatrix} \widetilde{u}_1(0,\widetilde{\rho}) & \sin\alpha \\ \widetilde{u}_1'(0,\widetilde{\rho}) & -\sqrt{\rho}\cos\alpha \end{vmatrix} = 0,$$

and the functions $\widetilde{u}_1(t,\widetilde{\rho})$ and $w_1(t,\widetilde{\rho})$ are linearly dependent on $\left[0, \frac{\pi}{2}\right]$. We can also prove that the functions $\widetilde{u}_2(t,\widetilde{\rho})$ and $w_2(t,\widetilde{\rho})$ are linearly dependent on $\left[\frac{\pi}{2}, \pi\right]$. Hence

$$\widetilde{u}_j(t,\widetilde{\rho}) = R_j w_j(t,\widetilde{\rho}) \quad (j = 1,2) \tag{10}$$

for some $R_1 \neq 0$ and $R_2 \neq 0$. We must show that $R_1 = R_2$. Suppose that $R_1 \neq R_2$. From the equalities (4) and (10), we have

$$\begin{aligned} \widetilde{u}(\frac{\pi}{2} - 0, \widetilde{\rho}) - \sqrt[3]{\widetilde{\rho}}\delta\widetilde{u}(\frac{\pi}{2} + 0, \widetilde{\rho}) &= \widetilde{u}_1(\frac{\pi}{2}, \widetilde{\rho}) - \sqrt[3]{\widetilde{\rho}}\delta\widetilde{u}_2(\frac{\pi}{2}, \widetilde{\rho}) \\ &= R_1 w_1(\frac{\pi}{2}, \widetilde{\rho}) - \sqrt[3]{\widetilde{\rho}}\delta R_2 w_2(\frac{\pi}{2}, \widetilde{\rho}) \\ &= \sqrt[3]{\widetilde{\rho}}\delta R_1 w_2(\frac{\pi}{2}, \widetilde{\rho}) - \sqrt[3]{\widetilde{\rho}}\delta R_2 w_2(\frac{\pi}{2}, \widetilde{\rho}) \\ &= \sqrt[3]{\widetilde{\rho}}\delta\left(R_1 - R_2\right) w_2(\frac{\pi}{2}, \widetilde{\rho}) = 0. \end{aligned}$$

Since $\delta_1\left(R_1 - R_2\right) \neq 0$ it follows that

$$w_2\left(\frac{\pi}{2}, \widetilde{\rho}\right) = 0. \tag{11}$$

By the same procedure, from equality (5), we can derive that

$$w_2'\left(\frac{\pi}{2}, \widetilde{\rho}\right) = 0. \tag{12}$$

From the fact that $w_2(t,\widetilde{\rho})$ is a solution of the differential Eq. (1) on $\left[\frac{\pi}{2}, \pi\right]$ and satisfies the initial conditions (11) and (12) it follows that $w_1(t,\widetilde{\rho}) = 0$ identically on $\left[\frac{\pi}{2}, \pi\right]$ (see [12, p. 12, Theorem 1.2.1]).

By using this method, we may also find

$$w_1\left(\frac{\pi}{2}, \widetilde{\rho}\right) = w_1'\left(\frac{\pi}{2}, \widetilde{\rho}\right) = 0.$$

From the latter discussions of $w_2(t,\widetilde{\rho})$ it follows that $w_1(t,\widetilde{\rho}) = 0$ identically on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. But this contradicts (6), thus completing the proof. □

The function $w(t,\rho)$ defined in introduction is a nontrivial solution of Eq. (1) satisfying conditions (2), (4) and (5). Putting $w(t,\rho)$ into (3), we get the characteristic equation

$$H(\rho) \equiv \sqrt{\rho} w(\pi,\rho)\cos\beta + w'(\pi,\rho)\sin\beta = 0. \tag{13}$$

By Theorem 2.1, the set of eigenvalues of boundary-value problem (1)-(5) coincides with the set of real roots of Eq. (13). Let $q_1 = \int_0^{\pi/2} |q(\tau)| d\tau$ and $q_2 = \int_{\pi/2}^{\pi} \left|q(\tau)\right| d\tau$.

Lemma 2.2. (1) *Let $\rho \geq 4q_1^2$. Then for the solution $w_1(t,\rho)$ of Eq. (8), the following inequality holds*:

$$\left|w_1(t,\rho)\right| \leq 2, \quad t \in \left[0, \frac{\pi}{2}\right]. \tag{14}$$

(2) *Let $\rho \geq \max\left\{4q_1^2, 4q_2^2\right\}$. Then for the solution $w_2(t,\rho)$ of Eq. (9), the following inequality holds*:

$$\left|w_2(t,\rho)\right| \leq \frac{4\sqrt{2}}{\sqrt[3]{q_1^2\delta}}, \quad t \in \left[\frac{\pi}{2}, \pi\right]. \tag{15}$$

Proof. Let $C_{1\rho} = \max_{\left[0,\frac{\pi}{2}\right]} \left|w_1(t,\rho)\right|$. Then from (8), for every $\rho > 0$, the following inequality holds:

$$C_{1\rho} \leq \left|\sin(\alpha - st)\right| + \frac{1}{s} C_{1\rho} q_1 \leq 1 + \frac{1}{s} C_{1\rho} q_1.$$

If $s \geq 2q_1$ then we get (14). Differentiating (8) with respect to t, we have

$$w_1'(t,\rho) = -s\cos(\alpha - st) - \int_0^t q(\tau)\cos s(t-\tau)\, w_1(\tau - \Delta(\tau), \rho) d\tau \tag{16}$$

From (16) and (14), for $s \geq 2q_1$, the following inequalities hold:

$$\left|w_1'(t,\rho)\right| \leq s + 2q_1 \leq s + s = 2s,$$

$$\frac{\left|w_1'(t,\rho)\right|}{s} \leq 2. \tag{17}$$

Let $C_{2\rho} = \max_{\left[\frac{\pi}{2},\pi\right]} \left|w_2(t,\rho)\right|$. Then from (9), (14) and (17), for $s \geq 2q_1$ and $s \geq 2q_2$, the following inequalities hold:

$$C_{2\rho} \leq \frac{4}{\sqrt[3]{4q_1^2\delta}} + \frac{1}{s} C_{2\rho} q_2,$$

$$C_{2\rho} \leq \frac{4\sqrt{2}}{\sqrt[3]{q_1^2\delta}}.$$

Hence, if $\rho \geq \max\left\{4q_1^2, 4q_2^2\right\}$, we get (15). □

Theorem 2.3. *The problem (1)-(5) has an infinite set of positive eigenvalues.*

Proof. Differentiating (9) with respect to t, we get

$$w_2'(t,\rho) = -\frac{\sqrt[3]{s}}{\delta} w_1\left(\frac{\pi}{2},\rho\right)\sin s\left(t-\frac{\pi}{2}\right) + \frac{w_1'\left(\frac{\pi}{2},\rho\right)}{\sqrt[3]{s^2}\delta}\cos s\left(t-\frac{\pi}{2}\right)$$
$$-\int_{\pi/2}^{t} q(\tau)\cos s\left(t-\tau\right) w_2(\tau-\Delta\left(\tau\right),\rho)d\tau \quad (s=\sqrt{\rho},\rho>0). \tag{18}$$

From (8), (9), (13), (16) and (18), we get

$$s\left[\frac{1}{s^{2/3}\delta}\left(\sin\left(\alpha-\frac{s\pi}{2}\right)-\frac{1}{s}\int_0^{\frac{\pi}{2}} q(\tau)\sin s(\frac{\pi}{2}-\tau)w_1(\tau-\Delta(\tau),\rho)d\tau\right)\cos\frac{s\pi}{2}\right.$$
$$-\frac{1}{s^{5/3}\delta}\left(s\cos\left(\alpha-\frac{s\pi}{2}\right)+\int_0^{\frac{\pi}{2}} q(\tau)\cos s(\frac{\pi}{2}-\tau)w_1(\tau-\Delta(\tau),\rho)d\tau\right)\sin\frac{s\pi}{2}$$
$$\left.-\frac{1}{s}\int_{\pi/2}^{\pi} q(\tau)\sin s(\pi-\tau)w_2(\tau-\Delta(\tau),\rho)d\tau\right]\cos\beta$$
$$+\left[-\frac{\sqrt[3]{s}}{\delta}\left(\sin\left(\alpha-\frac{s\pi}{2}\right)-\frac{1}{s}\int_0^{\frac{\pi}{2}} q(\tau)\sin s(\frac{\pi}{2}-\tau)w_1(\tau-\Delta(\tau),\rho)d\tau\right)\sin\frac{s\pi}{2}\right.$$
$$-\frac{1}{\sqrt[3]{s^2}\delta}\left(s\cos\left(\alpha-\frac{s\pi}{2}\right)+\int_0^{\frac{\pi}{2}} q(\tau)\cos s(\frac{\pi}{2}-\tau)w_1(\tau-\Delta(\tau),\rho)d\tau\right)\cos\frac{s\pi}{2}$$
$$\left.-\int_{\frac{\pi}{2}}^{\pi} q(\tau)\cos s(\pi-\tau)w_2(\tau-\Delta(\tau),\rho)d\tau\right]\sin\beta = 0. \tag{19}$$

There are four possible cases:

1. $\sin\alpha \neq 0, \sin\beta \neq 0$;
2. $\sin\alpha \neq 0, \sin\beta = 0$;
3. $\sin\alpha = 0, \sin\beta \neq 0$;
4. $\sin\alpha = 0, \sin\beta = 0$.

In this paper, we shall consider only case 1. The other cases may be considered analogically. Let ρ be sufficiently large. Then, by (14) and (15), the Eq. (19) may be rewritten in the form

$$\frac{\sqrt[3]{s}}{\delta}\left[\sin\left(\alpha-s\pi\right)\cos\beta-\cos\left(\alpha-s\pi\right)\sin\beta\right]+O(1)=0$$

or

$$\sqrt[3]{s}\sin\left(s\pi+\beta-\alpha\right)+O(1)=0. \tag{20}$$

Obviously, for large s, Eq. (20) has an infinite set of roots. Thus the theorem is proved. □

3. Asymptotic equalities for eigenvalues and eigenfunctions

Now, we begin to study asymptotic properties of eigenvalues and eigenfunctions. In the following we shall assume that s is sufficiently large. From (8) and (14), we get

$$w_1(t,\rho) = O(1) \quad \text{on} \quad [0, \frac{\pi}{2}]. \tag{21}$$

From (9) and (15), we get

$$w_2(t,\rho) = O(1) \quad \text{on} \quad [\frac{\pi}{2}, \pi]. \tag{22}$$

The existence and continuity of the derivatives $w'_{1s}(t,\rho)$ for $0 \le t \le \frac{\pi}{2}$, $|\rho| < \infty$, and $w'_{2s}(t,\rho)$ for $\frac{\pi}{2} \le t \le \pi$, $|\rho| < \infty$, follow from Theorem 1.4.1 in [12].

Lemma 3.1. *In case 1, the following asymptotic equalities hold:*

$$w'_{1s}(t,\rho) = O(1), \quad t \in [0, \frac{\pi}{2}], \tag{23}$$

$$w'_{2s}(t,\rho) = O(1), \quad t \in [\frac{\pi}{2}, \pi]. \tag{24}$$

Proof. By differentiation of (8) with respect to s, we get, by (21)

$$w'_{1s}(t,\rho) = -\frac{1}{s}\int_0^t q(\tau)\sin s(t-\tau) w'_{1s}\left(\tau - \Delta(\tau), \rho\right) + Z(t,\rho), \quad \left|Z(t,\rho)\right| \le Z_0. \tag{25}$$

Let $D_\rho = \max_{[0,\frac{\pi}{2}]}\left|w'_{1s}(t,\rho)\right|$. Then the existance of D_ρ follows from continuity of derivation for $t \in \left[0, \frac{\pi}{2}\right]$. From (25), we have

$$D_\rho \le \frac{1}{s} q_1 D_\rho + Z_0.$$

Now, let $s \ge 2q_1$. Then $D_\rho \le 2Z_0$ and the validity of the asymptotic formula (23) follows. Formula (24) may be proved analogically. □

Theorem 3.2. *Let n be a natural number. For each sufficiently large n, in case 1, there is exactly one eigenvalue of the problem (1)-(5) near $\left(n + \frac{\alpha-\beta}{\pi}\right)^2$.*

Proof. We consider the expression which is denoted by $O(1)$ in the Eq. (20):

$$-\frac{1}{\sqrt[3]{s^2}\delta}\int_0^{\frac{\pi}{2}} \sin\left(s(\pi-\tau)+\beta\right) q(\tau)\, w_1\left(\tau - \Delta(\tau), \rho\right) d\tau - \int_{\frac{\pi}{2}}^{\pi} \sin\left(s(\pi-\tau)+\beta\right) q(\tau)\, w_2\left(\tau - \Delta(\tau), \rho\right) d\tau.$$

If formulas (21)-(23) are taken into consideration, it can be shown by differentiation with respect to s that for large s this expression has bounded derivative. It is obvious that for large s, the roots of Eq. (20) are situated close to entire numbers. We shall show that, for large n, only one root of (20) lies near to each $n + \frac{\alpha-\beta}{\pi}$. Let us consider the function $J(s) = \sqrt[3]{s}\sin\left(s\pi + \beta - \alpha\right) + O(1)$. Its derivative, which has the form $J'(s) = \frac{1}{3\sqrt[3]{s^2}}\sin\left(s\pi+\beta-\alpha\right) + \sqrt[3]{s}\pi\cos\left(s\pi+\beta-\alpha\right) + O(1)$, does not vanish for s close to $n + \frac{\alpha-\beta}{\pi}$ for sufficiently large n. Thus our assertion follows by Rolle's Theorem. □

Let n be sufficiently large. In what follows we shall denote by $\rho_n = s_n^2$ the eigenvalue of the problem (1)-(5) situated near $\left(n + \frac{\alpha-\beta}{\pi}\right)^2$. Set $s_n = n + \frac{\alpha-\beta}{\pi} + \delta_n$. Then, from (20), it follows that $\delta_n = O\left(\frac{1}{n^{1/3}}\right)$. Consequently

$$s_n = n + \frac{\alpha-\beta}{\pi} + O\left(\frac{1}{n^{1/3}}\right). \tag{26}$$

The formula (26) make it possible to obtain asymptotic expressions for eigenfunction of the problem (1)-(5). From (8), (16) and (21), we get

$$w_1(t,\ \rho) = \sin(\alpha - st) + O\left(\frac{1}{s}\right), \tag{27}$$

$$w_1^{'}(t,\rho) = -s\cos(\alpha - st) + O(1). \tag{28}$$

From (9), (22), (27) and (28), we get

$$w_2(t,\rho) = \frac{\sin(\alpha - st)}{s^{2/3}\delta} + O\left(\frac{1}{s}\right). \tag{29}$$

By putting (26) in the (27) and (29), we derive that

$$u_{1n} = w_1(t,\rho_n) = \sin\left(\alpha + \frac{(\alpha-\beta)t}{\pi} - nt\right) + O\left(\frac{1}{n^{1/3}}\right),$$

$$u_{2n} = w_2(t,\rho_n) = \frac{\sin\left(\alpha + \frac{(\alpha-\beta)t}{\pi} - nt\right)}{\delta\left(n + \frac{\alpha-\beta}{\pi}\right)^{2/3}} + O\left(\frac{1}{n}\right).$$

Hence the eigenfunctions $u_n(t)$ have the following asymptotic representation:

$$u_n(t) = \begin{cases} \sin\left(\alpha + \frac{(\alpha-\beta)t}{\pi} - nt\right) + O\left(\frac{1}{n^{1/3}}\right) & \text{for} \quad t \in [0, \frac{\pi}{2}), \\ \frac{\sin\left(\alpha + \frac{(\alpha-\beta)t}{\pi} - nt\right)}{\delta\left(n + \frac{\alpha-\beta}{\pi}\right)^{2/3}} + O\left(\frac{1}{n}\right) & \text{for} \quad t \in (\frac{\pi}{2}, \pi]. \end{cases}$$

Under some additional conditions the more exact asymptotic formulas which depend upon the retardation may be obtained. Let us assume that the following conditions are fulfilled:

a) The derivatives $q'(t)$ and $\Delta''(t)$ exist and are bounded in $[0, \frac{\pi}{2}) \bigcup (\frac{\pi}{2}, \pi]$ and have finite limits $q'(\frac{\pi}{2} \pm 0) = \lim\limits_{t \to \frac{\pi}{2} \pm 0} q'(t)$ and $\Delta''(\frac{\pi}{2} \pm 0) = \lim\limits_{t \to \frac{\pi}{2} \pm 0} \Delta''(t)$, respectively.

b) $\Delta'(t) \leq 1$ in $[0, \frac{\pi}{2}) \bigcup (\frac{\pi}{2}, \pi]$, $\Delta(0) = 0$ and $\lim\limits_{t \to \frac{\pi}{2} + 0} \Delta(t) = 0$.

By using b), we have

$$t - \Delta(t) \geq 0,\ t \in [0, \frac{\pi}{2}); \tag{30}$$

$$t - \Delta(t) \geq \frac{\pi}{2},\ t \in (\frac{\pi}{2}, \pi]. \tag{31}$$

From (27), (29), (30) and (31), we have

$$w_1(\tau - \Delta(\tau), \rho) = \sin(\alpha - s(\tau - \Delta(\tau))) + O\left(\frac{1}{s}\right), \tag{32}$$

$$w_2(\tau - \Delta(\tau), \rho) = \frac{\sin(\alpha - s(\tau - \Delta(\tau)))}{s^{2/3}\delta} + O\left(\frac{1}{s}\right). \tag{33}$$

Putting these expressions into (19), we have

$$\frac{s^{1/3}}{\delta}\sin(\alpha-\beta-s\pi)+\frac{1}{2s^{2/3}\delta}\int_0^{\frac{\pi}{2}}\Big[\cos(\beta-\alpha+s\pi)\cos s\Delta(\tau)+\sin(\beta-\alpha+s\pi)\sin s\Delta(\tau)$$
$$-\cos(\beta-\alpha+s\pi)\cos s(2\tau-\Delta(\tau))-\sin(\beta-\alpha+s\pi)\sin s(2\tau-\Delta(\tau))\Big]q(\tau)d\tau$$
$$-\frac{1}{2s^{2/3}\delta}\int_{\frac{\pi}{2}}^{\pi}\Big[\cos(\beta-\alpha+s\pi)\cos s\Delta(\tau)+\sin(\beta-\alpha+s\pi)\sin s\Delta(\tau)$$
$$-\cos(\beta-\alpha+s\pi)\cos s(2\tau-\Delta(\tau))-\sin(\beta-\alpha+s\pi)\sin s(2\tau-\Delta(\tau))\Big]q(\tau)d\tau+O\Big(\frac{1}{s}\Big)=0. \tag{34}$$

Let

$$\begin{cases} K(t,s,\Delta(\tau))=\frac{1}{2\delta}\int_0^t q(\tau)\sin s\Delta(\tau)d\tau, \\ L(t,s,\Delta(\tau))=\frac{1}{2\delta}\int_0^t q(\tau)\cos s\Delta(\tau)d\tau. \end{cases} \tag{35}$$

It is obviously that these functions are bounded for $0\le t\le\pi,\ 0<s<+\infty$.

Under the conditions a) and b) the following formulas

$$\int_0^t q(\tau)\cos s(2\tau-\Delta(\tau))d\tau=O\Big(\frac{1}{s}\Big),\quad \int_0^t q(\tau)\sin s(2\tau-\Delta(\tau))d\tau=O\Big(\frac{1}{s}\Big) \tag{36}$$

can be proved by the same technique in Lemma 3.3.3 in [12]. From (34), (35) and (36), we have

$$s\sin(\beta-\alpha+s\pi)-\cos(\beta-\alpha+s\pi)L(\pi,s,\Delta(\tau))-\sin(\beta-\alpha+s\pi)K(\pi,s,\Delta(\tau))+O\Big(\frac{1}{s^{1/3}}\Big)=0.$$

Therefore, we obtain

$$\tan(\beta-\alpha+s\pi)=\frac{L(\pi,s,\Delta(\tau))}{s}+O\Big(\frac{1}{s^{4/3}}\Big).$$

Again if we take $s_n=n+\frac{\alpha-\beta}{\pi}+\delta_n$, then from (26),

$$\tan\Big(\beta-\alpha+\Big(n+\frac{\alpha-\beta}{\pi}+\delta_n\Big)\pi\Big)=\tan(n+\delta_n)\pi=\tan\delta_n\pi=\frac{L\Big(\pi,n+\frac{\alpha-\beta}{\pi},\Delta(\tau)\Big)}{\pi n}+O\Big(\frac{1}{n^{4/3}}\Big).$$

Hence, for large n, it follows that

$$\delta_n=\frac{L\Big(\pi,n+\frac{\alpha-\beta}{\pi},\Delta(\tau)\Big)}{n\pi}+O\Big(\frac{1}{n^{4/3}}\Big)$$

and finally

$$s_n=n+\frac{\alpha-\beta}{\pi}+\frac{L\Big(\pi,n+\frac{\alpha-\beta}{\pi},\Delta(\tau)\Big)}{n\pi}+O\Big(\frac{1}{n^{4/3}}\Big). \tag{37}$$

Thus, we have proven the following theorem.

Theorem 3.3. *If conditions a) and b) are satisfied, then the positive eigenvalues $\rho_n = s_n^2$ of the problem (1)-(5) have the asymptotic formula (37) for $n \to \infty$.*

We now may obtain a sharper asymptotic formula for the eigenfunctions. From (8) and (32), we have

$$w_1(t,\rho) = \sin(\alpha - st) - \frac{1}{s}\int_0^t q(\tau)\sin s(t-\tau)\sin s(\alpha - s(\tau - \Delta(\tau)))\,d\tau + O\left(\frac{1}{s^2}\right).$$

Thus, from (35) and (36), it follows that

$$w_1(t,\rho) = \sin(\alpha - st)\left[1 + \frac{K(t,\, s,\, \Delta(\tau))}{s}\right] - \cos(\alpha - st)\,\frac{L(t,s,\Delta(\tau))}{s} + O\left(\frac{1}{s^2}\right). \tag{38}$$

Replacing s by s_n and using (37), we have

$$u_{1n}(t) = w_1(t,\rho_n) = \sin\left(\alpha - \left(n + \frac{\alpha-\beta}{\pi}\right)t\right)\left[1 + \frac{K\left(t, n + \frac{\alpha-\beta}{\pi}, \Delta(\tau)\right)}{n}\right]$$

$$-\frac{\cos\left(\alpha - \left(n + \frac{\alpha-\beta}{\pi}\right)t\right)}{n\pi}\left[L\left(\pi, n + \frac{\alpha-\beta}{\pi}, \Delta(\tau)\right) - \pi L\left(t, n + \frac{\alpha-\beta}{\pi}, \Delta(\tau)\right)\right] + O\left(\frac{1}{n^2}\right). \tag{39}$$

From (16), (32) and (35), for $t \in \left[0, \frac{\pi}{2}\right]$, it follows that

$$\frac{w_1'(t,\rho)}{s} = -\cos(\alpha - st)\left[1 + \frac{K(t,\, s,\, \Delta(\tau))}{s}\right] - \frac{L(t,s,\Delta(\tau))\sin(\alpha - st)}{s} + O\left(\frac{1}{s^2}\right). \tag{40}$$

From (9), (33), (36), (38) and (40), for $t \in \left[\frac{\pi}{2}, \pi\right]$, we have

$$\begin{aligned} w_2(t,\rho) &= \frac{1}{s^{2/3}\delta}\left\{\sin\left(\alpha - \frac{s\pi}{2}\right)\left[1 + \frac{K\left(\frac{\pi}{2}, s, \Delta(\tau)\right)}{s}\right] - \frac{L\left(\frac{\pi}{2}, s, \Delta(\tau)\right)\cos\left(\alpha - \frac{s\pi}{2}\right)}{s} + O\left(\frac{1}{s^{8/3}}\right)\right\} \\ &\quad \times \cos s\left(t - \frac{\pi}{2}\right) - \frac{1}{s^{2/3}\delta}\left\{\cos\left(\alpha - \frac{s\pi}{2}\right)\left[1 + \frac{K\left(\frac{\pi}{2}, s, \Delta(\tau)\right)}{s}\right]\right. \\ &\quad \left. + \frac{L\left(\frac{\pi}{2}, s, \Delta(\tau)\right)\sin\left(\alpha - \frac{s\pi}{2}\right)}{s} + O\left(\frac{1}{s^{8/3}}\right)\right\}\sin s\left(t - \frac{\pi}{2}\right) \\ &\quad - \frac{1}{s^{5/3}\delta}\int_{\pi/2}^t q(\tau)\sin s(t-\tau)\sin(\alpha - s(\tau - \Delta(\tau)))\,d\tau + O\left(\frac{1}{s^2}\right) \\ &= \frac{\sin(\alpha - st)}{s^{2/3}\delta}\left[1 + \frac{K(t,s,\Delta(\tau))}{s}\right] - \frac{L(t,s,\Delta(\tau))\cos(\alpha - st)}{s^{5/3}\delta} + O\left(\frac{1}{s^2}\right). \end{aligned}$$

Now, replacing s by s_n and using (37), we have

$$u_{2n}(t) = w_2(t,\rho_n) = \frac{\sin\left(\alpha - \left(n + \frac{\alpha-\beta}{\pi}\right)t\right)}{n^{2/3}\delta}\left[1 + \frac{K\left(t, n + \frac{\alpha-\beta}{\pi}, \Delta(\tau)\right)}{n}\right]$$

$$+\frac{\cos\left(\alpha - \left(n + \frac{\alpha-\beta}{\pi}\right)t\right)}{n^{5/3}\delta\pi}\left[L\left(\pi, n + \frac{\alpha-\beta}{\pi}, \Delta(\tau)\right) - \pi L\left(t, n + \frac{\alpha-\beta}{\pi}, \Delta(\tau)\right)\right] + O\left(\frac{1}{n^2}\right). \tag{41}$$

Thus, we have proven the following theorem.

Theorem 3.4. *If conditions a) and b) are satisfied, then the eigenfunctions $u_n(t)$ of the problem (1)-(5) have the following asymptotic representation for $n \to \infty$:*

$$u_n(t) = \begin{cases} u_{1n}(t) \quad for \quad t \in [0, \frac{\pi}{2}), \\ \\ u_{2n}(t) \quad for \quad t \in (\frac{\pi}{2}, \pi], \end{cases}$$

where $u_{1n}(t)$ and $u_{2n}(t)$ defined as in (39) and (41) respectively.

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